Exercise sheet 10.

Name

Exercise group (tutor's name)

Deadline: **Friday, 10.1.2025, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Apply matrix-stability of van Daele K-theory and exercise 3 of the previous sheet for

$$A_n := \hat{\mathbb{M}}_4\mathbb{M}_k(\mathbb{M}_n(B)^+), \qquad A := \hat{\mathbb{M}}_4\mathbb{M}_k(\mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B)^+$$

for a $\mathbb{Z}/2$-graded C*-algebra B and $k \in \mathbb{N}_{\geq 1}$ to deduce that a corner embedding $B \to \mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B$ induces an isomorphism $\mathrm{K}(B) \cong \mathrm{K}(\mathbb{K}(\ell^2\mathbb{N}) \bar{\otimes} B)$. Here $\bar{\otimes}$ denotes any C*-completion of the algebraic tensor product. Actually, this C*-completion is unique, but this is not relevant to the exercise. You may assume that $A_n \subseteq A$ is a C*-subalgebra as needed for exercise 3 of last exercise sheet. Here $\hat{\mathbb{M}}_4$ denotes the algebra of 4×4-matrices with the grading described in exercise 1 of sheet 8.

On the rest of this sheet, let H be a functor from $\mathbb{Z}/2$-graded Banach algebras to Abelian groups that is exact and homotopy invariant, such as van Daele's K-theory. Let $H_k(A) := H(\mathrm{C}_0(\mathbb{R}^k, A))$; these are the groups that appear in the long exact sequence for H applied to an extension.

Exercise 2. Let $I \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} Q$ be an extension of $\mathbb{Z}/2$-graded Banach algebras that that splits by a homomorphism $s\colon Q \to E$. Show that the boundary map $H(\mathrm{C}_0(\mathbb{R}, Q)) \to H(I)$ has to vanish and deduce that the following map is invertible:

$$(H(i), H(s))\colon H(I) \oplus H(Q) \xrightarrow{\cong} H(E).$$

Briefly, exact homotopy invariant functors are split exact.

Exercise 3. Let $\mathbb{T} := \{z \in \mathbb{C},\ |z| = 1\}$ with the trivial grading. Let $\mathbb{K}$ denote $\mathbb{R}$ or $\mathbb{C}$ and let $\mathrm{C}_0(X) = \mathrm{C}_0(X, \mathbb{K})$. Let $d, j \in \mathbb{N}$. Let $\mathbb{T}^d$ be the d-fold Cartesian product, where $\mathbb{T}^0$ is just a single point.

1. Show that there is a split extension of Banach algebras $\mathrm{C}_0(\mathbb{R} \times \mathbb{T}^d) \rightarrowtail \mathrm{C}(\mathbb{T}^{d+1}) \twoheadrightarrow \mathrm{C}(\mathbb{T}^d)$.
2. Deduce that $H_j(\mathrm{C}(\mathbb{T}^{d+1})) \cong H_{j+1}(\mathrm{C}(\mathbb{T}^d)) \oplus H_j(\mathrm{C}(\mathbb{T}^d))$.
3. Use induction to express $H_j(\mathbb{T}^d)$ as a direct sum of $H_{j+k}(\mathbb{K})$.