Mathematisches Institut
Georg-August-Universität Göttingen
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Exercise sheet 11.

Name

Exercise group (tutor's name)

Deadline: **Friday, 17.1.2025, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. *(The point of this exercise is to prove that the standard recipe for describing inverses in the fundamental group of a space also describes the negation map on $H_1(A)$ for a homotopy invariant additive functor H.)* Let H be an additive, homotopy invariant functor from the category of $\mathbb{Z}/2$-graded Banach algebras to the category of Abelian groups and let A be a $\mathbb{Z}/2$-graded Banach algebra. Define the following grading-preserving endomorphisms of flip, $\iota_1, \iota_2\colon \mathrm{C}_0((0,1), A) \to \mathrm{C}_0((0,1), A)$:

$$\mathrm{flip}(f)(t) := f(1-t),$$
$$\iota_1(f)(t) := \begin{cases} f(2t) & \text{if } 0 \le t \le 1/2, \\ 0 & \text{if } 1/2 \le t \le 1, \end{cases}$$
$$\iota_2(f)(t) := \begin{cases} 0 & \text{if } 0 \le t \le 1/2, \\ f(2t-1) & \text{if } 1/2 \le t \le 1. \end{cases}$$

(Check for yourself that $\mathrm{flip}, \iota_1, \iota_2$ are well defined, grading-preserving homomorphisms, but there is no need to write this down in detail.)

1. Show that both ι_1 and ι_2 are homotopic to the identity map.
2. Observe that $\iota_1(f) \cdot \iota_2(g) = 0$ and $\iota_2(f) \cdot \iota_1(g) = 0$ for all $f, g \in \mathrm{C}_0((0,1), A)$. Deduce that
$$\mathrm{C}_0((0,1), A) \oplus \mathrm{C}_0((0,1), A) \to \mathrm{C}_0((0,1), A), \qquad (f, g) \mapsto \iota_1(f) + \iota_2(g),$$
is a well defined grading-preserving homomorphism. Show that the induced map
$$H_1(A) \oplus H_1(A) \cong H(\mathrm{C}_0((0,1), A) \oplus \mathrm{C}_0((0,1), A)) \to H(\mathrm{C}_0((0,1), A)) = H_1(A)$$
is the addition map $H_1(A) \oplus H_1(A) \to H_1(A)$, $(x, y) \mapsto x + y$.
3. Prove that the map $\mathrm{C}_0((0,1), A) \to \mathrm{C}_0((0,1), A)$, $f \mapsto \iota_1(f) + \iota_2(\mathrm{flip}(f))$, is homotopic to the zero map $f \mapsto 0$.
4. Deduce that flip induces the negation map $x \mapsto -x$ on the Abelian group $H(\mathrm{C}_0((0,1), A)$.

Exercise 2. *(This exercise deals with aspects of the proof of the K-theory long exact sequence.)* Let $I \overset{i}{\rightarrowtail} A \overset{p}{\twoheadrightarrow} B$ be an extension of $\mathbb{Z}/2$-graded Banach algebras. Define

$$C_p := \{(a, f) \in A \oplus \mathrm{C}_0([0,1), B) : p(a) = f(0)\}$$
$$Q := \{(g, f) \in \mathrm{C}_0([0,1), A) \oplus \mathrm{C}_0([0,1), B) : f(0) = p(g(0))\}.$$

1. Show that the map $\eta\colon Q \to C_p$, $(g, f) \mapsto (g(0), f)$, is surjective and identify its kernel with $\mathrm{C}_0((0,1), A)$.
2. Show that the inclusion $\varphi\colon \mathrm{C}_0((0,1), B) \to Q$, $f \mapsto (0, f)$, is a homotopy equivalence. More precisely, prove that the following map $\psi\colon Q \to \mathrm{C}_0((0,1), B)$ is inverse to φ up to homotopy. For $(g, f) \in Q$, $t \in [0,1]$, let $\psi(g, f)(t)$ be $p(g(1-2t))$ if $0 \le t \le 1/2$ and $f(2t)$ if $1/2 \le t \le 1$. So the remaining task is to check that $\psi \circ \varphi$ and $\varphi \circ \psi$ are homotopic to the identity map.

3. Show that φ is an isomorphism onto an ideal with $Q/\operatorname{im}(\varphi) \cong \mathrm{C}_0([0,1), B)$. (Together with the long exact sequence, this would prove that φ induces an isomorphism on any exact homotopy invariant functor, but we cannot use this argument while we are still proving the long exact sequence.)

4. Show that the composite map of $j\colon \mathrm{C}_0((0,1), A) \to Q$, $g \mapsto (g, 0)$, and ψ is homotopic to the composite of p_* with the reparametrisation map flip: $\mathrm{C}_0((0,1), A) \to \mathrm{C}_0((0,1), A)$ from the previous exercise. So after identifying $H(Q) \cong H_1(B)$, the inclusion map j induces the map $H_1(p)$ up to a minus sign, by the previous exercise.

(Observe that the composite map $\eta \circ \varphi\colon \mathrm{C}_0((0,1), B) \to C_p$ is the canonical map used to define the boundary map in the long exact sequence; there is nothing to write down here, this is just going through your notes and checking that it is so.) So the computations above finish the proof of the long exact sequence for homotopy-invariant exact functors.