Mathematisches Institut
Georg-August-Universität Göttingen
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Noncommutative geometry III
Winter semester 2024-2025

Exercise sheet 12.

Name

Exercise group (tutor's name)

Exercise	1	2	3	Σ
Points				

Deadline: **Friday, 24.1.2025, 12:00**.

Please use this page as a cover sheet and enter your name and tutor in the appropriate fields. Please staple your solutions to this cover sheet.

Exercise 1. Let A be a $\mathbb{Z}/2$-graded Banach algebra with grading automorphism α. Let α_2 and α_2' denote the involutions on $\mathbb{M}_2(A)$ defined by

$$\alpha_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \alpha(a) & \alpha(b) \\ \alpha(c) & \alpha(d) \end{pmatrix}, \qquad \alpha_2'\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \alpha(a) & -\alpha(b) \\ -\alpha(c) & \alpha(d) \end{pmatrix}.$$

Assume that A contains an odd involution e. Show that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & be \\ ec & ede \end{pmatrix}$$

is a grading-preserving isomorphism $(\mathbb{M}_2(A), \alpha_2) \cong (\mathbb{M}_2(A), \alpha_2')$.

Exercise 2. Let A and B be unital $\mathbb{Z}/2$-graded Banach algebras. These are *Morita equivalent* if and only if there are $m, n \in \mathbb{N}_{\geq 1}$ and even idempotent elements $p \in \mathbb{M}_m(A)$, $q \in \mathbb{M}_n(B)$ and isomorphisms

$$A \cong q\mathbb{M}_n(B)q, \qquad B \cong p\mathbb{M}_m(A)p.$$

Let A and B be Morita equivalent and let H be a stable, homotopy-invariant functor. Construct an isomorphism $H(A) \cong H(B)$.

The following exercise discusses the main step that is still missing to prove the Pimsner–Voiculescu exact sequence for the K-theory of crossed products.

Exercise 3. Let $A \subseteq \mathbb{B}(\mathcal{H})$ be a unital C*-subalgebra, that is, a norm-closed unital *-subalgebra. Let $\varphi\colon A \to A$ be an automorphism of A. Let $\Phi\colon A \to \mathbb{B}(\ell^2(\mathbb{N}, \mathcal{H}))$ be the representation defined by $(\Phi(a)f)(n) := \varphi^n(a)f(n)$ for all $a \in A$, $f \in \ell^2(\mathbb{N}, \mathcal{H})$, $n \in \mathbb{N}$. Define the unilateral shift $S\colon \ell^2(\mathbb{N}, \mathcal{H}) \to \ell^2(\mathbb{N}, \mathcal{H})$ by $(Sf)(n) = f(n-1)$ for $f \in \ell^2(\mathbb{N}, \mathcal{H})$, $n \in \mathbb{N}$. The *Toeplitz* C*-*algebra* $\mathcal{T}(A, \varphi)$ is the C*-subalgebra of $\mathbb{B}(\ell^2(\mathbb{N}, \mathcal{H}))$ generated by the above representation of A and S. Let $j\colon A \to \mathcal{T}(A, \varphi)$ be the map $a \mapsto \Phi(a)$. Let $\pi_0\colon \mathcal{T}(A, \varphi) \to \mathcal{T}(A, \varphi)$ be the identity map and let $\pi_1(x) = SxS^*$. The Toeplitz algebra proof of Bott periodicity shows that (π_0, π_1) is a quasi-homomorphism from $\mathcal{T}(A, \varphi)$ to $\mathbb{K} \bar{\otimes} A$ and that the map it induces on a $\mathbb{K}$-stable, split-exact, homotopy invariant functor is inverse to the map induced by j.

1. Show that $S^*\Phi(a)S = \Phi(\varphi(a))$ for all $a \in A$. (In fact, the Toeplitz C*-algebra is the universal C*-algebra generated by a copy of A and an isometry, subject to this relation.)
2. Show that $\mathcal{T}(A, \varphi)$ contains $\mathbb{K} \bar{\otimes} A$ as an ideal; this is the same as the closure of the *-algebra of finite-sized block matrices with entries in $A \subseteq \mathbb{B}(\mathcal{H})$.
3. Show that the map $H(A) \to H(A)$ induced by the corner embedding $A \to \mathbb{K} \bar{\otimes} A$ composed with the map induced by the quasi-homomorphism (π_0, π_1) is $\mathrm{id} - [\varphi]$.