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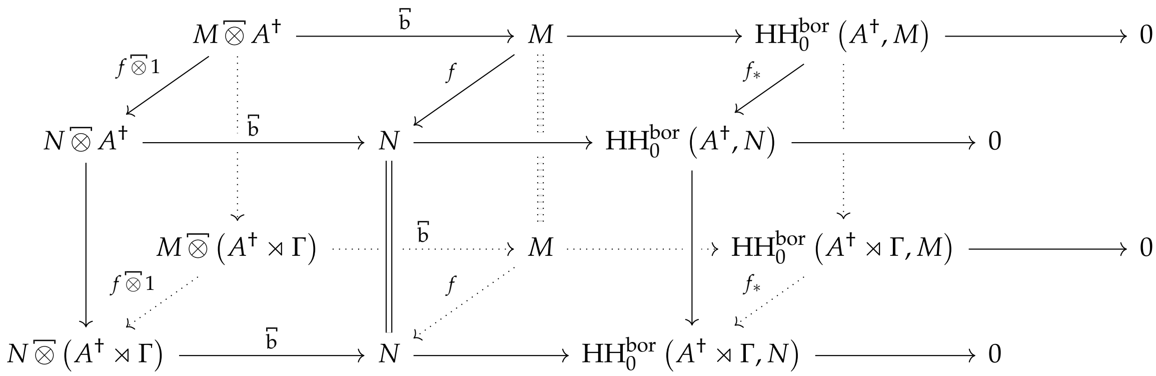
Towards Hochschild and Cyclic Homology of Dagger Algebras



Master Thesis in Mathematics
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by

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Introduction

Motivation

Fix a field k and let X be an affine k-variety with coordinate ring $\mathcal{O}[X]$. Assume that a finite group Γ acts on X by biregular maps or, equivalently, acts on $\mathcal{O}[X]$ by k-algebra automorphisms. Understanding the relation between the group action of Γ on X and the geometry of X naturally leads to the study of the quotient X/Γ. This is an affine k-variety with coordinate ring $\mathcal{O}[X]^\Gamma$, the subalgebra of Γ-invariant regular functions. However, there is a certain loss of control involved in the construction of this quotient. For instance, the quotient of a smooth k-variety is not necessarily smooth. Furthermore, in more general situations, these quotient constructions are not well-behaved.

We generalise our setting as follows. Suppose that A is a commutative k-algebra and that a finite group Γ acts on A by k-algebra automorphisms. This now corresponds, geometrically, to the case of a finite group Γ acting on an affine k-scheme $X = \mathrm{Spec}(A)$ by k-automorphisms. Instead of studying the affine k-scheme $X/\Gamma = \mathrm{Spec}\left(A^\Gamma\right)$ directly, we consider the *crossed product algebra* $A \rtimes \Gamma$. This ring encodes the algebra of the quotient construction more compactly. However, this will generally be a noncommutative ring. As such, the ideas and results of algebraic geometry do not apply. They are instead replaced by formal analogues, capturing the essence of geometric constructions.

For instance, schemes admit a well-defined geometric notion of smoothness, together with a deeply intertwined theory of tangent and cotangent spaces. Using the algebra-geometry dictionary at the heart of algebraic geometry, these concepts can be reinterpreted purely algebraically. This leads to formal notions of smoothness and differential forms, applicable not only to commutative rings but also to noncommutative rings. This constitutes the formalism of *Hochschild homology*, and of the derived theories of *cyclic* and *periodic cyclic homology*.

We are interested in crossed product algebras arising from affine k-varieties in positive characteristic. Specifically, suppose that k is the residue field of a complete discrete valuation ring V with uniformizer π. This is particularly useful if the field of fractions of V is of characteristic 0. For simplicity, we only consider finite group actions of commutative V-algebras, which naturally induce to finite group actions on commutative k-algebras. In the setting of a complete discrete valuation ring, there is a non-trivial topological component involved. Exploiting this perspective directly in the context of homological computations is, however, difficult, since topological constructions do not interact well with homological ones. Instead, we follow the example of [CCMT18] and use the formalism of *bornologies*. This allows us to make use of the underlying topological notions, while simultaneously preserving a well-behaved homological theory.

Overview of the Main Results

More concretely, we start with purely algebro-geometric considerations. Suppose that X is a smooth k-variety, with k a field of characteristic 0. As a consequence of the *Hochschild–Konstant–Rosenberg theorem*, the cyclic homology of $\mathcal{O}[X]$ is given in terms of the modules of the algebraic de Rham cohomology of X. This generalises to the case of a finite group action as follows. If a finite group Γ acts on X, it can be shown that the cyclic homology of the crossed product algebra $\mathcal{O}[X] \rtimes \Gamma$ is given in terms of the algebraic orbifold cohomology of X/Γ (see [Pon17, Remark 11.7]). An explicit and solely algebro-geometric approach is given in [BDN17]. They show how to compute the Hochschild, cyclic and periodic cyclic homology of $\mathcal{O}[X] \rtimes \Gamma$ for $k = \mathbb{C}$.

We explicitly work out that the Lefschetz principle (see [Ekl73]) holds for the results of [BDN17] on Hochschild, cyclic and periodic cyclic homology, which is to say that their arguments apply to any algebraically closed field of characteristic 0. We then prove that the assumption of algebraic closure is not necessary. This is done by reducing the general case of characteristic 0 to the algebraically closed case via base change. In summary, we prove the following.

Theorem (2.3.4). *Let $A = \mathcal{O}[X]$ be the coordinate ring of a smooth k-variety with k of characteristic 0. Suppose that a finite group Γ acts on X and let $\gamma_1, \ldots, \gamma_s$ be a set of representatives for the conjugacy classes of Γ. If $X_i = X^{\gamma_i}$, $A_i = \mathcal{O}[X_i]$ and $C_i = C_{\gamma_i}$, then*

$$\mathrm{HH}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \Omega^n(X_i)^{C_i},$$

$$\mathrm{HC}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\Omega^n(X_i)^{C_i} / \mathrm{d}\Omega^{n-1}(X_i)^{C_{\gamma_i}} \oplus \mathrm{H}^{n-2}_{\mathrm{dR}}(A_i)^{C_i} \oplus \mathrm{H}^{n-4}_{\mathrm{dR}}(A_i)^{C_i} \oplus \cdots \right),$$

and

$$\mathrm{HP}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\prod_{m=0}^{\infty} \mathrm{H}^{2m+n}_{\mathrm{dR}}(A_i)^{C_i} \right).$$

Now suppose that k is of positive characteristic and the residue field of a complete valuation ring V with field of fractions K of characteristic 0. [CCMT18] show that in the context of bornologies, the *weak completions* of Monsky–Washnitzer (see [MW68]) can be described as *bornological completions*. Monsky–Washnitzer used these weak completions to define a Weil cohomology theory of smooth affine varieties in positive characteristic. It is shown in [CCMT18] how this theory may be re-expressed entirely in terms of bornologies using the aforementioned identification. This required the definition of a suitable analogue of Hochschild homology for complete bornological V-algebras.

We extend these considerations and define *bornological Hochschild homology with coefficients*. By allowing the presence of coefficients in suitable bimodules, we can show that bornological Hochschild homology admits an axiomatic description.

Theorem (3.3.3, 3.3.4). *Let A be a complete bornological V-algebra.*

(a) Let $f\colon M \to N$ be a bounded A-bimodule map of complete bornological A-bimodules. Then there is an induced map $f_*\colon \mathrm{HH}^{\mathrm{bor}}_n(A, M) \to \mathrm{HH}^{\mathrm{bor}}_n(A, N)$. This defines a functor from the category of complete A-bimodules to the category of V-modules.

(b) Let M be a complete A-bimodule. Then $A \otimes_V M \otimes_V A$ carries a canonical A-bimodule structure. If $F = \overline{A \otimes_V M \otimes_V A}$, then there is a semi-split extension

$$0 \longrightarrow \ker(q) \longrightarrow F \xrightarrow{\ q\ } M \longrightarrow 0$$

of complete bornological A-bimodules, with $\mathrm{HH}^{\mathrm{bor}}_n(A, F) = 0$ for $n > 0$.

(c) Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a semi-split extension of complete bornological A-bimodules. Then there is a long exact sequence

$$\cdots \to \mathrm{HH}^{\mathrm{bor}}_{n+1}(A, M'') \to \mathrm{HH}^{\mathrm{bor}}_n(A, M') \to \mathrm{HH}^{\mathrm{bor}}_n(A, M) \to \mathrm{HH}^{\mathrm{bor}}_n(A, M'') \to \cdots$$

and the connecting morphisms $\mathrm{HH}^{\mathrm{bor}}_{n+1}(A, M'') \to \mathrm{HH}^{\mathrm{bor}}_n(A, M')$ are natural for morphism of extensions of complete bornological A-bimodules.

This description mirrors the corresponding characterisation of ordinary Hochschild homology as a derived functor. However, for bornological Hochschild homology, this axiomatic perspective does not follow immediately from general principles, since the categories in consideration are not abelian.

The computations of [BDN17] for the Hochschild homology of $A \rtimes \Gamma$ are based on certain decomposition results. It is established in [Lor92] that these results hold in much greater generality. More precisely, one can decompose the Hochschild homology of $A \rtimes \Gamma$ into summands corresponding to the conjugacy classes of Γ and consider the particular action of representatives on A. This reduces the computation of the Hochschild homology of $A \rtimes \Gamma$ to the computation of the Hochschild homology of A with coefficients encoding the action of Γ on A. This approach exploits the axiomatic point of view, identifying Hochschild homology with another homology theory constructed from group hyperhomology of Γ.

This yields an application of our axiomatic description. We consider the weak completion $A^\dagger$ of a commutative V-algebra of finite type A. If a finite group Γ acts on A, then there is an induced action of Γ on $A^\dagger$. Write $\gamma^\dagger\colon A^\dagger \to A^\dagger$ for the automorphism induced by $\gamma\colon A \to A$. With respect to this action, we denote by $A^\dagger_{\gamma^\dagger}$ the complete bornological $A^\dagger$-bimodules with a $\gamma^\dagger$-twisted right $A^\dagger$-module structure. The bornological Hochschild homology of $A^\dagger$ with coefficients in $A^\dagger_{\gamma^\dagger}$ is denoted for brevity by $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, \gamma^\dagger\right)$. We show the following.

Theorem (4.1.2, 4.2.6, 4.2.7). *Let A be a V-algebra and let Γ be a finite group acting on A by V-linear automorphisms. For each $n \geq 0$, there is an isomorphism*

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) \cong \bigoplus_{[\gamma]\in[\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma .$$

Furthermore, for each $\gamma \in \Gamma$ and $n \geq 0$, there are isomorphisms

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma \cong \mathbb{H}_n\Big(C_\gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, \gamma^\dagger\right)\Big) \cong \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, \gamma^\dagger\right)_{C_\gamma} .$$

For a torsion-free commutative V-algebra of finite type A, [CCMT18] prove that the bornological Hochschild homology of its weak completion $A^\dagger$ is given by the base change of the ordinary Hochschild homology of A along $A \to A^\dagger$. Thus, computations involving $A^\dagger$ can be reduced to computations involving only A.

Write A_γ for the A-bimodule, whose right A-module structure is twisted by γ. Denote the Hochschild homology of A with coefficients in A_γ by $\mathrm{HH}_n\left(A, \gamma\right)$. We prove the following.

Proposition (4.3.3). *Let A be a torsion-free commutative V-algebra of finite type. The natural homomorphism*

$$\mathrm{HH}_n\left(A, \gamma\right) \to \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, \gamma^\dagger\right)$$

induces an isomorphism

$$A^\dagger \otimes_A \mathrm{HH}_n\left(A, \gamma\right) \xrightarrow{\sim} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, \gamma^\dagger\right)$$

for all $n \geq 0$.

Recall that K is the field of fractions of V. Denote by $\overline{A} = K \otimes_V A$ the base change of a commutative V-algebra A along $V \to K$. Since this is a flat base change, there is an isomorphism $\overline{A} \otimes_A \mathrm{HH}_n\left(A, \gamma\right) \cong \mathrm{HH}_n\left(\overline{A}, \overline{\gamma}\right)$ for all $n \geq 0$. If we now assume that $\overline{A}$ is smooth, we can apply the earlier theorems in their full strength.

Structure of the Thesis

In Section 1, we introduce *Hochschild homology*, *cyclic* and *periodic cyclic homology* for arbitrary k-algebras. First, we define Hochschild homology abstractly as a derived functor, then relate it to *noncommutative differential forms* in Section 1.1. The latter allows, in particular, an efficient definition of both cyclic and periodic cyclic homology, which we spell out in Section 1.2. In Section 1.3 we formulate the *Hochschild–Kostant–Rosenberg theorem*, which identifies the Hochschild homology of a formally smooth commutative k-algebra with its modules of Kähler differentials, assuming that k is of characteristic 0. As a consequence, cyclic and periodic cyclic homology of formally smooth k-algebras can then be described in terms of algebraic de Rham cohomology.

In Section 2, we consider a finite group Γ acting on a smooth affine k-variety X of finite type. Equivalently, we may assume that Γ acts on its coordinate ring $A = \mathcal{O}[X]$ by k-algebra automorphisms. We then consider the *crossed product algebra* $A \rtimes \Gamma$. We will see in Section 2.1 that computing the Hochschild homology of $A \rtimes \Gamma$ can be reduced to considering the *twisted Hochschild homology* of A. The twist arises from the group action of Γ. Working in characteristic 0, we obtain a generalisation of the *Hochschild–Kostant–Rosenberg theorem*, assuming that Γ is a finite group. For this, we first consider the case of algebraically closed fields in Section 2.2, from which we deduce the case of non-algebraically closed fields in Section 2.3 via base change.

In Section 3 we turn towards positive characteristic. More precisely, we consider a discrete valuation ring V with residue field of positive characteristic. We study V-modules and V-algebras equipped with the additional structure of a *bornology*, which we introduce in Section 3.1. Bornological completions provide a new perspective on *weak completions*, or *dagger completions*, which were used by Monsky–Washnitzer to define a Weil cohomology theory for smooth affine varieties in positive characteristic. For this, we consider *linear growth bornologies* in Section 3.2, a notion that applies equally well to noncommutative V-algebras. This section culminates in the definition and characterisation of a bornological variant of Hochschild homology. This characterisation is proved in Section 3.3.

Section 4 combines the ideas and results presented throughout Sections 1 to 3. The aim is to compute the bornological Hochschild homology of the dagger completion of a crossed product algebra. The crossed product of interest arises from a finite group Γ acting on a commutative V-algebra A of finite type. We make some initial observations about such algebras in Section 4.1. In Section 4.2, we study the intricate relation between the bornological Hochschild homology of the crossed product algebra $A^\dagger \rtimes \Gamma$ and the group action of Γ on the bornological Hochschild homology of $A^\dagger$. Using the axiomatic characterisation of Section 3.3, we relate bornological Hochschild homology to group hyperhomology. We then observe in Section 4.3 that *twisted bornological Hochschild homology* can be related to ordinary twisted Hochschild homology in a straightforward way. In combination, we then argue in Section 4.4 how Sections 4.1 to 4.3 reduce computing the bornological Hochschild homology of $A^\dagger \rtimes \Gamma$ to the ordinary Hochschild homology of $A \rtimes \Gamma$. Assuming that the field of fractions of V is of characteristic 0 provides a natural base change, after which we can then apply the results of Section 2.

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1 Hochschild, Cyclic and Periodic Cyclic Homology

In Section 1.1, we introduce Hochschild homology, starting with the derived functor perspective. Following mainly [Mey21] we then explain how Hochschild homology can be defined using *noncommutative differentials forms* (see Definition 1.1.8 and Corollary 1.1.9). This approach highlights the role of Hochschild homology as a noncommutative analogue for algebraic differential forms of commutative algebras. We make this analogy precise by considering the *Hochschild–Kostant–Rosenberg theorem* (see Theorem 1.3.1), which we discuss in Section 1.3. This is complemented by introducing the closely related cyclic and periodic cyclic homology theories in Section 1.2, with cyclic homology serving as a noncommutative analogue of algebraic de Rham cohomology.

Fix a field k. Throughout, all undecorated tensor products are taken over k.

1.1 Hochschild Homology and Noncommutative Differential Forms

We assume familiarity with the formalism of homological algebra, in particular, the notion of derived functors. For notation and relevant results, we refer to [Wei94].

Let A be a unital k-algebra. We consider the category of (unital) A-bimodules ${}_A\mathsf{Mod}_A$. Following [Lod98, §1.1.0], we always understand an A-bimodule to be a symmetric k-bimodule endowed with a left and a right A-module structure such that $(am)a' = a(ma')$ for all $a, a' \in A$ and $m \in M$. Furthermore, we assume that these actions are compatible with the underlying symmetric k-bimodule structure.

Let A^{op} be the opposite algebra of A. The k-algebra $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ is the **enveloping algebra** of A. Observe that ${}_A\mathsf{Mod}_A$ is equivalent to the category $\mathsf{Mod}_{A^{\mathrm{e}}}$ of left A^{e}-modules. Since $(A^{\mathrm{e}})^{\mathrm{op}} \cong A^{\mathrm{e}}$, ${}_A\mathsf{Mod}_A$ is also equivalent to the category ${}_{A^{\mathrm{e}}}\mathsf{Mod}$ of right A^{e}-modules. In particular, ${}_A\mathsf{Mod}_A$ admits left (respectively, right) derived functors for any right (respectively, left) exact functor defined on ${}_A\mathsf{Mod}_A$.

Definition 1.1.1. Let M be an A-bimodule. For all $n \geq 0$, the group

$$\mathrm{HH}_n(A, M) = \mathrm{Tor}_n^{A^{\mathrm{e}}}(M, A)$$

is called the **n-th Hochschild homology of A with coefficients in M**. For $M = A$,

$$\mathrm{HH}_n(A) = \mathrm{HH}_n(A, A)$$

is the **n-th Hochschild homology of A**. ⌟

Definition 1.1.2. Let M be an A-bimodule. For all $n \geq 0$, the group

$$\mathrm{HH}^n(A, M) = \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, M)$$

is called the **n-th Hochschild cohomology of A with coefficients in M**. ⌟

Remark. The **n-th Hochschild cohomology of** A is $\mathrm{HH}^n(A) = \mathrm{HH}^n(A, A^\vee)$, for $A^\vee = \mathrm{Hom}_k(A, k)$ the k-dual space of A. This is the correct definition for Hochschild cohomology, so that it may be naturally related to cyclic cohomology, whereas the groups $\mathrm{HH}^n(A, A)$ are related to the deformation theory of A (see [Kha10, §3.1]). Neither theory nor cyclic cohomology will make an appearance outside this remark.

We will primarily consider the Hochschild homology of various k-algebras. However, having Hochschild cohomology defined paints a more complete theoretical picture. To see one such instance, we introduce so-called *Hochschild extensions.*

Definition 1.1.3. Let E be a k-algebra and M an A-bimodule. We say that E is a **Hochschild extension**

of A by M if there is a short exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow 0$$

so that $M^2 = 0$ relative to the product structure on E. Two Hochschild extensions E, E' of A by M are **equivalent** if there is an A-bimodule isomorphism $\varphi\colon E \to E'$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commute. ⌟

Proposition 1.1.4. *The set of equivalence classes of Hochschild extensions of A by M is in one-to-one correspondence with* $\mathrm{HH}^2(A, M)$.

Proof. This is [Wei94, Theorem 9.3.1]. □

In the remainder of Section 1.1, we explain how we can construct the Hochschild homology of A using a geometrically flavoured construction, called noncommutative differential forms. To this end, we follow a combination of [Mey21] and [Muk22].

To define noncommutative differential forms, we first consider *(noncommutative) derivations*.

Definition 1.1.5. Let M be an A-bimodule. We say that a k-linear map $D\colon A \to M$ is a **k-derivation on A with values in M**, or simply **(k-)derivation**, if the Leibniz rule

$$D(ab) = aD(b) + D(a)b$$

is satisfied for all $a, b \in A$. Denote the k-vector space of k-derivations by $\mathrm{Der}_k(A, M)$. ⌟

Remark. Let $m \in M$. Then $D_m\colon A \to M$, $a \mapsto am - ma$, is a derivation, called the **principal derivation associated to** m. If we let $\mathrm{PDer}_k(A, M) \leq \mathrm{Der}_k(A, M)$ be the submodule of principal derivations, then

$$\mathrm{HH}^1(A, M) \cong \frac{\mathrm{Der}_k(A, M)}{\mathrm{PDer}_k(A, M)}\,.$$

This is [Wei94, Lemma 9.2.1].

Noncommutative differential 1*-forms* now arise as a universal object classifying derivations on A.

Definition 1.1.6. Let $\mu\colon A \otimes A \to A$, $a \otimes b \mapsto ab$. The A-bimodule $\Omega^1_k(A) = \ker(\mu)$ is called the bimodule of **noncommutative differential 1-forms on A** and the map

$$\mathrm{d}\colon A \to \Omega^1_k(A), \quad a \mapsto 1 \otimes a - a \otimes 1\,,$$

is the **universal derivation on A**. ⌟

Using the structure homomorphism $k \to A$ of the k-algebra A, we let $\overline{A} = A/(k \cdot 1)$ for the remainder of Section 1.

Theorem 1.1.7. *The universal derivation* $\mathrm{d}\colon A \to \Omega^1_k(A)$ *defines a representation of the functor*

$$\mathrm{Der}_k(A, -)\colon {}_A\mathsf{Mod}_A \to \mathsf{Vect}_k\,.$$

The elements $\mathrm{d}a = \mathrm{d}(a)$*,* $a \in A$*, generate* $\Omega^1_k(A)$ *as a left* A*-module. Moreover, there is an isomorphism*

$$\Omega^1_k(A) \xrightarrow{\sim} A \otimes \overline{A}, \quad a\mathrm{d}b \mapsto a \otimes b\,,$$

of left A*-modules.*

Proof. See [Mey21, Proposition 15.2] and the preceding discussion. □

Definition 1.1.8. For $n \geq 1$, the n-fold tensor product

$$\Omega^n_k(A) = \Omega^1_k(A) \otimes_A \cdots \otimes_A \Omega^1_k(A)$$

is called the bimodule of **noncommutative differential n-forms**. For $n = 0$, we let $\Omega^0_k(A) = A$. ⌟

To ease notation, we will write $a_0\mathrm{d}a_1 \dots \mathrm{d}a_n$ for $a_0(\mathrm{d}a_1 \otimes \cdots \otimes \mathrm{d}a_n)$. By A-bilinearity and Theorem 1.1.7, the latter is one of the elementary tensors spanning $\Omega^n_k(A)$.

Corollary 1.1.9. *There is an isomorphism*

$$\Omega^n_k(A) \xrightarrow{\sim} A \otimes \overline{A}^{\otimes n}, \quad a_0\mathrm{d}a_1 \dots \mathrm{d}a_n \mapsto a_0 \otimes a_1 \otimes \cdots \otimes a_n\,,$$

of left A*-modules. The right* A*-module structure on* $A \otimes \overline{A}^{\otimes n}$ *is then explicitly given by*

$$\begin{aligned}(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n) \cdot b = a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes (a_n b) - a_0 \otimes a_1 \otimes \cdots \otimes (a_{n-1}a_n) \otimes b + \cdots \\ + (-1)^n (a_0 a_1) \otimes a_2 \otimes \cdots \otimes a_n \otimes b\,.\end{aligned}$$

Proof. This is [Mey21, Lemma 22.2]. □

To obtain homological insight from noncommutative differential forms, we are in need of appropriate boundary operators to define either a chain or cochain complex. Drawing from the commutative case, the canonical map

$$\mathrm{d}\colon \Omega^n_k(A) \to \Omega^{n+1}_k(A), \quad a_0\mathrm{d}a_1 \dots \mathrm{d}a_n \mapsto \mathrm{d}a_0\mathrm{d}a_1 \dots \mathrm{d}a_n\,,$$

is a natural candidate. However, in the noncommutative case, this yields an uninteresting cochain complex (d satisfies $\mathrm{d}^2 = 0$ since it is a derivation, hence $\mathrm{d}(1) = 0$).

Proposition 1.1.10. *The cohomology of the cochain complex* $\big(\Omega^\bullet_k(A), \mathrm{d}\big)$ *vanishes in all non-zero degrees and is isomorphic to* k *in degree* 0*.*

Proof. The noncommutative Poincaré Lemma [Tsy20, Lemma 2.1] proves that $\big(\Omega^\bullet_k(k), \mathrm{d}\big) \cong \big(\Omega^\bullet_k(A), \mathrm{d}\big)$ as differential graded algebras. By construction $\Omega^\bullet_k(k) \cong \Omega^0_k(k) = k$ and the assertion follows. □

Although the natural cochain complex structure on the noncommutative differential forms is not particularly insightful, there is a way for obtaining an interesting chain complex. For this, we define the **Hochschild boundary**

$$\mathrm{b}\colon \Omega^{n+1}_k(A) \to \Omega^n_k(A), \quad a_0\mathrm{d}a_1 \dots \mathrm{d}a_n\mathrm{d}a_{n+1} \mapsto (-1)^n[a_0\mathrm{d}a_1 \dots \mathrm{d}a_n, a_{n+1}]\,.$$

Here $[\cdot,\cdot]$ denotes the commutator. For the convenience of the reader, we include the straightforward computation from [Mey21, §22.1], which shows that $\mathrm{b}^2 = 0$.

Proposition 1.1.11. *The pair* $\big(\Omega^\bullet_k(A), \mathrm{b}\big)$ *is a chain complex.*

Proof. We have to check that $\mathrm{b}^2 = 0$. Note that the only essential alterations in the formula defining the Hochschild boundary are concerned with the last occurring differential. Hence, we may abbreviate

$a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n \mathrm{d}a_{n+1}$ as $\omega \mathrm{d}a_n \mathrm{d}a_{n+1}$ for our purpose. Now we compute

$$\begin{aligned}
\mathrm{b}(\omega \mathrm{d}a_n \mathrm{d}a_{n+1}) &= (-1)^n [\omega \mathrm{d}a_n, a_{n+1}] \\
&= (-1)^n (\omega \mathrm{d}a_n \cdot a_{n+1} - a_{n+1} \omega \mathrm{d}a_n) \\
&= (-1)^n (\omega \mathrm{d}(a_n a_{n+1}) - (\omega \cdot a_n) \mathrm{d}a_{n+1} - a_{n+1} \omega \mathrm{d}a_n)
\end{aligned}$$

and then observe that

$$\begin{aligned}
\mathrm{b}^2(\omega \mathrm{d}a_n \mathrm{d}a_{n+1}) &= (-1)^{n-1}(-1)^n ([\omega, a_n a_{n+1}] - [\omega \cdot a_n, a_{n+1}] - [a_{n+1}\omega, a_n]) \\
&= -(\omega \cdot (a_n a_{n+1}) - (a_n a_{n+1})\omega) + ((\omega \cdot a_n) \cdot a_{n+1}) - a_{n+1}(\omega \cdot a_n)) \\
&\quad + ((a_{n+1}\omega) \cdot a_n - a_n(a_{n+1}\omega)) \\
&= 0
\end{aligned}$$

by bilinearity. □

We have now introduced the necessary terminology involving noncommutative differential forms to relate this construction to Hochschild homology.

Recall that by the general theory of derived functors, any resolution of A by projective A^{e}-modules may be used to compute $\mathrm{HH}_n(A) = \mathrm{Tor}_n^{A^{\mathrm{e}}}(A, A)$. Hence, to see that $\mathrm{HH}_n(A)$ may be computed using the complex $\big(\Omega_k^\bullet(A), \mathrm{b}\big)$, it suffices to find such a resolution which upon tensoring by A over A^{e}, results in a chain complex that is quasi-isomorphic to $\big(\Omega_k^\bullet(A), \mathrm{b}\big)$. In fact, we can even find a chain complex arising in this way which is isomorphic to $\big(\Omega_k^\bullet(A), \mathrm{b}\big)$.

The resolution in question is the *normalised bar resolution*.

Definition 1.1.12. For $n \geq 0$, let $\overline{\mathrm{B}}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A$ and

$$\begin{aligned}
\mathrm{b}' \colon \overline{\mathrm{B}}_n(A) &\to \overline{\mathrm{B}}_{n-1}(A), \\
a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} &\mapsto \sum_{j=0}^{n} (-1)^j a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes a_{n+1}\,.
\end{aligned}$$

The complex $\big(\overline{\mathrm{B}}_\bullet(A), \mathrm{b}'\big)$ is the **normalised bar resolution**. ⌟

Lemma 1.1.13. *The bar resolution is a well-defined projective resolution of A as an A^{e}-module.*

Proof. This is established during the proof of [Mey21, Theorem 17.4]. □

Instead of considering the *normalised* bar resolution $\big(\overline{\mathrm{B}}_\bullet(A), \mathrm{b}'\big)$, one could also work with the **unnormalised bar resolution** $\big(\mathrm{B}_\bullet(A), \mathrm{b}'\big)$ instead. The n-th chain group of the unnormalised bar resolution is simply $\mathrm{B}_n(A) = A \otimes A^{\otimes n} \otimes A$. Homologically speaking, this does not make a difference, since the normalised and unnormalised bar resolutions are quasi-isomorphic (see [Lod98, Proposition 1.1.15]). However, the normalised variant is more easily related to noncommutative differential forms.

Theorem 1.1.14. *For all $n \geq 0$, the n-th homology of $\big(\Omega_k^\bullet(A), \mathrm{b}\big)$ is isomorphic to $\mathrm{HH}_n(A)$.*

Proof. We claim that the chain complex obtained by tensoring $\big(\overline{\mathrm{B}}_\bullet(A), \mathrm{b}'\big)$ with A as A^{e}-modules is isomorphic to $\big(\Omega_k^\bullet(A), \mathrm{b}\big)$. By our earlier remarks on derived functors, this implies the statement of the theorem.

We first observe that $A \otimes_{A^{\mathrm{e}}} M$ is isomorphic to the commutator quotient $M/[A, M]$ of M. An explicit pair of isomorphisms is given by

$$A \otimes_{A^{\mathrm{e}}} M \to M/[A, M], \quad a \otimes m \mapsto [am]\,,$$

and

$$M/[A,M] \to A \otimes_{A^e} M, \quad [m] \mapsto 1 \otimes m\,.$$

Using this identification, we can compute $A \otimes_{A^e} \overline{\mathrm{B}}_n(A)$. We claim that

$$A \otimes_{A^e} \overline{\mathrm{B}}_n(A) \cong \overline{\mathrm{B}}_n(A)/[A, \overline{\mathrm{B}}_n(A)] \cong A \otimes \overline{A}^{\otimes n}$$

along

$$\begin{gathered}\overline{\mathrm{B}}_n(A)/[A,\overline{\mathrm{B}}_n(A)] \to A \otimes \overline{A}^{\otimes n},\\ [a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}] \mapsto (a_{n+1}a_0) \otimes a_1 \otimes \cdots \otimes a_n\,.\end{gathered}$$

To see this, consider

$$\begin{gathered}A \otimes \overline{A}^{\otimes n} \to \overline{\mathrm{B}}_n(A)/[A,\overline{\mathrm{B}}_n(A)],\\ a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto [a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1]\,.\end{gathered}$$

The above maps are mutually inverse, since

$$[(a_{n+1}a_0) \otimes a_1 \otimes \cdots \otimes a_n \otimes 1] = [a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}]$$

by the construction of the commutator quotient.

It remains to check that these isomorphisms are compatible with the given boundary maps. Hence, we have to show that the diagram

$$\begin{array}{ccc} A \otimes \overline{A}^{\otimes n} & \xrightarrow{\quad \mathrm{b} \quad} & A \otimes \overline{A}^{\otimes (n-1)} \\ \cong\downarrow & & \downarrow\cong \\ A \otimes_{A^e} \overline{\mathrm{B}}_n(A) & \xrightarrow[1_A \otimes \mathrm{b}']{} & A \otimes_{A^e} \overline{\mathrm{B}}_{n-1}(A) \end{array}$$

commutes. For this, we first unpack the Hochschild boundary, using the explicit right action we established in Corollary 1.1.9. Thus, we find that

$$\mathrm{b}(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = (-1)^n (a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1} + \sum_{j=0}^{n-1} (-1)^j a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n\,.$$

Along the isomorphism to the commutator quotient complex, this expression is identified with

$$(-1)^n [(a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1] + \sum_{j=0}^{n-1} (-1)^j [a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes 1]\,.$$

On the other hand, tracing the isomorphism $A \otimes_{A^e} \overline{\mathrm{B}}_n(A) \cong \overline{\mathrm{B}}_n(C)/[A, \overline{\mathrm{B}}_n(A)]$, we see that the explicit boundary operator in the commutator quotient complex is simply the original formula applied to equivalence classes. Therefore, we find that

$$\mathrm{b}'([a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1]) = \sum_{j=0}^{n} (-1)^j [a_0 \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_n \otimes 1]\,.$$

The summands agree verbatim in the range $0 \leq j \leq n-1$, and for $j = n$ the commutator relation yields

$$[(a_n a_0) \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1] = [a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n]$$

as needed. □

Remark. In fact, given any projective resolution $P_\bullet \to A$ of A as an A^{e}-module, the proof of Theorem 1.1.14 shows that we may compute $\mathrm{HH}_n(A)$ as the homology of the chain complex $P_\bullet/[A, P_\bullet]$ of commutator quotients.

In light of Theorem 1.1.14, we let $(\mathrm{C}_\bullet(A), \mathrm{b}) = \big(\Omega^\bullet_k(A), \mathrm{b}\big)$ be the **Hochschild complex of A**. More generally, given an A-bimodule M, the **Hochschild complex of A with coefficients in M** is the complex obtained by tensoring the unnormalised bar resolution with M, considered as a right A^{e}-module. We denote this complex by $(\mathrm{C}_\bullet(A, M), \mathrm{b})$. In analogy to Theorem 1.1.14, this complex computes the Hochschild homology of A with coefficients in M (cf. [Lod98, Proposition 1.1.13] and [Wei94, Corollary 9.1.5]).

1.2 The Cyclic Double Complex

In Section 1.1 we observed the existence of two separate boundary operators for noncommutative differential forms: On the one hand, the canonically defined $\mathrm{d}\colon \Omega^n_k(A) \to \Omega^{n+1}_k(A)$ and, on the other hand, the Hochschild boundary $\mathrm{b}\colon \Omega^n_k(A) \to \Omega^{n-1}_k(A)$. We also observed that the associated cochain complex $\big(\Omega^\bullet_k(A), \mathrm{d}\big)$ provides no homological insight, whereas the associated chain complex $\big(\Omega^\bullet_k(A), \mathrm{b}\big)$ computes the Hochschild homology of A.

Nonetheless, we can exploit the presence of two opposing boundary operators by studying their potential for the construction of a double complex. Define the **Karoubi operator** as

$$\kappa = 1 - [\mathrm{d}, \mathrm{b}] = 1 - (\mathrm{db} + \mathrm{bd}),$$

which measures the anticommutativity of b and d. More explicitly, the Karoubi operator acts on a noncommutative differential form $\omega \mathrm{d}a_n \in \Omega^n_k(A)$ as

$$\begin{aligned}
\kappa(\omega \mathrm{d}a_n) &= \omega \mathrm{d}a_n - (\mathrm{d}((-1)^{n-1}[\omega, a_n]) + \mathrm{b}(\mathrm{d}\omega \mathrm{d}a_n)) \\
&= \omega \mathrm{d}a_n - (-1)^{n-1}\mathrm{d}([\omega, a_n]) - (-1)^n[\mathrm{d}\omega, a_n] \\
&= \omega \mathrm{d}a_n - (-1)^n([\mathrm{d}\omega, a_n] - \mathrm{d}([\omega, a_n])) \\
&= \omega \mathrm{d}a_n - (-1)^n[\mathrm{d}a_n, \omega] \\
&= (-1)^{n-1}\mathrm{d}a_n \cdot \omega,
\end{aligned}$$

where all occurring commutators are graded commutators. Note that this does not interfere with our earlier usage of a commutator when defining the Hochschild boundary, since elements of A have degree 0 by definition. In particular, we conclude that

$$\kappa(\mathrm{d}a_0 \ldots \mathrm{d}a_{n-1}\mathrm{d}a_n) = (-1)^{n-1}\mathrm{d}a_n\mathrm{d}a_0 \ldots \mathrm{d}a_{n-1}.$$

If we now let the **Connes operator** be

$$\mathrm{B} = \sum_{j=0}^{n} \kappa^j \mathrm{d},$$

we observe that

$$\mathrm{B}(a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n) = \sum_{j=0}^{n} (-1)^{jn} \mathrm{d}a_j \ldots \mathrm{d}a_n \mathrm{d}a_0 \ldots \mathrm{d}a_{j-1} .$$

Among others, the following elementary identities are established in [Mey21, §23.1]. We repeat the ones crucial for us.

Lemma 1.2.1. *The Connes operator satisfies*

$$\mathrm{B}^2 = 0 \quad \textit{and} \quad [\mathrm{B}, \mathrm{b}] = 0 .$$

Proof. We start by showing that the Karoubi operator κ commutes with both d and b. Indeed, we find that

$$\kappa \mathrm{d} = (1 - (\mathrm{db} + \mathrm{bd}))\mathrm{d} = \mathrm{d} - \mathrm{dbd} - \mathrm{bd}^2 = \mathrm{d} - \mathrm{d}^2\mathrm{b} - \mathrm{dbd} = \mathrm{d}(1 - (\mathrm{db} - \mathrm{bd})) = \mathrm{d}\kappa ,$$

since $\mathrm{d}^2 = 0$. The same holds true for b, since $\mathrm{b}^2 = 0$ as well.

Hence, we deduce that

$$\mathrm{B}^2 = \sum_{j=0}^{n} \mathrm{B}(\kappa^j \mathrm{d}) = \sum_{j=0}^{n} \sum_{j'=0}^{n} \kappa^{j'} \mathrm{d} \kappa^j \mathrm{d} = \sum_{j=0}^{n} \sum_{j'=0}^{n} \kappa^{j'+j} \mathrm{d}^2 = 0 .$$

Moreover, note that, on the one hand,

$$\mathrm{bB} = \sum_{j=0}^{n} \mathrm{b}\kappa^j \mathrm{d} = \sum_{j=0}^{n} \kappa^j (\mathrm{bd}) ,$$

while, on the other hand,

$$\mathrm{Bb} = \sum_{j=0}^{n} \kappa^j (\mathrm{db}) = \sum_{j=0}^{n} \kappa^j (1 - \kappa - \mathrm{dd}) = \sum_{j=0}^{n} (\kappa^j - \kappa^{j+1}) - \mathrm{bB} = (1 - \kappa^{n+1}) - \mathrm{bB} = -\mathrm{bB}$$

as $\kappa^{n+1} = 1$. Thus, $[\mathrm{B}, \mathrm{b}] = \mathrm{Bb} + \mathrm{bB} = 0$. □

Lemma 1.2.1 implies that

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} \\
\Omega^3_k(A) & \xleftarrow[\mathrm{B}]{} & \Omega^2_k(A) & \xleftarrow[\mathrm{B}]{} & \Omega^1_k(A) & \xleftarrow[\mathrm{B}]{} & A \\
\downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} & & \\
\Omega^2_k(A) & \xleftarrow[\mathrm{B}]{} & \Omega^1_k(A) & \xleftarrow[\mathrm{B}]{} & A & & \\
\downarrow{\scriptstyle \mathrm{b}} & & \downarrow{\scriptstyle \mathrm{b}} & & & & \\
\Omega^1_k(A) & \xleftarrow[\mathrm{B}]{} & A & & & & \\
\downarrow{\scriptstyle \mathrm{b}} & & & & & & \\
A & & & & & &
\end{array}$$

is a double complex $\big(\Omega^\bullet_k(A), \mathrm{b}, \mathrm{B}\big)$. Such double complexes are also called **mixed complexes**. This naturally leads to *cyclic homology*.

Definition 1.2.2. The double complex $\big(\Omega^\bullet_k(A), \mathrm{b}, \mathrm{B}\big)$ is called the **cyclic bicomplex**. Its total complex is the **cyclic complex of A**, denoted by $(\mathrm{CC}_\bullet(A), \mathrm{b} + \mathrm{B})$, with its n-th homology defining the **n-th cyclic**

homology of A, denoted by $\mathrm{HC}_n(A)$. ⌟

Explicitly, the n-th chain group $\mathrm{CC}_n(A)$ of the cyclic complex is

$$\Omega_k^n(A) \oplus \Omega_k^{n-2}(A) \oplus \cdots \oplus \Omega_k^{n \bmod 2}(A)\,.$$

Its n-th differential is componentwise the sum of the maps b and B except in the first component, where it is b, and for even n in the last component, where it is B.

Embed the Hochschild complex as the first column of the cyclic complex. This gives a morphism of chain complexes $\mathrm{I}\colon \mathrm{C}_\bullet(A) \to \mathrm{CC}_\bullet(A)$ from the Hochschild complex of A into the cyclic complex of A. Shifting the cyclic bicomplex to the right induces a shift operator $\mathrm{S}\colon \mathrm{CC}_\bullet(A) \to \mathrm{CC}_\bullet(A)[-2]$ on the cyclic complex of A. The maps I and S give rise to **Connes' SBI sequence**, relating Hochschild and cyclic homology.

Proposition 1.2.3. *There is a long exact sequence in homology*

$$\cdots \longrightarrow \mathrm{HC}_n(A) \xrightarrow{\ \mathrm{S}\ } \mathrm{HC}_{n-2}(A) \xrightarrow{\ \mathrm{B}\ } \mathrm{HH}_{n-1}(A) \xrightarrow{\ \mathrm{I}\ } \mathrm{HC}_{n-1}(A) \longrightarrow \cdots$$

with B *induced by the Connes operator.*

Proof. The maps $\mathrm{I}\colon \mathrm{C}_\bullet(A) \to \mathrm{CC}_\bullet(A)$ and $\mathrm{S}\colon \mathrm{CC}_\bullet(A) \to \mathrm{CC}_\bullet(A)[-2]$ evidently define a short exact sequence

$$0 \longrightarrow \mathrm{C}_\bullet(A) \xrightarrow{\ \mathrm{I}\ } \mathrm{CC}_\bullet(A) \xrightarrow{\ \mathrm{S}\ } \mathrm{CC}_\bullet(A)[-2] \longrightarrow 0$$

of chain complexes. Thus, there is an induced long exact sequence in homology

$$\cdots \longrightarrow \mathrm{HC}_n(A) \xrightarrow{\ \mathrm{S}\ } \mathrm{HC}_n(A)[-2] \xrightarrow{\ \delta\ } \mathrm{HH}_{n-1}(A) \xrightarrow{\ \mathrm{I}\ } \mathrm{HC}_{n-1}(A) \longrightarrow \cdots\ ,$$

where $\mathrm{HC}_n(A)[-2] = \mathrm{HC}_{n-2}(A)$ by definition and δ is defined via a standard diagram chase. In more detail, consider the partial diagram of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(A) & \xrightarrow{\ \mathrm{I}\ } & \mathrm{CC}_n(A) & \xrightarrow{\ \mathrm{S}\ } & \mathrm{CC}_n(A)[-2] & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{b}} & & \Big\downarrow & & \Big\downarrow & & \\
0 & \longrightarrow & \mathrm{C}_{n-1}(A) & \xrightarrow{\ \mathrm{I}\ } & \mathrm{CC}_{n-1}(A) & \xrightarrow{\ \mathrm{S}\ } & \mathrm{CC}_{n-1}(A)[-2] & \longrightarrow & 0
\end{array}$$

and let $x = (x_{n-2}, x_{n-4}, \dots) \in \mathrm{CC}_{n-2}(A) = \mathrm{CC}_n(A)[-2]$, with $x_{n-i} \in \Omega_k^{n-i}(A)$, be an n-cocycle of the shifted complex. The connecting morphism is constructed by first choosing any preimage along $\mathrm{S}\colon \mathrm{CC}_n(A) \to \mathrm{CC}_n(A)[-2]$, considering its image along the boundary map $\mathrm{CC}_n(A) \to \mathrm{CC}_{n-1}(A)$ and then checking that the latter lies in the image of $\mathrm{I}\colon C_{n-1}(A) \to \mathrm{CC}_{n-1}(A)$. This will then define an $(n-1)$-cocycle.

Along the shift map S, we choose a preimage of the element $(0, x_{n-2}, x_{n-4}, \dots)$. The boundary operator of the cyclic complex maps this element to $(\mathrm{B}(x_{n-2}), \mathrm{b}(x_{n-2}) + \mathrm{B}(x_{n-4}), \dots)$. Since I embeds the Hochschild complex in the first coordinate, we see that the connecting morphism associates $\mathrm{B}(x_{n-2})$ to x. □

The subtleties in the definition of the boundary for the cyclic complex disappear if we instead consider *periodic cyclic homology*. For this, let

$$\mathrm{CP}_0(A) = \prod_{m=0}^{\infty} \Omega_k^{2m}(A) \quad \text{and} \quad \mathrm{CP}_1(A) = \prod_{m=0}^{\infty} \Omega_k^{2m+1}(A)\,.$$

Then $\mathrm{b}+\mathrm{B}$ can be considered as both a map $\mathrm{CP}_0(A) \to \mathrm{CP}_1(A)$ and as a map $\mathrm{CP}_1(A) \to \mathrm{CP}_0(A)$. Moreover, $(\mathrm{b}+\mathrm{B})^2 = 0$ by Lemma 1.2.1. Thus, $(\mathrm{CP}_\bullet(A), \mathrm{b}+\mathrm{B})$ is a 2-periodic chain complex.

Definition 1.2.4. The complex $(\mathrm{CP}_\bullet(A), \mathrm{b}+\mathrm{B})$ is the **periodic cyclic complex of A**. Its n-th homology is the **n-th periodic cyclic homology of A**, denoted by $\mathrm{HP}_n(A)$. ⌟

Remark. An analogue of cyclic and periodic cyclic homology can be defined for any mixed complex.

1.3 Computations for Smooth Commutative Algebras

We close this section by briefly discussing computations of Hochschild, cyclic and periodic cyclic homology for $A = \mathcal{O}[X]$ the coordinate ring of a smooth k-variety. For this, we assume k to be of characteristic 0.

This can be deduced from the more general case of smooth k-algebras. Following [Lod98, Proposition 3.2.4], a k-algebra A is **smooth** if, for every commutative A-algebra C and ideals $I \leq C$ with $I^2 = 0$, the induced map $\mathrm{Hom}_k(A, C) \to \mathrm{Hom}_k(A, C/I)$ of sets of k-algebra homomorphisms is surjective. This agrees with A being **formally smooth** as a k-algebra in the sense of [Sta23, Tag 00TI]. Furthermore, if $A = \mathcal{O}[X]$, then [Sta23, Tag 00TN] shows that this notion agrees with the geometric notion of smoothness for k-varieties X.

Denote by $\Omega^1_{A/k} = \Omega^1_k(A)/\Omega^1_k(A)^2$ the module of **Kähler differentials of A**, or **algebraic differential 1-forms of A**, and by $\Omega^n_{A/k}$ its n-th exterior power. Let $\mathrm{d}\colon \Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ be the exterior differential. The n-th cohomology of the cochain complex $\left(\Omega^\bullet_{A/k}, \mathrm{d}\right)$ is the **n-th algebraic de Rham cohomology of A**, denoted by $\mathrm{H}^n_{\mathrm{dR}}(A)$.

Theorem 1.3.1 (Hochschild–Kostant–Rosenberg). *Let A be a smooth commutative k-algebra. The projection maps*

$$\pi_n\colon \Omega^n_k(A) \to \Omega^n_{A/k}, \quad a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n \mapsto \frac{1}{n!} a_0 \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_n\,,$$

assemble into a morphism of mixed complexes

$$\pi_\bullet\colon \left(\Omega^\bullet_k(A), \mathrm{b}, \mathrm{B}\right) \to \left(\Omega^\bullet_{A/k}, 0, \mathrm{d}\right),$$

which induces an isomorphism of cochain complexes

$$\pi_\bullet\colon (\mathrm{HH}_\bullet(A), \mathrm{B}) \xrightarrow{\sim} \left(\Omega^\bullet_{A/k}, \mathrm{d}\right).$$

Proof. The first assertion is [Lod98, Proposition 2.3.4] and does not require smoothness. The second assertion is part of [Lod98, Theorem 3.4.4], which also provides an explicit inverse. □

Corollary 1.3.2. *Let A be a smooth commutative k-algebra. Then*

$$\mathrm{HC}_n(A) \cong \Omega^n_{A/k}/\mathrm{d}\Omega^{n-1}_{A/k} \oplus \mathrm{H}^{n-2}_{\mathrm{dR}}(A) \oplus \mathrm{H}^{n-4}_{\mathrm{dR}}(A) \oplus \cdots$$

and

$$\mathrm{HP}_n(A) \cong \prod_{m=0}^{\infty} \mathrm{H}^{2m+n}_{\mathrm{dR}}(A)\,.$$

Proof. By Theorem 1.3.1, the mixed complex $\left(\Omega^\bullet_k(A), \mathrm{b}, \mathrm{B}\right)$ defining cyclic and periodic cyclic homology is quasi-isomorphic to the mixed complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\
\Omega^3_{A/k} & \xleftarrow[\mathrm{d}]{} & \Omega^2_{A/k} & \xleftarrow[\mathrm{d}]{} & \Omega^1_{A/k} & \xleftarrow[\mathrm{d}]{} & A \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \\
\Omega^2_{A/k} & \xleftarrow[\mathrm{d}]{} & \Omega^1_{A/k} & \xleftarrow[\mathrm{d}]{} & A & & \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & & & \\
\Omega^1_{A/k} & \xleftarrow[\mathrm{d}]{} & A & & & & \\
\downarrow{\scriptstyle 0} & & & & & & \\
A & & & & & &
\end{array}$$

made from the Kähler differentials of A. By our earlier observations following Definition 1.2.2, the cyclic homology of the above mixed complex is computed by the cyclic complex with chain groups

$$\Omega^n_{A/k} \oplus \Omega^{n-2}_{A/k} \oplus \cdots \oplus \Omega^{n \bmod 2}_{A/k}$$

and differentials defined componentwise, with the zero map in the first component and the exterior differential in all the other components. Thus, the first component yields the summand $\Omega^n_{A/k}/\mathrm{d}\Omega^{n-1}_{A/k}$. The remaining summands compute the appropriately indexed algebraic de Rham cohomologies $\mathrm{H}^{n-2}_{\mathrm{dR}}(A), \mathrm{H}^{n-4}_{\mathrm{dR}}(A), \ldots$, which completes the proof of the statement about cyclic homology.

The result for periodic cyclic homology is almost immediate. The periodic cyclic complex of the mixed complex of Kähler differentials has as boundary operators

$$\prod_{m=0}^{\infty} \Omega^{2m}_{A/k} \to \prod_{m=0}^{\infty} \Omega^{2m+1}_{A/k}, \quad (x_0, x_2, x_4, \ldots) \mapsto (\mathrm{d}x_0, \mathrm{d}x_2, \mathrm{d}x_4, \ldots),$$

and

$$\prod_{m=0}^{\infty} \Omega^{2m+1}_{A/k} \to \prod_{m=0}^{\infty} \Omega^{2m}_{A/k}, \quad (x_1, x_3, x_5, \ldots) \mapsto (0, \mathrm{d}x_1, \mathrm{d}x_3, \ldots).$$

As we are considering homogeneous maps, kernels and cokernels are computed coordinate-wise. Moreover, the associated quotients are componentwise, hence the periodic cyclic homology is of the form claimed. □

2 Hochschild Homology of Crossed Product Algebras

Let A be a ring and Γ a group acting on A by automorphisms. The associated **crossed product algebra** $A \rtimes \Gamma$ is additively generated by elements of the form $a\gamma$, $a \in A$ and $\gamma \in \Gamma$, with the convolution product

$$a\gamma \cdot b\delta = a\gamma(b)\gamma\delta\,.$$

Suppose that A is a $\mathbb{C}$-algebra and Γ acts on A by $\mathbb{C}$-algebra endomorphism. For $A = \mathcal{O}[X]$ the coordinate ring of a complex smooth affine variety and Γ a finite group acting on X by biregular morphisms, [BDN17] describe the Hochschild homology, cyclic homology, and periodic cyclic homology of the crossed product algebra $A \rtimes \Gamma$. We extend their results to not necessarily algebraically closed fields of characteristic 0.

In Section 2.1 we start by reducing the computations to *g-twisted Hochschild homology of A* (see Definition 2.1.1 and equation (1)). In Section 2.2 we treat the general case of algebraically closed fields of characteristic 0, showing that the arguments of [BDN17] can be adapted to this case. This is an instance of the Lefschetz principle (see [Ekl73]), which we spell out in detail. In Section 2.3 we then explain how to extend their main results to non-algebraically closed fields of characteristic 0 via base change.

As we are working over different base fields throughout Section 2, tensor products will be, in contrast to Section 1, appropriately decorated. In Section 2.3 we will also repurpose the notation $\overline{A}$ from Section 1.

2.1 Decompositions and Twisted Hochschild Homology

As in Section 1, let k be a field, A a unital k-algebra and A^{e} its enveloping algebra. The explicit left A^{e}-module structure on A is given by $(a_1 \otimes a_2) \cdot a = a_1 a a_2$. Given a k-linear endomorphism $g\colon A \to A$ of A, we consider A as a twisted right A^{e}-module A_g by defining $a \cdot (a_1 \otimes a_2) = a_2 a g(a_1)$.

For $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in A^{\otimes_k (n+1)}$, let

$$\begin{aligned}\mathrm{b}_g(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 g(a_1) \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}\,.\end{aligned}$$

This is well-defined, since g is k-linear. For $n \geq 0$, let $\mathrm{C}_n\left(A, g\right) = A \otimes_k (A/k \cdot 1)^{\otimes_k n}$. Then b_g descends to define a differential $b_g\colon \mathrm{C}_{n+1}\left(A, g\right) \to \mathrm{C}_n\left(A, g\right)$, as g is a k-linear endomorphism of A.

Definition 2.1.1. [BDN17, Definition 1.1] The complex $\left(\mathrm{C}_\bullet\left(A, g\right), \mathrm{b}_g\right)$ is the ***g*-twisted Hochschild complex of *A***. Its n-th homology is the ***n*-th *g*-twisted Hochschild homology of *A***, denoted by $\mathrm{HH}_n\left(A, g\right)$. ⌟

If g is the identity on A, $\mathrm{HH}_n\left(A, g\right) = \mathrm{HH}_n(A)$ is the ordinary Hochschild homology of A, computed from the standard complex $\left(\mathrm{C}_\bullet\left(A, g\right), \mathrm{b}_g\right) = (\mathrm{C}_\bullet(A), \mathrm{b})$. For general g, we find that

$$\mathrm{HH}_n\left(A, g\right) = \mathrm{HH}_n\left(A, A_g\right) \cong \mathrm{Tor}_n^{A^{\mathrm{e}}}\left(A_g, A\right)$$

for all $n \geq 0$. That is, the g-twisted Hochschild homology of A can be identified with the Hochschild homology of A with coefficients in A_g and consequently with certain Tor groups (cf. Section 1.1 and [Lod98, Chapter 1] or [Wei94, Chapter 9]). The latter relies on A being a k-algebra for a field k.

Suppose that A is a commutative k-algebra of finite type. Then, by acting on the first factor in the complex $\left(\mathrm{C}_\bullet\left(A, g\right), \mathrm{b}_g\right)$, each $\mathrm{HH}_n\left(A, g\right)$ inherits the structure of an A-module. Moreover, we can define an $A^{\mathrm{e}} = A \otimes_k A$-module structure on both A and A_g by setting

$$(a_1 \otimes a_2) \cdot a = g(a_1) a_2 a$$

for $a_1, a_2 \in A$ and $a \in A$ or $a \in A_g$. This gives each $\operatorname{Tor}_n^{A^{\mathrm{e}}}(A_g, A)$ the structure of an $A \otimes_k A$-module, hence also each $\mathrm{HH}_n(A, g)$.

Consider now the case of a crossed product algebra $A \rtimes \Gamma$, arising from a finite group Γ acting on the commutative k-algebra A by k-algebra automorphisms.

Now we define, for each conjugacy class of Γ, a subcomplex of the bar complex $(\mathrm{C}_\bullet(A \rtimes \Gamma), \mathrm{b})$. For this, fix $\gamma \in \Gamma$ and consider the linear span of the tensors $a_0\gamma_0 \otimes \cdots \otimes a_n\gamma_n \in \mathrm{C}_n(A \rtimes \Gamma)$ such that $\gamma_0 \cdots \gamma_n \in [\gamma]$, with $[\gamma]$ the conjugacy class of γ. Denote this subgroup by $\mathrm{C}_n(A \rtimes \Gamma)_\gamma$. The boundary map $\mathrm{b}\colon \mathrm{C}_n(A \rtimes \Gamma) \to \mathrm{C}_{n-1}(A \rtimes \Gamma)$ restricts to $\mathrm{b}\colon \mathrm{C}_n(A \rtimes \Gamma)_\gamma \to \mathrm{C}_{n-1}(A \rtimes \Gamma)_\gamma$, hence $(\mathrm{C}_\bullet(A \rtimes \Gamma)_\gamma, \mathrm{b})$ is a subcomplex. We write $\mathrm{HH}_n(A \rtimes \Gamma)_\gamma$ for its n-th homology. This gives a decomposition

$$\mathrm{HH}_n(A \rtimes \Gamma) \cong \bigoplus_{[\gamma] \in [\Gamma]} \mathrm{HH}_n(A \rtimes \Gamma)_\gamma \cong \bigoplus_{[\gamma] \in [\Gamma]} \mathrm{HH}_n(A, \gamma)^{C_\gamma} \tag{1}$$

for each $n \geq 0$ (cf. [BDN17, Proposition 1.7], with its preceding discussion, and [Lor92], which also covers the infinite case). Here $[\Gamma]$ is the set of conjugacy classes of Γ and C_γ is the centraliser of $\gamma \in \Gamma$. The above reduces the computation of the Hochschild homology of $A \rtimes \Gamma$ to understanding the γ-twisted Hochschild homology of A, viewing $\gamma \in \Gamma$ as a k-linear endomorphism $\gamma\colon A \to A$.

2.2 The Case of Algebraically Closed Fields

Throughout Section 2.2 assume k to be algebraically closed of characteristic 0.

Let $g\colon \mathcal{O}[X] \to \mathcal{O}[X]$ be a k-algebra endomorphism. The **g-twisted Connes–Hochschild–Kostant–Rosenberg map** is

$$\chi_g\colon \mathrm{HH}_n(\mathcal{O}[X], g) \to \Omega^n(X^g)$$

induced by

$$\chi_g\colon \mathcal{O}[X]^{\otimes_k (n+1)} \to \Omega^n(X^g), \quad a_0 \otimes \cdots \otimes a_n \mapsto \frac{1}{n!}\, a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n|_{X^g}\ .$$

By abuse of notation, we write $g\colon X \to X$ for the regular morphism of varieties corresponding to $g\colon \mathcal{O}[X] \to \mathcal{O}[X]$. For X and the fixed point subvariety X^g, we denote by $\Omega^n(X) = \Omega_{\mathcal{O}[X]/k}$ and $\Omega^n(X^g) = \Omega^n_{\mathcal{O}[X^g]/k}$ the respective modules of algebraic n-forms. Moreover, $a_0\mathrm{d}a_1 \ldots \mathrm{d}a_n|_{X^g}$ is the restriction of the algebraic n-form $a_0\mathrm{d}a_1 \ldots \mathrm{d}a_n \in \Omega^n(X)$ to $\Omega^n(X^g)$, induced by the k-algebra homomorphism $\mathcal{O}[X] \to \mathcal{O}[X^g]$ corresponding to the inclusion morphism $X^g \to X$. This is to say that $a_0\mathrm{d}a_1 \ldots \mathrm{d}a_n|_{X^g}$ is the pullback of $a_0\mathrm{d}a_1 \ldots \mathrm{d}a_n$ along the inclusion $X^g \to X$.

For ease of notation, we will now assume that A is a commutative k-algebra of finite type, equipped with a k-algebra endomorphism $g\colon A \to A$. This suffices as we are only interested in the case that $A = \mathcal{O}[X]$, for X a smooth affine variety over k, and $g\colon \mathcal{O}[X] \to \mathcal{O}[X]$ an endomorphism as above.

A Vanishing Condition

Let $\operatorname{Spec}_{\max}(A)$ be the maximal spectrum of A. We first deduce a vanishing condition for $\mathrm{HH}_n(A, g)$, the n-th g-twisted Hochschild homology A of Definition 2.1.1, localised at a maximal ideal $\mathfrak{m} \in \operatorname{Spec}_{\max}(A)$.

Lemma 2.2.1. *If $\mathfrak{m} \in \operatorname{Spec}_{\max}(A)$ is such that $g^{-1}(\mathfrak{m}) \neq \mathfrak{m}$, then $\mathrm{HH}_n(A, g)_{\mathfrak{m}} = 0$ for all $n \geq 0$.*

Remark. For $k = \mathbb{C}$, this is [BDN17, Corollary 1.6].

In order to adapt Lemma 2.2.1 to the general case of algebraically closed fields, we first identify the scheme-theoretic product $\mathrm{Spec}_{\max}(A \otimes_k A)$ with the set-theoretic product $\mathrm{Spec}_{\max}(A) \times \mathrm{Spec}_{\max}(A)$. This identification notably fails for non-algebraically closed fields.

Let $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Spec}_{\max}(A)$ be two maximal ideals. Denote by $\chi_{\mathfrak{m}_1} \colon A \to K_{\mathfrak{m}_1}$ and $\chi_{\mathfrak{m}_2} \colon A \to K_{\mathfrak{m}_2}$ the canonical homomorphisms onto finite field extensions $K_{\mathfrak{m}_1}$ and $K_{\mathfrak{m}_2}$ of k with kernels $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. Since k is algebraically closed, these induce isomorphisms $\chi_{\mathfrak{m}_1} \colon A/\mathfrak{m}_1 \to k$ and $\chi_{\mathfrak{m}_1} \colon A/\mathfrak{m}_2 \to k$. Thus, we may associate to any $(\mathfrak{m}_1, \mathfrak{m}_2) \in \mathrm{Spec}_{\max}(A) \times \mathrm{Spec}_{\max}(A)$ the homomorphism $\chi_{\mathfrak{m}_1} \otimes \chi_{\mathfrak{m}_2} \colon A \otimes_k A \to k$, which corresponds to a point of $\mathrm{Spec}_{\max}(A \otimes_k A)$.

Conversely, consider a point of $\mathrm{Spec}_{\max}(A \otimes_k A)$, i.e. a surjection $\chi \colon A \otimes_k A \to k$ with kernel $\mathfrak{M}$, a maximal ideal of $A \otimes_k A$. Write $\iota_1 \colon A \to A \otimes_k A$ and $\iota_2 \colon A \to A \otimes_k A$ for the two canonical inclusions. Since $A \otimes_k A$ is the coproduct of k-algebras, we find $\chi_1, \chi_2 \colon A \to k$ such that $\chi = \chi_1 \otimes \chi_2$. Moreover, $\chi\iota_1 = \chi_1$ and $\chi\iota_2 = \chi_2$. Since A is a finite type k-algebra, so is $A \otimes_k A$, hence [Kem11, Proposition 1.2] shows that both $\mathfrak{m}_1 = \iota_1^{-1}(\mathfrak{M})$ and $\mathfrak{m}_2 = \iota_2^{-1}(\mathfrak{M})$ are maximal. Observe that

$$\mathfrak{m}_i = \iota_i^{-1}(\mathfrak{M}) = \iota_i^{-1}\big(\ker(\chi)\big) = \ker(\chi_1)$$

for $i = 1, 2$. Then $\mathfrak{m}_1 \otimes A + A \otimes \mathfrak{m}_2 \leq \mathfrak{M}$ by construction. Since the residue field of $\mathfrak{m}_1 \otimes A + A \otimes \mathfrak{m}_2$ is $k \otimes_k k \cong k$, we conclude that $\mathfrak{m}_1 \otimes A + A \otimes \mathfrak{m}_2 = \mathfrak{M}$.

The proof of [BDN17, Corollary 1.6] now goes through verbatim.

Completions and Local Hochschild Homology

We will now consider completions of A-modules with respect to certain filtrations. A **filtration** of an A-module M is a decreasing chain of submodules

$$M = F_0 M \supset F_1 M \supset F_2 M \supset \cdots \supset F_n M \supset \cdots .$$

To each filtration there is an associated **completion**, which is defined by

$$\widehat{M} = \varprojlim M/F_n M .$$

We say that M is **complete** if the natural map $M \to \widehat{M}$ is an isomorphism.

We specialise to considering I-adic completions of A and A-modules. Recall that given an ideal $I \leq A$ and an A-module M, its ***I*-adic completion** is

$$\widehat{M}_I = \varprojlim M/I^n M .$$

Furthermore, for two A-modules M and N, their **completed tensor product** is

$$M \mathbin{\widehat{\otimes}_A} N = \varprojlim M/I^n M \otimes_A N/I^n N .$$

Suppose now that $g \colon A \to A$ preserves the chosen $I \leq A$ in the sense that $g^{-1}(I) \subseteq I$. Then g induces a well-defined endomorphism $\widehat{g}_I \colon \widehat{A}_I \to \widehat{A}_I$. Completion of the complex $\big(\mathrm{C}_\bullet(A, g), b_g\big)$ produces a complex $\big(\widehat{\mathrm{C}}_\bullet(\widehat{A}_I, \widehat{g}_I), b_{\widehat{g}_I}\big)$ with $\widehat{\mathrm{C}}_n(\widehat{A}_I, \widehat{g}_I) = \widehat{A}_I \mathbin{\widehat{\otimes}_k} \big(\widehat{A}_I / k \cdot 1\big)^{\widehat{\otimes}_k n}$, $n \geq 0$, and an induced differential $b_{\widehat{g}_I} \colon \widehat{\mathrm{C}}_n(\widehat{A}_I, \widehat{g}_I) \to \widehat{\mathrm{C}}_{n-1}(\widehat{A}_I, \widehat{g}_I)$.

Definition 2.2.2. The complex $\big(\widehat{\mathrm{C}}_\bullet(\widehat{A}_I, \widehat{g}_I), b_{\widehat{g}_I}\big)$ is the **local *g*-twisted Hochschild complex of *A***. Its n-th homology is the ***n*-th local *g*-twisted homology of *A***, denoted by $\mathrm{HH}_n^{\mathrm{loc}}(\widehat{A}_I, \widehat{g}_I)$. ⌟

We introduce this variant of the Hochschild homology for the localised study of the g-twisted homology. For its application, we have to restrict the algebras A in consideration.

Suppose that A is an R-algebra for R some commutative k-algebra. We say that A is a **finite type *R*-algebra** if R is of finite type and A is finitely generated as an R-module. Assuming that A is a finite type R-algebra guarantees a well-behaved completion functor (see [AM69, §10]). These assumptions are satisfied, in particular, if $R = A$ is already a commutative k-algebra of finite type.

We obtain the following.

Lemma 2.2.3. *For all $n \geq 0$, $\mathrm{HH}_n\left(A, g\right)$ is finitely generated as an A-module. The natural map*

$$\mathrm{HH}_n\left(A, g\right) \to \mathrm{HH}_n^{\mathrm{loc}}\left(\widehat{A}_I, \widehat{g}_I\right)$$

induces an isomorphism

$$\widehat{A}_I \otimes_A \mathrm{HH}_n\left(A, g\right) \xrightarrow{\sim} \mathrm{HH}_n^{\mathrm{loc}}\left(\widehat{A}_I, \widehat{g}_I\right)$$

of A-modules for all $n \geq 0$.

Remark. For $k = \mathbb{C}$, this is [BDN17, Theorem 1.10].

The proof requires establishing the existence of an **admissible resolution** of $\widehat{A}_I$, that is, a resolution admitting bounded k-linear contractions. Here, bounded is a purely algebraic notion. More precisely, a map $\phi\colon V \to W$ of filtered complexes V and W is **bounded** if there exists an integer k such that $\phi(F_n V) \subseteq F_{n-k} W$ for all n. Such contractions are constructed in [KNS98, Lemma 3] and their existence does not depend on the chosen base field.

The proof of [BDN17, Theorem 1.10] can now be repeated for general algebraically closed fields.

Computations with Koszul Complexes

As a final computational ingredient, we consider *Koszul complexes*.

Definition 2.2.4. Let R be a commutative k-algebra, M an R-module, E a finite-dimensional k-vector space, and $f\colon E \to R$ a k-linear map. The associated **Koszul complex** $(\mathbb{K}_\bullet, \partial) = \left(\mathbb{K}_\bullet(M, E, f), \partial\right)$ has as n-th chain group

$$\mathbb{K}_n = \mathbb{K}_n(M, E, f) = M \otimes_k \bigwedge\nolimits^n E$$

and as differential

$$\partial(m \otimes (v_{i_1} \wedge \cdots \wedge v_{i_n})) = \sum_{j=1}^{n} (-1)^{j-1} f(v_{i_j}) m \otimes (v_{i_1} \wedge \cdots \wedge \widehat{v_{i_j}} \wedge \cdots \wedge v_{i_n}),$$

where $m \in M$ and $e_1, \ldots, e_r$ are a basis of V. ⌟

We are especially interested in the case where $M = \mathcal{O}[E]$ is the ring of regular functions on E. Here E will geometrically correspond to the (Zariski) tangent space of X at some point. We view E as an affine space over k, hence $\mathcal{O}[E]$ is a ring of polynomials over k in $\dim E$ variables. This uses that k is algebraically closed. Denote by $i\colon E^\vee \to \mathcal{O}[E]$ the canonical embedding of the dual space $E^\vee$ into $\mathcal{O}[E]$, considering a linear functional on E as a regular function on E.

If $g\colon E \to E$ is a linear endomorphism of E we let $f = i \circ (g^\vee - 1)\colon E^\vee \to \mathcal{O}[E]$. Denote by g also the endomorphism of $\mathcal{O}[E]$ induced by g. This gives rise to a Koszul complex $\left(\mathbb{K}_\bullet(\mathcal{O}[E], E^\vee, f), \partial\right)$. These complexes compute the g-twisted Hochschild homology of $\mathcal{O}[E]$.

Lemma 2.2.5. *Fix a basis $e_1, \ldots, e_r$ of E. The map $\kappa_V\colon \left(\mathbb{K}_\bullet(\mathcal{O}[E], E^\vee, f), \partial\right) \to \left(\mathrm{C}_\bullet\left(\mathcal{O}[E], g\right), b_g\right)$ defined by*

$$\kappa_V(a \otimes (e_{i_1} \wedge \cdots \wedge e_{i_n})) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) a \otimes i(e_{i_{\sigma(1)}}) \otimes \cdots \otimes i(e_{i_{\sigma(n)}})$$

is a quasi-isomorphism.

Remark. For $k = \mathbb{C}$, this is [BDN17, Corollary 2.11].

To prove Lemma 2.2.5 one shows that the above Koszul complex may be obtained from tensoring a projective resolution of $\mathcal{O}[E]$ as left $\mathcal{O}[E]^{\mathrm{e}}$-module. This projective resolution is built by reducing to the case of $\mathcal{O}[E] = k[x]$. Here a direct computation can be carried out completing the argument. No particular properties of $\mathbb{C}$ are used besides being algebraically closed to identify maximal ideals of $\mathcal{O}[E]$ with points of E.

Finally, replacing the unhandy bar complexes by concrete Koszul complexes allows us to compute the g-twisted Hochschild homology of E in terms of the Kähler differentials of the subspace E^g of g-fixed points.

Lemma 2.2.6. *Let $g\colon E \to E$ be linear and assume that $g - 1\colon E/\ker(g-1) \to E/\ker(g-1)$ is injective. Then for $E^g = \ker(g-1)$, the restriction $\mathcal{O}[E] \to \mathcal{O}\left[E^g\right]$ defines isomorphisms*

$$\mathrm{res}_{\mathrm{HH}}\colon \mathrm{HH}_n\left(\mathcal{O}[E], g\right) \xrightarrow{\sim} \mathrm{HH}_n\left(\mathcal{O}\left[E^g\right]\right)$$

and hence the g-twisted Connes–Hochschild–Kostant–Rosenberg map χ_g defines isomorphisms

$$\chi_g\colon \mathrm{HH}_n\left(\mathcal{O}[E], g\right) \xrightarrow{\sim} \Omega^n\left(E^g\right)$$

for all $n \geq 0$.

Remark. For $k = \mathbb{C}$, this is [BDN17, Lemma 2.12].

Using Lemma 2.2.5 reduces the question to the computation of homologies arising from certain Koszul complexes. Studying these complexes is a question of linear algebra and re-using earlier results about Koszul complexes of polynomial rings. The technical assumption about g is necessary to unravel how g interacts with the inclusion $E^\vee \to \mathcal{O}[E]$. In particular, the proofs remain unchanged in the case of a general algebraically closed base field.

One now combines the results obtained using Koszul complexes with the local g-twisted Hochschild homology. The following is then immediate.

Lemma 2.2.7. *Let $g\colon E \to E$ be linear and assume that $g - 1\colon E/\ker(g-1) \to E/\ker(g-1)$ is injective. Let $E^g = \ker(g-1)$ and $\mathfrak{m} \leq \mathcal{O}[E]$ the maximal ideal corresponding to functions vanishing at 0. Then the restriction $\widehat{\mathcal{O}}[E]_{\mathfrak{m}} \to \widehat{\mathcal{O}}\left[E^g\right]_{\mathfrak{m}}$ defines an isomorphism*

$$\widehat{\mathrm{res}}_{\mathrm{HH}}\colon \mathrm{HH}_n^{\mathrm{loc}}\left(\widehat{\mathcal{O}}[E]_{\mathfrak{m}}, \widehat{g}_{\mathfrak{m}}\right) \xrightarrow{\sim} \mathrm{HH}_n^{\mathrm{loc}}\left(\widehat{\mathcal{O}}\left[E^g\right]_{\mathfrak{m}}\right)$$

and $\widehat{\chi}_g = 1 \otimes \chi_g$ gives an isomorphism

$$\widehat{\chi}_g\colon \widehat{\mathcal{O}}[E]_{\mathfrak{m}} \otimes_{\mathcal{O}[E]} \mathrm{HH}_n\left(\mathcal{O}[E], g\right) \xrightarrow{\sim} \widehat{\mathcal{O}}[E]_{\mathfrak{m}} \otimes_{\mathcal{O}[E]} \Omega^n\left(E^g\right)$$

for all $n \geq 0$.

Remark. For $k = \mathbb{C}$, this is [BDN17, Corollary 2.15].

Deducing the Hochschild Homology

After due preparation, we return to the case of a smooth affine k-variety X, and $g\colon \mathcal{O}[X] \to \mathcal{O}[X]$ a k-algebra endomorphism of its coordinate ring.

Suppose that g is induced by a finite group acting on X. In particular, g has finite order. Then by

[Edi92, Proposition 3.4], its fixed point subvariety $X^g \subseteq X$ is again smooth, since we assume that k is of characteristic 0. Furthermore, [Edi92, Proposition 3.2] computes the tangent space of $x \in X^g$ as $T_x X^g = \left(T_x X\right)^{T_x(g)} = \ker(T_x(g) - 1)$ where $T_x(g)$ denotes the induced tangent map of $g\colon X \to X$.

Finally, observe that no non-zero element of $T_x X$ can at the same time be fixed by $T_x(g)$ and lie in the image of $T_x(g) - 1$. Indeed, if $T_x(g)(v) = v$ and $v = \left(T_x(g) - 1\right)(w)$, then considering its algebraic norm, we see that

$$nv = \sum_{i=0}^{n-1} T_x(g)^i(v) = \sum_{i=0}^{n-1} \left(T_x(g)^{i+1} - T_x(g)^i\right)(w) = 0$$

for $n = |g|$ the order of g. Thus, $T_x(g)$ induces an injective endomorphism of $T_x X / T_x X^g$.

The above shows that the technical assumptions made in the following proposition are negligible in our case of interest, hence may be safely ignored in the sequel. These assumptions are, however, necessary in order to apply Lemma 2.2.7 for E the tangent space at a fixed point of g.

We give the full proof of [BDN17, Proposition 2.16], while expanding on some details and simplifying the final step.

Proposition 2.2.8. *Let X be a smooth affine k-variety and g a k-algebra endomorphism of $\mathcal{O}[X]$.*

Suppose that X^g is again a smooth affine variety and, for every fixed point $x \in X^g$, its tangent space $T_x X^g$ is the kernel of $T_x(g) - 1$ and $T_x(g) - 1$ induces an injective endomorphism of $T_x X / T_x X^g$.

Then the g-twisted Connes–Hochschild–Kostant–Rosenberg map χ_g induces isomorphisms

$$\chi_g\colon \operatorname{HH}_n\left(\mathcal{O}[X], g\right) \xrightarrow{\sim} \Omega^n\left(X^g\right)$$

for all $n \geq 0$.

Remark. For $k = \mathbb{C}$, this is [BDN17, Proposition 2.16].

Proof. For clarity, we write χ_g^X for the Connes–Hochschild–Kostant–Rosenberg map within the proof. We will prove that χ_g^X is an isomorphism by showing that for every maximal $\mathfrak{m} \leq \mathcal{O}[X]$, its completion $\widehat{\left(\chi_g^X\right)_{\mathfrak{m}}}$ is an isomorphism. Since the completion factors as $\mathcal{O}[X] \to \mathcal{O}[X]_{\mathfrak{m}} \to \widehat{\mathcal{O}}[X]_{\mathfrak{m}}$ and the map $\mathcal{O}[X]_{\mathfrak{m}} \to \widehat{\mathcal{O}}[X]_{\mathfrak{m}}$ is faithfully flat by [Sta23, Tag 00MC], this reduces the assertion to a well-known criterion: An R-module homomorphism $\varphi\colon M \to N$ is an isomorphism if and only if its localisation $\varphi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ at each maximal ideal $\mathfrak{m} \leq R$ is an isomorphism.

Let us denote $A = \mathcal{O}[X]$ and fix a maximal ideal $\mathfrak{m} \leq A$. We consider

$$\widehat{A}_{\mathfrak{m}} \otimes_A \operatorname{HH}_n\left(A, g\right) \cong \widehat{\operatorname{HH}}_n\left(A, g\right)_{\mathfrak{m}} \xrightarrow{\widehat{\left(\chi_g^X\right)_{\mathfrak{m}}}} \widehat{\Omega}^n\left(X^g\right)_{\mathfrak{m}} \cong \widehat{A}_{\mathfrak{m}} \otimes_A \Omega^n\left(X^g\right)$$

in two cases, distinguished by whether the geometric morphism $g\colon X \to X$ fixes the point corresponding to $\mathfrak{m}$ or not. Since the points of X are identified with the maximal ideals of A, this leaves us to consider if $g^{-1}(\mathfrak{m}) = \mathfrak{m}$ or $g^{-1}(\mathfrak{m}) \neq \mathfrak{m}$.

First, assume that $g^{-1}(\mathfrak{m}) \neq \mathfrak{m}$. By Lemma 2.2.1, we then conclude that $\operatorname{HH}_n\left(A, g\right)_{\mathfrak{m}} = 0$, hence $\widehat{\operatorname{HH}}_n\left(A, g\right)_{\mathfrak{m}} = 0$, since the localisation is dense in the completion. Additionally, we claim that $\Omega^n\left(X^g\right)_{\mathfrak{m}} = 0$, which implies that $\widehat{\Omega}^n\left(X^g\right)_{\mathfrak{m}} = 0$ as before. For the first claim, note that $X^g \subseteq X$ is a closed immersion by [Edi92, Proposition 3.1], hence $\mathcal{O}[X] \to \mathcal{O}\left[X^g\right]$ is a quotient map by [Sta23, Tag 01QN]. If we let $I \leq \mathcal{O}[X]$ be the ideal defining $\mathcal{O}\left[X^g\right]$, the conormal sequence reads

$$I/I^2 \longrightarrow \Omega^1(X) \otimes_A \mathcal{O}\left[X^g\right] \longrightarrow \Omega^1\left(X^g\right) \longrightarrow 0$$

with $\Omega^1(X) = \Omega^1_{A/k} = \Omega^1_{\mathcal{O}[X]/k}$ and $\Omega^1(X^g) = \Omega_{\mathcal{O}[X^g]/k}$. If $g^{-1}(\mathfrak{m}) \neq \mathfrak{m}$, then $\mathfrak{m}$ does not correspond to a closed point of X^g, hence the induced ideal in $\mathcal{O}[X^g]$ has to be the zero ideal. Thus, localising the conormal sequence at such $\mathfrak{m}$ implies that $\Omega^1(X^g)_{\mathfrak{m}} = 0$ by exactness, since $\mathcal{O}[X^g]_{\mathfrak{m}} = 0$. Thus, for $g^{-1}(\mathfrak{m}) \neq \mathfrak{m}$ we observe that $\widehat{(\chi^X_g)}_{\mathfrak{m}}$ is an isomorphism, as both sides vanish.

Now, assume that $g^{-1}(\mathfrak{m}) = \mathfrak{m}$. Write $\mathfrak{m} = \mathfrak{m}_x$ with $x \in X^g$. Denote by $E = T_x X = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ the tangent space of X at x. Since we assume X to be smooth, there are natural isomorphisms

$$\tan\colon \widehat{\mathcal{O}}[X]_{\mathfrak{m}_x} \xrightarrow{\sim} \widehat{\mathcal{O}}[E]_{\mathfrak{m}_0}$$

and

$$\tan\colon \widehat{\Omega}^n(X^g)_{\mathfrak{m}_x} \xrightarrow{\sim} \widehat{\Omega}^n\left(E^{T_x(g)}\right)_{\mathfrak{m}_0},$$

which are discussed in [Liu06, §4, §6]. On the right, $\mathfrak{m}_0$ corresponds to the origin of E as a k-vector space. We consider the induced tangent map $T_x(g)\colon E \to E$, which is well-defined, as g is assumed to fix $\mathfrak{m}_x$. The second identification uses our assumption $T_x X^g = \ker\left(T_x(g) - 1\right) = E^{T_x(g)}$.

The canonical isomorphism from Lemma 2.2.3 now gives rise to a commuting diagram

$$\begin{array}{ccc}
\widehat{A}_{\mathfrak{m}} \otimes_A \mathrm{HH}_n(A, g) & \xrightarrow{1\otimes\chi^X_g} & \widehat{A}_{\mathfrak{m}} \otimes_A \Omega^n(X^g) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{HH}^{\mathrm{loc}}_n\left(\widehat{A}_{\mathfrak{m}}, \widehat{g}_{\mathfrak{m}}\right) & \xrightarrow{\widehat{(\chi^X_g)}_{\mathfrak{m}}} & \widehat{\Omega}^n(X^g)_{\mathfrak{m}} \\
\tan\downarrow & & \downarrow\tan \\
\mathrm{HH}^{\mathrm{loc}}_n\left(\widehat{\mathcal{O}}[E]_{\mathfrak{m}}, \widehat{T_x(g)}_{\mathfrak{m}}\right) & \xrightarrow{\widehat{(\chi^E_{T_x(g)})}_{\mathfrak{m}}} & \widehat{\Omega}^n\left(E^{T_x(g)}\right)_{\mathfrak{m}}
\end{array}$$

where all vertical arrows are isomorphisms. Thus, it suffices to check that the completion of $\chi^E_{T_x(g)}$ at the bottom is an isomorphism to complete the proof. Since $T_x(g)\colon E \to E$ is a k-linear endomorphism so that $T_x(g)\colon E/E^{T_x(g)} \to E/E^{T_x(g)}$ is injective, Lemma 2.2.7 applies. This shows that the completion of $\chi^E_{T_x(g)}$ is an isomorphism, which concludes the proof. □

Combining Proposition 2.2.8 with the decomposition from equation (1), we complete our computation of the Hochschild homology of $A \rtimes \Gamma$ for a finite group Γ.

Theorem 2.2.9. *Let $A = \mathcal{O}[X]$ be the coordinate ring of a smooth k-variety. Assume that k is of characteristic* 0 *and algebraically closed. Suppose that a finite group Γ acts on X and let $\gamma_1, \dots, \gamma_s$ be a set of representatives for the conjugacy classes of Γ. If $X_i = X^{\gamma_i}$ and $C_i = C_{\gamma_i}$, then*

$$\mathrm{HH}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \Omega^n(X_i)^{C_i}.$$

Since Γ is finite, both the cyclic and the periodic cyclic homology of $A \rtimes \Gamma$ admit analogous decompositions (see [BDN17, Proposition 1.7] and the preceding discussion). The cyclic and periodic cyclic theories can now be deduced from Theorem 2.2.9 as in Corollary 1.3.2.

Corollary 2.2.10. *Let $A = \mathcal{O}[X]$ be the coordinate ring of a smooth k-variety. Assume that k is of characteristic* 0 *and algebraically closed. Suppose that a finite group Γ acts on X and let $\gamma_1, \dots, \gamma_s$ be a set of representatives for* *the conjugacy classes of* Γ. *If* $X_i = X^{\gamma_i}$, $A_i = \mathcal{O}[X_i]$ *and* $C_i = C_{\gamma_i}$, *then*

$$\mathrm{HC}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\Omega^n(X_i)^{C_i} / \mathrm{d}\Omega^{n-1}(X_i)^{C_{\gamma_i}} \oplus \mathrm{H}^{n-2}_{\mathrm{dR}}(A_i)^{C_i} \oplus \mathrm{H}^{n-4}_{\mathrm{dR}}(A_i)^{C_i} \oplus \cdots \right)$$

and

$$\mathrm{HP}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\prod_{m=0}^{\infty} \mathrm{H}^{2m+n}_{\mathrm{dR}}(A_i)^{C_i} \right).$$

Remark. For $k = \mathbb{C}$, Theorem 2.2.9 and Corollary 2.2.10 are [BDN17, Proposition 2.18].

2.3 Extension to Non-Algebraically Closed Fields

Now assume k not to be algebraically closed, but still of characteristic 0. Fix an algebraic closure $\overline{k}$.

Assume that B is a unital k-algebra equipped with a k-linear endomorphism $h\colon B \to B$. Denote its base change by $\overline{B} = \overline{k} \otimes_k B$. Then there is an induced $\overline{k}$-linear endomorphism $1_{\overline{k}} \otimes h\colon \overline{k} \otimes_k B \to \overline{k} \otimes_k B$, which we write as $\overline{h}\colon \overline{B} \to \overline{B}$.

Let X be a smooth affine variety over k and denote by $\overline{X}$ its base change along $\mathrm{Spec}\left(\overline{k}\right) \to \mathrm{Spec}(k)$. As X is affine, $X = \mathrm{Spec}(\mathcal{O}[X])$, hence $\overline{X} = \mathrm{Spec}\left(\overline{k} \otimes_k \mathcal{O}[X]\right) = \mathrm{Spec}\left(\overline{\mathcal{O}[X]}\right)$. Moreover, $\overline{X}$ is smooth by definition (see [Liu06, §4, §6]). If $\mathcal{O}[X]$ is in addition endowed with a k-linear endomorphism g of finite order, then $\overline{\mathcal{O}[X]}$ admits a corresponding $\overline{k}$-linear endomorphism $\overline{g}$ of finite order.

Thus, Proposition 2.2.8 identifies the $\overline{g}$-twisted homology of $\overline{\mathcal{O}[X]} = \mathcal{O}\left[\overline{X}\right]$ with the Kähler differentials of its fixed point subvariety $\overline{X}^{\overline{g}}$. More precisely, the twisted Connes–Hochschild–Kostant–Rosenberg map $\chi_{\overline{g}}$ induces isomorphisms

$$\chi_{\overline{g}}\colon \mathrm{HH}_n\left(\overline{\mathcal{O}[X]}, \overline{g}\right) \xrightarrow{\sim} \Omega^n\left(\overline{X}^{\overline{g}}\right)$$

for all $n \geq 0$.

We first show that $\chi_{\overline{g}}$ being an isomorphism implies that $1_{\overline{k}} \otimes \chi_g$ is an isomorphism as well, by appropriately identifying their domains and codomains. Let us start with these identifications.

Lemma 2.3.1. *Let* $A = \mathcal{O}[X]$ *and* $g\colon A \to A$ *as above. Then*

$$\overline{A} \otimes_A \left(A \otimes_k (A/k \cdot 1)^{\otimes_k n} \right) \to \overline{A} \otimes_{\overline{k}} \left(\overline{A}/\overline{k} \cdot 1 \right)^{\otimes_{\overline{k}} n},$$
$$(\lambda \otimes a) \otimes (a_0 \otimes a_1 \otimes \cdots \otimes a_n) \mapsto (\lambda \otimes (aa_0)) \otimes ((1 \otimes a_1) \otimes \cdots \otimes (1 \otimes a_n)),$$

induces an isomorphism

$$\overline{A} \otimes_A \mathrm{HH}_n(A, g) \xrightarrow{\sim} \mathrm{HH}_n\left(\overline{A}, \overline{g}\right).$$

Proof. Since $k \to \overline{k}$ is a field extension, hence faithfully flat, the base change $A \to \overline{A}$ is faithfully flat as well by [Sta23, Tag 00HI]. Hence, tensoring with $\overline{A}$ over A commutes with taking homology. Thus, it suffices to prove that the given map extends to an isomorphism of chain complexes. This is immediate, as the canonical identifications

$$\overline{A} \otimes_A \left(A \otimes_k (A/k \cdot 1)^{\otimes_k n} \right) \cong \overline{A} \otimes_{\overline{k}} \left(\overline{A} \otimes_k (A/k \cdot 1) \right)^{\otimes_{\overline{k}} n} \cong \overline{A} \otimes_{\overline{k}} \left(\overline{A}/\overline{k} \cdot 1 \right)^{\otimes_{\overline{k}} n}$$

together define the given map. Since g is k-linear and $\overline{g} = 1_{\overline{k}} \otimes g$ acts identically on the $\overline{k}$-factor, the above isomorphism commutes with both b_g and $b_{\overline{g}}$. ☐

Lemma 2.3.2. *Let $A = \mathcal{O}[X]$ and $g\colon A \to A$ as above. Then*

$$\overline{A} \otimes_A \Omega^n\left(X^g\right) \to \Omega^n\left(\overline{X^g}\right),$$
$$(\lambda \otimes a) \otimes (a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n) \mapsto (\lambda \otimes (aa_0))\mathrm{d}(1 \otimes a_1) \ldots \mathrm{d}(1 \otimes a_n)\,,$$

is an isomorphism. Furthermore, there is an isomorphism $\overline{X^g} \xrightarrow{\sim} \overline{X}^{\overline{g}}$ such that the diagram

$$\begin{array}{ccc} \overline{X^g} & \xrightarrow{\ \sim\ } & \overline{X}^{\overline{g}} \\ & \searrow \qquad \swarrow & \\ & \overline{X} & \end{array}$$

commutes.

Proof. For $n = 1$, [Sta23, Tag 00RV] shows that

$$\overline{A} \otimes_A \Omega^1_{\mathcal{O}[X^g]/k} \cong \overline{k} \otimes_k \Omega^1_{\mathcal{O}[X^g]/k} \cong \Omega^1_{\mathcal{O}[\overline{X^g}]/\overline{k}}\,,$$

since $\overline{k} \otimes_k \mathcal{O}\left[X^g\right] = \mathcal{O}\left[\,\overline{X^g}\,\right]$. This isomorphism is explicitly defined by the formula corresponding to the case $n = 1$ as given. The general case then follows by construction of $\Omega^n\left(X^g\right)$ and $\Omega^n\left(\overline{X}^{\overline{g}}\right)$ as exterior powers of $\Omega^1\left(X^g\right)$ and $\Omega^1\left(\overline{X}^{\overline{g}}\right)$, respectively.

We first show that $\mathcal{O}\left[\,\overline{X^g}\,\right] \cong \mathcal{O}\left[\overline{X}^{\overline{g}}\right]$ or, equivalently, that $\overline{X^g} \cong \overline{X}^{\overline{g}}$. For the last assertion, we observe that $X^g \subseteq X$ as well as $\overline{X}^{\overline{g}} \subseteq \overline{X}$ can be described as fibre products

$$\begin{array}{ccc} X^g & \hookrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 1\times g} \\ X & \xrightarrow[\Delta_X]{} & X \times X \end{array} \quad \text{and} \quad \begin{array}{ccc} \overline{X}^{\overline{g}} & \hookrightarrow & \overline{X} \\ \downarrow & & \downarrow{\scriptstyle 1\times \overline{g}} \\ \overline{X} & \xrightarrow[\Delta_{\overline{X}}]{} & \overline{X} \times \overline{X} \end{array},$$

respectively (see [Edi92, Proposition 3.1]). As fibre products are preserved by base change and $\overline{g} = 1_{\overline{k}} \otimes g$ algebraically, we conclude that $\overline{X^g} \cong \overline{X}^{\overline{g}}$. In particular, the naturally induced morphism $\overline{X^g} \to \overline{X}^{\overline{g}}$ is an isomorphism so that the diagram

$$\begin{array}{ccc} \overline{X^g} & \longrightarrow & \overline{X}^{\overline{g}} \\ & \searrow \qquad \swarrow & \\ & \overline{X} & \end{array}$$

commutes. This concludes the proof. □

Lemma 2.3.3. *Let $A = \mathcal{O}[X]$ and $g\colon A \to A$ as above. Then*

$$1_{\overline{k}} \otimes \chi_g \colon \overline{A} \otimes_A \mathrm{HH}_n\left(A, g\right) \to \overline{A} \otimes_A \Omega^n\left(X^g\right)$$

is an isomorphism.

Proof. At the level of chain complexes, the g-twisted Connes–Hochschild–Kostant–Rosenberg map χ_g is the composition of

$$A \otimes_k (A/k \cdot 1)^{\otimes_k n} \to \Omega^n(X), \quad a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n \mapsto \frac{1}{n!} a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_n\,,$$

with the restriction

$$\Omega^n(X) \to \Omega^n(X^g), \quad a_0 \mathrm{d}a_1 \dots \mathrm{d}a_n \mapsto a_0 \mathrm{d}a_1 \dots \mathrm{d}a_n|_{X^g} .$$

The latter is induced by the k-algebra homomorphisms $\mathcal{O}[X] \to \mathcal{O}\left[X^g\right]$ corresponding to the inclusion $X^g \hookrightarrow X$. Analogously, we may describe $\chi_{\overline{g}}$.

Consider now the diagram

$$\begin{array}{ccccc}
\overline{A} \otimes_A \left(A \otimes_k (A/k \cdot 1)^{\otimes_k n}\right) & \longrightarrow & \overline{A} \otimes_A \Omega^n(X) & \longrightarrow & \overline{A} \otimes_A \Omega^n\left(X^g\right) \\
\downarrow & & \downarrow & & \downarrow \\
\overline{A} \otimes_{\overline{k}} \left(\overline{A}/\overline{k} \cdot 1\right)^{\otimes_{\overline{k}} n} & \longrightarrow & \Omega^n\left(\overline{X}\right) & \longrightarrow & \Omega^n\left(\overline{X^g}\right) \\
\| & & \| & & \uparrow \\
\overline{A} \otimes_{\overline{k}} \left(\overline{A}/\overline{k} \cdot 1\right)^{\otimes_{\overline{k}} n} & \longrightarrow & \Omega^n\left(\overline{X}\right) & \longrightarrow & \Omega^n\left(\overline{X}^{\overline{g}}\right)
\end{array}$$

obtained from these factorisations and using the second part of Lemma 2.3.2. The top row defines $1_{\overline{k}} \otimes \chi_g$, whereas the bottom row defines $\chi_{\overline{g}}$. Finally, we know all vertical maps to be isomorphisms by Lemma 2.3.1 and Lemma 2.3.2. Thus, if we show that the diagram commutes, the result follows, as we know that $\chi_{\overline{g}}$ is an isomorphism by Proposition 2.2.8.

The lower part of the diagram commutes by the second part of Lemma 2.3.2. The top-left square commutes, since

$$(\lambda \otimes aa_0) \otimes \left(\frac{1}{n!}\mathrm{d}(1 \otimes a_1) \dots \mathrm{d}(1 \otimes a_n)\right) = (\lambda \otimes a)\left(\frac{1}{n!}(1 \otimes a_0)\right)\mathrm{d}(1 \otimes a_1) \dots \mathrm{d}(1 \otimes a_n)$$

by bilinearity and the induced algebra structure on $\overline{A} = \bar{k} \otimes_k A$. The top-right square commutes as

$$\overline{\pi}(\lambda \otimes (aa_0))\mathrm{d}(\overline{\pi}(1 \otimes a_1)) \dots \mathrm{d}(\overline{\pi}(1 \otimes a_n)) = (\lambda \otimes a\pi(a_0))\mathrm{d}(1 \otimes \pi(a_1)) \dots \mathrm{d}(1 \otimes \pi(a_n)),$$

where $\pi \colon \mathcal{O}[X] \to \mathcal{O}[X^g]$ corresponds to $X^g \to X$ and $\overline{\pi} = 1 \otimes \pi$. Here, $\overline{\pi}(\lambda \otimes (aa_0)) = \lambda \otimes \pi(aa_0)$ can be identified with $\lambda \otimes a\pi(a_0)$, since the quotient map π is A-linear. ☐

As mentioned in the proof of Lemma 2.3.1 the base change $A \to \overline{A}$ is faithfully flat. In particular, tensoring with $\overline{A}$ over A reflects isomorphisms: If $\varphi \colon M \to N$ is a map of A-modules M, N so that $1 \otimes \varphi \colon \overline{A} \otimes_A M \to \overline{A} \otimes_A N$ is an isomorphism, then φ is an isomorphism.

As Lemma 2.3.3 shows that $1_{\overline{k}} \otimes \chi_g$ is an isomorphism, we conclude that χ_g was an isomorphism to begin with. Thus,

$$\chi_g \colon \mathrm{HH}_n\left(A, g\right) \xrightarrow{\sim} \Omega\left(X^g\right)$$

for non-algebraically closed fields as well, generalising Proposition 2.2.8. The analogues of Theorem 2.2.9 and Corollary 2.2.10 follow immediately as well.

Theorem 2.3.4. *Let $A = \mathcal{O}[X]$ be the coordinate ring of a smooth k-variety. Assume that k is of characteristic* 0*. Suppose that a finite group Γ acts on X and let $\gamma_1, \ldots, \gamma_s$ be a set of representatives for the conjugacy classes of Γ. If $X_i = X^{\gamma_i}$, $A_i = \mathcal{O}[X_i]$ and $C_i = C_{\gamma_i}$, then*

$$\mathrm{HH}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \Omega^n(X_i)^{C_i},$$

$$\mathrm{HC}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\Omega^n(X_i)^{C_i} / \mathrm{d}\Omega^{n-1}(X_i)^{C_{\gamma_i}} \oplus \mathrm{H}^{n-2}_{\mathrm{dR}}(A_i)^{C_i} \oplus \mathrm{H}^{n-4}_{\mathrm{dR}}(A_i)^{C_i} \oplus \cdots \right),$$

and

$$\mathrm{HP}_n(A \rtimes \Gamma) \cong \bigoplus_{i=1}^{s} \left(\prod_{m=0}^{\infty} \mathrm{H}^{2m+n}_{\mathrm{dR}}(A_i)^{C_i} \right).$$

3 Bornologies and Dagger Algebras

We fix a complete discrete valuation ring V with uniformizer π, residue field $k = V/\pi V$ and field of fractions $K = V[\pi^{-1}]$. This is the notation of [CCMT18].

For a V-algebra A, its *weak completion* à la Monsky–Washnitzer [MW68] is a carefully chosen subalgebra of the π-adic completion of A. In algebraic geometry, they allow the definition of a Weil cohomology theory for smooth affine varieties in positive characteristic. Following [CCMT18], we reinterpret weak completions as *bornological completions* (see Definition 3.1.5).

To this end, we start by introducing the notion of *bornological modules* (see Definition 3.1.1) in Section 3.1. We then explain in Section 3.2 how these completions can be realised as certain bornological completions (see Theorem 3.2.9). This allows the usage of bornological tools to study weak completions. Additionally, we cover the basic constructions needed to develop an analogue of Hochschild homology, *bornological Hochschild homology* (see Definition 3.3.1), which we define in Section 3.3. We close this section by discussing an axiomatic characterisation of bornological Hochschild homology (see Theorem 3.3.3).

3.1 Background on Bornological Modules

We introduce the language of *bornological modules* as presented in [CCMT18, §2]. To keep the exposition brief, we focus on those constructions and results needed for doing homological algebra in the bornological setting.

Definition 3.1.1. Let M be a V-module. A **bornology** on M is a collection of subsets $\mathcal{B}$ of M, called **bounded** subsets, satisfying the following properties:

- If $m \in M$, then $\{m\} \in \mathcal{B}$;
- If $B \in \mathcal{B}$ and $B' \subseteq B$, then $B' \in \mathcal{B}$;
- If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cup B_2 \in \mathcal{B}$.

A **convex bornology** on M additionally satisfies the following:

- If $B \in \mathcal{B}$, then $\langle B \rangle \in \mathcal{B}$, for $\langle B \rangle$ the V-submodule generated by B.

A V-module together with a chosen convex bornology is a **bornological V-module**. If M and N are two bornological V-modules, we say that a V-module homomorphism $f\colon M \to N$ is **bounded**, if for every bounded subset $B \subseteq M$, $f(B) \subseteq N$ is a bounded subset. ⌟

From here on, a bornology on a V-module M will always refer to a convex bornology on M. The distinction in Definition 3.1.1 was made to highlight the additional role the V-module structure plays in the present setting: Convexity implies that the V-module operations are automatically bounded.

Let M be a V-module. Then we can always equip M with the **fine bornology**, having as bounded sets all subsets of finitely generated V-submodules.

Lemma 3.1.2. *Let M, N be V-modules. The fine bornology defines a bornology on M and N such that any V-module map $f\colon M \to N$ is bounded.*

If M is a finitely generated V-module, the fine bornology is the unique bornology on M.

Proof. Let $\mathcal{B}$ be defined as the set

$$\mathcal{B} = \{B \subseteq M \,|\, \text{there exists a finitely generated } V\text{-submodule } M' \text{ such that } B \subseteq M'\}.$$

For every $m \in M$, the V-submodule $\langle m \rangle$ is finitely generated and contains m. If $B' \subseteq B$ and $B \subseteq M'$ with M' a finitely generated V-submodule, then $B' \subseteq M'$. Hence, B' is bounded as well. For $B_1 \subseteq M'_1$,

$B_2 \subseteq M_2'$ with finitely generated V-submodules M_1', M_2', we have $B_1 \cup B_2 \subseteq M_1' \cup M_2' \subseteq M_1 + M_2$. Since $M_1 + M_2$ is finitely generated as a sum of finitely generated submodules, $B_1 \cup B_2 \in \mathcal{B}$. Finally, if $B \subseteq M'$ for a finitely generated submodule M', then $\langle B \rangle \subseteq M'$, since $\langle B \rangle$ is the submodule generated by B. In particular, $\langle B \rangle \in \mathcal{B}$. This shows that the fine bornology is a bornology on M. If $B \subseteq M$ is bounded in the fine bornology on M, there is a finitely generated V-submodule $M' \leq M$ such that $B \subseteq M'$. As f is a V-module map, $f(M') \leq N$ is a finitely generated V-submodule. Since $B \subseteq M'$, $f(B) \subseteq f(M')$. This shows that $f(B) \subseteq N$ is bounded in the fine bornology on N. Thus, f is bounded.

Now assume M to be finitely generated and denote the fine bornology on M by $\mathcal{B}_f$. Suppose that $\mathcal{B}$ is any bornology on M. If $m_1, \dots, m_n \in M$, then $\{m_1, \dots, m_n\} \in \mathcal{B}$ and hence the finitely generated submodule $\langle m_1, \dots, m_n \rangle$ is also $\mathcal{B}$-bounded. Since any subset of $\langle m_1, \dots, m_n \rangle$ is $\mathcal{B}$-bounded as well, $\mathcal{B}_f \subseteq \mathcal{B}$. Conversely, let $B \in \mathcal{B}$. Then $B \subseteq \langle B \rangle \leq M$. As V is a discrete valuation ring, hence Noetherian, $\langle B \rangle$ is finitely generated as submodule of the finitely generated V-module M. Thus, B is contained in a finitely generated submodule and therefore $B \in \mathcal{B}_f$. Thus, $\mathcal{B}_f = \mathcal{B}$ as claimed. □

We now move towards bornological completions. *Cauchy sequences* and *convergent sequences* of bornological V-modules are defined relative to the π-adic topology of V.

Definition 3.1.3. Let M be a bornological V-module and $(x_n)_n$ a sequence in M.

Fix a bounded subset $B \subseteq M$. We say that $(x_n)_n$ is ***B*-Cauchy** if there is a sequence $(\delta_n)_n$ in V such that $(\delta_n)_n$ is a π-adic nullsequence and $x_m - x_n \in \delta_\ell \cdot B$ for all $n, m \geq \ell$. We say that $(x_n)_n$ ***B*-converges** to $x \in M$ if there is a sequence $(\delta_n)_n$ in V such that $(\delta_n)_n$ is a π-adic nullsequence and $x_n - x \in \delta_n \cdot B$ for all $n \geq 0$.

We say that $(x_n)_n$ is **Cauchy** if $(x_n)_n$ is B-Cauchy for some bounded subset B. Similarly, $(x_n)_n$ **converges** to $x \in M$ if $(x_n)_n$ B-converges to x for some bounded subset B. ⌟

Definition 3.1.4. Let M be a bornological V-module. We say that M is **separated** if each convergent sequence in M has a unique limit. If M is separated, we say that M is **complete** if, for every bounded subset $B \subseteq M$, there is a bounded subset $B' \subseteq M$ such that B-Cauchy sequences are B'-convergent. ⌟

The *completion* of a bornological V-module is now naturally characterised by a universal property.

Definition 3.1.5. Let M be bornological V-module. The **completion** of M is a complete bornological V-module $\overline{M}$ equipped with a bounded map $b\colon M \to \overline{M}$ such that, for every bounded map $f\colon M \to N$, with N a complete bornological V-module, there is a unique bounded map $\overline{f}\colon \overline{M} \to N$ so that the diagram

$$\begin{array}{ccc} \overline{M} & \overset{\overline{f}}{\dashrightarrow} & N \\ & {\scriptstyle b}\nwarrow \quad \nearrow{\scriptstyle f} & \\ & M & \end{array}$$

commutes. ⌟

Upon identifying the category of bornological V-modules with the category of inductive systems of V-module with injective transition maps as in [CCMT18, Proposition 2.5], one sees that bornological completions always exist and are explicitly described by [CCMT18, Proposition 2.14]. The construction uses that the category of complete bornological V-modules is equivalent to the category of inductive systems of π-adically complete V-modules and injective transition maps given by [CCMT18, Proposition 2.10].

We close this section by introducing bornological constructions which we ultimately need to obtain a suitable homology theory for bornological V-algebras (see Definition 3.2.1). As this theory will be a variation of Hochschild homology, we consider *bornological tensor products*.

For notational simplicity, undecorated tensor products of any kind will always refer to tensor products taken over V.

Definition 3.1.6. Let M, N be V-modules. Their **bornological tensor product** is a bornological V-module $M \otimes N$ together with a bounded V-bilinear map $\otimes\colon M \times N \to M \otimes N$ such that, for every bounded V-bilinear map $f\colon M \times N \to L$ into a bornological V-module L, there is a unique bounded V-linear map $\overline{f}\colon M \otimes N \to L$ such that the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad\otimes\quad} & M \otimes N \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \overline{f}} & \\ & L & \end{array}$$

commutes. ⌟

Bornological tensor products exist by [CCMT18, Lemma 2.17] and are given by the algebraic tensor product endowed with the **tensor product bornology**, which is generated by the images of $A \otimes B$ for $A \leq M$ and $B \leq N$ bounded V-submodules.

However, if we consider two complete V-modules M and N, their bornological tensor product is not necessarily complete. This is a common phenomenon and the reason for our definition of completed tensor products in Section 2.2. An account of the type of problems arising from taking naive tensor products of complete structures is given in [Mey08, Section 3]. Instead, we introduce the *complete bornological tensor product*.

Definition 3.1.7. Let M, N be complete V-modules. Their **complete bornological tensor product** is a complete bornological V-module $M \mathbin{\overline{\otimes}} N$ together with a bounded V-bilinear map $\otimes\colon M \times N \to M \mathbin{\overline{\otimes}} N$ such that for every bounded V-bilinear map $f\colon M \times N \to L$ into a complete bornological V-module L, there is a unique bounded V-linear map $\overline{f}\colon M \mathbin{\overline{\otimes}} N \to L$ such that the diagram

$$\begin{array}{ccc} M \times N & \xrightarrow{\quad\otimes\quad} & M \mathbin{\overline{\otimes}} N \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \overline{f}} & \\ & L & \end{array}$$

commutes. ⌟

Complete bornological tensor products always exist and are given by the bornological completion of the bornological tensor product. This is [CCMT18, Lemma 2.17].

There is a bornological variant of adjoint associativity for the bornological tensor product. For this, let M, N be bornological V-modules and equip the V-module $\operatorname{Hom}(M, N)$ with the **equibounded bornology**, where $S \subseteq \operatorname{Hom}(M, N)$ is bounded if, for every bounded $B \subseteq M$, the set $f(b)$, $f \in S$, $b \in B$, is bounded. If N is a complete bornological module, so is $\operatorname{Hom}(M, N)$ in the equibounded bornology. See [Mey04, §2.1] or the discussion preceding [CCMT18, Proposition 2.19].

Lemma 3.1.8. *If M and N are bornological V-modules, then there is an isomorphism*

$$\overline{M \otimes N} \cong \overline{M} \mathbin{\overline{\otimes}} \overline{N}\ .$$

Moreover, if M_1, M_2, M_3 are bornological V-modules, there is a natural isomorphism

$$\overline{M_1} \mathbin{\overline{\otimes}} \left(\overline{M_2} \mathbin{\overline{\otimes}} \overline{M_3} \right) \cong \left(\overline{M_1} \mathbin{\overline{\otimes}} \overline{M_2} \right) \mathbin{\overline{\otimes}} \overline{M_3}\ .$$

Proof. Fix a complete bornological V-module L. By definition, $\operatorname{Hom}\left(\overline{M \otimes N}, L\right)$ classifies bounded

V-linear maps $M \otimes N \to L$, whereas $\operatorname{Hom}\left(\overline{M} \overline{\otimes} \overline{N}, L\right)$ classifies bounded V-linear maps $\overline{M} \otimes \overline{N} \to L$. By adjoint associativity,

$$\operatorname{Hom}(M \otimes N, L) \cong \operatorname{Hom}(M, \operatorname{Hom}(N, L)) \cong \operatorname{Hom}\left(\overline{M}, \operatorname{Hom}(N, L)\right),$$

since $\operatorname{Hom}(N, L)$ is complete in the equibounded bornology, and furthermore,

$$\operatorname{Hom}\left(\overline{M}, \operatorname{Hom}(N, L)\right) \cong \operatorname{Hom}\left(N, \operatorname{Hom}\left(\overline{M}, L\right)\right) \cong \operatorname{Hom}\left(\overline{N}, \operatorname{Hom}\left(\overline{M}, L\right)\right),$$

since $\operatorname{Hom}\left(\overline{M}, L\right)$ is complete as well. Thus,

$$\operatorname{Hom}(M \otimes N, L) \cong \operatorname{Hom}\left(\overline{M} \otimes \overline{N}, L\right),$$

from which the first assertion follows.

For the second assertion, consider bornological V-modules M_1, M_2, M_3. Since the canonical isomorphism

$$M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3$$

is bounded, there is an induced isomorphism of completions

$$\overline{M_1 \otimes (M_2 \otimes M_3)} \cong \overline{(M_1 \otimes M_2) \otimes M_3}.$$

Using the first part of the lemma, we conclude that

$$\overline{M_1} \overline{\otimes} \overline{(M_2 \otimes M_3)} \cong \overline{M_1 \otimes (M_2 \otimes M_3)} \cong \overline{(M_1 \otimes M_2) \otimes M_3} \cong \overline{M_1 \otimes M_2} \overline{\otimes} \overline{M_3}.$$

Since $\overline{M_1 \otimes M_2} \cong \overline{M_1} \overline{\otimes} \overline{M_2}$ and $\overline{M_2 \otimes M_3} \cong \overline{M_2} \overline{\otimes} \overline{M_3}$, the result follows. □

By Lemma 3.1.8, any choice for representing an iterated complete bornological tensor product is equivalent. Hence, we will suppress this distinction in our notation.

A foundational notion of homological algebra is that of a short exact sequence or extension. To properly define this notion as a bornological one, we require compatibility of the algebraic constructions with the bornological ones.

Definition 3.1.9. Let $q\colon M \to N$ be a bounded map of bornological V-modules. Then q is a **bornological quotient map** if every bounded subset of N is of the form $q(B)$ for a bounded subset $B \subseteq M$. ⌟

Definition 3.1.10. We say that

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is an **extension** of bornological V-modules if it is a short exact sequence of V-modules such that $M' \leq M$ carries the subspace bornology and g is a bornological quotient map. It is a **split extension** if g admits a bounded V-linear section. ⌟

Lemma 3.1.11. *Let N be a complete bornological V-module and*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

an extension of complete bornological V-modules.

Then $g \overline{\otimes} 1_N$ is a bornological quotient map such that $g \overline{\otimes} 1_N = \operatorname{coker}\left(f \overline{\otimes} 1_N\right)$. Moreover, $\ker\left(g \overline{\otimes} 1_N\right)$ is *the bornological closure of* $\mathrm{im}\,(f \overline{\otimes} 1_N)$. *If* g *admits a bounded* V*-linear section, then so does* $g \overline{\otimes} 1_N$.

Proof. This is [CCMT18, Lemma 2.21]. ☐

Lemma 3.1.12. *If* $q\colon M \to N$ *is a bornological quotient map of bornological modules, then* $\overline{q}\colon \overline{M} \to \overline{N}$ *is a bornological quotient map of complete bornological modules.*

Proof. This is [CCMT18, Lemma 2.22]. ☐

3.2 Weak Completions as Bornological Completions

As for the case of algebraic Hochschild homology, we are not interested in merely bornological V-modules, but rather *bornological* V*-algebras*, as covered in [CCMT18, §3].

Definition 3.2.1. Let A be a unital associative V-algebra. Then A is a **bornological V-algebra** if A is a bornological V-module and, whenever $B_1, B_2 \subseteq A$ are bounded, so is $B_1 \cdot B_2 \subseteq A$. ⌟

The notion of bornological V-algebras allows an elegant redefinition of *weak completions* in the sense of Monsky–Washnitzer [MW68]. To see this, we first recall the original definition of Monsky–Washnitzer. Their weak completions are defined only for commutative V-algebras A of finite type. More precisely, the **weak completion $A^\dagger$**, or **dagger completion**, of A is the subset of its π-adic completion

$$\widehat{A} = \varprojlim A/\pi^n A$$

consisting of formal power series in π of the form

$$b = \sum_{n=0}^{\infty} b_n \pi^n$$

with $b_n \in M^{\kappa_n}$, where M is a finitely generated V-submodule of A containing 1 and such that $\kappa_n \leq c(n+1)$ for a constant $c > 0$ depending on b.

This growth restriction gives rise to a canonical bornology on $A^\dagger$.

Lemma 3.2.2. *Let* $A^\dagger$ *be the weak completion of a commutative* V*-algebra of finite type. Define a bornology on* $A^\dagger$ *with bounded subsets those* $B \subseteq A^\dagger$*, for which there are a finitely generated* V*-submodule* $M \leq A^\dagger$ *containing* 1 *and a positive constant* $c > 0$ *such that, for* $a \in B$ *of the form*

$$a = \sum_{n=0}^{\infty} a_n \pi^n\,,$$

we have that $a_n \in M^{\kappa_n}$ *with* $\kappa_n \leq c(n+1)$*, define a bornology on* $A^\dagger$.

Proof. Let $a \in A^\dagger$. Then, by definition, there is a finitely generated V-submodule M and a fixed positive constant $c > 0$ such that $a_n \in M^{\kappa_n}$ for $\kappa_n \leq c(n+1)$. Thus, $\{a\}$ is bounded. If $B \subseteq A^\dagger$ is bounded, then so is any subset $B' \subseteq B$, since the condition for boundedness is still satisfied elementwise for each subcollection of elements of B.

Now, let $B_1, B_2 \subseteq A^\dagger$ be bounded. By definition, there are finitely generated V-submodules M_1, M_2 and fixed positive constants $c_1, c_2 > 0$ realising growth restrictions for elements of M_1 and M_2, respectively. Let $M = \langle M_1, M_2 \rangle$ and $c = \max\{c_1, c_2\}$. Since $M_1, M_2 \leq M$ with M finitely generated and $c_1, c_2 \leq c$, $B_1 \cup B_2$ is bounded.

Finally, let $B \subseteq A^\dagger$ be bounded with the boundedness given by a finitely generated V-submodule M and a positive constant $c > 0$. We show that for $b_1, b_2, b \in B$ and $\lambda \in V$, both $b_1 + b_2$ and λb satisfy the same growth restriction. First, note that

$$b_1 + b_2 = \sum_{n=0}^{\infty} b_{1,n}\pi^n + \sum_{n=0}^{\infty} b_{2,n}\pi^n = \sum_{n=0}^{\infty} (b_1+b_2)_n \pi^n ,$$

where $(b_1 + b_2)_n$ and $b_{1,n} + b_{2,n}$ differ at most by a multiple of π. Since $1 \in M$ and $b_{1,n}, b_{2,n} \in M^{\kappa_n}$, $(b_1 + b_2)_n \in M^{\kappa_n}$ for $\kappa_n \leq c(n+1)$. Second, write $\lambda = u\pi^m$ with $m \geq 0$ and u a unit of V. Then

$$\lambda b = (u\pi^m) \sum_{n=0}^{\infty} b_n \pi^n = \sum_{n=m}^{\infty} (ub_{n-m})\pi^n .$$

Then $ub_{n-m} \in M^{\kappa_{n-m}}$ with $\kappa_{n-m} \leq c(n-m+1)$, since M is a V-submodule of $A^\dagger$. As $m \geq 0$, we also have that $c(n-m+1) \leq c(n+1)$. Thus, λb satisfies the appropriate growth restriction. □

Let A be a commutative V-algebra of finite type. We define a bornology on A such that its corresponding bornological completion (in the sense of Definition 3.1.5) is isomorphic to the dagger completion $A^\dagger$ of A. For this, we define a *spectral radius* for bornological V-algebras. Since we only consider bornological V-algebras, we use the modifications for the definitions and results of [CCMT18] as explicitly considered in [MM19].

Definition 3.2.3. [MM19, Definition 3.1] Let A be a bornological V-algebra and let $M \subseteq A$ be a bounded V-submodule. For $r \geq 1$ and $\epsilon = |\pi|$, set

$$r \star M = r^{\lceil \log_\epsilon(r) \rceil} M .$$

The **(truncated) spectral radius $\rho(M)$** is the infimum of all $r \geq 1$ such that $\sum_{n\geq 1} r^{-n} \star M$ is bounded. If there is no such r, then $\rho(M) = \infty$. ⌟

Remark. As mentioned above, we exclude $0 < r < 1$ from our definition. In these cases, the $\star$-operation is only well-defined for bornological V-algebras, with K the field of fractions of V, for which multiplication by π admits a bounded inverse. As we do not need this case for our present purposes, we exclude it altogether.

Proposition 3.2.4. *Let A be a bornological V-algebra. The following are equivalent:*

- *(a) $\rho(M) = 1$ for all bounded V-submodules M;*
- *(b) $\sum_{n=0}^{\infty} \pi^n M^{cn+d}$ is bounded for all bounded V-submodules M and $c, d \in \mathbb{N}$;*
- *(c) $\sum_{n=0}^{\infty} \pi^n M^{n+1}$ is bounded for all bounded V-submodules M;*
- *(d) any bounded subset of A is contained in a bounded V-submodule M with $\pi M^2 \subseteq M$.*

Proof. Compare [CCMT18, Proposition 3.1.3] and [MM19, Proposition 3.4]. □

As made evident by Proposition 3.2.4(b), the spectral radius estimate given corresponds to a linear growth condition. Moreover, Proposition 3.2.4(c) is reminiscent of the growth condition used to define the canonical bornology on $A^\dagger$. This motivates the definition of the *linear growth bornology*.

Definition 3.2.5. [CCMT18, Definition 3.1.6] Let A be a bornological V-algebra with bornology $\mathcal{B}$. The **linear growth bornology** $\mathcal{B}_{\mathrm{lg}}$ on A is the smallest bornology containing $\mathcal{B}$ such that $\rho(M) = 1$ for all V-submodules M. A subset of A has **linear growth** with respect to $\mathcal{B}$ if it is contained in $\mathcal{B}_{\mathrm{lg}}$. The bornological V-algebra A equipped with $\mathcal{B}_{\mathrm{lg}}$ is denoted by A_{lg} and its completion by $\overline{A_{\mathrm{lg}}}$. ⌟

Corollary 3.2.6. *The fine bornology on V is the linear growth bornology. In particular, $\overline{V_{\mathrm{lg}}} = V$.*

Proof. By Lemma 3.1.2, the fine bornology is the unique bornology on V. It also makes V into a bornological V-algebra. By Proposition 3.2.4(d), it now suffices to show that for every V-submodule M,

$\pi M^2 \subseteq M$. As the V-submodules of V are precisely the ideals of V, $M^2 \subseteq VM \subseteq M$ and $\pi M \subseteq M$ are immediate. Thus, the fine bornology on V is the linear growth bornology as claimed.

The second assertion holds in general for bornological completions of V-modules equipped with the fine bornology. ☐

Proposition 3.2.4 yields a characterisation of subsets of linear growth.

Lemma 3.2.7. *Let A be a bornological V-algebra and $B \subseteq A$. The following are equivalent:*

(a) B has linear growth;

(b) B is contained in $\sum_{n=0}^{\infty} \pi^n S^{n+1}$ for some bounded subset $S \subseteq A$;

(c) B is contained in $\sum_{n=0}^{\infty} \pi^n S^{cn+d}$ for some bounded subset $S \subseteq A$ and $c, d \in \mathbb{N}$.

Proof. Compare [CCMT18, Lemma 3.1.10] and [MM19, Lemma 3.6]. ☐

Since the linear growth bornology is constructed as the smallest bornology subject to a spectral radius constraint, the corresponding completion satisfies an appropriate universal property.

Proposition 3.2.8. *Let A and B be bornological V-algebras. Assume that B is complete and that $\rho(M) = 1$ for all bounded V-submodules $M \leq B$. For every bounded map $\varphi\colon A \to B$, there is a unique bounded homomorphism $\overline{\varphi}\colon \overline{A_{\mathrm{lg}}} \to B$ such that the diagram*

$$\begin{array}{ccc} \overline{A_{\mathrm{lg}}} & \overset{\overline{\varphi}}{\dashrightarrow} & B \\ & \nwarrow \quad \nearrow_{\varphi} & \\ & A & \end{array}$$

commutes. If $\rho(M) = 1$ for all bounded V-submodules A, then $A = A_{\mathrm{lg}}$.

Proof. This is [CCMT18, Proposition 3.1.15]. ☐

We can now reinterpret weak completions as bornological completions, using the linear growth bornology associated to the fine bornology.

Theorem 3.2.9. *Let A be a commutative V-algebra of finite type, equipped with the fine bornology. The canonical map $A \to A^\dagger$ extends uniquely to an isomorphism of bornological algebras $\overline{A_{\mathrm{lg}}} \xrightarrow{\sim} A^\dagger$.*

Proof. This is [CCMT18, Theorem 3.2.1]. ☐

With Theorem 3.2.9 in mind, we can apply results about linear growth completions to dagger completions and vice versa. We collect the most important instances of this in the following two lemmata.

Lemma 3.2.10. *Let A be a commutative V-algebra of finite type and let $J \leq A$ be an ideal. Denote by $JA^\dagger \leq A^\dagger$ the ideal J generates in $A^\dagger$. The natural map $A^\dagger / JA^\dagger \to (A/J)^\dagger$ is an isomorphism.*

Proof. This is [CCMT18, Lemma 3.2.4]. ☐

Lemma 3.2.11. *Let A and B be bornological V-algebras.*

(a) The multiplication on A is bounded in the linear growth bornology and extends to a bornological V-algebra structure on $\overline{A_{\mathrm{lg}}}$.

(b) If $\varphi\colon A \to B$ is a bounded unital algebra homomorphism, then there is an induced bounded unital homomorphism $\overline{\varphi_{\mathrm{lg}}}\colon \overline{A_{\mathrm{lg}}} \to \overline{B_{\mathrm{lg}}}$. If φ is a bornological quotient map, so is $\overline{\varphi_{\mathrm{lg}}}$.

(c) $(A \otimes B)_{\mathrm{lg}} = A_{\mathrm{lg}} \otimes B_{\mathrm{lg}}$ and therefore $\overline{(A \otimes B)_{\mathrm{lg}}} \cong \overline{A_{\mathrm{lg}}} \mathbin{\overline{\otimes}} \overline{B_{\mathrm{lg}}}$.

Proof. Part (a) is [CCMT18, Lemma 3.1.12], part (b) is [CCMT18, Proposition 3.1.17] and part (c) is [CCMT18, Proposition 3.1.25]. ☐

For a general V-algebra A carrying the fine bornology, we will always denote the completion $\overline{A_{\mathrm{lg}}}$ by $A^\dagger$. This includes cases for which the conclusion of Theorem 3.2.9 does not hold, but will simplify our notation considerably.

3.3 Axiomatic Characterisation of Hochschild Homology of Bornological Algebras

Let us now consider a homology theory for bornological algebras. In order to take full advantage of the bornological structure, we define *bornological Hochschild homology*. Let A be a complete bornological V-algebra and M a complete bornological A-bimodule. Define $\mathrm{C}_n^{\mathrm{bor}}(A, M) = M \mathbin{\overline{\otimes}} A^{\overline{\otimes} n}$ and let $\overline{\mathrm{b}}\colon \mathrm{C}_n^{\mathrm{bor}}(A, M) \to \mathrm{C}_{n-1}^{\mathrm{bor}}(A, M)$ be the completion of the bounded V-linear map

$$\mathrm{b}(m \otimes a_1 \otimes \cdots \otimes a_n) = (ma_1) \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n (a_n m) \otimes a_1 \otimes \cdots \otimes a_{n-1}\,.$$

Definition 3.3.1. The complex $\left(\mathrm{C}_\bullet^{\mathrm{bor}}(A, M), \overline{\mathrm{b}}\right)$ is the **bornological Hochschild complex of A with coefficients in M**. Its n-th homology is the **n-th bornological Hochschild homology of A with coefficients in M**, denoted by $\mathrm{HH}_n^{\mathrm{bor}}(A, M)$. ⌟

If $A = M$, we denote by $\mathrm{HH}_n^{\mathrm{bor}}(A)$ the homology groups $\mathrm{HH}_n^{\mathrm{bor}}(A, A)$. This notion coincides with the Hochschild homology for complete bornological V-algebras defined in [CCMT18, Definition 4.1.1]. As noted there, the bornological Hochschild complex for a V-algebra carrying the fine bornology agrees with the ordinary Hochschild complex, since any bornological module is complete in the fine bornology. We write $\mathrm{C}_\bullet(A, M)$ and $\mathrm{HH}_n(A, M)$ in this case, following our notation from Section 1.

Section 1.1 shows that over a field, algebraic Hochschild homology may be defined as a derived functor. This implies, in particular, that Hochschild homology can be characterised axiomatically, since derived functors are universal δ-functors (in the sense of [Wei94, Definition 2.1.4]; see [Wei94, Theorem 2.4.7] for a proof of this assertion). For Hochschild homology, a more explicit formulation of such a characterisation is given by [Mac75, Theorem X.4.1].

Choosing the appropriate notion of short exact sequences yields a generalisation to bornological Hochschild homology. For this we consider a variation of the split extensions introduced in Definition 3.1.10, called *semi-split extensions*.

Definition 3.3.2. Let

$$0 \longrightarrow M' \xrightarrow{\ f\ } M \xrightarrow{\ g\ } M'' \longrightarrow 0$$

be an extension of bornological A-bimodules. We say that this extension is **semi-split** if g admits a bounded V-linear section. ⌟

Theorem 3.3.3. *Let A be a complete bornological V-algebra.*

(a) Let $f\colon M \to N$ be a bounded A-bimodule map of complete bornological A-bimodules. Then there is a bounded chain map $f \otimes 1\colon \mathrm{C}_\bullet(A, M) \to \mathrm{C}_\bullet(A, N)$, a chain map $\overline{f \otimes 1}\colon \mathrm{C}_\bullet^{\mathrm{bor}}(A, M) \to \mathrm{C}_\bullet^{\mathrm{bor}}(A, N)$ and an induced map $f_*\colon \mathrm{HH}_n^{\mathrm{bor}}(A, M) \to \mathrm{HH}_n^{\mathrm{bor}}(A, N)$ for each $n \geq 0$. This defines a functor from the category of complete A-bimodules to the category of V-modules.

(b) Let X be a V-module. Then $A \otimes X \otimes A$ carries a canonical A-bimodule structure such that

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A, \overline{A \otimes X \otimes A}\right) = 0$$

for $n > 0$.

(c) Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a semi-split extension of complete bornological A-bimodules. Then there is a long exact sequence

$$\cdots \longrightarrow \mathrm{HH}^{\mathrm{bor}}_{n+1}\left(A, M''\right) \longrightarrow \mathrm{HH}^{\mathrm{bor}}_{n}\left(A, M'\right) \longrightarrow \mathrm{HH}^{\mathrm{bor}}_{n}(A, M) \longrightarrow \mathrm{HH}^{\mathrm{bor}}_{n}\left(A, M''\right) \longrightarrow \cdots$$

and the connecting morphisms $\mathrm{HH}^{\mathrm{bor}}_{n+1}\left(A, M''\right) \to \mathrm{HH}^{\mathrm{bor}}_{n}\left(A, M'\right)$ are natural for morphism of extensions of complete bornological A-bimodules.

Proof. For (a), observe that $f \otimes 1\colon \mathrm{C}_\bullet(A, M) \to \mathrm{C}_\bullet(A, N)$ defines a chain map. As f is bounded, so is $f \otimes 1$. Thus, its completion $\overline{f \otimes 1}\colon \mathrm{C}^{\mathrm{bor}}_\bullet(A, M) \to \mathrm{C}^{\mathrm{bor}}_\bullet(A, N)$ defines a chain map of the bornological Hochschild complexes. This gives an induced map of Hochschild homology groups. Functoriality follows from functoriality of completions for bounded maps.

Given a V-module X, as in (b), define an A-bimodule structure on $A \otimes X \otimes A$ by

$$(a_1 \otimes a_2) \cdot (a \otimes x \otimes a') = (a_1 a) \otimes x \otimes (a' a_2)\,.$$

Its bornological completion $\overline{A \otimes X \otimes A}$ is then a complete bornological A-bimodule. To see that $\mathrm{HH}^{\mathrm{bor}}_{n}\left(A, \overline{A \otimes X \otimes A}\right) = 0$ for $n > 0$, we show that $\mathrm{C}^{\mathrm{bor}}_\bullet\left(A, \overline{A \otimes X \otimes A}\right)$ admits a bounded chain contraction.

First, observe that the bounded homorphism

$$\begin{gathered}(A \otimes X \otimes A) \otimes A^{\otimes n} \to X \otimes \left(A \otimes A^{\otimes n} \otimes A\right),\\ (a \otimes x \otimes a') \otimes (a_1 \otimes \cdots \otimes a_n) \mapsto x \otimes (a' \otimes a_1 \otimes \cdots \otimes a_n \otimes a)\end{gathered}$$

admits a bounded inverse

$$\begin{gathered}X \otimes \left(A \otimes A^{\otimes n} \otimes A\right) \to (A \otimes X \otimes A) \otimes A^{\otimes n},\\ x \otimes (a' \otimes a_1 \otimes \cdots \otimes a_n \otimes a) \mapsto (a \otimes x \otimes a') \otimes (a_1 \otimes \cdots \otimes a_n)\,.\end{gathered}$$

The induced boundary map on the right is

$$(1 \otimes \mathrm{b}')(x \otimes a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i x \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for b' the boundary map of the (unnormalised) bar resolution, whose n-th chain group is $\mathrm{B}_n(A) = A \otimes A^{\otimes n} \otimes A$. The latter admits a well-known contraction

$$s'\colon A \otimes A^{\otimes n} \otimes A \to A \otimes A^{\otimes n+1} \otimes A, \quad a_0 \otimes \cdots \otimes a_{n+1} \mapsto 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}\,,$$

which is bounded. This yields a bounded contraction for the chain complex $X \otimes \mathrm{B}_\bullet(A)$. By Lemma 3.1.8, we conclude that

$$\overline{X \otimes \mathrm{B}_n(A)} \cong \overline{(A \otimes X \otimes A) \otimes A^{\otimes n}} \cong \overline{A \otimes X \otimes A}\, \overline{\otimes}\, A^{\overline{\otimes} n} = \mathrm{C}^{\mathrm{bor}}_n\left(A, \overline{A \otimes X \otimes A}\right).$$

Completing $1 \otimes s'$ then shows that $\overline{X \otimes \mathrm{B}_\bullet(A)}$, and consequently $\mathrm{C}^{\mathrm{bor}}_\bullet\left(A, \overline{A \otimes X \otimes A}\right)$, admit a bounded contraction as well.

For (c), let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be a semi-split extension of complete bornological A-bimodules. From an application of Lemma 3.1.11 in combination with Lemma 3.1.8, we deduce that

$$0 \longrightarrow M' \mathbin{\overline{\otimes}} A^{\overline{\otimes} n} \longrightarrow M \mathbin{\overline{\otimes}} A^{\overline{\otimes} n} \longrightarrow M'' \mathbin{\overline{\otimes}} A^{\overline{\otimes} n} \longrightarrow 0$$

remains a semi-split extension. By (a), we know that any bounded A-bimodule map gives rise to a chain map of the bornological Hochschild complexes. Hence, the above yields a short exact sequence

$$0 \longrightarrow \mathrm{C}^{\mathrm{bor}}_{\bullet}(A, M') \longrightarrow \mathrm{C}^{\mathrm{bor}}_{\bullet}(A, M) \longrightarrow \mathrm{C}^{\mathrm{bor}}_{\bullet}(A, M'') \longrightarrow 0$$

of bornological Hochschild complexes. This then gives the desired long exact sequence. Moreover, a morphism of extensions defines a morphism of the associated short exact sequences of bornological Hochschild complexes, which proves naturality of the long exact sequence. □

Theorem 3.3.3 is supplemented by the following.

Lemma 3.3.4. *Let M be a complete bornological A-bimodule. Then there is a semi-split extension*

$$0 \longrightarrow N \longrightarrow F \longrightarrow M \longrightarrow 0$$

of complete bornological A-bimodules, such that $\mathrm{HH}^{\mathrm{bor}}_n(A, F) = 0$ for $n > 0$.

Proof. Let $F = \overline{A \otimes M \otimes A}$, viewing the A-bimodule F as a V-module. By part (b) of Theorem 3.3.3, we then have $\mathrm{HH}^{\mathrm{bor}}_n(A, F) = 0$ for $n > 0$.

We now show that the canonical bounded surjection

$$q\colon A \otimes M \otimes A \to M, \quad a \otimes m \otimes a' \mapsto ama',$$

gives rise to an extension of the desired form.

First, note that q is a bornological quotient map. Ultimately, this will also be a consequence of the associated extension being semi-split, but we find it instructive to prove this independently.

Let $B \subseteq M$ be bounded. By convexity, we may assume without loss of generality that B is a bounded V-submodule. Then $A \otimes B \otimes A \subseteq A \otimes M \otimes A$ is bounded in the tensor product bornology. Furthermore, $q(A \otimes B \otimes A) = B$, which proves that q is a bornological quotient map. Thus, by Lemma 3.1.12, its completion $\overline{q}$, which is a bounded map from F to $\overline{M} = M$, is a bornological quotient map as well.

Equipping the kernel $N = \ker(\overline{q}) \leq F$ with the subspace bornology yields an extension

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\overline{q}} M \longrightarrow 0$$

of complete bornological modules. Note that

$$s\colon M \to A \otimes M \otimes A, \quad m \mapsto 1 \otimes m \otimes 1$$

is a bounded V-linear section of q, hence its completion $\overline{s}$ is a bounded V-linear section of $\overline{q}$. Thus, the extension is semi-split. □

In tandem with Lemma 3.3.4, Theorem 3.3.3 characterises bornological Hochschild homology axiomatically in the following sense. First, part (a) and (c) of Theorem 3.3.3 show that together, the $\mathrm{HH}_n^{\mathrm{bor}}(A,-)$ behave like a δ-functor (in the sense of [Wei94, Definition 2.1.1]). Second, Lemma 3.3.4 shows, essentially, that $\mathrm{HH}_n^{\mathrm{bor}}(A,-)$ is coeffacable (in the sense of [Gro57, §2.2]) for $n > 1$. The sought-after conclusion, that coeffacable δ-functors are universal (in the sense of [Wei94, Definition 2.1.4]), does not apply verbatim, since the categories we are considering are, in general, not abelian. However, the given characterisation suffices for our purposes.

More precisely, suppose we are given any family of functors $\mathrm{H}_n(-)$ from the category of complete bornological A-bimodules to the category of V-modules, satisfying the conclusions of Theorem 3.3.3. Furthermore, assume that, for any complete A-bimodule M, there is a natural isomorphism $\mathrm{H}_0(M) \xrightarrow{\sim} \mathrm{H}_0^{\mathrm{bor}}(A,M)$. We can then use dimension shifting to prove that $\mathrm{H}_n(M) \xrightarrow{\sim} \mathrm{H}_n^{\mathrm{bor}}(A,M)$ for all $n \geq 1$ as well.

First, choose a semi-split extension

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

as constructed in Lemma 3.3.4. Observe that we may choose $F = \overline{A \otimes M \otimes A}$, hence $\mathrm{HH}_n^{\mathrm{bor}}(A,F) = 0$ and $\mathrm{H}_n(F) = 0$ for $n > 0$. Now the long exact sequences associated to the above semi-split extension provide a commuting diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{HH}_1^{\mathrm{bor}}(A,M) & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}(A,K) & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}(A,F) & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}(A,M) & \longrightarrow 0 \\
& \downarrow & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{H}_1(M) & \longrightarrow & \mathrm{H}_0(K) & \longrightarrow & \mathrm{H}_1(F) & \longrightarrow & \mathrm{H}_0(M) & \longrightarrow 0
\end{array}$$

with exact rows. Here we use that the isomorphism $\mathrm{H}_0(M) \xrightarrow{\sim} \mathrm{H}_0^{\mathrm{bor}}(A,M)$ is assumed to be natural. From this, we define the map $\mathrm{HH}_1^{\mathrm{bor}}(A,M) \to \mathrm{H}_1(M)$, which is an isomorphism by the Five Lemma. In higher degrees, the long exact sequences above also yield isomorphisms $\mathrm{HH}_{n+1}^{\mathrm{bor}}(A,M) \cong \mathrm{HH}_n^{\mathrm{bor}}(A,K)$ and $\mathrm{H}_{n+1}(K) \cong \mathrm{H}_n(M)$ for $n > 1$. We then conclude by induction that $\mathrm{HH}_n^{\mathrm{bor}}(A,M) \cong \mathrm{H}_n(M)$ for all complete bornological A-bimodules M.

4 The Case of Crossed Product Dagger Algebras

Assume that a finite group Γ acts on a V-algebra A by V-linear automorphisms.

In Section 4.1 we consider $A \rtimes \Gamma$ bornologically and record elementary observations on its dagger completion and the corresponding bornological Hochschild homology. In particular, $(A \rtimes \Gamma)^\dagger \cong A^\dagger \rtimes \Gamma$ (see Lemma 4.1.1). Using the additional structure provided by the group action, we relate the bornological Hochschild homology of $A^\dagger \rtimes \Gamma$ to certain hyperhomology groups in Section 4.2. This is done by employing the machinery of group hyperhomology coupled with the axiomatic characterisation proved in Section 3.3. In particular, using the full strength of the Γ-grading on $A^\dagger \rtimes \Gamma$, we reduce to *twisted bornological Hochschild homology* (see Definition 4.3.1). The latter is defined in Section 4.3 and can be expressed in terms of the twisted Hochschild homology of A (see Proposition 4.3.3).

Finally, in Section 4.4 we spell out in detail how the results of Sections 4.1 to 4.3 reduce the computation of the bornological Hochschild homology of $A^\dagger \rtimes \Gamma$ to the ordinary Hochschild homology of $A \rtimes \Gamma$.

4.1 Dagger Completions of Crossed Product Algebras

Consider the algebraic crossed product $A \rtimes \Gamma$. As a V-module, $A \rtimes \Gamma = \bigoplus_{\gamma \in \Gamma} A\gamma$. We make $A \rtimes \Gamma$ into a bornological V-module by equipping it with the direct sum bornology of the fine bornologies on each A-summand. This turns $A \rtimes \Gamma$ into a bornological V-algebra. To see this, first note that a bounded subset of $A \rtimes \Gamma$ is of the form $\sum_{\gamma\in\Gamma} B_\gamma \gamma$, for $B_\gamma \subseteq A$ bounded. Hence, it suffices to check that $(B_\gamma\gamma)\cdot(B_\delta\delta) \subseteq A(\gamma\delta)$ is bounded. Observe that

$$(B_\gamma\gamma)\cdot(B_\delta\delta) = \{(a\gamma)\cdot(b\delta) \mid a,b \in A\} = \{(a\gamma(b))(\gamma\delta) \mid a,b \in A\} = (B_\gamma \cdot \gamma(B_\delta))(\gamma\delta)\,.$$

Therefore, $(B_\gamma\gamma)\cdot(B_\delta\delta)$ is bounded, since A is a bornological V-algebra with respect to the fine bornology and each $\gamma\colon A \to A$ is bounded by Lemma 3.1.2.

Lemma 4.1.1. *Let A be a commutative V-algebra of finite type. There is an isomorphism $(A \rtimes \Gamma)^\dagger \cong A^\dagger \rtimes \Gamma$ and, for each $n \geq 2$, an isomorphism $\left(A^\dagger \rtimes \Gamma\right)^{\overline{\otimes}n} \cong \left(A^\dagger\right)^{\overline{\otimes}n} \rtimes \Gamma^n$.*

Proof. Let $\gamma \in \Gamma$ act on A as V-algebra endomorphism $\gamma\colon A \to A$. Then each $\gamma \in \Gamma$ acts on $A^\dagger$ via its dagger completion $\gamma^\dagger\colon A^\dagger \to A^\dagger$ by Lemma 3.2.11(b). The crossed product $A^\dagger \rtimes \Gamma$ is now constructed with respect to this action. To show that $(A \rtimes \Gamma)^\dagger$ and $A^\dagger \rtimes \Gamma$ are isomorphic, we use their respective universal properties.

The bornology on $A^\dagger \rtimes \Gamma$ is the direct sum bornology of the bornologies on $A^\dagger$. This means that a bounded subset is of the form $\sum_{\gamma\in\Gamma} B_\gamma\gamma$ with $B_\gamma \subseteq A^\dagger$ bounded. As Γ is finite, any bounded subset is contained in a bounded V-submodule of the form $\sum_{\gamma\in\Gamma} M\gamma$, for M a bounded Γ-invariant V-submodule $M \leq A^\dagger$. To see this, take $M = \sum_{\gamma\in\Gamma} \gamma^\dagger\big(\sum_{\gamma\in\Gamma}\langle B_\gamma\rangle\big)$.

We can use this explicit description of the bornology of $A^\dagger \rtimes \Gamma$ to show that each bounded V-submodule of $A^\dagger \rtimes \Gamma$ has spectral radius 1. First, consider a bounded V-submodule of the form $\sum_{\gamma\in\Gamma} M\gamma$, with $M \leq A^\dagger$ bounded and Γ-invariant. Then

$$\Big(\sum\nolimits_{\gamma\in\Gamma} M\gamma\Big)^{n+1} \subseteq \sum\nolimits_{\gamma\in\Gamma} M^{n+1}\gamma\,,$$

since M is Γ-invariant. This implies that

$$\sum\nolimits_{n=0}^\infty \pi^n \Big(\sum\nolimits_{\gamma\in\Gamma} M\gamma\Big)^{n+1} \subseteq \sum\nolimits_{n=0}^\infty \pi^n \sum\nolimits_{\gamma\in\Gamma} M^{n+1}\gamma = \sum\nolimits_{\gamma\in\Gamma}\Big(\sum\nolimits_{n=0}^\infty \pi^n M^{n+1}\Big)\gamma$$

is bounded, since $\sum_{n=0}^{\infty} \pi^n M^{n+1} \subseteq A^\dagger$ is bounded by assumption. As any bounded V-submodule of $A^\dagger \rtimes \Gamma$ is contained in one of the considered form, Proposition 3.2.4(b) applies. This shows that the natural map $A \rtimes \Gamma \to A^\dagger \rtimes \Gamma$ extends to a bounded map $(A \rtimes \Gamma)^\dagger \to A^\dagger \rtimes \Gamma$ by Proposition 3.2.8.

Conversely, consider the structure maps $\iota_A \colon A \to A \rtimes \Gamma$ and $\iota_\Gamma \colon \Gamma \to (A \rtimes \Gamma)^\times$. By Proposition 3.2.8, there is an induced bounded map $\iota_A^\dagger \colon A^\dagger \to (A \rtimes \Gamma)^\dagger$. Additionally, the natural map $A \rtimes \Gamma \to (A \rtimes \Gamma)^\dagger$ provides a map $\iota_\Gamma^\dagger \colon \Gamma \to \left((A \rtimes \Gamma)^\dagger\right)^\times$. The crossed product algebra $A \rtimes \Gamma$ is uniquely determined by requiring that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & A \rtimes \Gamma \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\iota_\Gamma(\gamma)(-)\iota_\Gamma(\gamma^{-1})} \\ A & \xrightarrow[\iota_A]{} & A \rtimes \Gamma \end{array} \qquad \begin{array}{ccc} a & \longmapsto & \iota_A(a) \\ \downarrow & & \downarrow \\ \gamma(a) & \longmapsto & \iota_A(\gamma(a)) = \iota_\Gamma(\gamma)\iota_A(a)\iota_\Gamma(\gamma^{-1}) \end{array}$$

commutes as a diagram of V-algebra homomorphisms. Completing this diagram shows that there is a map $A^\dagger \rtimes \Gamma \to (A \rtimes \Gamma)^\dagger$, which is moreover bounded by construction.

The pair of bounded V-linear maps $(A \rtimes \Gamma)^\dagger \to A^\dagger \rtimes \Gamma$ and $A^\dagger \rtimes \Gamma \to (A \rtimes \Gamma)^\dagger$ are mutually inverse, since their restrictions to $A^\dagger$ and Γ agree. This gives the desired isomorphism.

Now, we combine the first assertion with Lemma 3.1.8 and Lemma 3.2.11(c) to conclude that

$$(A^\dagger \rtimes \Gamma)^{\overline{\otimes} n} \cong \left((A \rtimes \Gamma)^\dagger\right)^{\overline{\otimes} n} \cong \left((A \rtimes \Gamma)^{\otimes n}\right)^\dagger \cong \left(A^{\otimes n} \rtimes \Gamma^n\right)^\dagger \cong \left(A^{\otimes n}\right)^\dagger \rtimes \Gamma^n \cong \left(A^\dagger\right)^{\overline{\otimes} n} \rtimes \Gamma^n . \qquad \square$$

By Lemma 4.1.1, the n-th chain group of the bornological Hochschild complex of $A^\dagger \rtimes \Gamma$ is given by

$$\mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) = \left(A^\dagger \rtimes \Gamma\right)^{\overline{\otimes}(n+1)} \cong \left(A^\dagger\right)^{\overline{\otimes}(n+1)} \rtimes \Gamma^{n+1} .$$

In analogy to Section 2.1, we consider for any $\gamma \in \Gamma$ a subcomplex of $\left(\mathrm{C}_\bullet\left(A^\dagger \rtimes \Gamma\right), \overline{\mathrm{b}}\right)$. For this, consider the subgroup $\mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma$ indexed by $(n+1)$-tuples $(\gamma_0, \ldots, \gamma_n)$ such that $\gamma_0 \cdots \gamma_n \in [\gamma]$. This defines a subcomplex $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma, \overline{\mathrm{b}}\right)$ of $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right), \overline{\mathrm{b}}\right)$. Denote its n-homology by $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma$.

Corollary 4.1.2. *There is an isomorphism*

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) \cong \bigoplus_{[\gamma] \in [\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma$$

for each $n \geq 0$.

Proof. This is immediate from Lemma 4.1.1, since

$$\mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) \cong \left(A^\dagger\right)^{\overline{\otimes}(n+1)} \rtimes \Gamma^{n+1} = \bigoplus_{(\gamma_0, \ldots, \gamma_n) \in \Gamma^{n+1}} \left(A^\dagger\right)^{\overline{\otimes}(n+1)} (\gamma_0, \ldots, \gamma_n) = \bigoplus_{[\gamma] \in [\Gamma]} \mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma .$$

$\square$

Thus, computing the bornological Hochschild homology of $A^\dagger \rtimes \Gamma$ is reduced to computing the homology groups $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma$, where γ runs through a set of representatives for the conjugacy classes of Γ.

4.2 Bornological Hochschild Homology and Group Hyperhomology

In Section 2.1 we saw that the computation of the algebraic Hochschild homology may be further reduced to understanding certain invariant subsets of twisted Hochschild homology groups. But studying invariants is done by studying group cohomology rather than group homology. Since we are interested in bornological Hochschild homology, this seems to present a slight complication. This apparent issue, however, is entirely artificial. In characteristic 0, the modules of invariants and coinvariants are isomorphic via the algebraic norm map (see [Wei94, Proposition 6.1.10]). As we assumed the case of characteristic 0 throughout Section 2, this was of no concern then.

In order to properly generalise the results of Section 2.1 to bornological Hochschild homology, we adapt the strategy of [Lor92]. This approach makes use of group hyperhomology, for which we refer to [Wei94, §5.7 and §6.1] and [Bro82, §VII.5].

Given a finite group Γ and a chain complex of $V[\Gamma]$-modules $C_\bullet$, we denote the n-th hyperhomology group by $\mathbb{H}_n(\Gamma, C_\bullet)$. Since Γ is assumed to be finite, the spectral sequence computing $\mathbb{H}_n(\Gamma, C_\bullet)$ collapses to $\mathrm{H}_n(C_\bullet)_\Gamma$. Thus, the hyperhomology groups are simply given by the module of coinvariants of $\mathrm{H}_n(C_\bullet)$. However, approaching coinvariants through the lens of hyperhomology allows making use of the axiomatic characterisation of bornological Hochschild homology given by Theorem 3.3.3.

More precisely, we show

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) \cong \mathbb{H}_n\left(\Gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, M\right)\right)$$

for all complete bornological $A^\dagger \rtimes \Gamma$-bimodules M.

Following [Lor92, §2.6], we define a $V[\Gamma]$-module structure on the bornological Hochschild complexes $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, M\right), \overline{\mathrm{b}}\right)$ and $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right), \overline{\mathrm{b}}\right)$ for M any complete bornological $A^\dagger \rtimes \Gamma$-module. For the first complex, this is done by acting on the ordinary Hochschild complex $(\mathrm{C}_\bullet(A, M), \mathrm{b})$ as

$$\gamma(m \otimes a_1 \otimes \cdots \otimes a_n) = \gamma m \gamma^{-1} \otimes \gamma(a_1) \otimes \cdots \otimes \gamma(a_n)$$

and then taking the completion of this action, equipping $V[\Gamma]$ with the fine bornology. The case of $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right), \overline{\mathrm{b}}\right)$ is entirely analogous. These actions are compatible with the bornological Hochschild boundaries, hence descend to a $V[\Gamma]$-module structure on the bornological Hochschild homology groups.

First, we compute $\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$ in terms of $\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)$ and the group action on the bornological Hochschild homology. For this, we adapt the proof of [Lor92, Lemma 2.4(a)] to our setting.

Lemma 4.2.1. *Let M be a complete bornological $A^\dagger \rtimes \Gamma$-bimodule. There is a natural isomorphism*

$$\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) \cong \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)_\Gamma.$$

Proof. The inclusion $A^\dagger \to A^\dagger \rtimes \Gamma$ gives rise to a commuting diagram

$$\begin{array}{ccccccc}
M \mathbin{\overline{\otimes}} A^\dagger & \xrightarrow{\overline{\mathrm{b}}} & M & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right) & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
M \mathbin{\overline{\otimes}} \left(A^\dagger \rtimes \Gamma\right) & \xrightarrow[\overline{\mathrm{b}}]{} & M & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) & \longrightarrow & 0
\end{array}$$

showing that $\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)$ surjects onto $\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$. This surjection factors through the associated coinvariants as $\varphi\colon \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)_\Gamma \to \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$.

To see this, recall that the coinvariants are obtained by dividing out the $V[\Gamma]$-submodule generated by elements of the form $\gamma \cdot m - m = \gamma m \gamma^{-1} - m$ for $\gamma \in \Gamma$ and $m \in M$. As γ is invertible in $V[\Gamma]$, we may equivalently consider the submodule generated by elements of the form $\gamma m - m\gamma$ for $\gamma \in \Gamma$ and $m \in M$. These are, by definition, contained in the image of $\overline{\mathrm{b}}\colon M \overline{\otimes} (A^\dagger \rtimes \Gamma) \to M$, providing the claimed factorisation.

In order to construct an inverse, consider the canonical surjection

$$\psi\colon M \to \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, M\right) \to \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, M\right)_\Gamma .$$

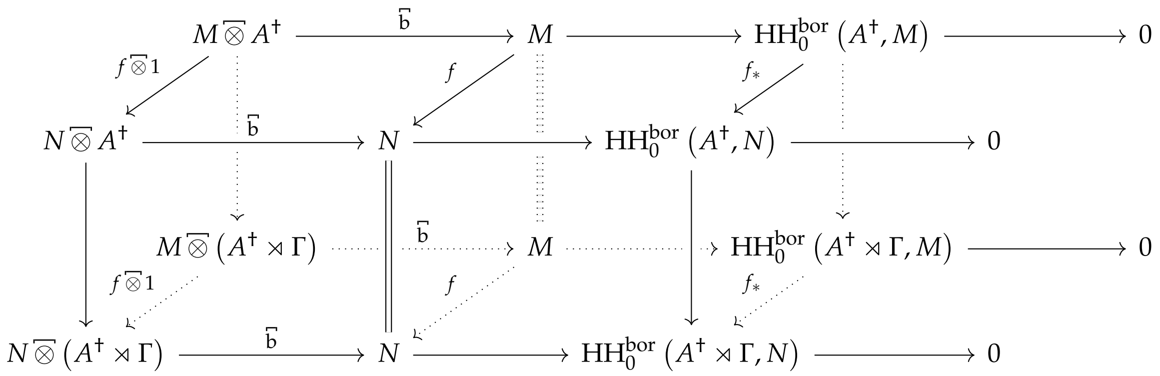
We claim that ψ factors through $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$. So let $a\gamma \in A^\dagger \rtimes \Gamma$, $a \in A^\dagger$ and $\gamma \in \Gamma$. We have to show that $\psi((a\gamma)m) = \psi(m(a\gamma))$ for all $m \in M$. This holds, as

$$\psi(m(a\gamma)) = \psi(m(\gamma\gamma^{-1})(a\gamma)) = \psi(\gamma^{-1}(a\gamma)m\gamma) = \psi((a\gamma)m) .$$

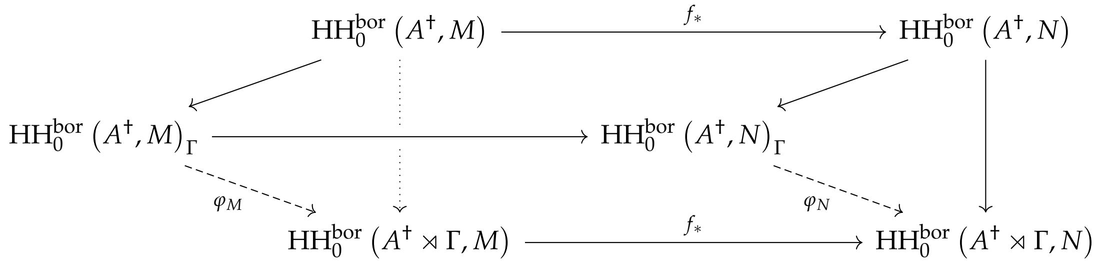
Here, we first use $\gamma^{-1}(a\gamma) \in A^\dagger$ and that the equality holds there, since we factor through $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger, M\right)$. Surjecting onto the coinvariants then yields the last equality, as we act on M by conjugation.

Since φ is induced by the identity and ψ is induced by the canonical quotient projections, they define a pair of mutually inverse homomorphisms.

For naturality, we first recall that any bounded A-bimodule map $f\colon M \to N$ induces a morphism of the bornological Hochschild complexes. This morphism of complexes is compatible with the one induced by $A^\dagger \to A^\dagger \rtimes \Gamma$. Thus, we obtain a commuting diagram



defining the bornological Hochschild homology groups of interest. The rightmost face then gives rise to another diagram



with $\varphi_M\colon \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)_\Gamma \to \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$ and $\varphi_N\colon \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, N\right)_\Gamma \to \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, N\right)$ the isomorphisms from above. The back commutes by the above, whereas the left, right and upper faces commute by construction of the coinvariants as quotients. Since $\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right) \to \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M\right)_\Gamma$ is surjective, we conclude that the lower face commutes as well. This proves naturality. ☐

Next, we establish that certain hyperhomology groups with coefficients in bornological Hochschild complexes vanish. We reconstruct the explicit chain map used in the proof of [Lor92, Proposition 2.6(a)] in Appendix A and then check that it continues to work when considered bornologically.

Lemma 4.2.2. *Let X be a V-module. Then*

$$\mathbb{H}_n\left(\Gamma, \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right)\right) = 0$$

for $n > 0$.

Proof. By Lemma A.1, the maps

$$\begin{aligned}\varphi_n\colon ((A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \otimes A^{\otimes n} &\to ((A \rtimes \Gamma) \otimes X \otimes A) \otimes A^{\otimes n}) \otimes V[\Gamma]\,,\\ (a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n) &\mapsto (\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1}\end{aligned}$$

assemble into an isomorphism of $V[\Gamma]$-complexes

$$\varphi_\bullet\colon\ \mathrm{C}_\bullet(A, (A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \xrightarrow{\sim} \mathrm{C}_\bullet(A, (A \rtimes \Gamma) \otimes X \otimes A) \otimes V[\Gamma]\,.$$

Here we use the $V[\Gamma]$-structure from before for the left hand side, but endow the right with the induced $V[\Gamma]$-structure. The components of the inverse are explicitly given by

$$\begin{aligned}\psi_n\colon ((A \rtimes \Gamma) \otimes X \otimes A) \otimes A^{\otimes n}) \otimes V[\Gamma] &\to ((A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \otimes A^{\otimes n}\,,\\ (a\gamma \otimes x \otimes b) \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta &\mapsto (\delta(a)\delta\gamma \otimes x \otimes b\delta^{-1}) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n))\,.\end{aligned}$$

Observe that the φ_n and ψ_n are bounded. This is the case, since we are considering bornological algebras with a bounded group action by Γ. Thus, taking bornological completions shows that

$$\overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma))} \cong \overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes A) \otimes V[\Gamma]} \cong \overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes A)}\,\overline{\otimes}\, V[\Gamma]\,,$$

by Lemma 3.1.8 and using that $V[\Gamma]$ is complete with respect to the fine bornology. Moreover, it holds true that

$$\overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes A)}\,\overline{\otimes}\, V[\Gamma] = \overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes A)} \otimes V[\Gamma]$$

since Γ is finite. This follows more generally for any complete bornological V-module. Indeed, since $M \otimes V[\Gamma] \cong \bigoplus_{\gamma \in \Gamma} M$ as bornological V-modules and the direct sum of complete bornological modules is complete, $M \otimes V[\Gamma] = M \,\overline{\otimes}\, V[\Gamma]$. Now observe that

$$\overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma))} \cong \mathrm{C}^{\mathrm{bor}}_n\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right)$$

as well as

$$\overline{\mathrm{C}_n(A, (A \rtimes \Gamma) \otimes X \otimes A)} \cong \mathrm{C}^{\mathrm{bor}}_n\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes A^\dagger}\right),$$

by combining Lemma 4.1.1 with Lemma 3.1.8. Thus, we just showed that

$$\mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right) \cong \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes A^\dagger}\right) \otimes V[\Gamma]$$

as $V[\Gamma]$-complexes, which gives an isomorphism

$$\mathbb{H}_n\left(\Gamma, \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right)\right) \cong \mathbb{H}_n\left(\Gamma, \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes A^\dagger}\right) \otimes V[\Gamma]\right)$$

of hyperhomology groups. Since $\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},\overline{(A^{\dagger}\rtimes\Gamma)\otimes X\otimes A^{\dagger}}\right)$ carries the trivial $V[\Gamma]$-complex structure, Shapiro's lemma ([Bro82, Proposition 6.2]) implies that

$$\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},\overline{(A^{\dagger}\rtimes\Gamma)\otimes X\otimes A^{\dagger}}\right)\otimes V[\Gamma]\right)\cong\mathbb{H}_n\left(1,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},\overline{(A^{\dagger}\rtimes\Gamma)\otimes X\otimes A^{\dagger}}\right)\right)$$

with 1 the trivial group. These groups vanish for $n>0$, proving the claim. □

Following the strategy of [Lor92, Proposition 2.6], we combine Lemma 4.2.1 and Lemma 4.2.2 to identify the bornological Hochschild homology of $A^{\dagger}\rtimes M$ with certain hyperhomology groups with coefficients in bornological Hochschild complexes. In our case, this now makes use of the results of Section 3.3.

Theorem 4.2.3. *Let M be a complete bornological $A^{\dagger}\rtimes\Gamma$-bimodule. There is an isomorphism*

$$\mathrm{HH}^{\mathrm{bor}}_n\left(A^{\dagger}\rtimes\Gamma,M\right)\cong\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\right)$$

for all $n\geq 0$.

Proof. To prove the claimed isomorphism, it suffices to check that there is a natural isomorphism $\mathrm{HH}^{\mathrm{bor}}_0\left(A^{\dagger}\rtimes\Gamma,M\right)\cong\mathbb{H}_0\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\right)$ and that $\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},-\right)\right)$ satisfies the conclusions of Theorem 3.3.3. This was remarked at the end of Section 3.3.

Since $\mathbb{H}_0\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\right)\cong\mathrm{HH}^{\mathrm{bor}}_0\left(A^{\dagger},M\right)_{\Gamma}$, by the spectral sequence computing the left hand side, Lemma 4.2.1 provides a natural isomorphism $\mathrm{HH}^{\mathrm{bor}}_0\left(A^{\dagger}\rtimes\Gamma,M\right)\cong\mathbb{H}_0\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\right)$. Lemma 4.2.2 gives the required vanishing. Finally, any semi-split extension

$$0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$$

of complete bornological $A^{\dagger}\rtimes\Gamma$-bimodules is also a semi-split extension of complete bornological $A^{\dagger}$-bimodules. This gives a short exact sequence of bornological Hochschild complexes

$$0\longrightarrow\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M'\right)\longrightarrow\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\longrightarrow\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M''\right)\longrightarrow 0$$

as established in the proof of Theorem 3.3.3, part (3). Thus, there is a long exact sequence

$$\begin{aligned}\cdots\longrightarrow\mathbb{H}_{n+1}\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M''\right)\right)&\longrightarrow\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M'\right)\right)\\&\longrightarrow\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M\right)\right)\longrightarrow\mathbb{H}_n\left(\Gamma,\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger},M''\right)\right)\longrightarrow\cdots\end{aligned}$$

of hyperhomology groups. □

In the end, we want to apply Theorem 4.2.3 in the case of $M=A^{\dagger}\rtimes\Gamma$. Note that $A^{\dagger}\rtimes\Gamma$ carries a natural Γ-grading, meaning that $A^{\dagger}\rtimes\Gamma$ decomposes as $A^{\dagger}\rtimes\Gamma=\bigoplus_{\gamma\in\Gamma}A^{\dagger}\gamma$. More generally, a **complete bornological Γ-graded $A^{\dagger}\rtimes\Gamma$-bimodule** is a complete bornological $A^{\dagger}\rtimes\Gamma$-bimodule M which can be written as $M=\bigoplus_{\gamma\in\Gamma}M_{\gamma}$ for M_{γ} complete bornological $A^{\dagger}\rtimes\Gamma$-modules such that $(A^{\dagger}\gamma)M_{\delta}\subseteq M_{\gamma\delta}$ and $M_{\gamma}(A^{\dagger}\delta)\subseteq M_{\gamma\delta}$ for all $\gamma,\delta\in\Gamma$. Morphisms of complete bornological Γ-graded $A^{\dagger}\rtimes\Gamma$ are required to preserve the grading.

Corollary 4.1.2 generalises to the bornological Hochschild homology of $A^{\dagger}\rtimes\Gamma$ with coefficients in Γ-graded bimodules. For this, we rewrite the n-th chain group of $\mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger}\rtimes\Gamma,M\right)$, with M a complete bornological Γ-graded $A^\dagger \rtimes \Gamma$-bimodule, as

$$\mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) = M \mathbin{\overline{\otimes}} \left(A^\dagger \rtimes \Gamma\right)^{\overline{\otimes} n} \cong \overline{M \otimes (A \rtimes \Gamma)^{\otimes n}}$$

using Lemma 4.1.1 and Lemma 3.1.8. For each $\gamma \in \Gamma$, we can define a subgroup of $M \otimes (A \rtimes \Gamma)^{\otimes n}$ spanned by elements of the form $m_{\gamma_0} \otimes a_1\gamma_1 \otimes \cdots \otimes a_n\gamma_n$ with $\gamma_0 \cdots \gamma_n \in [\gamma]$. Taking completions defines a subcomplex $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma, \bar{\mathrm{b}}\right)$ of $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right), \bar{\mathrm{b}}\right)$. Denote its n-th homology by $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)$.

Corollary 4.2.4. *Let M be a complete bornological Γ-graded $A^\dagger \rtimes \Gamma$-bimodule. There is an isomorphism*

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) \cong \bigoplus_{[\gamma] \in [\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma$$

for each $n \geq 0$.

Proof. Since direct sums commute with completions, the given algebraic decomposition immediately gives the appropriate decomposition of $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right), \bar{\mathrm{b}}\right)$. The claim follows by taking homology. □

With this in mind, both Lemma 4.2.1 and Lemma 4.2.2 generalise well to the graded case.

Lemma 4.2.5. *Let M be a complete bornological Γ-graded $A^\dagger \rtimes \Gamma$-bimodule and let X be a Γ-graded V-module.*

(a) There is a natural isomorphism

$$\mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma \cong \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right)_\Gamma$$

for every $\gamma \in \Gamma$.

(b) There is a Γ-grading on $\overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}$ such that

$$\mathbb{H}_n\left(\Gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}_{[\gamma]}\right)\right) = 0$$

for each $\gamma \in \Gamma$ and for $n > 0$.

Proof. For part (a), we recall that Γ acts on the M factor in $\left(\mathrm{C}_\bullet\left(A^\dagger, M\right), \bar{\mathrm{b}}\right)$ by conjugation. Thus, the Γ-action stabilizes $M_{[\gamma]}$. This shows that $\left(\mathrm{C}_\bullet\left(A^\dagger, M_{[\gamma]}\right), \bar{\mathrm{b}}\right)$ is a complex of $V[\Gamma]$-modules. Moreover, by construction, $\mathrm{C}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma = M_{[\gamma]}$. Together with the way the subcomplex $\left(\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma, \bar{\mathrm{b}}\right)$ is defined, this gives a commuting diagram

$$\begin{array}{ccccccc}
M_{[\gamma]} \mathbin{\overline{\otimes}} A^\dagger & \xrightarrow{\ \bar{\mathrm{b}}\ } & M_{[\gamma]} & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right) & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
M \mathbin{\overline{\otimes}} \left(A^\dagger \rtimes \Gamma\right)_\gamma & \xrightarrow[\ \bar{\mathrm{b}}\ ]{} & M_\gamma & \longrightarrow & \mathrm{HH}_0^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma & \longrightarrow & 0
\end{array}$$

with exact rows. From here on we can continue as in the proof Lemma 4.2.1.

For part (b), we define a Γ-grading on $(A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)$ by

$$\left((A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)\right)_\gamma = \bigoplus_{\substack{\delta, \epsilon, \eta \\ \delta\epsilon\eta = \gamma}} A\delta \otimes V_\epsilon \otimes A\eta\,.$$

Taking completion of these sums gives a decomposition for $\overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}$. Note that

$$\mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right)\right) = \bigoplus_{[\gamma]\in[\Gamma]} \mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}_{[\gamma]}\right)\right),$$

since completed tensor products commute with direct sums and hyperhomology is additive. As

$$\mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}\right)\right) = 0$$

by Lemma 4.2.2, we conclude that

$$\mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, \overline{(A^\dagger \rtimes \Gamma) \otimes X \otimes (A^\dagger \rtimes \Gamma)}_{[\gamma]}\right)\right) = 0$$

for all $\gamma \in \Gamma$ and $n > 0$. □

Using Lemma 4.2.5, we strengthen Theorem 4.2.3 in case of Γ-graded bimodules.

Theorem 4.2.6. *Let M be a complete bornological Γ-graded $A^\dagger \rtimes \Gamma$-bimodule. There are isomorphisms*

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma \cong \mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right)\right)$$

for all $\gamma \in \Gamma$ and $n \geq 0$.

Proof. We consider the groups $\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma$ as part of a functor: Associate to a Γ-graded complete bornological $A^\dagger \rtimes \Gamma$-bimodule the sum

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) \cong \bigoplus_{[\gamma]\in[\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma .$$

If we consider only Γ-graded bimodules, the results of Theorem 3.3.3 continue to hold true. Now Lemma 4.2.5(a) shows that the isomorphism of Theorem 4.2.3 is compatible with the given decompositions. Thus, following the proof strategy outlined at the end of Section 3.3, we can inductively show that the isomorphisms

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right) \cong \mathbb{H}_n(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, M\right))$$

of Theorem 4.2.3 are in fact induced by a family of isomorphisms

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma, M\right)_\gamma \cong \mathbb{H}_n(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, M_\gamma\right)) .$$

Here, γ runs through a set of representatives of the conjugacy classes of Γ. □

The hyperhomology group in Theorem 4.2.6 can be further simplified. For this, we reconstruct the explicit chain map used in the proof of [Lor92, Proposition 2.6(b)] in Appendix A and then consider it in the bornological context.

Proposition 4.2.7. *Let M be a complete bornological Γ-graded $A^\dagger \rtimes \Gamma$-bimodule. There are isomorphisms*

$$\mathbb{H}_n\left(\Gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right)\right) \cong \mathbb{H}_n\left(C_\gamma, C_\bullet^{\mathrm{bor}}\left(A^\dagger, M_\gamma\right)\right)$$

for all $\gamma \in \Gamma$ and $n \geq 0$.

Proof. As in the proof of Lemma 4.2.2, we start with algebraic considerations first. Fix $\gamma \in \Gamma$ and consider the complex

$$\mathrm{C}_\bullet\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma]\,.$$

Acting on the right factor turns this into a complex of $V[\Gamma]$-modules. By Lemma A.2, the maps

$$\begin{aligned}\varphi_n\colon \left(M_\gamma \otimes A^{\otimes n}\right) \otimes_{V[C_\gamma]} V[\Gamma] &\to M_{[\gamma]} \otimes A^{\otimes n}\,,\\ m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta &\mapsto (\delta^{-1} m_\gamma \delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n))\end{aligned}$$

assemble into an isomorphism

$$\varphi_\bullet\colon \mathrm{C}_\bullet\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma] \xrightarrow{\sim} \mathrm{C}_\bullet\left(A, M_{[\gamma]}\right)$$

of $V[\Gamma]$-complexes. The inverse ψ_n of φ_n is defined as follows. Let $[\gamma] = \{\gamma_1, \ldots, \gamma_c\}$ and for each $\gamma_i \in [\gamma]$, choose a fixed $\delta_i \in \Gamma$ such that $\delta_i \gamma \delta_i^{-1} = \gamma_i$. Now let

$$\begin{aligned}\psi_n\colon M_{[\gamma]} \otimes A^{\otimes n} &\to \left(M_\gamma \otimes A^{\otimes n}\right) \otimes_{V[C_\gamma]} V[\Gamma]\,,\\ m_i \otimes (a_1 \otimes \cdots \otimes a_n) &\mapsto (\delta_i^{-1} m_i \delta_i) \otimes (\delta_i^{-1}(a_1) \otimes \cdots \otimes \delta_i^{-1}(a_n)) \otimes \delta_i^{-1}\,.\end{aligned}$$

We view the $V[C_\gamma]$-tensor products as quotients of V-tensor products and consider the corresponding quotient bornology. Note that then both φ_n and ψ_n are bounded, since we are given bornological algebras endowed with bounded group actions. Taking bornological completions now shows that

$$\overline{\mathrm{C}_n\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma]} \cong \overline{\mathrm{C}_n\left(A, M_{[\gamma]}\right)}\,.$$

There is an isomorphism

$$\overline{\mathrm{C}_n\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma]} \cong \overline{\mathrm{C}_n\left(A, M_\gamma\right)} \otimes_{V[C_\gamma]} V[\Gamma]\,.$$

Indeed, as $V[C_\gamma]$-modules,

$$\mathrm{C}_n\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma] \cong \bigoplus_{\delta \in C_\gamma \backslash \Gamma} \mathrm{C}_n\left(A, M_\gamma\right)$$

and therefore

$$\overline{\mathrm{C}_n\left(A, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma]} \cong \bigoplus_{\delta \in C_\gamma \backslash \Gamma} \overline{\mathrm{C}_n\left(A, M_\gamma\right)} \cong \overline{\mathrm{C}_n\left(A, M_\gamma\right)} \otimes_{V[C_\gamma]} V[\Gamma]\,.$$

Using Lemma 3.2.11(c) and Lemma 3.1.8, we also conclude that

$$\overline{\mathrm{C}_n\left(A, M_\gamma\right)} \cong \mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger, M_\gamma\right)$$

as well as

$$\overline{\mathrm{C}_n\left(A, M_{[\gamma]}\right)} \cong \mathrm{C}_n^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right)\,.$$

Thus, there is an isomorphism

$$\mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma] \cong \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, M_{[\gamma]}\right)$$

of $V[\Gamma]$-complexes. Shapiro's lemma ([Bro82, Proposition 6.2]) then gives an isomorphism

$$\mathbb{H}_n\Big(C_\gamma, \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, M_\gamma\right)\Big) \cong \mathbb{H}_n\Big(\Gamma, \mathrm{C}^{\mathrm{bor}}_n\left(A^\dagger, M_\gamma\right) \otimes_{V[C_\gamma]} V[\Gamma]\Big) \cong \mathbb{H}_n\Big(\Gamma, \mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, M_{[\gamma]}\right)\Big)$$

of hyperhomology groups, as claimed. □

4.3 Complements on Twisted Bornological Hochschild Homology

Let $g\colon A \to A$ be a V-algebra homomorphism. As in Section 2.1 we consider the A-bimodule A_g, with right A^{e}-module structure given by $a \cdot (a_1 \otimes a_2) = a_2 a g(a_1)$. Since g is bounded in the fine bornology, we can also endow $A^\dagger$ with a twisted $A^\dagger$-bimodule structure. Denote this $(A^{\mathrm{e}})^\dagger$-module by $A^\dagger_{g^\dagger}$. We write $\mathrm{C}^{\mathrm{bor}}_n\left(A^\dagger, g^\dagger\right) = \mathrm{C}^{\mathrm{bor}}_n\left(A^\dagger, A^\dagger_{g^\dagger}\right)$.

Definition 4.3.1. The complex $\left(\mathrm{C}^{\mathrm{bor}}_\bullet\left(A^\dagger, g^\dagger\right), \overline{\mathrm{b}}_{g^\dagger}\right)$ is the **$g^\dagger$-twisted bornological Hochschild complex of $A^\dagger$**. Its n-th homology is the **n-th $g^\dagger$-twisted bornological Hochschild homology of $A^\dagger$**, denoted by $\mathrm{HH}^{\mathrm{bor}}_n\left(A^\dagger, g^\dagger\right)$. ⌟

We use the strategy of [CCMT18, Proposition 4.1.7(a)] to prove an analogue of Lemma 2.2.3 for $g^\dagger$-twisted bornological Hochschild homology. For this, we need a flatness result.

Lemma 4.3.2. *Let A be a commutative V-algebra of finite type. The canonical map $A \to A^\dagger$ is flat.*

Proof. This is [CCMT18, Lemma 4.1.4]. □

Proposition 4.3.3. *Let A be a torsion-free commutative V-algebra of finite type. The natural homomorphism*

$$\mathrm{HH}_n\left(A, g\right) \to \mathrm{HH}_n\left(A^\dagger, g^\dagger\right)$$

induces an isomorphism

$$A^\dagger \otimes_A \mathrm{HH}_n\left(A, g\right) \xrightarrow{\sim} \mathrm{HH}^{\mathrm{bor}}_n\left(A^\dagger, g^\dagger\right)$$

for all $n \geq 0$.

Proof. Consider the (unnormalised) bar resolutions $\left(\mathrm{B}_\bullet(A), \mathrm{b}'\right)$ of A and $\left(\mathrm{B}^{\mathrm{bor}}_\bullet\left(A^\dagger\right), \overline{\mathrm{b}'}\right)$ of $A^\dagger$, respectively. Let M be an $(A^{\mathrm{e}})^\dagger$-module. The proof of [CCMT18, Proposition 4.1.7(a)] shows that the natural chain map

$$M \otimes_{A^{\mathrm{e}}} \mathrm{B}_\bullet(A) \to M \otimes_{(A^{\mathrm{e}})^\dagger} \mathrm{B}^{\mathrm{bor}}_\bullet\left(A^\dagger\right)$$

is a quasi-isomorphism. We claim that for $M = A^\dagger_{g^\dagger}$, the quasi-isomorphism specialises to the asserted isomorphism.

First, we show that the homology of $A^\dagger_{g^\dagger} \otimes_{A^{\mathrm{e}}} \mathrm{B}_\bullet(A)$ is given by $A^\dagger \otimes_A \mathrm{HH}_n\left(A, g\right)$. For this, we observe that

$$A^\dagger \otimes_A A_g \to A^\dagger_{g^\dagger}, \quad a \otimes b \mapsto ab$$

is an isomorphism of A^{e}-modules. This is the case, since the twisted right $A^\dagger$-module structure of $A^\dagger_{g^\dagger}$ is the extension of the twisted right A-module structure of A_g. Thus,

$$A^\dagger_{g^\dagger} \otimes_{A^{\mathrm{e}}} \mathrm{B}_\bullet(A) \cong A^\dagger \otimes_A \otimes A_g \otimes_{A^{\mathrm{e}}} \mathrm{B}_\bullet(A) \cong A^\dagger \otimes_A \mathrm{C}_\bullet\left(A, g\right)$$

is a chain isomorphism. Since $A^\dagger$ is a flat A-module by Lemma 4.3.2, the n-th homology of the last complex is given by $A^\dagger \otimes_A \mathrm{HH}_n\left(A, g\right)$.

Second, we show that the homology of $A^\dagger_{g^\dagger} \otimes_{(A^{\mathrm{e}})^\dagger} \mathrm{B}^{\mathrm{bor}}_\bullet(A^\dagger)$ is given by $\mathrm{HH}^{\mathrm{bor}}_n\left(A^\dagger, g^\dagger\right)$. For this, we first rewrite A_g as a quotient of A^{e}. Note that the twisted multiplication map

$$\mu_g\colon A^{\mathrm{e}} \to A_g, \quad a \otimes b \mapsto a g(b)$$

is a surjective A^{e}-module map. Thus, for $J = \ker(\mu_g)$ we obtain an isomorphism $A^{\mathrm{e}}/J \cong A_g$ of A^{e}-modules. We consider the tensor product $A_g \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}})^{\dagger}$ as an $(A^{\mathrm{e}})^{\dagger}$-module, by acting on the right. With the aid of Lemma 3.2.10, this yields an isomorphism

$$A_g \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}})^{\dagger} \cong (A^{\mathrm{e}}/J) \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}})^{\dagger} \cong (A^{\mathrm{e}})^{\dagger}/J(A^{\mathrm{e}})^{\dagger} \cong (A^{\mathrm{e}}/J)^{\dagger} \cong (A_g)^{\dagger}$$

of $(A^{\mathrm{e}})^{\dagger}$-modules. Note that $(A_g)^{\dagger}$ is by construction the $(A^{\mathrm{e}})^{\dagger}$-module, with right structure the extension of the right structure on A_g. That is, $(A_g)^{\dagger} = A^{\dagger}_{g^{\dagger}}$. Thus, as $(A^{\mathrm{e}})^{\dagger}$-modules, $A_g \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}})^{\dagger} \cong A^{\dagger}_{g^{\dagger}}$. This now implies that

$$A^{\dagger}_{g^{\dagger}} \otimes_{(A^{\mathrm{e}})^{\dagger}} \mathrm{B}^{\mathrm{bor}}_{\bullet}(A^{\dagger}) \cong A_g \otimes_{A^{\mathrm{e}}} (A^{\mathrm{e}})^{\dagger} \otimes_{(A^{\mathrm{e}})^{\dagger}} \mathrm{B}^{\mathrm{bor}}_{\bullet}(A^{\dagger}) \cong A_g \otimes_{A^{\mathrm{e}}} \mathrm{B}^{\mathrm{bor}}_{\bullet}(A^{\dagger})$$

is a chain isomorphism. The n-th chain group can be rewritten as

$$A_g \otimes_{A^{\mathrm{e}}} \mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) \cong (A^{\mathrm{e}}/J) \otimes_{A^{\mathrm{e}}} \mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) \cong \mathrm{B}^{\mathrm{bor}}_n(A^{\dagger})/J\,\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger})\,.$$

We note that $J\,\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) = (J\,\mathrm{B}_n(A))\,\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger})$ and that $\mathrm{B}_n(A)$ is a commutative V-algebra of finite type, since A is. Recall that $\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) = \left(A^{\dagger}\right)^{\overline{\otimes}(n+1)}$ by definition and that $A^{\dagger} \cong \overline{A}_{\mathrm{lg}}$ by Theorem 3.2.9. Thus, using Lemma 3.2.11(c) we deduce that

$$\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) = \left(A^{\dagger}\right)^{\overline{\otimes}(n+2)} \cong \left(\overline{A}_{\mathrm{lg}}\right)^{\overline{\otimes}(n+2)} \cong \overline{\left(A^{\otimes(n+1)}\right)}_{\mathrm{lg}} \cong (\mathrm{B}_n(A))^{\dagger}\,.$$

Thus, Lemma 3.2.10 applies to produce an isomorphism

$$\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger})/J\,\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) = \mathrm{B}^{\mathrm{bor}}_n(A^{\dagger})/(J\,\mathrm{B}_n(A))\,\mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) \cong (\mathrm{B}_n(A)/J\,\mathrm{B}_n(A))^{\dagger}\,.$$

Algebraically,

$$\mathrm{B}_n(A)/J\,\mathrm{B}_n(A) \cong (A^{\mathrm{e}}/J) \otimes_{A^{\mathrm{e}}} \mathrm{B}_n(A) \cong A_g \otimes_{A^{\mathrm{e}}} \mathrm{B}_n(A) \cong \mathrm{C}_n(A,g)\,.$$

Taking completions then shows that

$$A^{\dagger}_{g^{\dagger}} \otimes_{(A^{\mathrm{e}})^{\dagger}} \mathrm{B}^{\mathrm{bor}}_n(A^{\dagger}) \cong \mathrm{C}_n(A,g)^{\dagger} \cong \mathrm{C}^{\mathrm{bor}}_n\left(A^{\dagger}, g^{\dagger}\right).$$

Here, the last isomorphism is the chain of isomorphisms

$$\mathrm{C}^{\mathrm{bor}}_n\left(A^{\dagger}, g^{\dagger}\right) = \left(A^{\dagger}\right)^{\overline{\otimes}(n+1)} \cong \left(\overline{A}_{\mathrm{lg}}\right)^{\overline{\otimes}(n+1)} \cong \overline{\left(A^{\otimes(n+1)}\right)}_{\mathrm{lg}} \cong \mathrm{C}_n(A,g)^{\dagger}$$

provided, again, by Theorem 3.2.9 and Lemma 3.2.11(c).

Now note that the intermediate identifications make only use of natural quotient maps, hence assemble to define an isomorphism

$$A^{\dagger}_{g^{\dagger}} \otimes_{(A^{\mathrm{e}})^{\dagger}} \mathrm{B}^{\mathrm{bor}}_{\bullet}(A^{\dagger}) \cong \mathrm{C}^{\mathrm{bor}}_{\bullet}\left(A^{\dagger}, g^{\dagger}\right)$$

of chain complexes. The n-th homology of the last complex is given by $\mathrm{HH}^{\mathrm{bor}}_n\left(A^{\dagger}, g^{\dagger}\right)$ as claimed. □

4.4 Hochschild Homology of Crossed Product Dagger Algebras

We now combine the results of Section 4.2 and Section 4.3. View $A^\dagger \rtimes \Gamma$ as a Γ-graded complete bornological $A^\dagger \rtimes \Gamma$-bimodule. In combination, Corollary 4.2.4, Theorem 4.2.6 and Proposition 4.2.7 show that

$$\begin{aligned}\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) &\cong \bigoplus_{[\gamma]\in[\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right)_\gamma \\ &\cong \bigoplus_{[\gamma]\in[\Gamma]} \mathbb{H}_n\Big(\Gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, \left(A^\dagger \rtimes \Gamma\right)_{[\gamma]}\right)\Big) \\ &\cong \bigoplus_{[\gamma]\in[\Gamma]} \mathbb{H}_n\Big(C_\gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, \left(A^\dagger \rtimes \Gamma\right)_{\gamma}\right)\Big)\end{aligned}$$

Note that $\left(A^\dagger \rtimes \Gamma\right)_\gamma = A^\dagger\gamma$. As an $A^\dagger$-bimodule, we have that $A^\dagger\gamma \cong A^\dagger_{\gamma^\dagger}$. Since C_γ is finite,

$$\mathbb{H}_n\Big(C_\gamma, \mathrm{C}_\bullet^{\mathrm{bor}}\left(A^\dagger, \left(A^\dagger \rtimes \Gamma\right)_{\gamma}\right)\Big) \cong \mathrm{HH}_n^{\mathrm{bor}}\Big(A^\dagger, \left(A^\dagger \rtimes \Gamma\right)_\gamma\Big)_{C_\gamma} \cong \mathrm{HH}_n^{\mathrm{bor}}\Big(A^\dagger, \gamma^\dagger\Big)_{C_\gamma}$$

for each $[\gamma] \in [\Gamma]$. Thus,

$$\mathrm{HH}_n^{\mathrm{bor}}\left(A^\dagger \rtimes \Gamma\right) \cong \bigoplus_{[\gamma]\in[\Gamma]} \mathrm{HH}_n^{\mathrm{bor}}\Big(A^\dagger, \gamma^\dagger\Big)_{C_\gamma}.$$

By Proposition 4.3.3, this reduces the computation of the bornological Hochschild homology of $A^\dagger \rtimes \Gamma$ entirely to the computation of the twisted Hochschild homology of A. From Section 2.1 we know this to be equivalent to computing the Hochschild homology of $A \rtimes \Gamma$.

If A is, moreover, torsion-free as a V-algebra, Proposition 4.3.3 shows that

$$A^\dagger \otimes_A \mathrm{HH}_n\left(A, \gamma\right) \cong \mathrm{HH}_n^{\mathrm{bor}}\Big(A^\dagger, \gamma^\dagger\Big).$$

Assume now that K, the field of fractions of V, is of characteristic 0. Along the base change $V \to K$, we can then take full advantage of the results established in Section 2. In our setting, this is the most natural choice of a base change to characteristic 0. Let $\overline{A} = K \otimes A$, which is a commutative K-algebra of finite type and is endowed with a group action by Γ as well. Write $\overline{\gamma}\colon \overline{A} \to \overline{A}$ for the endomorphism induced by $\gamma\colon A \to A$. Since the base change $V \to K$ is flat, we can show that

$$\overline{A} \otimes_A \mathrm{HH}_n\left(A, \gamma\right) \cong \mathrm{HH}_n\left(\overline{A}, \overline{\gamma}\right)$$

analogously to Lemma 2.3.1. If we now assume $\overline{A}$ to be smooth, $\mathrm{HH}_n\left(\overline{A}, \overline{\gamma}\right)$ is computed in Section 2.3.

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A Two Isomorphisms of Chain Complexes

We work in the setting of Section 4.2. Lemma A.1 and Lemma A.2 reconstruct the explicit isomorphisms of $V[\Gamma]$-complexes used in the proof of [Lor92, Proposition 2.6].

Lemma A.1. *Let X be a V-module. There is an isomorphism*

$$\mathrm{C}_\bullet(A, (A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \xrightarrow{\sim} \mathrm{C}_\bullet(A, (A \rtimes \Gamma) \otimes X \otimes A) \otimes V[\Gamma]$$

of $V[\Gamma]$-complexes.

Proof. For each $n \geq 0$, let

$$\begin{aligned} \varphi_n \colon ((A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \otimes A^{\otimes n} &\to \big((A \rtimes \Gamma) \otimes X \otimes A) \otimes A^{\otimes n}\big) \otimes V[\Gamma]\,, \\ (a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n) &\mapsto (\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1}\,. \end{aligned}$$

Here we equip the left hand side with the $V[\Gamma]$-structure from before, whereas on the right hand side Γ acts trivially on the first factor. Then

$$\begin{aligned} &\varphi_n\big(g \cdot [(a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n)]\big) \\ &= \varphi_n\big((g(a)g\gamma \otimes x \otimes b\delta g^{-1}) \otimes (g(a_1) \otimes \cdots \otimes g(a_n))\big) \\ &= ((\delta g^{-1})(g(a))(\delta g^{-1})g\gamma \otimes x \otimes b) \otimes ((\delta g^{-1})(g(a_1)) \otimes \cdots \otimes (\delta g^{-1})(g(a_n))) \otimes (\delta g^{-1})^{-1} \\ &= (\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes g\delta^{-1} \\ &= g \cdot [(\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1}] \\ &= g \cdot \varphi_n\big((a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n)\big) \end{aligned}$$

shows that each φ_n is Γ-equivariant. To check that $\varphi_\bullet$ is a chain map, we compute

$$\begin{aligned} &(\varphi_{n-1}\mathrm{b})((a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n)) \\ &= \varphi_{n-1}\Big((a\gamma \otimes x \otimes b\delta(a_1)\delta) \otimes (a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n-1} (a\gamma \otimes x \otimes b\delta) \otimes (\cdots \otimes a_i a_{i+1} \otimes \cdots) \\ &\quad + (-1)^n (a_n a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_{n-1})\Big) \\ &= (\delta(a)\delta\gamma \otimes x \otimes b\delta(a_1)) \otimes (\delta(a_2) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1} \\ &\quad + \sum_{i=1}^{n-1} (\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\cdots \otimes \delta(a_i a_{i+1}) \otimes \cdots) \otimes \delta^{-1} \\ &\quad + (-1)^n (\delta(a_n a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_{n-1})) \otimes \delta^{-1} \end{aligned}$$

and

$$\begin{aligned} &((\mathrm{b} \otimes 1)\varphi_n)((a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n)) \\ &= (\mathrm{b} \otimes 1)\big((\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1}\big) \\ &= (\delta(a)\delta\gamma \otimes x \otimes b\delta(a_1)) \otimes (\delta(a_2) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1} \\ &\quad + \sum_{i=1}^{n-1} (\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\cdots \otimes \delta(a_i)\delta(a_{i+1}) \otimes \cdots) \otimes \delta^{-1} \\ &\quad + (-1)^n (\delta(a_n)\delta(a)\delta\gamma \otimes x \otimes b\delta) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_{n-1})) \otimes \delta^{-1}\,. \end{aligned}$$

These two expressions agree, since each $\gamma \in \Gamma$ defines a V-algebra homomorphism of A by assumption.

Finally, the maps

$$
\begin{aligned}
\psi_n \colon \big((A \rtimes \Gamma) \otimes X \otimes A) \otimes A^{\otimes n}\big) \otimes V[\Gamma] &\to ((A \rtimes \Gamma) \otimes X \otimes (A \rtimes \Gamma)) \otimes A^{\otimes n}\,,\\
(a\gamma \otimes x \otimes b) \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta &\mapsto (\delta(a)\delta\gamma \otimes x \otimes b\delta^{-1}) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n))
\end{aligned}
$$

define inverses. Indeed, on generators we see that

$$
\begin{aligned}
&(\psi_n\varphi_n)((a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n))\\
&= \psi_n((\delta(a)\delta\gamma \otimes x \otimes b) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)) \otimes \delta^{-1})\\
&= ((\delta^{-1}(\delta(a))(\delta^{-1}\delta\gamma)) \otimes x \otimes b(\delta^{-1})^{-1}) \otimes (\delta^{-1}(\delta(a_1)) \otimes \cdots \otimes \delta^{-1}(\delta(a_n)))\\
&= (a\gamma \otimes x \otimes b\delta) \otimes (a_1 \otimes \cdots \otimes a_n)
\end{aligned}
$$

and that

$$
\begin{aligned}
&(\varphi_n\psi_n)((a\gamma \otimes x \otimes b) \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta)\\
&= \varphi_n((\delta(a)\delta\gamma \otimes x \otimes b\delta^{-1}) \otimes (\delta(a_1) \otimes \cdots \otimes \delta(a_n)))\\
&= ((\delta^{-1}(\delta(a))(\delta^{-1}\delta\gamma)) \otimes x \otimes b \otimes (\delta^{-1}(\delta(a_1)) \otimes \cdots \otimes \delta^{-1}(\delta(a_n))) \otimes (\delta^{-1})^{-1}\\
&= (a\gamma \otimes x \otimes b) \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta\,.
\end{aligned}
$$

Thus, $\varphi_\bullet$ is an isomorphism of $V[\Gamma]$-complexes. □

Lemma A.2. *Let M be a Γ-graded $A \rtimes \Gamma$-bimodule. For each $\gamma \in \Gamma$, there is an isomorphism*

$$
\mathrm{C}_\bullet\,(A, M_\gamma) \otimes_{V[C_\gamma]} V[\Gamma] \xrightarrow{\sim} \mathrm{C}_\bullet\,(A, M_{[\gamma]})
$$

of $V[\Gamma]$-complexes.

Proof. Fix $\gamma \in \Gamma$. If $m_\gamma \in M_\gamma$ and $\delta \in \Gamma$ so that $\delta^{-1}\gamma\delta = \gamma'$, then $\delta^{-1}m_\gamma\delta \in M_{\gamma'} \leq M_{[\gamma]}$. Thus, for each $n \geq 0$, let

$$
\begin{aligned}
\varphi_n \colon \big(M_\gamma \otimes A^{\otimes n}\big) \otimes_{V[C_\gamma]} V[\Gamma] &\to M_{[\gamma]} \otimes A^{\otimes n}\,,\\
m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta &\mapsto (\delta^{-1}m_\gamma\delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n))
\end{aligned}
$$

The right side carries the $V[\Gamma]$-structure from before and the left side is equipped with the induced $V[\Gamma]$-structure. Equivariance is immediate from

$$
\begin{aligned}
&\varphi_n(g \cdot [m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta])\\
&= \varphi_n(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes (\delta g^{-1}))\\
&= ((\delta g^{-1})^{-1} m_\gamma (\delta g^{-1})) \otimes ((\delta g^{-1})^{-1}(a_1) \otimes \cdots \otimes (\delta g^{-1})^{-1}(a_n))\\
&= (g(\delta^{-1}m_\gamma\delta)g^{-1}) \otimes (g(\delta^{-1}(a_1)) \otimes \cdots \otimes g(\delta^{-1}(a_n)))\\
&= g \cdot [(\delta^{-1}m_\gamma\delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n))]\\
&= g \cdot \varphi_n(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta)\,.
\end{aligned}
$$

For $\epsilon \in C_\gamma$ we also have that

$$\begin{aligned}
&\varphi_n(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes (\epsilon\delta)) \\
&= ((\epsilon\delta)^{-1} m_\gamma (\epsilon\delta)) \otimes ((\epsilon\delta)^{-1}(a_1) \otimes \cdots \otimes (\epsilon\delta)^{-1}(a_n)) \\
&= (\delta^{-1}(\epsilon^{-1} m_\gamma \epsilon)\delta) \otimes (\delta^{-1}(\epsilon^{-1}(a_1)) \otimes \cdots \otimes (\delta^{-1}(\epsilon^{-1}(a_1))) \\
&= \varphi_n((\epsilon^{-1} m_\gamma \epsilon) \otimes (\epsilon^{-1}(a_1) \otimes \cdots \otimes \epsilon^{-1}(a_n)))
\end{aligned}$$

which shows that the φ_n are $V[C_\gamma]$-balanced. To see that $\varphi_\bullet$ is a chain map, we note that

$$\begin{aligned}
&(\varphi_{n-1}(\mathsf{b} \otimes 1))(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta) \\
&= \varphi_{n-1}\Big((m_\gamma a_1) \otimes (a_2 \otimes \cdots \otimes a_n) \otimes \delta \\
&\quad + \sum_{i=1}^{n-1} (-1)^i m_\gamma \otimes (\cdots \otimes a_i a_{i+1} \otimes \cdots) \otimes \delta \\
&\quad + (-1)^n (a_n m_\gamma) \otimes (a_1 \otimes \cdots \otimes a_{n-1}) \otimes \delta\Big) \\
&= (\delta^{-1}(m_\gamma a_1)\delta) \otimes (\delta^{-1}(a_2) \otimes \cdots \otimes \delta^{-1}(a_n)) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (\delta^{-1} m_\gamma \delta) \otimes (\cdots \otimes \delta^{-1}(a_i a_{i+1}) \otimes \cdots) \\
&\quad + (-1)^n (\delta^{-1}(a_n m_\gamma)\delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_{n-1}))
\end{aligned}$$

and

$$\begin{aligned}
&(\mathsf{b}\varphi_n)(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta) \\
&= \mathsf{b}((\delta^{-1} m_\gamma \delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n)) \\
&= ((\delta^{-1} m_\gamma \delta)\delta^{-1}(a_1)) \otimes (\delta^{-1}(a_2) \otimes \cdots \otimes \delta^{-1}(a_n)) \\
&\quad + \sum_{i=1}^{n-1} (-1)^i (\delta^{-1} m_\gamma \delta) \otimes (\cdots \otimes \delta^{-1}(a_i)\delta^{-1}(a_{i+1}) \otimes \cdots) \\
&\quad + (-1)^n (\delta^{-1}(a_n)(\delta^{-1} m_\gamma \delta)) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_{n-1}))\,.
\end{aligned}$$

The final expressions agree, since the δ^{-1} are V-algebra homomorphisms and the $V[\Gamma]$-structure of $M_{[\gamma]}$ is defined as the restriction of its $A \rtimes \Gamma$-structure.

To obtain an inverse of $\varphi_\bullet$, we will define an inverse ψ_n of each φ_n. Let $[\gamma] = \{\gamma_1, \ldots, \gamma_c\}$ and for each $\gamma_i \in [\gamma]$, choose a fixed $\delta_i \in \Gamma$ such that $\delta_i \gamma \delta_i^{-1} = \gamma_i$. We then define

$$\begin{aligned}
\psi_n \colon M_{[\gamma]} \otimes A^{\otimes n} &\to \big(M_\gamma \otimes A^{\otimes n}\big) \otimes_{V[C_\gamma]} V[\Gamma]\,, \\
m_i \otimes (a_1 \otimes \cdots \otimes a_n) &\mapsto (\delta_i^{-1} m_i \delta_i) \otimes (\delta_i^{-1}(a_1) \otimes \cdots \otimes \delta_i^{-1}(a_n)) \otimes \delta_i^{-1}\,.
\end{aligned}$$

It remains to check that φ_n and ψ_n are mutually inverse. We start with

$$(\psi_n\varphi_n)(m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta) = \psi_n((\delta^{-1} m_\gamma \delta) \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n)))\,.$$

Since $\delta^{-1}\gamma\delta \in [\gamma]$, there is a unique $\gamma_j \in [\gamma]$ such that $\delta^{-1}\gamma\delta = \gamma_i = \delta_i \gamma \delta_i^{-1}$. Note that this implies that

$\delta\delta_i \in C_\gamma$. Now $\delta^{-1} m_\gamma \delta \in M_{\gamma_i}$, from which we deduce that

$$
\begin{aligned}
&\psi_n(\delta^{-1} m_\gamma \delta \otimes (\delta^{-1}(a_1) \otimes \cdots \otimes \delta^{-1}(a_n))) \\
&= (\delta_i^{-1}(\delta^{-1} m_\gamma \delta)\delta_i) \otimes (\delta_i^{-1}(\delta^{-1}(a_1)) \otimes \cdots \otimes \delta_i^{-1}(\delta^{-1}(a_n))) \otimes \delta_i^{-1} \\
&= ((\delta\delta_i)^{-1}) m_\gamma (\delta\delta_i)) \otimes ((\delta\delta_i)^{-1}(a_1) \otimes \cdots \otimes (\delta\delta_i)^{-1}(a_n)) \otimes \delta_i^{-1} \\
&= m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes (\delta\delta_i\delta_i^{-1}) \\
&= m_\gamma \otimes (a_1 \otimes \cdots \otimes a_n) \otimes \delta\,,
\end{aligned}
$$

since we consider a $V[C_\gamma]$-balanced tensor product. Conversely, we find that

$$
\begin{aligned}
&(\varphi_n \psi_n)(m_i \otimes (a_1 \otimes \cdots \otimes a_n)) \\
&= \varphi_n((\delta_i^{-1} m_i \delta_i) \otimes (\delta_i^{-1}(a_1) \otimes \cdots \otimes \delta_i^{-1}(a_n)) \otimes \delta_i^{-1}) \\
&= ((\delta_i^{-1})^{-1}(\delta_i^{-1} m_i \delta_i)\delta_i^{-1}) \otimes (((\delta_i^{-1})^{-1})(\delta_i^{-1}(a_1)) \otimes \cdots \otimes ((\delta_i^{-1})^{-1})(\delta_i^{-1}(a_n))) \\
&= m_i \otimes (a_1 \otimes \cdots \otimes a_n)\,.
\end{aligned}
$$

Thus, $\varphi_\bullet$ is an isomorphism of chain complexes. ☐