Groupoid correspondences and K-theory

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Topological correspondences

Correspondences and K-theory
Outline

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Plan of talk

1. Some basics about groupoids with Haar system and C*-correspondences.
2. Various notions of groupoid morphisms in literature.
3. Topological correspondences.
4. KK-theory via unbounded operators.
5. Constructing some unbounded odd KK-cycles using topological correspondences.
Definition

A groupoid \( G \) is a small category in which every arrow is invertible.

We denote the base space of groupoid \( G \) by \( G^{(0)} \) and the arrow space by \( G^{(1)} \).

Important terminology: inverse map, range map and source map.
Topological groupoids

Definition

$G$ is a topological groupoid if $G^{(0)}$ and $G^{(1)}$ are topological spaces, all the above maps and the composition are continuous.

Examples

1. A topological space $X$, where $X^{(0)} = X$ and $X^{(1)} = \text{identity arrows} = X$.
2. A topological group $W$, where $W^{(0)} = \{\star\}$ and $W^{(1)} = W$.
3. Action groupoid.
Action of a groupoid

**Definition: Left action**

$G$ is said to act on $X$ if there is an open map $\rho : X \to G^{(0)}$, called anchor map, such that,

1. $\rho$ is an open surjection,
2. If $G \times_{s,\rho} X := \{(g, x) \in G \times X : s(g) = \rho(x)\}$, then $\exists$ map $m : G \times_{s,\rho} X \to X$ satisfying

   $$m(g, (h, x)) = m(gh, x),$$

   $$m(id(g), x) = x,$$

   $\forall g, h \in G^{(1)}$ and $x \in X$. 

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**Groupoid correspondences and (some) KK-theory**

Rohit Dilip Holkar

**Introduction and warm-up**

**Groupoids and their actions**

Haar groupoids

**C***- algebra of a Haar groupoid

Various notions of groupoid morphisms

**Definition**

**Towards a main theorem**

**First theorem**

**Correspondences and K-theory**

**KK-theory via unbounded operators**

**Summary**
Action of a groupoid

Examples

1. A space $Y$ acting on another space $X$:
   A map $f : X \rightarrow Y$ can be viewed as an anchor map for
   trivial action $y \cdot x = x$, for all $x \in X$, and all $y \in Y$.

2. A group action on a space.

3. Right (or left) multiplication of a groupoid $G$ on itself is
   a right (resp. left) action of $G$ on itself. The anchor
   map is the source map $s$ (resp. range map $r$) in this
   case.
Haar groupoid

Definition

A groupoid $G$ is called locally compact Haar groupoid if,

1. $G^{(0)}$ is locally compact Hausdorff subspace of $G$,
2. Topology of $G$ has a countable basis consisting of relatively compact Hausdorff subsets.
3. For every $u \in G^{(0)}$, $G^u := r^{-1}(u)$ is locally compact Hausdorff in the relative topology inherited from $G$,
4. $G$ admits a left Haar system $\{\alpha^u\}_{u \in G^{(0)}}$. 
Haar groupoid: Left Haar system

A heuristic definition

A left Haar system for a locally compact groupoid $G$ is a family of positive Borel measures $\{\alpha^u\}_{u \in G(0)}$, where each $\alpha^u$ is defined on $G^u$, such that following conditions hold:

1. Support condition.
2. Continuity conditions for the measure.
3. Invariance under the right multiplication.
Haar groupoid

Examples

1. A second countable, locally compact, Hausdorff group with a Haar measure is a Haar groupoid.

2. If \((H, \beta^h)\) is a left Haar groupoid then \((H, \beta_h)\) is a right Haar groupoid where we define

\[
\int f(x)d\beta_h(x) := \int f(x^{-1})d\beta^h(x^{-1}).
\]
Let $G$ is locally compact Haar groupoid. For $f, g \in C_c(G)$ define:

1. Convolution: $f \ast g(x) := \int_{G^r(x)} f(y) \cdot g(y^{-1}x) \, d\lambda(y)$.
2. Involution: $f^*(x) = f^{-1}(x^{-1})$.
3. $I$-norm:

Let $\|f\|_{I,r} = \sup_{u \in G^{(0)}} \int_{G^u} |f(t)| \, d\lambda^u(t)$

$\|f\|_{I,s} = \sup_{u \in G^{(0)}} \int_{G^u} |f(t)| \, d\lambda_u(t)$

and $\|f\|_I := max\{\|f\|_{I,r}, \|f\|_{I,s}\}$. 
**Theorem**

Let $G$ be a locally compact Haar groupoid. Then $C_c(G)$ is a separable, normed $*$-algebra under the convolution multiplication and the $l^1$-norm and the involution is isometric.

**Definition**

The $C^*$-enveloping algebra $C^*(G)$ of $C_c(G)$ above is defined as the $C^*$-algebra of $G$. 

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**C*- algebra of a Haar groupoid**
Various notions of groupoid morphisms

**Definition: Usual morphism**

A *usual* morphism from $G$ to $H$ is a functor between the groupoids, when they are considered as small categories.

**Definition: Hilsum-Skandalis morphism**

A Hilsum-Skandalis morphism from $G$ to $H$ is given by a topological space $X$ with a left $G$-action and a right $H$-action, such that

1. the actions of $G$ and $H$ commute,
2. the right $H$-action is free and proper,
3. the anchor map $X \to G^{(0)}$ for the left action induces a homeomorphism $X/H \cong G^{(0)}$. 
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Various notions of groupoid morphisms

**Definition: Morita equivalence (Muhly-Renault-Williams)**

A Morita equivalence between two topological groupoids $G$ and $H$ is given by a space $X$ with $G \curvearrowright X$, $X \curvearrowleft H$ and the following conditions are satisfied,

1. the actions are principal,
2. the actions commute i.e. $g(xh) = (gx)h$ for all $g \in G$, $x \in X$ and $h \in H$,
3. the anchor map for the right action induces a bijection between $G/X$ to $H^{(0)}$,
4. the anchor map for the left action induces a bijection between $X\setminus H$ and $G^{(0)}$.

A M.E. from $G$ to $H$ produces strong M.E. from $C^*(G)$ to $C^*(H)$.
Various notions of groupoid morphisms

**Definition: Algebraic morphism (Buneci and Stachura; 2005)**

An algebraic morphism from $G$ to $H$ is a right action of $G$ on $H$ that commutes with the right action of $H$ on itself by right multiplication.

**Use of algebraic morphism (Buneci and Stachura)**

An algebraic morphism from groupoid $G$ to groupoid $H$ induces a $*$-homomorphism $C^*(G) \rightarrow \mathcal{M}(C^*(H))$. 
Various notions of groupoid morphisms

Definition: Generalized algebraic morphism

A generalized algebraic morphism from $G$ to $H$ is a topological space $X$ and actions $G \acts X$ and $X \acts H$ such that following conditions hold,

1. Actions of $G$ and $H$ on $X$ commute i.e. $g(xh) = (gx)h$ for all $a \in G$, $x \in X$ and $h \in H$,
2. The right action $X \acts H$ is principal.

Examples

1. A Morita equivalence.
2. An algebraic morphism.
Various notions of groupoid morphisms

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**Examples**

1. A Morita equivalence.
2. An algebraic morphism.
**Left** \( H \)-Haar space

For a (left) Haar groupoid \((H, \beta)\) a left \( H \)-Haar space is a tuple \((X, \{\lambda_h\}_{h \in H^0})\) where \(X \home H\), \(\rho : X \to H^0\) is the anchor map and \(\{\lambda_h\}_{h \in H^0}\) is a family of Borel measures that satisfy,

1. \(\text{supp}(\lambda_h) = \rho^{-1}(h) := X^h, \quad \forall h \in H^0\),

2. \(H\)-invariance:

\[
\int_{X_{r(h)}} f(xh) d\lambda_{r(h)}(x) = \int_{X_{s(y)}} f(\tilde{x}) d\lambda_{s(y)}(\tilde{x})
\]

For all \(f \in C_c(X)\)

3. **Continuity:** The map \(\Lambda : C_c(X) \to C_c(G^0)\) given by \(\Lambda(f)(h) = \int_X f(x) d\lambda_{r(h)}(x)\) is continuous surjection.
Topological correspondence

**Definition**

A correspondence from groupoid \( (G, \lambda) \) to groupoid \( (H, \mu) \) is a triple \( (X, \alpha, \Delta) \) where,

1. \( X \) is a \( G \)-\( H \) bispace and action of \( H \) is proper.
2. \( \alpha \) is a \( H \)-invariant family of measures on \( X \).
3. \( \Delta : G \times X \to \mathbb{R}^+ \) is a continuous function such that for each \( u \in H^{(0)} \)

\[
\int \int F(g, x) \, d\lambda(g) \, d\alpha^u(x) = \int \int F(g^{-1}, gx) \Delta(g^{-1}, gx) \, d\lambda(g) \, d\alpha^{r(u)}(x)
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In short a topological correspondences from \( G \) to \( H \) is a space with a family of measures that is \( G \)-invariant and \( H \)-quasi invariant.
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In short a topological correspondences from \(G\) to \(H\) is a space with a family of measures that is \(G\)-invariant and \(H\)-quasi invariant.
Composition

Let \((X, \alpha, \Delta_1) : (G_1, \lambda_1) \rightarrow (G_2, \lambda_2)\) and \((Y, \beta, \Delta_2) : (G_2, \lambda_2) \rightarrow (G_3, \lambda_3)\) be correspondences, their composition consists of

A space with left \(G_1\) and right \(G_3\) action,

\[\Omega := (X \times_{G(0)} Y)/G_2\]

Family of measure on this space

Does it exit...?

Yes!

Adjoining function for the left action.

Adjoining function

\[\Delta_{12}(\gamma, [x, y]) = \Delta_1(\gamma, x)\]

This makes sense as \(\Delta_1\) is invariant under \(G_2\) action.
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Composition

**Family of measures**

1. \((X \times_{G(0)} Y) := Z\) carries a family \(G_3\)-invariant family of measures, which is \(\alpha \times_{G(0)} \beta\).

2. A \(G_3\)-equivariant continuous positive function \(b\) on \(Z\) that is a 0-cocycle \(b\) in \(G_3\)-equivariant continuous \(\mathbb{R}^+\) cohomology of groupoid \(Z \times \Omega Z\). It is special function.

3. \(\pi: (X \times_{G(0)} Y) \to \Omega\) be the projection map.

\[
\lambda(f)[x, y] = \int f(x\gamma, \gamma^{-1}y) \, d\lambda_2(\gamma)
\]

is a system of measures along \(\pi\).

4. Take measure \(\mu\) on \(\Omega\) such that \(b(\alpha \times_{G(0)} \beta) = \mu \circ \lambda\)

The function \(b\) has to satisfy some technical computation conditions.
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The function \(b\) has to satisfy some technical computation conditions.
Composition of topological correspondences

**Definition (Composition)**

For correspondences $(X, \alpha, \Delta_1) : (G_1, \lambda_1) \to (G_2, \lambda_2)$ and $(Y, \beta, \Delta_2) : (G_2, \lambda_2) \to (G_3, \lambda_3)$ a composed correspondence $(\Omega, \mu, \Delta_{1,2}) : (G_1, \lambda_1) \to (G_3, \lambda_3)$ is defined by:

1. Space $\Omega := (X \times_{G_2} Y)/G_2$,

2. Family of measure $\mu = \{ \mu^u \}_{u \in G_2^{(0)}}$ is a family of measure that lifts to $b\alpha \times \beta$ on $Z$ for some $b' \in HH_{G_3}^0(Z \ast Z, \mathbb{R}^*_+)$ satisfying condition that $d^0(b) = \Delta$.

3. AdJOINING function $\Delta_{1,2}$ is the one given by equation.
Bicategory of topological correspondences

We use definition of bicategory introduced by Bénabou.

Object: a groupoid \((G, \alpha)\) goes to its \(C^*\)-algebra \(C^*(G, \alpha)\)

1-arrow: a correspondences \((X, \lambda)\) from \((G, \alpha)\) to \((H, \beta)\) goes to a \(C^*\)-correspondence \(\mathcal{H}(X, \lambda)\) from \(C^*(G, \alpha)\) to \(C^*(H, \beta)\).

2-arrow: A \(G\)-\(H\) invariant measure preserving function between the spaces that give the correspondence.

Proposition (R.D.H.)
Locally compact Haar groupoids with above given data form a bicategory.
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**Proposition (R.D.H.)**

Locally compact Haar groupoids with above given data form a bicategory.
**C*-correspondences**

**Definition**

A C*-correspondence from a C*-algebra $A$ to C*-algebra $B$ is a $B$ Hilbert module $\mathcal{H}$ with an action of $A$ such and the actions commute.

**Definition**

A C*-correspondence $\mathcal{H}$ from $A$ to $B$ is proper if $A$ acts on $\mathcal{H}$ by compact operators.

C*-correspondence form a bicategory.

- **Objects:** C*-algebras
- **1-arrows:** C*-correspondences
- **2-arrows:** Equivariant isomorphisms of Hilbert modules
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- Objects: C*-algebras
- 1-arrows: C*-correspondences
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$C^*$-correspondences

**Definition**

A $C^*$-correspondence from a $C^*$-algebra $A$ to $C^*$-algebra $B$ is a $B$ Hilbert module $\mathcal{H}$ with an action of $A$ such and the actions commute.

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A $C^*$-correspondence $\mathcal{H}$ from $A$ to $B$ is proper if $A$ acts on $\mathcal{H}$ by compact operators.

$C^*$-correspondence form a bicategory.

**Objects:** $C^*$-algebras

**1-arrows:** $C^*$-correspondences

**2--arrows:** Equivariant isomorphisms of Hilbert modules
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- **Objects:** C*-algebras
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**C*-correspondences**

**Definition**

A C*-correspondence from a C*-algebra $A$ to C*-algebra $B$ is a $B$ Hilbert module $\mathcal{H}$ with an action of $A$ such and the actions commute.

**Definition**

A C*-correspondence $\mathcal{H}$ from $A$ to $B$ is proper if $A$ acts on $\mathcal{H}$ by compact operators.

C*-correspondence form a bicategory.

- **Objects**: C*-algebras
- **1-arrows**: C*-correspondences
- **2-arrows**: Equivariant isomorphisms of Hilbert modules
Towards a main theorem

Let \((X, \lambda, \Delta)\) be correspondence from \((G, \alpha)\) to \((H, \beta)\).
\(C_c(X)\) can be made into a \(C_c(G)\)-\(C_c(H)\) bimodule and it has a \(C_c(H)\)-valued bilinear form for which the formulae are defined as follows:

If \(\phi \in C_c(G)\) and \(\psi \in C_c(H)\) and \(f \in C_c(X)\) then,

- **Left action:** \((\phi \cdot f)(x) :=\)
  \[
  \int_{G^r(x)} \phi(\gamma^{-1}x)f(x) \Delta^{1/2}(\gamma', \gamma'^{-1}x) \, d\alpha(\gamma')
  \]

- **Right action:** \((f \cdot \psi)(x) := \int_{H^s(x)} f(x\gamma^{-1})\phi(\gamma) \, d\beta(t)\)

- **Inner product:** \(\langle f, g \rangle(\gamma) := \int_{X^r(h)} f(x)g(x\gamma) \, d\lambda(x)\)
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First theorem

Theorem (R.D.H.)

Let \((G, \alpha)\) and \((H, \beta)\) be locally compact Haar groupoids and \((X, \lambda, \Delta)\) be a measured correspondence. Then \(C_c(X)\) can be naturally completed into a \(C^*\)-correspondence from \(C^*(G)\) to \(C^*(H)\).

This construction works for reduced \(C^*\)-algebras, too.

Theorem (J. Renault & R.D.H.)

A topological correspondence going to a \(C^*\)-correspondence is a morphism of bicategories.
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**Theorem (R.D.H.)**

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Examples

1. A space map $f : X \rightarrow Y$ is a measured correspondence as $\gamma X \chi$.

2. A group homomorphism $G \rightarrow H$ is a measured correspondence $G \cdot H$.

3. A groupoid action $G \rtimes X$ is a measured correspondence $G \cdot X$.

4. If $G \cdot X \sim H$ is a Morita equivalence then $X$ can be made into $H$-Haar space.

Examples
One (important!) more example

- $(G, \lambda, \sigma)$ be a measured groupoid
- $(H, \mu, \tau)$ be a measured subgroupoid of $G$ and
- $(K, \nu)$ be a subgroupoid of $G$

- $\delta_G$ and $\delta_H$ denote modular functions of measures $\sigma \circ \lambda$ and $\tau \circ \mu$.

Lemma

If data above is given then $(\delta_G, \delta_H, G, \lambda^{-1})$ is a correspondence from $(H, \mu)$ to $(K, \nu)$.

What does the lemma tell?

- If $H = G$ and $(X_{Point}, \lambda)$ is a correspondence, then composition of correspondence in the lemma and $X$ is process of induction a representation.

- If $K = G$ and $(Y_{Point}, \lambda')$ is a correspondence then the composition of correspondence in lemma and $Y$ is process of restriction of representation.
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If data above is given then $(\frac{\delta_G}{\delta_H}, G, \lambda^{-1})$ is a correspondence from $(H, \mu)$ to $(K, \nu)$.

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Outline

Topological correspondences

Correspondences and K-theory
**R-equivariant $C^*$-correspondence**

Let $A$ be a $C^*$-algebra and $R$ be a group.

**Definition**

The $C^*$-algebra $A$ is called a $R$-algebra if there is a continuous homomorphism $R \to \text{Aut}(A)$.

Let $A$ and $B$ be $R$-$C^*$-algebras.

**Definition**

A $C^*$-correspondence $E$ from $A$ to $B$ is called $R$-equivariant if $R$-acts on $E$ by unitary operators and

- $t(eb) = (te)(tb)$
- $\langle re_1, re_2 \rangle = r\langle e_1, e_2 \rangle$
- $(ra)e = a(re)$
K-theory via unbounded operators

Definition

Let $B$ be a $C^*$-algebra and $E$ be a $C^*$-$B$-module. A densely defined closed operator $D : \text{Dom}(D) \to E$ is called regular if $D^*$ is densely defined in $E$ and $1 + DD^*$ has dense range.

- Such $D$ is automatically $B$-linear and $\text{Dom}(D)$ is a $B$-submodule of $E$.
- There are two operators related to this operators:
  - Resolvent: $r(D) := (1 + D^*D)^{-1/2}$
  - Bounded transform: $b(D) := D(1 + D^*D)^{-1/2}$
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K-theory via unbounded operators

Let $R$ be a group.

Definition (Unbounded equivariant KK-cycle, (Baaj-Julg,1983))

A $R$-equivariant, odd unbounded bimodule from a $R$-algebra $A$ to $R$-algebra $B$ is tuple $(E,D)$ where $E$ is an equivariant $A$-$B$ bimodule together with an unbounded regular operator $D$ on $E$ such that:

1. $[D,a] \in \text{End}(E)$ for all $a$ in a dense subalgebra of $A$,
2. $a \cdot r(D) \in \mathbb{K}_B(E)$, for all $a$ in a dense subalgebra of $A$.
3. The map $r \mapsto rDr^{-1}$ is strictly continuous map $R \rightarrow \text{End}(E)$.
Let $G$ be a groupoid and $R$ be a group.

Some definitions

A 0-cocycle: is a continuous function on $G^{(0)}/G$.
Set of $R$-valued 0-cocycles is denoted by $Z^0(G; R)$.

A 1-cocycle: is a continuous homomorphism from $G$ to $R$.
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For us— a cocycle would always mean a positive 1-cocycle

Definition

A cocycle $c \in Z^1(G; \mathbb{R})$ is regular if $\ker(c) = H$ admits a Haar system and exact if it is regular and the map

$$r \times s : G' \to G^{(0)} \times \mathbb{R}$$

$$\gamma \mapsto (r(\gamma), c(\gamma))$$

is a quotient map onto its image.
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Lemma (Renault, 1980)

Let \((G, \lambda)\) be a groupoid and \(c\) be a cocycle on \(G\) for each \(t \in R\) cocycle \(c\) gives an automorphism \(U_t\) of \(\ast\)-algebra \(C_c(G)\) by formula, \(U_t(f)(\gamma) = e^{itc(\gamma)}f(\gamma)\). This automorphism extends to an automorphism of \(C^*(G)\).

Proposition (Mesland, 2011)

If \((G, \lambda), (H, \alpha), c, U_t\) be as above then \(U_t\) extends to one parameter group of unitaries in \(C^*(H)\) (resp. \(C^*_r(H)\)). Further more \(\mathcal{H}(G)\) is a \(R\)-equivariant \(C^*(H)\) (resp. \(C^*_r(H)\)) a module.

Corollary

Let \((G, \lambda, \tau), (H, \mu, \tau) (K, \nu)\) be as in the example then above mentioned \(R\) action makes the \(C^*\)-correspondence related to \((\delta_G, G, \lambda^{-1})\) into a \(R\)-equivariant correspondence from \(C^*(H)\) to \(C^*(K)\).

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Here is the Hilbert module
The unbounded operator

Proposition (Mesland, 2011)
Let $G$, $c$ and $K$ be as above. Then the operator

$$D : C_c(G) \to C_c(G)$$

$$f(\gamma) \mapsto c(\gamma)f(\gamma)$$

is a $C_c(K)$-linear derivation of $C_c(G)$ considered as a bimodule over itself. Moreover, it extends to a self adjoint regular operator on the $C^*(K)$-Hilbert module $\mathcal{H}(G)$ (similar for $C^*(K)$).
**Theorem**

Let \((G, \lambda)\) be a second countable locally compact Hausdorff groupoid, \(c\) be a real 1-exact cocycle on \(G\) and \((H, \alpha)\) be an open subgroupoid of \(G\) such that \(H(0) = G(0)\). Then for a quasi-invariant measure \(\sigma\) and \(\tau\) on \(G\) and \(H\) respectively the operator \(D\) in last proposition, makes the \(\mathbb{R}\)-equivariant correspondence \((\mathcal{H}(G), D)\) into an odd \(\mathbb{R}\)-equivariant unbounded bimodule from \(C^*(H)\) to \(C^*(K)\). Similar statement holds for \((\mathcal{H}_r(G), D_r)\) from \(C^*_r(H)\) to \(C^*_r(K)\).

**Sketch of proof:** \(H \subseteq G\) is open hence we can realise \(C_c(H) \subseteq C_c(G)\) by extending functions by zero outside their domain. A similar statement holds for reduced \(C^*-\)algebras.
Proof:

Given \( \phi \in C_c(H) \subseteq C_c(G) \) we use to see that 
\([D, \phi]g = D(f \ast g)\) for all \( g \in C_c(G) \). Hence using the 
same proposition for each \( \phi \) the commutator \([D, \phi]\) is 
bounded.

It is cleat that The map \( r \mapsto rDr^{-1} \) is strictly 
continuous map \( R \to \text{End}(E) \).

Only thing to be proven is that \( \phi(1 + DD^*)^{-1} \) has 
\( C^*(K) \)-compact resolvant.

This operator acts on a \( g \in C_c(G) \) as

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\phi(1 + DD^*)^{-1} \circ g(\omega) = 
\int_G f(\gamma)(1 + c^2(\gamma^{-1}\omega))^{-1} \Delta(\gamma, \gamma^{-1}\omega) \, d\alpha^{r(\omega)}(\gamma)
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\( \Delta = \delta_H/\delta_G \) is the adjoining function.
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$\Delta = \delta_H/\delta_G$ is the adjoining function.
Proof continued:

- Take $k(\gamma, [\omega]) := \phi(\gamma)(1 + c^2(\omega))^{-1} \Delta(\gamma, \omega)$

Define $K_n = r_G(\text{supp}(\phi) \times \mathbb{R}) \cap \bar{c}^{-1}([-n, n]) \subseteq G/K$

Due to exactness of $c$, $K_n$s are compact.

- Take functions $e_nC(G/K, [0, 1])$ such that $f = 1$ on $K_n$ and zero outside $K_{n+1}$. Define

$$k^n(\gamma, [\omega]) := e_n[\omega]k(\gamma, [\omega])$$

Each $k^n$ is compact.

- Finally we show that $\{k^n\}_n$ is a Cauchy sequence in $l$-norm.

It can be seen easily that this sequence converges to

$$\phi(\gamma)(1 + c^2(\omega))^{-1} \Delta(\gamma, \omega).$$
Proof continued:

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Proof continued:

- Take \( k(\gamma, [\omega]) := \phi(\gamma)(1 + c^2(\omega))^{-1} \Delta(\gamma, \omega) \)

Define \( K_n = r_G(\text{supp}(\phi) \times \mathbb{R}) \cap c^{-1}([-n, n]) \subseteq G/K \)
Due to exactness of \( c \), \( K_n \)'s are compact.

- Take functions \( e_n C(G/K, [0, 1]) \) such that \( f = 1 \) on \( K_n \) and zero outside \( K_{n+1} \). Define

\[
k^n(\gamma, [\omega]) := e_n[\omega]k(\gamma, [\omega])
\]

Each \( k^n \) is compact.

- Finally we show that \( \{k^n\}_n \) is a Cauchy sequence in \( l \)-norm.

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Corollary: Mesland’s construction of odd KK-cycles, 2011

Let $G$ be a locally compact Hausdorff groupoid and $c: G^{(0)} \to \mathbb{R}$ be an exact cocycle. The operator $D$ as above makes the $C^*$-correspondence $\mathcal{H}(G)$ from $C^*(G)$ to $C^*(\ker(c))$ into an odd $\mathbb{R}$-equivariant unbounded bimodule. A similar statement holds for reduced $C^*$-algebras.

Example

Non-commutative torus
Some remarks

Getting cocycles: quasi-invariant measures, corresponding modular functions.
Summary

1. Topological correspondences from $G$ to $H$ is a $G$-$H$ bispace with a $H$ invariant and $G$ quasi invariant family of measures indexed by $H^{(0)}$.

2. A topological correspondence between groupoids induce a $C^*$-algebraic correspondence between the $C^*$-algebras of the groupoids.

3. For a groupoid $G$ and its two appropriate subgroupoids $H$ and $K$ a cocycle make the correspondence $C^*(H)\mathcal{H}(G)C^*(K)$ into a $\mathbb{R}$-equivariant correspondence.

4. Using the same cocycle we obtained an odd element of $\mathbb{R}$-equivariant unbounded Kasparov theory of the pair $(C^*(H), C^*(K))$. 
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