NOTES OF THE TALKS OF THE CONFERENCE TOPOLOGY AND ANALYSIS IN INTERACTION CORTONA 2011

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NOTES TAKEN BY PIERRE ALBIN

1. Bunke: The topological content of the eta invariant

Start with a manifold M, spin^c, form the Dirac operator $\eth_M$ Assume you have a complex vector bundle $V \longrightarrow M$ and form the twisted Driac operator $\eth_M \otimes V$, and now can produce a number, the reduced eta invariant. Think of this as an association

$$V \mapsto \overline{\eta}(\eth_M \otimes V) \in \mathbb{R}/\mathbb{Z}$$

Of course this requires choosing a metric, a spin^c connection. a connection on the bundle.

To define the reduced eta invariant, start with the unreduced eta invariant

$$\eth(\eth_M \otimes V) = \mathrm{Tr}(\mathrm{sign}(\eth_M \otimes V)|\eth_M \otimes V|^{-s})\big|_{s=0}$$

a zeta-regularization. For the reduced one, take

$$\overline{\eta} = \frac{\eta + \dim \ker(\eth_M \otimes V)}{2} \in \mathbb{R}/\mathbb{Z}.$$

Note that this is not topological yet, becuase it requires geometric data.

All examples of topological invariants that can be obtained from the eta invariant can be obtained by the following universal construction. We will construct a bordism invariant.

Fix a bordism theory, and since we want to talk about Dirac operators it should be finer than spin^c. Choose a map from some space B to $BSpin^c$ from here get MB a Thom spectrum. So given a space X can talk about $MB_*(X)$, the B-bordism theory of X.

Want to construct a map

$$MB_n(X) \xrightarrow{\eta^{an}} Q_n(B, X)$$

and a corresponding topological map (using homotopy theory) η^{top} and then there will be an index theorem saying that these coincide.

What is $Q_n(B, X)$, it's a certain quotient:

$$\frac{\mathrm{Hom}(K^0(B \times X), K_{n+1}\mathbb{Q}/\mathbb{Z}(*))}{\mathrm{Im}(MB\mathbb{Q}_{n+1}(X))}$$

where the image is from mapping

(1.1)

$$MB_{\text{continuous}}\mathbb{Q}_{n+1} \xrightarrow{\text{unit of K-theory}} K\mathbb{Q}_{n+1}(MB \wedge X_+) \xrightarrow{\text{Thom isomorphism}} K_{n+1}(B \times X)$$

followed by evaluation, which lands in $K_{n+1}\mathbb{Q}(*)$ and then you mod out by $\mathbb{Z}$.

Now let's construct η^{an}, this uses the geometric version of this bordism theory. (These invariants will actually only be defined on the torsion part of $MB_n(X)$.) Let's take an element

$$x \in MB_n(X)_{\text{tor}}$$

Goemetrically, this is a manifold M together with two maps $M \xrightarrow{f} B$ which classifies the *spin*^{c} structure, and a map $g : M \longrightarrow X$, Then we need to define an equivalence. So assume you have ℓ copies of these representatives, and a null-bordism together with maps that extend f and g, F and G.

Choose a class $v \in K^0(B \times X)$ and choose $V \longrightarrow M$, so that the class $[v]$ is equal to the pull-back under (F, G) of $v \in K^0(M)$. Choose geometry, so that the geometry near the bundary is the same near each of the ℓ copies. (Formally, a $\mathbb{Z}_\ell$ equiv. on ∂W.) Now can form a formal APS-type boundary value problem and take the index and form

$$[\frac{1}{\ell} \operatorname{ind}(\eth_W \otimes V)] \in \mathbb{Q}/\mathbb{Z}$$

The map $v \mapsto [\ldots]$ represents η^{an}. Notice that we need to factor out by some things. The $\mathbb{Z}$ is because different choices might change the index by an integer. There's also the choice of null-bordism. For instance, can always add a closed manifold disjoint to the bordism, this gets taken care of by the denominator in the definition of $Q_n(B, X)$. After factoring out these things, end up with a well-defined invariant.

The relation with the eta-invariant follows from the $\mathbb{Z}_\ell$ index theorem of Freed-Melrose. It's equal to

$$-\overline{\eta}(\eth_M \otimes V_M) + \frac{1}{\ell} \int_W \widehat{A} \wedge \operatorname{Ch}(\nabla^V) \in \mathbb{R}/\mathbb{Z}.$$

Next, let us construct the topological index. We have this bordism theory $MB_n(X)$, and we can rationalize it and get a Bockstein sequence:

$$\ldots \longrightarrow MB\mathbb{Q}/\mathbb{Z}_{n+1}(X) \longrightarrow MB_n(X) \longrightarrow MB\mathbb{Q}_n(X) \longrightarrow \ldots$$

So given $x \in MB_n(X)$ that maps to zero in $MB\mathbb{Q}_n(X)$ we can find a pre-image $\widetilde{x} \in MB\mathbb{Q}/\mathbb{Z}_{n+1}(X)$. Following $\widetilde{x}$ through the maps analogous to (1.1), we gat an element $\widehat{x} \in K\mathbb{Q}/\mathbb{Z}_{n+1}(B \times X)$ The map

$$K^0(B \times X) \ni v \mapsto \langle v, \widehat{x} \rangle \in K\mathbb{Q}/\mathbb{Z}_{n+1}(*)$$

is the topological eta invariant.

Theorem 1.1. $\eta^{an} = \eta^{top}$

This is a version of the $\mathbb{Z}/\ell\mathbb{Z}$ theorem of Freed-Melrose.

The first thing to do to prove this is to note that establishing this for $\mathbb{Z}/\ell\mathbb{Z}$ for all ℓ, establishes it for $\mathbb{Q}/\mathbb{Z}$.

Next that we can using mapping cylinders: Have the map $S^n \xrightarrow{\ell} S^n \longrightarrow C_\ell$ a suspension of the Moore space.

To translate the topological construction to analysis, need to put some geometry on the topological objects. So use $MB_{n+2}(X \wedge C_\ell) \cong MB\mathbb{Z}/\mathbb{Z}_{n+1}(X)$ Picture: Bordism W, form $W \times \mathbb{S}^1$, now glue-in $\mathbb{S}^2 \setminus \ell$holes, and can extend the map to C_ℓ (by using ℓ copies of the inclusion map from the circle to the cone over the circle.

Taking the Kasparov class of the Dirac operator $\widetilde{W}$ (the manifold that resulted from the gluing construction applied to W), get an element in $K_{n+2}(X_+ \wedge_\ell)$. End up with an element in $K_{n+2}(C_\ell) = \mathbb{Z}/\ell\mathbb{Z}$. Now can twist with a bundle V.

Next, note that analytically there's no need to change the K-homology class if one wants to do surgery and allow non-compact manifolds, e.g., with cylindrical ends. This allows us to have the maps factor through $\mathbb{S}^1$, and then use the index theorem for cylindrical ends. This can be calculated using APS.

Now let us mention some special cases.

A classical case $B = * = X$, the topological eta is the Adams e invariant, and the analytic η is equal to this by the APS index formula.

Take $B = BSpin^c$ and $X = BU(n)^\delta$ (δ indicates discrete), then the analytic eta corresponds to ρ-invariants, and the index theorem for flat bundles gives the equality with the topological eta.

Take $B = *$ and $X = BGl(\mathbb{C}^\delta)^+$, get an extension of the Adams e-invariant which was studied by Jones and Westburg.

The motivation for this was that together with Nuamann, have invariants of string bordisms $B = BString = MO\langle 8\rangle$, $X = *$, had invariants involving modular forms.

Most recently, with Growley and Goette, take $B = BSpin$, X is the homotopy fiber of the map $BSU(w) \xrightarrow{\mathrm{Ch}_2} K$

Let me come to the second part. Just want to mention one interesting aspect. If you look at this formula

$$\eta^{an}(x)(v) - \overline{\eta}(\eth_{M\otimes V\big|_M} + [\frac{1}{\ell}\int_W \widehat{A}(\widehat{\nabla}) \wedge \mathrm{Ch}(\nabla^V)]$$

to calculate, need to choose a lift. In some cases can get an intrinsic formula, question was why? can one get one in general?

The answer is yes. There's a generalization of a connection. Have a map

$$K^0(B \times X) \xrightarrow{(f,g)^*} K^0(M)$$

but have to choose connections etc. Want to get a refined map into differential K-theory
$$K^0(B \times X) \xrightarrow{\mathcal{G}} \widehat{K}^0(M)$$
call this a 'geometrization.' (This is clear when we have a classifying space of a principal bundle.)

Want to have this map consistent with
$$K^0(B \times X) \xrightarrow{\mathrm{Ch} \wedge \widehat{A}} HP\mathbb{Q}^0(B \times X) \xrightarrow{C_{\mathcal{G}}} \Omega P^0_{cl}(M) \xleftarrow{R \wedge \widehat{A}(\nabla)} \widehat{K}^0(M).$$

Given this map, can get an intrinsic index formula: Start with $[V] \in K^0(B \times X)$ get $\mathcal{G}(V)$, and the difference is a differential form $a(\kappa)$ and we can rewrite
$$[\frac{1}{\ell} \int_W \widehat{A}(\widetilde{\nabla}) \wedge \mathrm{Ch}(\nabla^V) \equiv \ _{\mathrm{mod\ Im}(MB\mathbb{R}_{n+1}(X))} \int M\widehat{A}(\widetilde{\nabla}) \wedge \kappa$$
(all $\widehat{A}$ should be corrected $\widehat{A}$, because it's *spin*^{c}) so end up with an intrinsic formula.

The theorem is that, after fixing a *spin*^{c} structure on a closed manifold, then this geometrization map exists.

2. Wang: Stringy product in orbifold K-theory

joint work with Jianxun Hu.

Try to find cycles for orbifold K-theory.

2.1. **Background review.** orbifolds have orbifold charts $(\widetilde{U}_L, G_L)$ more fancy can talk about proper Étale groupoid which consists of $\mathcal{G}_1$, $\mathcal{G}_0$ and source and target maps.

Can also define orbifold vector bundles: Over an orbifold chart, have the quotient of an equivariant vector bundle, or from the groupoid point of view have an orbifold vector bundle.

Theorem: $K^*_{orb}(X) \cong K^*(\mathcal{G})$ (groupoid K-theory)

There's a de Rham cohomology, and again $H^*_{dR}(X) \cong H^*_{dR}(\mathcal{G})$ and by Sataki, these are isomorphic to the singular cohomology of the underlying space. There's a Chern character from the orbifold K-theory to de Rham cohomology

$$K^*_{orb}(X) \xrightarrow{\mathrm{Ch}} K^*_{orb}(X)$$

but this is not an isomorphism over $\mathbb{C}$, so instead should use the delocalized Chern character of Baum-Connes: There's a groupoid IX whose objects are objects of $\mathcal{G}$ with source and target equal, and arrows induced from $\mathcal{G}$, you take the disjoint union of $X_{(g)}$ with (g) the conjugacy classes of elements in $\mathcal{G}$. Note that there's a natural evaluation map $e : IX \longrightarrow X$, and we have

$$\begin{tikzcd}
K^*_{orb}(X) \arrow[rr, "\mathrm{Ch}_{\mathrm{deloc}}"] \arrow[dr, "e^*"'] & & H^*_{dR}(IX) \\
& K^*_{orb}(IX) \arrow[ur, "\mathrm{Ch}_\Phi"'] &
\end{tikzcd}$$

with Ch_Φ defined via a decomposition of a bundle into eigenbundles, as any complex vector bundle over IX admits an automorphism. This $\mathrm{Ch}_{\mathrm{deloc}}$ is an isomorphism over $\mathbb{C}$.

Now define k-sectors: a groupoid whose objects are objects in $\mathcal{G}$ with k arrows with source and target this object, get $I^{[k]}(X)$.

For any almost complex orbifold X, define Chen-Ruan cohomology: $H^*_{dR}(IX, \mathbb{C})$ with new product, the Chen-Ruan product. Let's call the evaluation maps

$$\begin{tikzcd}
E^{[2]} \arrow[dr] & & & \\
& I^{[2]}X \arrow[rr, "e_1"] \arrow[dr, "e_{12}"] \arrow[dd, "e_2"] & & IX \arrow[dd, "e"] \\
& & IX \arrow[dr, "e"] & \\
& IX \arrow[rr, "e"] & & X
\end{tikzcd}$$

Define

$$w_1 \circ_{CR} w_2 = (e_{12})_*(e_1^* w_1 \wedge e_2^* w_2 \wedge \chi(E^{[2]})) \in H^*_{dR}(IX, \mathbb{C})$$

where $\chi(E^{[2]})$ is the cohomology Euler class of the obstruction bundle $E^{[2]}$.

To get $E^{[2]}$, define

$$\mathcal{M}_{(0,3)}(X) = \text{ moduli space of map from orbisphere with three marked points to } X$$

This can be identified with $I^{[2]}X$, and $E^{[2]}$ is the cokernel of the map $\overline{\partial} \otimes (e \circ e_2)^* T_x X$ (this is the obstruction bundle to the orbifold being smooth).

A few years later Adem-Ruan-Zhang defined a product on $K_{orb}(IX)$, $*_{ARZ}$, which is formally the same, but with the K-theoretic Euler class $\lambda_{-1}(E^{[2]})$ instead of the cohomology Euler class.

Can we define this twisted product on the orbifold K-theory $K^*_{orb}(X)$ itself so that the delocalized Chern character is an isomorphism?

Now to define the stringy product on the orbifold K-theory $K^*_{orb}(X)$ itself:

2.2. **Intrinsic description of $E^{[2]}$.** Point is that the previous description of $E^{[2]}$ comes from nonlinear things, and we'd rather have a description in terms of the original orbifold data.

The connected components are labeled by equivalence classes of conjugacy pairs (g_1, g_2) and there's an obvious commutative diagram

$$\begin{tikzcd}
X_{(g_1,g_2)} \arrow[rr, "e_1"] \arrow[dr, "e_{12}"] \arrow[dd, "e_2"'] & & X_{(g_1)} \arrow[dd, "e_{(g_1)}"] \\
& X_{(g_1 g_2)} \arrow[dr, "e_{(g_1 g_2)}"] & \\
X_{(g_2)} \arrow[rr, "e_{(g_2)}"] & & X
\end{tikzcd}$$

Let

$$N = \bigsqcup_{(g)} N_{e,(g)}$$

each normal bundle has an action of Φ and hence an eigenbundle decomposition:

$$N_{e,(g)} = \oplus_{\theta \in (0,1)} N_e(\theta),$$

where Φ acts $N_e(\theta)$ by the multiplication of $e^{2\pi i \theta}$ for $\theta \in \mathbb{Q} \cap (0,1)$. Define

$$N_{e,\Phi} = \bigoplus_{\theta \in \mathbb{Q} \cap (0,1)} \theta N_e(\theta) \in K^*_{orb}(IX, \mathbb{Q}),$$

and

$$N_{e,\Phi^{-1}} = \bigoplus_{\theta \in \mathbb{Q} \cap (0,1)} (1-\theta) N_e(\theta) \in K^*_{orb}(IX, \mathbb{Q}),$$

Theorem 2.1.
$$E^{[2]} = e_1^* N_{e_1} \oplus e_2^* N_{e,\Phi} \oplus e_{12}^* N_{e,\Phi^{-1}} \ominus N$$
in $K^*_{orb}(IX, \mathbb{Q})$.

Remark:

(1) For almost complex orbifolds whose local groups are all abelian, this intrinsic description was obtained by Bohui Chen and Shengda Hu.
(2) After combining the like terms, $e_1^* N_{e_1} \oplus e_2^* N_{e,\Phi} \oplus e_{12}^* N_{e,\Phi^{-1}} \ominus N$ is a genuine vector bundle, which can be identified with the obstruction bundle $E^{[2]}$.

2.3. **Stringy product in orbifold K-theory.** We kill-off the torsion.

$$\begin{tikzcd}
K^*_{orb}(IX) \otimes \mathbb{C} \arrow[r, "\mathrm{Ch}_\Phi"] & H^*_{dR}(IX, \mathbb{C}) = \bigoplus_{(g)} H^*_{dR}(X_{(g)}) \\
K_{orb}(X) \otimes \mathbb{C} \arrow[u, "e^*"] \arrow[ur, "\mathrm{Ch}_{\mathrm{deloc}}", "\cong"']
\end{tikzcd}$$

because $\mathrm{Ch}_{\mathrm{deloc}}$ is an isomorphism, we get a decomposition
$$K^*_{orb}(X) \otimes \mathbb{C} = \bigoplus_{\langle g \rangle} K_{orb}(X, \langle g \rangle) \otimes \mathbb{C}$$

Lemma 2.2. *For* $\alpha_{(g)} \in K_{orb}(X, \langle g \rangle)$, *we have*
$$\mathrm{Ch}_\Phi(e^*_{(h)}\alpha_{(g)}) = \begin{cases} 0 & (h) \neq (g) \\ \mathrm{Ch}_{deloc}(\alpha_{(g)}) & else \end{cases}$$

Define
$$e^{\#}(\sum_{(g)} \alpha_{(g)}) = \sum e^*_{(g)} \alpha_{(g)}$$

Lemma 2.3. *1)* $\mathrm{Ch}_\Phi \circ e^{\#} = \mathrm{Ch}_\Phi \circ e^* = \mathrm{Ch}_{deloc}$
2) e^* *admits a left inverse, the componentwise (modified) push-forward,* $e_{\#}$. *Formally,*
$$(e_{(g)})_{\#}(\beta_{(g)}) = (e_{(g)})_*\Big(\frac{\beta_{(g)}}{\lambda_{-1}(N_{(g)})}\Big),$$
for $\beta_{(g)} \in K^*_{orb}(X_{(g)})$.

Definition 2.4. *The stringy product on* $K^*_{orb}(X) \otimes \mathbb{C}$ *is*
$$\alpha_{(g_1)} \circ \alpha_{(g_2)} = (e_{\#})_{(g_1 g_2)}(e^{\#}\alpha_{(g_1)} *_{ARZ} e^{\#}\alpha_{(g_2)})$$

Now, given a complex vector bundle with an automoprhism Φ, let
$$E = \bigoplus_{m_j} E_j$$

with Φ-action given by the multiplication $e^{2\pi i m)j}$. Define

$$\mathcal{T}(E,\Phi) = \prod_j \mathcal{T}(E_j)^{m_j},$$

where $\mathcal{T}(E_j)^{m_j}$ is the characteristic class associated to $\left(\frac{1-e^x}{x}\right)^{m_j}$

Then

$$\mathcal{T}(E^{[2]} \oplus N) \wedge e_{12}^* \mathcal{T}(N,\Phi) = e_1^* \mathcal{T}(N,\Phi) \wedge e_2^* \mathcal{T}(N,\Phi)$$

Define a modified delocalized Chern character

$$\widetilde{\mathrm{Ch}}_{\mathrm{deloc}} = \mathrm{Ch}_{\mathrm{deloc}} \wedge \mathcal{T}(N_e,\Phi)$$

Theorem 2.5. *$\widetilde{\mathrm{Ch}}_{deloc}$ is an isomorphism $K^*_{orb}(X) \otimes \mathbb{C}$ with the product $\circ$ and $H^*_{dR}(IX)$ with the Chen-Ruan product.*

2.4. **Example.** The first example is the orbifold $[\bullet/G]$ given by a point $\{pt\}$ with a trivial action of a finite group G. Then $K_{orb}[\bullet/G] \cong R(G)$, the representation ring of G. Under the delocalized Chern character

$$ch_{deloc} : (R(G) \otimes \mathbb{C}, \circ) \longrightarrow (\mathcal{C}(G), *)$$

is a ring isomorphism where $(\mathcal{C}(G), *)$ is the ring of complex valued class functions on G with the convolution product.

The second example is the conjugation groupoid of a finite group G: its inertia orbifold is represented by the conjugation action of G on the set of pairs of commuting elements in G.

On the orbifold K-theory

$$K_G(G) \cong \bigoplus_{\{(g):\text{conjugacy classes of } G\}} R(Z_G(g)),$$

where $Z_G(g)$ is the centralizer of g in G, there is a well-known Pontryagin product:

$$\bullet_G : K_G(G) \times K_G(G) \xrightarrow{\pi_1^* \times \pi_2^*} K_G(G \times G) \xrightarrow{m_*} K_G(G),$$

where $\pi_1, \pi_2 : G \times G \to G$ are the obvious projections, $m : G \times G \to G$ is the group multiplication.

The stringy product is a new product different to the Pontryagin product. For an abelian group G, the inertia orbifold is given by the quotient of $G \times G$ by the conjugation. Then the stringy product corresponds to the convolution product in the second variable and the Pontryagin product corresponds to the convolution product in the first variable.

3. Hanke: Homotopy groups of moduli spaces of metrics of positive scalar curvature

joint with Botvinnik, Schick, and Walsh.

I'm concerned with closed smooth manifolds, also connected. In particular, I'm interested in whether or not M admits a metric of positive scalar curvature and, if it does, in the space $Riem^+(M)$ of all such, endowed with the $\mathcal{C}^\infty$ topology.

Questions: Is the space not empty? If yes, want to study global properties, e.g., $\pi_k(Riem^+(M))$. Related questions for the Moduli space obtained by moding out by the diffeomorphism group.

There are results by Hitchin, Lawson-Michelson, ... about low values of k.

Theorem 3.1. *Given $d > 0$, can construct a Riemannian manifold (M, h) with positive scalar curvature such that*

$$\pi_{4q}(Riem^+(M)/\operatorname{Diff}(M)) \neq 0$$

for $1 \leq q \leq d$.

Remark 3.2. *These manifolds are non-spin, and odd dimensional.*

Remark 3.3. *The elements in this construction are maps $\mathbb{S}^k \longrightarrow Riem^+(M)/Diff(M)$ that do not lift to families $\mathbb{S}^k \longrightarrow Riem^+(M)$.*

Schick and Crowley have constructed non-trivial elements of $\pi_k(Riem^+(M))$ that however are trivial after passing to the quotient.

3.1. preparation.

Because $\operatorname{Diff}(M)$ does not act freely on $Riem^+(M)$, pass to subgroup

$$\operatorname{Diff}_{x_0}(M) = \{\phi \in \operatorname{Diff}_(M) : \phi(x_0) = x_0, T_{x_0}\phi = \mathrm{id}\}$$

which we point out does act freely.

Next we do a simple translation: Given

$$f : \mathbb{S}^k \longrightarrow B\operatorname{Diff}_{x_0}(M) = Riem(M)/\operatorname{Diff}_{x_0}(M)$$

we can consider the following diagram: This map classifies a bundle, say $M - E \longrightarrow \mathbb{S}^k$, by pulling-back a universal bundle $Riem(M) \times_{\operatorname{Diff}_{x_0}(M)} M$, so there is an induced map

$$F : E \longrightarrow Riem(M) \times_{\operatorname{Diff}_{x_0}(M)} M$$

Note that F corresponds to a smooth family of Riemannian metrics on M.

The proof of the theorem proceeds in three steps:
1) Construct maps f representing non-trivial elements in $\pi_k(B\operatorname{Diff}_{x_0}(M))$
2) Construct smooth families of positive scalar curvature metrics on the resulting bundles (note that this yields non-trivial elements in $\pi_k(Riem^+(M)/\operatorname{Diff}_{x_0}(M))$)
3) Make sure that these elements survive passing to the full quotient

First treat the first two points for $M = \mathbb{S}^n$. Start with a very specific example:

Theorem 3.4. *a) [Farrell-Hsiung, 1976]*
For $n \gg k$,

$$\pi_k(B\operatorname{Diff}_{x_0}(\mathbb{S}^n)) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & n \text{odd}, k = 4q \\ 0 & else \end{cases}$$

b) [Hatcher]
Non-zero elements can be realized by bundles

$$\mathbb{S}^n - E \longrightarrow \mathbb{S}^k$$

so that $E = N \cup_{\partial N} N$ where

$$\mathbb{D}^n - N \longrightarrow \mathbb{S}^k$$

and furthermore N has a fibrewise Morse function $\mu : N \longrightarrow [0,1]$ whose minimum is the 'south pole' of the disk, $\mu(1) = \partial N$, and we have two more critical points with index m, $m+1$, $m \leq n-3$. (due to Igusa and Goette)

Use Gromov-Lawson surgery principle which says that when attaching a handle of codimension at least three, one can extend the metric with positive scalar curvature.

Now apply a family version of Gromov-Lawson-Gajer handle attachment theorem (p.s.c. metrics can be extended over handles of codimension at least three). This gives a metric of positive scalar curvature on each fiber $\mathbb{D}^n \subseteq N$, and then one can just double and take the same metric on the 'upside-down part.' This takes care of the first two steps in the proof for $M = \mathbb{S}^n$.

For more general manifolds (M, h) with $n = \dim M$ odd, and $scal(h) > 0$. We can jazz up this construction in the following way: Consider $E \longrightarrow \mathbb{S}^k$ and do a fibrewise connected sum construction with the trivial fiber with bundle M. We can extend the metrics of positive scalar curvature to the resulting bundle, $\overline{E}$. Now when you consider the classifying map of this bundle to $B\operatorname{Diff}_{x_0}(\mathbb{S}^n \# M) = B\operatorname{Diff}_{x_0}(M)$, f, and the claim is that this map represents a non-zero class in $\pi_k(B\operatorname{Diff}_{x_0}(M))$. To check this, we use one more property of the Hatcher bundle: it's non-triviality can be detected by non-trivial Franz-Reidemeister torsion invariants, which are additive, so that the resulting thing is still non-trivial.

The last step deals with the third step in the proof.
Notice that this gives non-zero elements in $\pi_k(Riem^+(M)/\operatorname{Diff}_{x_0}(M))$ and so we just need to control what happens to this element after passing to the full quotient. Have to understand

$$\pi_k(Riem^+(M)/\operatorname{Diff}_{x_0}(M)) \longrightarrow \pi_k(Riem^+(M)/\operatorname{Diff}(M)) \tag{3.1}$$

which means understanding the cone singularities where the isotropy types change.

Recall that by Myers-Steenrod, the isometries on (M, g) form a compact Lie group, so the map (3.1) can be studied by the Leray-Serre spectral sequence if these isometry groups (=isotropy groups) of $\mathrm{Diff}(M)$ acting on $Riem(M)$ are finite.

This is done by picking a very particular closed manifold M. One that does not carry a non-trivial $\mathbb{S}^1$-action. Recall (Atiyah-Hirzebruch) that if M is spin and has non-zero $\widehat{A}$, then there are no non-trivial $\mathbb{S}^1$ actions. Remember that we needed that M admitted a positive scalar curvature metric, so we can't assume that.

Fortunately, there's a recent result of Herrera and Herrera (2009):
If $\pi_2(M)$ and $\pi_4(M)$ are finite, $\pi_1(M)$ does not contain $\mathbb{Z}$ in its center, and a higher $\widehat{A}$ genus is different from zero, then no non-trivial $\mathbb{S}^1$ action exists on M. There are manifolds of odd dimension with metrics of positive scalar curvature satisfying these conditions.

4. Bismut: Bott-Chern cohomology and Riemann-Roch formula

I will first of all explain the theorem I will try to prove:

4.1. The main statement. Let $\pi : M \longrightarrow S$ be a proper holomorphic map of complex manifolds, let F be a holomorphic vector bundle on M. I define $R\pi_* F$ to be the direct image of F (so this actually is a sheaf on S, and we will assume that this sheaf is locally free). You can think of this as being just the Dolbeault cohomology of the fibers. This cohomology of the fibers you assume forms a vector bundle on the base.

The theorem I will prove is that the Chern character of the direct image is the integral along the fibers of $\mathrm{Td}(TX)\,\mathrm{Ch}(F)$. To make this even more historical I will add this extra 'BC', here for Bott-Chern.

This equality takes place in $H_{BC}^{=}(S, C)$ and so all of these characteristic classes have to be decorated by BC.

Let me make a remark on this statement here:
- Let me just say that $H_{BC}^{=}$ of a complex manifold is a refinement of de Rham chomology.
- When the complex manifolds are projective, or compact Kähler, this is just Riemann-Roch-Grothendieck.
-This theorem also refines on the families index theorem of Atiyah-Singer.

The main point of the theorem is that we get away without any projectivity or Kähler assumption.

Let me explain the general philosophy of the proof:
1)Using fiberwise Hodge theory and its deformations, we will produce analytically a family of forms $[\alpha_t]$ with $t > 0$ forms on S that represent the LHS of the equality in the proper way. What one would like to do is to let $t \to 0$ in these forms. However, in the elliptic world, this does not work in general (for non-Kähler manifolds).
So the idea is to deform in the hypoellitptic category, get forms that still represent the class you want

$$\alpha_t \xrightarrow{\text{first deformation}} \alpha_{b,t} \xrightarrow{\text{second deformation}} \beta_t \xrightarrow{\text{let } t \to 0} \text{obtain RHS}$$

Fact: Let me explain what the hypoelliptic Laplacian is:
Let X be a compact Riemannian manifold, $\mathcal{X}$ the total space of its tangent bundle with fibre $\widehat{TX}$. Then you add the harmonic oscillator on the fibers, and the generator of the geodesic flow:

$$\frac{1}{2b^2}(-\Delta^{\widehat{TX}} + |Y|^2 - n) - \tfrac{1}{b}\nabla_Y + \dots$$

This turns out not to be good enough, and we will need to replace $|Y|^2$ with $|Y|^4$.

Just a remark so you can see how wonderful the hypoelliptic world is: one can define two norms $\|\|$ and $|||\;|||$ so that

$$|||Y|||^2 = \|Y\|^4$$

so on the hypoelliptic world you can allow yourself to write whatever you like and so to prove the theorems you couldn't prove before.

4.2. **Bott-Chern cohomology.** M a compact complex manifold.

Definition 4.1. $H^{p,q}_{BC}$ *is*

$$\frac{\ker d \cap \Omega^{(p,q)}}{\partial\overline{\partial}\Omega^{(p-1,q-1)}}$$

and then $H^{=}_{BC}$ *is the direct sum over* $p = q$

Remarks:
1) If M is compact, these are finite dimensional.
2) M compact Kähler, this coincides with de Rham cohomology.
3) There is a map $H_{BC} \longrightarrow H_{dR}$ and examples where this is not an isomorphism, so generally a refinement.
4) Can define characteristic classes with values in $H^{=}_{BC}$ for holomorphic vector bundles: Let E be a holomorphic vector bundle with antiholomorphic connection $\nabla^{E''}$, fix a metric and hence a Chern connection, $\nabla^{E'}$, define ∇^E to be the sum of these two connections. And for any invariant polynomial define

$$[P(-R^E/2i\pi)] \in H^{=}_{BC}(M;\mathbb{C}).$$

The class of this form is independent of the metric g^E. This is the refinement of the Chern-Simons theory.

Final remark: Actually when you look at these components of the holomoprhic connections, we can write $(\nabla^{E''})^2 = 0$, $(\nabla^{E'})^2 = 0$ and hence $R^E = [\nabla^{E'}, \nabla^{E''}]$. The similarity with $\overline{\partial} + \overline{\partial}^*$ is no accident! The fact that you have this definition of Bott-Chern characteristic classes comes from a 'double transgression formalism' coming precisely from the decomposition $\nabla^E = \nabla^{E''} + \nabla^{E'}$.

4.3. **Two extreme cases.** Let us now consider two extreme cases of the theorem above:

First, consider the case where S is reduced to a point. In this case, the theorem is the Riemann-Roch-Hirzebruch formula:

$$\chi(X, F) = \int_X \mathrm{Td}(TX)\,\mathrm{Ch}(F)$$

Second, when the fiber is reduced to a point, $M = S$. In this case the theorem becomes a tautology, $1 = 1$, again a known result!

It would actually be entirely wrong to think that these extreme cases do not teach us anything. Since we are interested in proving a Riemann-Roch theorem which is local on the base, in the context of the first case, ultimately, at a technical level, the problem will be to prove a local index theorem on a non-Kähler manifold using a Laplacian of the kind $[\overline{\partial}, \overline{\partial}^*]$.

It's well-known that the elliptic local index theorem only works in this context for Kähler manifolds. This proof becomes impossible in the elliptic category, so the point of moving to the hypoelliptic category while respecting the complex structure of the manifold is to get a proof of the local index theorem without destroying the $\overline{\partial}$, $\overline{\partial}^*$ structure. (Normally what you would do would modify $\overline{\partial} + \overline{\partial}^*$ in the smooth category, and this would take you out of Bott-Chern chomology, so ultimately yielding a weaker theorem)

Let us give an example of this in the context of the second extreme case. From the analytic point of view, the data is a Kähler form ω^S, a $(1,1)$ form smooth on S. The forms you would produce using the elliptic theory are

$$\alpha_t = \exp(\frac{\overline{\partial}\partial i\omega^{\mathbb{S}}}{t})$$

and you see that $\alpha_t = 1$ in $H^{=}_{BC}(\mathbb{S},\mathbb{C})$. However as $t \to 0$, α_t does not converge, unless $\overline{\partial}\partial\omega^S = 0$ or better $\omega^S = 0$.

4.4. **Elliptic Hodge theory and RRH.** X a compact manifold, ω^X a $(1,1)$ form (not necessarily closed) which defines a metric g^{TX}. Let (F, g^F) be a holomorphic Hermitian vector bundle, $D^X = \overline{\partial}^X + \overline{\partial}^{X*}$. To prove a local index theorem you start with

$$\chi(X,F) = \mathrm{Str}(\exp(-tD^{X,2})).$$

Note here

$$D^{X,2} = -\frac{1}{2}\Delta - (\overline{\partial}^X\partial^X i\omega^X)^c + \dots.$$

We know that $\Lambda(T^{*(0,1)}X)$ is a $c\ell\,(T_{\mathbb{R}}X)$ Clifford module. We have an isomorphism of vector spaces $c : \Lambda\,(T^*_{\mathbb{R}}X) \to c\,(T_{\mathbb{R}}X)$. The second term on the RHS above is of length 4 in the Clifford algebra, and as this is greater than 2, it destroys the possibility of getting a local index theorem.

Thus when $\overline{\partial}^X\partial^X i\omega^X \neq 0$ there is no local index theorem. One can find a leading term

$$\exp(\frac{\overline{\partial}^X\partial^X i\omega^X}{t})$$

in the heat kernel, and this diverges as $t \to 0$.

This is the reason why we are going to move to a hypoelliptic theory.

Let me explain the strategy of the proof of the main theorem:

4.5. **Elliptic superconnection forms.** Let

$$X - M \longrightarrow S$$

Let ω^M be a $(1,1)$ form on M which is a non-closed Kähler form inducing a metric g^{TX} on TX. Let g^F be a Hermitian metric on F.

Let $\overline{\partial}^M$ be the Dolbeault operator on M

$$\overline{\partial}^M = \overline{\partial}^H + \overline{\partial}^X + \dots$$

which we view as a superconnection, i.e., we distinguish between horizontal and vertical differentiation.

Define $A'' = \overline{\partial}^M$ and $A' =$ 'adjoint' of A'' (really the standard Hodge theoretic adjoint in the fibre directions and in the base replaces $\overline{\partial}^S$ with ∂^S), then A' is again a superconnection. Let $A = A'' + A'_t$, with t a parameter such that $\omega^M \mapsto \frac{1}{t}\omega^M$, and let

$$\alpha_t = \mathrm{Str}(\exp(-A_t^2)).$$

Theorem 4.2. *1)* $\alpha_t \in \Omega^{=}(S, \mathbb{C})$.
2) α_t *is closed.*
3) $\mathrm{Ch}(R\pi_* F) = [\alpha_t]$.

Proof. Standard. As $t \to \infty$,

$$\alpha_\infty = \mathrm{Ch}(R\pi_* F, g^{R\pi_* F}) + \mathcal{O}\left(1/t\right).$$

As $t \to 0$, except when $\overline{\partial}^M \partial^M \omega^M = 0$, α_t has a singular asymptotic expansion with leading term $\exp(\frac{\overline{\partial}^X \partial^X i\omega^X}{t})$ □

4.6. **From elliptic to hypoelliptic.** Just consider the case of one manifold X. On the total space $\mathcal{X}$ of the tangent bundle with fibre $\widehat{TX}$, look at the anti-holomorphic forms

$$\Omega^{(0,\cdot)}(\mathcal{X}, \pi^*(\Lambda(T^*X) \otimes F))$$

Consider $\overline{\partial}^{\mathcal{X}}$ and add to it interior multiplication by y (the tautological section of $\widehat{TX} \simeq TX$) divided by b^2, i.e.,

$$\frac{1}{b^2}\mathfrak{i}_y.$$

Let

$$A'' = \overline{\partial}^{\mathcal{X}} + \frac{1}{b^2}\mathfrak{i}_y$$

and get $(A'')^2 = 0$.

Let A' be the adjoint with respect to a Hermitian form on the Dolbeault complex,

$$\eta(s, s') = \pm \int_X \int_{\widehat{TX}} \langle s \wedge * e^{-i\omega^X} s' \rangle,$$

where ω^X is viewed as a form on the base (Hodge Hermitian product in the direction of $\widehat{TX}$, wedge product on X, with a sign included to make it Hermitian).

Now construct the corresponding hypoelliptic Laplacian $[A'', A']$,

$$[A'', A'] = \frac{1}{2b^2}(-\Delta^{\widehat{TX}} + |Y|^2 - n) - \frac{1}{b}\nabla_Y - \overline{\partial}^X \partial^X i\omega^X + \ldots,$$

with $Y = y + \overline{y}$.

Families case: same thing, $\alpha_{b,t} \in \Omega^{=}(S, \mathbb{C})$ are closed forms in the same Bott-Chern cohomology class in $H^{=}_{BC}(S, \mathbb{C})$ as the elliptic α_t (this is hard!!).

Now try again to let $t \to 0$, note that still have a divergence! The point is that in the formula above the bad term, $\overline{\partial}^X \partial^X i\omega^X$ is still there.

How to cure the sick man?

Notice that there is no requirement that ω^X be degenerate. So we replace ω^X with $|Y|^2\omega^X$. Note that if $|Y| = 0$, then this metric is completely degenerate. Note that with this metric,

$$|\|Y\||^2 = |Y|^2|Y|^2 = |Y|^4.$$

So the main effect in the formula for the hypoelliptic Laplacian is to replace $\overline{\partial}^X \partial^X i\omega^X$ with $|Y|^2\overline{\partial}^X \partial^X i\omega^X$

So now multiply the operator by t, this problem term becomes $t^2|Y|^2\overline{\partial}^X \partial^X i\omega^X$ (if $|Y|^2$ were not there, would only have a single factor of t), and so we end up with a non-singular term.

So now we do a homotopy in the hypoelliptic world from ω^X to $|Y|^2\omega^X$ the corresponding $[\alpha_t]$ do not change. And now our new forms β_t has a limit as $t \to 0$, which is (almost!) the RHS of the Riemann-Roch-Grothendieck formula.

5. Reich: Farrell-Jones conjecture for $SL_n(\mathbb{Z})$

joint work with Bartels, Lück, Rüping

Assembly maps: ($\mathbb{K}$ will refer to algebraic K-theory)

$$H_n(BG, \mathbb{K}(\mathbb{Z})) \xrightarrow{A^K} K_n(\mathbb{Z}G)$$

$$H_n(BG, \mathbb{K}(\mathbb{Z})) \xrightarrow{A^L} L_n(\mathbb{Z}G)$$

$$H_n(BG, \mathbb{K}^{top}(\mathbb{Z})) \xrightarrow{A^{K^{top}}} K_n(C_r^* G)$$

Conjecture: G torsion-free implies these maps are isomorphisms for all $n \in \mathbb{Z}$.

Borel conjecture: M, N closed aspherical then homotopy equivalence implies homeo.

Fact 1: A^k isomorphism for $n \leq 1$, A^L isomorphism implies Borel conjecture for manifolds M with $\pi_1(M) = G$ of dimension ≥ 5.

Novikov conjecture: $M \xrightarrow{f} BG$, $x \in H^*(BG, \mathbb{Q})$

$$\mathrm{sign}_x(M, f) = \langle \mathcal{L}(M) \cup f^* x, [M] \rangle$$

is a homotopy invariant.

Fact 2: If we know that $A^{K^{top}} \otimes \mathbb{Q}$ is injective then we have that $A^L \otimes \mathbb{Q}$ injective and this implies the Novikov conjecture.

Maybe should give a more concrete idea about the first map. Recall that in algebraic K-theory

$$K_0(R)$$

is the Grothendieck group of finitely generated projective R-modules

$$K_1(R) = GL(R)_{Ab}$$

There is a natural map

$$K_0(\mathbb{Z}) \longrightarrow K_0(\mathbb{Z}G)$$

whose cokernel is denoted $\widetilde{K}_0(\mathbb{Z}G)$.

The cokernel of the map

$$G_{Ab} \oplus K_1(\mathbb{Z}) \longrightarrow K_1(\mathbb{Z}G)$$

is the Whitehead group of G.

Conjecture: If G is torsion-free

$$\begin{aligned} K_{-n}(\mathbb{Z}G) &= 0, \text{ for } n > 0 \\ \widetilde{K}_0(\mathbb{Z}G) &= 0 \\ Wh(G) &= 0 \end{aligned} \tag{5.1}$$

Fact 3: A^K surjective $n \leq 1$ implies (5.1)

There are generalized assembly maps which clarify the picture in the general case: Take an arbitrary group G and a unital ring R ($EG(VCyc)$

refers to group actions whose isotropy groups are allowed to be virtually cyclic)

$$H_n^G(EG(VCyc), \mathbb{K}_R) \xrightarrow{A^K_{VCyc}} K_n(RG)$$

$$H_n^G(EG(VCyc), \mathbb{L}_R) \xrightarrow{A^L_{VCyc}} L_n(RG)$$

$$H_n^G(EG(Fin), \mathbb{K}^{top}_{\mathbb{C}}) \xrightarrow{A^{K^{top}}_{Fin}} K^{top}_n(C^*_r G)$$

Conjecture(Farrell-Jones): A^K_{VCyc} and A^L_{VCyc} are iso.
Conjecture(Baum-Connes): $A^{K^{top}}_{Fin}$ iso.

Fact 4: $A^{K^{top}}_{Fin} \otimes \mathbb{Q}$ injective implies $A^L_{VCyc} \otimes \mathbb{Q}$ injective, implies the Novikov conjecture

Fact 5:A^K_{VCyc} isomorphism for $n \leq 0$, $R = \mathbb{Q}$ implies

$$\operatorname{colim}_{\substack{H \subseteq G \\ \text{finite}}} K_0(\mathbb{Q}H) \xrightarrow{\cong} K_0(\mathbb{Q}G)$$

Theorem 5.1 (Main theorem)**.** *The F-J conjecture in K- and L-theory holds for $SL_n(\mathbb{Z})$ and $GL_n(\mathbb{Z})$.*

Remark: BC-Conjecture open for $SL_n(\mathbb{Z})$, $n \geq 4$.

Addendum: Main theorem holds also for the Farrell-Jones conjecture with coefficients, which is stronger than the fibered Farrell-Jones conjecture which in turn is stronger than the usual Farrell-Jones conjecture.

One can allow twisted rings: rings with a group acting on them, $K_n(R$ twisted product$G)$.

Corollary 5.2. *FJ conjecture with coefficients holds for subgroups, finite index 'overgroups' of $GL_n(R)$ with rings R, whose underlying additive group is finitely generated.*

Examples:
i) $GL_n(\mathcal{O})$ with $\mathcal{O}$ the ring of integers in a number field, though you don't really need the condition that this is integrally closed.
ii) Hence all arithmetic groups (safe side: this means those coming from affine algebraic groups over $\mathbb{Q}$).

To be fair, should mention that Farrell-Jones proved Borel conjecture for torsion-free discrete subgroups of $GL_n(\mathbb{R})$, though now we have the full picture.

Let's talk a little about the inheritance properties

Theorem 5.3 (BL Echterhoff, . . .)**.** *Farrell-Jones conjecture with coefficients is stable with respect to passage to:*
i) subgroups
ii) finite products
iii) colimits over directed systems of groups (structure maps not necessarily injective)

iv) pull-back under group homomorphisms: If $H \xrightarrow{\phi} G$, $A_{\mathcal{F}}$ *isomorphism for* G *implies* $A_{\phi^*\mathcal{F}}$ *isomorphism for* H.
v) 'transitivity in families'

Given these inheritance properties can see how the main theorem implies the corollary: $GL_n(R) = Aut_R(R^n)$, this sits inside $Aut_{\mathbb{Z}}(R^n)$, and then one can decompose $R^n = \mathbb{Z}^k \oplus F$, with F finite, and then use this, etc.

Theorem 5.4 (summarizing the state-of-the-art)**.** *The Farrell-Jones conjecture is known for:*
i) hyperboilic groups [BLR]
ii) CAT(0)-groups [BL]
iii) virtual poly-$\mathbb{Z}$ *groups [BFL]*
iv) cocompact discrete subgroups of almost connected Lie groups [FJ]

Let me just make one or two remarks about proofs. All of these proofs use that the group in question acts (cocompactly) on some geometry; one uses dynamical properties of the geometry (e.g., geodesic flow)

For $SL_n(\mathbb{Z})$, use the action on the inner products on $\mathbb{R}^n$, but the problem here is not cocompact!

6. Wahl: Higher rho invariants for the signature operator

Higher APS index theory

Lott defined higher eta's, Leichtnam-Piazza proved a higher APS theorem, and this was subsequently extended by Wahl.

The setting is an oriented closed Riemannian manifold M and a Galois covering $\widetilde{M} \xrightarrow{r} M$. We then form the Mischenko-Fomenko bundle

$$P = \widetilde{M} \underset{\Gamma}{\times} C^*\Gamma \longrightarrow M$$

where here $C^*\Gamma$ is the un-reduced C^*-algebra of Γ.

Now say we have a Dirac operator D associated to a Dirac bundle $E \longrightarrow M$, we get a higher Dirac operator D_P just by twisting the bundle E by P, these higher Dirac operators have index in $K_*(C^*\Gamma)$.

To get numerical invariants out of this want to look at Connes-Moscovici.

To get a smooth structure, we need to choose an algebra

$$\mathbb{C}\Gamma \subseteq \mathcal{A}_\infty \subseteq \mathcal{A} = C^*\Gamma$$

which is 'smooth' in that, e.g., $\mathcal{A}_\infty$ is closed under the holomorphic functional calculus and is Frechet.

For instance, if $\Gamma = \mathbb{Z}$, then $C^*\Gamma = C(\mathbb{S}^1)$ and we can take $\mathcal{A}_\infty = \mathcal{C}^\infty(\mathbb{S}^1)$. Other examples include the Connes-Moscovici algebra and the Puschnigg algebra, the latter is smaller than the former, which makes it more useful.

In the construction of the index theorem, you mimic the classical case; so first of all you define the differential forms:

$$(\Omega^*\mathcal{A}_\infty, d)$$

which is a differential algebra. Then you get $H^*_{dR}(\mathcal{A}_\infty)$, and a Chern character

$$\begin{tikzcd} K_*(\mathcal{A}) \arrow[rr] \arrow[dr, "\cong"] & & H^*_{dR}(\mathcal{A}_\infty) \\ & K_*(\mathcal{A}_\infty) \arrow[ur] & \end{tikzcd}$$

Then the index formula is formally the same

$$\mathrm{Ch}(\mathrm{ind}\, D_P) = \int_M \widehat{A}(M)\,\mathrm{Ch}(E/S)\,\mathrm{Ch}(P) \in H^*_{dR}(\mathcal{A}_\infty).$$

The second Chern character here, $\mathrm{Ch}(P)$ lives in $\Omega^*(M) \otimes \Omega_*(\mathcal{A}_\infty)/\overline{[\cdot,\cdot]}$. Connes-Moscovici did a more general index theorem, though actually this one is sufficient for what they were proving (the Novikov conjecture for Gromov hyperbolic groups).

One can also get secondary invariants:

Assume M odd-dimensional, and there exists smooth symmetric integral operator A acting on $L^2(M, E \otimes P)$, such that $D + A$ is invertible. (Note that hence $\mathrm{ind}(D) = 0$, that's why the invariants are called secondary.)

Can define $\eta(D,A)$ as an element of $\Omega^*(\mathcal{A}_\infty)/\overline{[\cdot,\cdot]+d}$ which looks formally like the eta form from the families index theorem, so that e.g.,

$$\eta^{[0]}(D,A) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\,\mathrm{Tr}((D+A)e^{-t(D+A)^2})\,dt$$

Now we can define the rho-invariant, it's just like the eta invariant but in a different space, let

$$\Omega_*^{\langle e\rangle}(\mathcal{A}_\infty) = \{g_0 dg_1 \cdots dg_m : g_i \in \Gamma, g_0 \cdots g_m = e\}$$

and then

$$\rho(D,A) = [\eta(D,A)] \in (\Omega^*(\mathcal{A}_\infty)/\overline{[\cdot,\cdot]+d})/\overline{\Omega_*^{\langle e\rangle}\mathcal{A}_\infty}$$

Next assume that M is an even dimensional manifold with cylindrical end and that D_P near the boundary has the form

$$c\ell\,(dx)\,(\partial_x - \begin{pmatrix} D_P^\partial & 0 \\ 0 & D_P^\partial \end{pmatrix})$$

where D_P^∂ acts on $L^2(\partial M, (E^+ \otimes P)\big|_\partial)$.

Choose $A = A^\partial$ such that $D_P^\partial + A$ is invertible, and define

$$D_P(A) = D_P - \phi c\ell\,(dx) \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

Then this operator is Fredholm.

The higher APS index theorem looks again as usual:

$$\mathrm{Ch}(\mathrm{ind}(D_P(A))) = \int_M \widehat{A}(M)\,\mathrm{Ch}(E/S)\,\mathrm{Ch}(P) - \eta(D_P^\partial, A)$$

the first term on the RHS lives in $\Omega_*^{\langle e\rangle}\mathcal{A}_\infty$, and so vanishes in the quotient.

The delocalized version would be then

$$\mathrm{Ch}^{deloc}(\mathrm{ind}(D_P(A))) = -\rho(D_P^\partial, A) \mod \overline{\Omega_*^{\langle e\rangle}\mathcal{A}_\infty}.$$

Motivated by a previous application of higher index theory of Leichtnam-Piazza, to positive scalar curvature metrics.

The part of the surgery exact sequence we're interested in is: N a closed manifold of dimension n odd,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & L_{n-1}(\mathbb{Z}\Gamma) & \longrightarrow & S(N) & \longrightarrow & \cdots \\ & & \big\downarrow{\scriptstyle \mathrm{ind}} & & \big\downarrow{\scriptstyle \rho} & & \\ \cdots & \longrightarrow & K_*(C^*\Gamma) & \longrightarrow & \Omega^*\mathcal{A}_\infty/(\ldots\Omega_*^{\langle e\rangle}(\mathcal{A}_\infty)) & \longrightarrow & \cdots \end{array}$$

We do not know if this square commutes!

Here $S(N)$ refers to equivalence classes of $(M \xrightarrow{f} N)$ where

$$(M_1 \xrightarrow{f_1} N) \sim (M_2 \xrightarrow{f_2} N) \iff (Z \xrightarrow{F} N \times [0,1] \text{ htpy equiv.}), F\big|_{\{0,1\}} = (f_1, f_2)$$

You have the signature operator D acting on the forms

$$\Omega^*(M \cup N^{op}, P_M \cup P_N)$$

Fix P_N and pull-back by f to get P_M. Hilsum and Skandalis showed that the index of this operator vanishes. They defined a family of operators on this space D_t such that D_0 is D, the signature operator, and D_1 is invertible.

There was a modification of this proof by Piazza-Schick. They show that one can choose $D_t = D + A_t$, with A_t as above. Then one can just define the ρ invariant of $M \xrightarrow{f} N$ as $\rho(D, A_1)$.

Main result: This is well-defined on the surgery structure set $S(N)$.

In the remaining time, we will just go over some more properties of this higher rho invariant.

The proof, as you might imagine, involves the higher APS index theorem. Idea is to show that the construction of Hilsum-Skandalis extends to manifolds with boundary, and then interpolate between the construction of Hilsum-Skandalis and the construction of Piazza-Schick.

Properties:

i) Piazza-Schick: $(\mathrm{Tr}_{\langle e\rangle} - \mathrm{Tr}_1)\rho(D, A) = \rho^{(2)}(M) - \rho^{(2)}(N)$

ii) Product formula: $\rho(M \times X, f \times \mathrm{id}) = \rho(M, f)\,\mathrm{Ch}(\mathrm{ind}(D_P^X))$

iii) Piazza-Schick: $\rho = 0$ if Γ is torsion-free and satisfies maximal Baum-Connes.

To apply these results (in the spirit of previous ones of Chang-Weinberger), one would like to know that the diagram commutes. Say that you have $a \in L_{n-1}(\mathbb{Z}\Gamma)$ with $\mathrm{Ch}^{deloc}(a) \neq 0$ then

$$\rho(a^k b) = k\,\mathrm{Ch}^{deloc}(a) + \rho(b)$$

would give you mutually different elements.

7. Davis: Torus bundles over lens spaces and topological K-theory

I'm going to do some computations. My advisor claimed that examples are more important than theory, and the theorems are there to unify the examples.

This is joint work with Wolfgang Lück.

Hoping that some of the computational ideas are useful; I'll state two computational ideas that I think are worth remembering.

The first is that fibers of assembly maps are well-behaved (we often expect them to be trivial, and that's well-behaved). The second is that if you have a group with torsion, this is not well-behaved. E.g., to compute $K_*(C_r^*\Gamma)$ one needs to compute $K_*(\underline{B}\Gamma)$, not well behaved. I.e., one reason that it is hard to compute $K_*(C_r^*\Gamma)$ is that along the way one has to compute $K_*(\underline{B}\Gamma)$.

(Baum asked a question, hilarity ensued)

Suppose we have a prime p, and we have a map

$$\mathbb{Z}_p \longrightarrow Aut(\mathbb{Z}^n)$$

such that $\mathbb{Z}_p$ acts freely on $\mathbb{Z}^n \setminus 0$. Then we have an action of $\mathbb{Z}_p$ on $\mathbb{T}^n$. If $n = k(p-1)$ then there are p^k fixed points.

Let Γ be the semidirect product of $\mathbb{Z}^n$ with $\mathbb{Z}_p$. Let $\mathcal{P}$ be the conjugacy classes of subgroups of order p, then $|\mathcal{P}| = p^k$.

Note that Γ acts on $\mathbb{R}^n$ and $\mathbb{R}^n = \underline{E}\Gamma$ (here $\underline{E}\Gamma$ is the universal space for actions with finite isotropy groups). Thus $\underline{B}\Gamma = (\underline{E}\Gamma)/\Gamma = \mathbb{R}^n/\Gamma = \mathbb{T}^n/\mathbb{Z}_p$ is not a manifold.

Definition: $r_i = \mathrm{rank} H_i(\mathbb{T}^n)^{\mathbb{Z}_p}$

Notice that the Baum-Connes conjecture is true for Γ by results of Higson-Kasparov. Thus we know that

$$K_m(C_r^*\Gamma) \cong K_m^\Gamma(\underline{E}\Gamma) = K_m^{\mathbb{Z}_p}(\mathbb{T}^n).$$

The computation turns out to be very nice, despite computational principle two, perhaps since the coinvariants $K_*(C_r^*\mathbb{Z}^n)_{\mathbb{Z}_p}$ turn out to be torsion free (a Tate cohomology computation).

Theorem 7.1.
a) $K_*(C_r^*\Gamma)$ *is free Abelian*
b) $K_1(C_r^*\Gamma) \cong K_1(C_r^*\Gamma)^{\mathbb{Z}_p} = \mathbb{Z}^{\sum_{m\,odd} r_m}$
c) $K_0(C_r^*\Gamma) \cong \oplus_p \widetilde{R}(\mathbb{Z}_p) \oplus \mathbb{Z}^{\sum_{m\,even} r_m}$

Along the way needed to compute $H_*(\Gamma)$, $H_*(\underline{B}\Gamma)$, the 'coinvariants' $K_*(C_r^*\mathbb{Z}^n)_{\mathbb{Z}_p}$, had to almost compute $K_*(\underline{B}\Gamma)$, and also $K_*^{\mathbb{Z}_p}(\mathbb{T}^n) \longrightarrow K_*(\mathbb{T}^n/\mathbb{Z}_p)$

Corollary of computation, for p odd: Γ satisfies Gromov-Lawson-Rosenberg property:

> If you have a manifold with fundamental group Γ of dimension at least 5 and the manifold is closed and spin, then the

manifold admits a positive scalar curvature metric iff the index of the Dirac operator is zero in $K_*(C_r^*\Gamma)$

Notice that Schick showed that the direct product $\mathbb{Z}^4 \times \mathbb{Z}_3$ does not satisfy the GLR-property, and here the twisted product does.

Now, we are going to switch to surgery theory. (p is odd) Suppose now that we have a lens space or more generally that $\mathbb{Z}_p$ acts freely on the sphere $\mathbb{S}^\ell$, then we can consider the fiber bundle

$$\mathbb{T}^n \longrightarrow \mathbb{S}^l \underset{\mathbb{Z}_p}{\times} \mathbb{T}^n \longrightarrow L^\ell$$

on π_1 :

$$1 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \longrightarrow \mathbb{Z}_p \longrightarrow 1.$$

Theorem 7.2.
a) $\mathcal{S}_{\ell+n}(B\Gamma) \cong \mathbb{Z}[\frac{1}{p}]^{(p-1)p^k}$
b) $\mathcal{S}(\mathbb{S}^\ell \times_{\mathbb{Z}_p} \mathbb{T}^n) \longrightarrow L_n(\mathbb{Z}^n)^{\mathbb{Z}_p} \times \mathcal{S}_{\ell+n}(B\Gamma)$

(These are geometric structure sets, actually groups, but the addition is hard to state.)

To define the first term: Note that one can take the assembly map

$$H_*(X; \underline{L}(\mathbb{Z})) \longrightarrow L_*(\mathbb{Z}\pi_1 X)$$

and then define the homotopy groups of the fiber to be $\mathcal{S}_*(X)$.

To bring back K-theory, which is somewhat friendlier than L-theory, one could do the same thing starting with

$$K_*(X) \longrightarrow K_*(C_r^*\pi_1 X)$$

and define $\mathcal{S}_*^K(X)$ (Roe might call this K-theory of $D_\Gamma^* X$)

Note

$$\mathcal{S}_*(X) = H_{*+1}^\Gamma(\widetilde{X} \longrightarrow \bullet; \underline{L}) \text{ and}$$
$$\mathcal{S}_*^K(X) = H_{*+1}^\Gamma(\widetilde{X} \longrightarrow \bullet; \underline{K}^{\text{top}})$$

where the definition of the right hand side is due to Davis-Lück and the identification of left and right is due to Hambleton-Pedersen.

For X a m-manifold,

$$\mathcal{S}(X) = \frac{M \longrightarrow X \text{ simple homotopy equivalence}}{\sim}$$

with equivalence

$$\begin{tikzcd} M \arrow[dr, "f"] \arrow[dd, "\approx"'] & \\ & X \\ N \arrow[ur, "g"] & \end{tikzcd}$$

Denote the one-connective cover

$$\underline{L}\langle 1\rangle(\mathbb{Z}) \longrightarrow \underline{L}(\mathbb{Z})$$

The surgery exact sequence:

$$\ldots \longrightarrow \mathcal{S}(X) \longrightarrow H_m(X; \underline{L}\langle 1\rangle(\mathbb{Z}) \longrightarrow L_m(\mathbb{Z}\pi_1 X)$$

satisfies:
1) $\mathcal{S}(X)$ has a group structure
2) $\mathcal{S}(X) \longrightarrow \mathcal{S}_m(X)$ hom.
3) $\pi_1(X) \longrightarrow \mathbb{Z}_p$, m odd

Follows from Atiyah-Singer, $\rho(X) \in \mathbb{Q} \otimes \widetilde{R}(\mathbb{Z}_p)$

Crowley-Macko $\rho : \mathcal{S}(X) \longrightarrow \mathbb{Q} \otimes \widetilde{R}(\mathbb{Z}_p)$ is a homomorphism.

To illustrate the first computational principle:
There's a push-out diagram

$$\begin{tikzcd} \sqcup \Gamma \times_P EP \arrow[r] \arrow[d] & E\Gamma \arrow[d] \\ \sqcup \Gamma/p \arrow[r] & \underline{E}\Gamma \end{tikzcd}$$

and hence $\mathcal{S}_*(B\Gamma) \cong_{\leftarrow} \oplus \mathcal{S}_*(BP)$.

Recall that $L[\frac{1}{2}](\mathbb{Z}) \cong KO[\frac{1}{2}]$. Suppose now that P is cyclic of odd prime order.

There are two approaches to the computation of $\mathcal{S}_*(BP)$:
Approach a) Try to compute the K-theory structure set of BP, which was

$$K_k^P(EP \longrightarrow \bullet) \longrightarrow K_{k-1}^P(EP) \longrightarrow K_{k-1}^P(\bullet)$$

Notice that $K_{k-1}^P(EP) = (\mathbb{Z}/p^\infty)^{p-1}$ with $\mathbb{Z}/p^\infty = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, and $K_k^P(\bullet) = \mathbb{Z}^p$, so the other group, which by the computational principles should be as simple as possible, is equal to $\mathbb{Z}[\frac{1}{p}]^{p-1}$.

No time to go through approach b, but it involves fake Lens spaces, spaces homotopy equivalent to lens spaces. It was shown that these are classified by the pair Reidemeister torsion and rho invariant. Notice that this is topological $\mathcal{S}(L^\ell)$, and not the smooth structure set. The smooth one is much harder, and apparently can not be given a group structure.

8. Carron: Subelliptic boundary conditions

joint with Ballmann and Brüning
Want to generalize some boundary conditions of APS and Epstein.

8.1. **APS Boundary conditions.**

Let M be a compact Riemannian manifold with boundary ∂M. (In fact can generalize to complete manifolds with some conditions on infinity.) Assume that

$$D^{\pm} : \mathcal{C}^{\infty}(M, E^{\pm}) \longrightarrow \mathcal{C}^{\infty}(M, E^{\mp})$$

is a supersymmetric Driac operator. The ina tubular nieghborhood of the boundary $[0, \varepsilon) \times \partial M$,

$$D = T \cdot (\nabla_{\partial_t} + A) + \text{ l.o.t.}$$

where

$$A^{\pm} : \mathcal{C}^{\infty}(\partial M, E^{\pm}) \longrightarrow \mathcal{C}^{\infty}(\partial M, E^{\pm})$$

is a Dirac-type operator on $E^{\pm} \longrightarrow \partial M$.

Let $Q^+_>$ be the projection onto the space spanned by eigensections corresponding to positive eigenvalues, and let P^+ be a pseudodifferential projection such that $Q^+_> - P^+$ has order -1.

Then with

$$W^{1,2}_{\ker P^+} = \left\{ \sigma \in W^{1,2}(M, E^+), P^+ \sigma = 0 \right\}$$

we get a Fredholm oeprator $D^+_P : W^{1,2}_{\ker P^+} \longrightarrow L^2$ and a relative index theorem

$$\operatorname{ind} D^+_P - \operatorname{ind} D^+_{\ker Q^+_>} = \operatorname{ind}(\ker P^+, \operatorname{Im} Q^+_>)$$

where this is the index of a Fredholm pair in the sense of Kato in L^2 or any Sobolev space.

There is a nice gerenalization by Epstein to Kähler manifolds:

8.2. **Epstein boundary condition.**

Now let M be a Kähler manifold with pseudoconvex boundary, that is to say that we have $\rho : \overline{M} \longrightarrow (-\infty, 0]$ smooth bdf with $\frac{1}{i} \partial \overline{\partial} \rho > 0$ along the boundary.

Consider the operator

$$D = \overline{\partial} + \overline{\partial}^* : \mathcal{C}^{\infty}(M, \Lambda^{0,\text{even}}) \longrightarrow \mathcal{C}^{\infty}(M, \Lambda^{0,\text{odd}})$$

Any $\sigma \in \Lambda^{0,*}$ can be decomposed

$$\sigma = \sum_{j=0}^{n} \sigma_{0,j}, \quad \sigma_j \in \overline{\partial} \rho \wedge \alpha_j + \beta_j$$

and the $\overline{\partial}$ Neumann boundary condition is to demand $\alpha_j = 0$ for all $j \geq 1$.

Epstein boundary condition

$$\sigma^{\text{even}} = \sum \sigma_{2j}, \quad \text{with } \sigma_{2j} = \beta_{2j}, \text{ for } j \geq 1$$

and σ_0 boundary value of a holomorphic function in $L^2(\partial M, \Lambda^{\text{even}}$

This yields B^{even}. To get B^{odd}, have

$$\sigma = \sum \sigma_{2j+1} = \sigma_1 + \sigma' \quad \text{ with } \sigma' \ \overline{\partial}\text{-Neumann boundary conditions}$$

$$\text{and } \sigma_1 = \overline{\partial}\rho \wedge \alpha_1 + \beta_1, \text{ with } \alpha_1 \text{ boundary value of holomorphic function}$$

Epstein showed that

$$D_{B^{\text{even}}} : W^{1,2}_{B^{\text{even}}} \longrightarrow L^2$$

has a closed Fredholm extension with domain contained in $W^{1/2,2}$. There's a relative index theorem

$$\operatorname{ind} D^{\text{even}}_{B^{\text{even}}} - \operatorname{ind} D^{\text{even}}_{\ker Q^+_>} = \text{“}\operatorname{ind}(B^{\text{even}}, \operatorname{Im} Q^+_>)\text{”}$$

The issue is that this pair of spaces is not a Fredholm pair on L^2. The problem is that $B^{\text{even}} + \operatorname{Im} Q^+_>$ is not closed.

The plan is to introduce a hybrid Sobolev space where these do form a Fredholm pair.

8.3. **More on boundary conditions.** Back to the general setting of a manifold with boundary with a Dirac-type operator.

D has two extremal closed extensions:

$D_{\min}$ with domain

$$W^{1,2}_0 = \{\sigma \in W^{1,2} : \sigma\big|_{\partial M} = 0\}$$

and $D_{\max}$ with domain

$$\{\sigma \in L^2 : D\sigma \in L^2\}$$

A result of Booss-Baunbek and Furutani (constant coeff near boundary, Carron et al. in more generality)

$$\mathcal{D}_{\max}/\mathcal{D}_{\min} \cong \check{H}$$

where

$$\check{H} = \left\{ \sum a_\lambda \phi_\lambda : \sum_{\lambda > 0} |a_\lambda|^2 (1 + |\lambda|^2)^{-1/2} + \sum_{\lambda \leq 0} |a_\lambda|^2 (1 + |\lambda|^2)^{1/2} < \infty \right\}$$
$$= \{\phi \in H^{-1/2} : Q_{\leq}\phi \in H^{1/2}\}$$

We get a trace map (surjective)

$$\mathcal{R} : \mathcal{D}_{\max} \longrightarrow \check{H}$$

given by restriction to the boundary, and all closed extensions of $\mathcal{D}_{\min}$ are given by closed subspaces B of $\check{H}$,

$$\mathcal{D}(D_B) = \{\sigma \in \mathcal{D}_{\max} : R\sigma \in B\}.$$

On $\check{H}$, there is a non-degenerate skew Hermitian form

$$\omega(x, y) = \langle x, Ty \rangle = \langle Q_> x, Q_> Ty \rangle + \langle Q_\leq x, Q_\leq Ty \rangle$$

and the adjoint domain B^a corresponds to the orthogonal complement of B with respect to this product.

8.4. **Subelliptic boundary conditions.** What should be in general an elliptic boundary condition? A condition for which you have elliptic regularity.

We call B regular if

$$\sigma \in L^2, D\sigma \in L^2, R\sigma \in B \implies \sigma \in W^{1,2}$$

and elliptic if both B and B^a are regular.

It turns out that B is elliptic iff

$$B, B^a \subseteq H^{1/2}(\partial M).$$

Let $t \in (0,1]$, we call $B \subseteq \check{H}$ t-elliptic if

$$\sigma \in L^2, D\sigma \in L^2, R\sigma \in B \cup B^a \implies \sigma \in W^{t,2}$$

or equivalently

$$B, B^a \subseteq H^{t-1/2}(\partial M).$$

Examples: APS boundary conditions are elliptic (=1-elliptic) and Epstein boundary conditions are 1/2-elliptic. This latter statement requires a lot of work in the non-Kähler case.

Theorem 8.1. *Let B be a t-elliptic boundary condition, the operator*

$$D_B : \mathcal{D}(D_B) \longrightarrow L^2$$

is Fredholm.

If $B = B^+ \oplus B^-$, then

$$\operatorname{ind}(D^+_{B^+}) - \operatorname{ind}(D^+_{\ker Q^+_{\geq}}) = \operatorname{ind}_{\check{H}}(B^+, \operatorname{Im} Q^+_{>}).$$

Remark: Let C be the Calderon projection, i.e.,

$$\ker D_{\max} = \{\sigma \in L^2 : D\sigma = 0\} R(\ker D_{\max}) = \check{\mathcal{C}} \subseteq \check{H}$$

and C is the L^2-projection onto $\check{\mathcal{C}}$. Then D_B is Fredholm precisely when $(B^+, \check{\mathcal{C}})$ is a Fredholm pair of subspaces of $\check{H}$.

We get a full description of t-elliptic boundary conditions up to a finite dimensional subspace that lives in $H^{t-1/2}$: B will be a graph in $Q_< H^{1/2} \times Q_> H^{t-1/2}$

8.5. **Possible application.** This can be extended to complete Riemannian manifolds; Take M complete Kähler of finite volume with $-b^2 \leq K \leq -a^2$, $\overline{\partial} + \overline{\partial}^*$ is Fredholm in its domain (Yeganefar), what is $\chi_{L^2}(M, \theta)$?

M has a compactification $\overline{M}$, and we want to compare this index with $\chi(\overline{M}, \theta)$, in fact want to show they are equal.

9. Goette: Generalized Kreck-Stolz invariants

joint work with Crowley in arXiv:1012.5237

Example of what Bunke talked about. Don't be scared, but pretend for a moment we are still in Ulrich's talk. Setup (àla Bunke): $B = BSpin$ and X is the homotopy fiber of the second Chern class as a map

$$Sp(1) = BSU(2) \longrightarrow H(\mathbb{Q}, 4)$$

There is one preferred lift.

What we're really thinking about is:
Let M be a closed oriented spin (smooth) manifold of dimension $4k+3$ and let $V \longrightarrow M$ be a quaternionic (ie $\mathbb{H}$-) line bundle, and $c_2(V)$ is torsion Now, in order to get a preferred lift here, assume that $H^3(M;\mathbb{Q}) = 0$.

Definition 9.1.

$$t_M(V) = \frac{1}{\alpha_k}(\frac{\eta+h}{2}(D^V) - (\eta+h)(D) + \widehat{A}(M, \nabla^{TM})(\widehat{c}_2 \cdot \mathrm{Ch}')(V, \nabla^V))$$

where a_k is 1 if k is even and 2 if k is odd, $\widehat{c}_2$ is a differential form whose differential equals $c_2(V, \nabla^V)$, $2 - \mathrm{Ch} = c_2 \cdot \mathrm{Ch}'$ for $\mathbb{H}$-line bundles it exists because c_2 is torsion and it is unique up to an exact term because of the vanishing of $H^3(M;\mathbb{Q})$

This would be η^{an} in Bunke's notation.

This has a description in terms of null bordisms. Let $M = \partial W$, where W is compact, spin, etc. $V = \overline{V}\big|_M$

$$\ldots \longrightarrow H^3(M, \mathbb{Q}) \longrightarrow H^4(W, M; \mathbb{Q}) \longrightarrow H^4(W; \mathbb{Q}) \longrightarrow H^4(M; \mathbb{Q}) \longrightarrow \ldots$$

The element $c_2(\overline{V})$ of $H^4(W;\mathbb{Q})$ maps to zero, so comes from an element $\bar{c}_2(\overline{V})$ in $H^4(W, M;\mathbb{Q})$, which is essentially unique. Then

$$\tau_M(V) = -\frac{1}{a_k}\left(\widehat{A}(TW)(\bar{c}_2 \cdot \mathrm{Ch}')(\overline{V})\right)[W, M] \in \mathbb{Q}/\mathbb{Z}$$

and so APS implies $\tau_M(V) = t_M(V)$.

As a corollary, $t_M(V)$ is independent of 'geometrization' (in the sense of Bunke); also $\tau_M(V)$ is independent of W, $\overline{V}$, and $t_M(V) \in \mathbb{Q}/\mathbb{Z}$.

Remark: There are some analogous invariants: ρ-invariants, Kreck-Stolz invariants

In fact the original definition of ρ-invariant is similar to the bordism discussion above.

Regarding Kreck-Stolz invariants: Assume $L \longrightarrow M$ is a $\mathbb{C}$-line bundle with $c_1(L)^2$ torsion then get $s_M(L)$ and s_2 and s_3 are similar to the invariants above (these generate the others),

$$s_M(L) = a_{k+1} t_M(L \oplus L^*) \in \mathbb{Q}/\mathbb{Z}.$$

These were invented to determine the diffeomorphism-type of certain 7-manifolds; the t invariant have a similar origin.

Consider a 2-connected, compact, smooth 7-manifold M with $\pi_3(M)$ torsion (so in particular $H^3(M) = 0$ and $H^4(M)$ is torsion).

What does the invariant look like here? Pick W with $\partial W = M$, then

$$\tau_M(V) = -\frac{1}{24}\left(\bar{c}_2(\overline{V}) \cdot \left(\tfrac{1}{2}p_1(TW) + c_2(\nabla)\right)\right)[W, M]$$

Crowley defined, in his thesis,

$$q_M(a) = 12 \cdot \tau_M(V)$$

where $a = c_2(V) \in H^4(M, \mathbb{Z})$.

Remarks:

1) q_M is independent of V,

2) q_M is quadratic on

3) $\text{link}_M(a, b) = q_M(a + b) - q_M(a) - q_M(b)$

4) $\text{link}_M(a, \frac{p_1}{2}) = q_M(a) - q_M(-a)$

5) $H^4(M; \mathbb{Z}_2)$ acts on the set of quadratic forms satisfying (2)-(4)

Theorem 9.2 (Crowley)**.** *If M, N are as above, then $M \setminus \{*\}$ is diffeomorphic to $N \setminus \{*\}$ iff there exists an isomorphism $H^4(M) \longrightarrow H^4(N)$ compatible with q_M and q_N.*

If you do note drop the point, you need to add an exotic sphere to one of the manifolds before they are diffeomorphic.

To see what this looks like in practice:

Consider

$$D^4 \longrightarrow W_{n,p} \longrightarrow \mathbb{S}^4$$

with Euler class n and $\frac{1}{2}p_1 = p$ ($p \equiv n \mod 2$.) Let $V \longrightarrow \mathbb{S}^4$ be a $\mathbb{H}$-line bundle with $c_2(V) = k$. Let $M_{n,p} = \partial W_{n,p}$ then

$$t_M(\pi^* V) = \frac{k(p+k)}{24n}, \quad \text{so } q([k]) = \frac{k(p+k)}{2n}$$

Remark: There is a short exact sequence

$$0 \longrightarrow \mathbb{Z}_{12} \cong \pi_6(\mathbb{S}^3) \longrightarrow [M, BSU(2)] \xrightarrow{c_2} H^4(M) \longrightarrow 0$$

Maybe I can state another small result here:

Call a homeomorphism between manifolds as above

$$h : M \longrightarrow N$$

'exotic' if h is not homotopic to an almost diffeomorphism (here almost means take out one point). There is something called a Kirby-Siebermann invariant $KS(h) \in H^4(M)$ such that h is exotic iff $KS(h) \neq 0$

Theorem 9.3. *link$(KS(h), c_2(V)) = t_M(h^*V) - t_N(V)$ (or some other combination of $KS(h)$ and $c_2(V)$)*

And now for something completely different.

Take a complex line bundle over a real manifold and call the zero set of a section (transverse to the zero section) a divisor. Note that it inherits an orientation on its normal bundle from the complex structure on the line bundle.

Definition 9.4. *A quaternionic divisor is a codimension* 4 *submanifold* $X \subseteq M$ *with a given quaternionic structure on the normal bundle* $N \longrightarrow X$.

Given $V \longrightarrow M$, $X = s^{-1}(0)$

Let $\mathbb{H} - V$, where $\mathbb{H}$ represents the trivial quaternionic bundle, this is a K-theoretic direct image of the trivial $\mathbb{R}$-bundle on X,

$$(\mathbb{H} - V)\big|_X \cong \Sigma^+ N - \Sigma^- N \longrightarrow X$$

We know from Bismut-Zhang that

$$t_M(E) = -\frac{1}{a_k}\left(\frac{\eta + h}{2}(D_X) - \int_M \widehat{A}(TM)\alpha_X \operatorname{Ch}'(E)\right)$$

where α_X satisfies $d\alpha_X = \delta_X$

On to some surprises:

Theorem 9.5 (Crowley-G.)**.** *If M is a stably-framed of dimension $4k+3$, with $k \geq 2$ and $V \longrightarrow M$ is pulled-back by $\xi : M \longrightarrow \mathbb{S}^4$, and $X = \xi^{-1}(x_0)$ then X is also stably framed and it turns out that*

$$t_M(V) + e(X) = 0$$

where e is the Adams e-invariant for framed manifolds.

This links Kreck-Stolz invariants to e invariants.

The proof is by computation:

$$\begin{aligned} t_M(V) + e(X) &= \frac{1}{a_k}\left(\int_M \widehat{A}(TM)\alpha_X \operatorname{Ch}'(E) - \widetilde{A} \cdot \delta_X\right) \\ &= \frac{1}{a_k}\left(\int_M \alpha_X \operatorname{Ch}'(V) + d(\widetilde{A}\alpha_X \operatorname{Ch}'(V))\right) \end{aligned}$$

the first term on the right is the integral of a differential form of degree 7, hence vanishes if the dimension of the manifold is greater than 7. Notice this also shows that the sum is not zero in dimension 7, it's not clear what this means in the context of Bunke's theory!

10. Steimle: K-theory obstructions to fibering manifolds

Leading question: Suppose you have $M \xrightarrow{f} B$ a map between closed CAT manifolds (where CAT is either Diff or Top), when is f homotopic to the projection map of a CAT-fiber bundle?

Example: Note that if f a homotopy equivalence, then f fibers iff it is homotopic to a CAT isomorphism. Indeed, homotopy notes the dimension so the fiber is finite, and considering the fundamental group, the fiber is a point, which implies the result.

Example: The only case where we have a complete answer to this question is $B = \mathbb{S}^1$, where it was workerd out by Browder-Levine, Farrell, and Siebenmann. Assume that $f : M^n \longrightarrow \mathbb{S}^1$ is π_1-onto and $n \geq 6$. We can lift f to $\overline{f} : \widetilde{M} \longrightarrow \mathbb{R}$, if f fibers then so does $\overline{f}$ and so using contractibility of $\mathbb{R}$, $\widetilde{M}$ is homotopic to the fiber of $\overline{f}$. There's a theorem of Farrell: f fibers iff (i) $\widetilde{M} \cong$ finite complex, and (ii) $\tau_{fib} \in Wh(\pi_1 M)$ vanishes. (Recall $Wh(\pi_1 M) = GL(\mathbb{Z}\pi_1 M)_{ab}/\langle \pm g \rangle$.)

Goal: Define obstructions to the general fibering problem in algebraic K-theory.

10.1. Review of Whitehead torsion.

Why care? We want to classify manifolds and we need invariants that are not homotopy invariant. An intermediate goal is to classify manifolds up to simple homotopy equivalence, which is where Whitehead torsion comes in.

Start with a homotopy equivalence between CW-complexes

$$h : X \xrightarrow{\cong} Y$$

and obtain $\tau(h) \in Wh(\pi_1 Y)$. Idea is to look at chain complexes and the behavior of h on bases. If the Whitehead torsion vanishes, h is called a simple homotopy equivalence.

Properties:

a) If $h \simeq h'$ then $\tau(h) = \tau(h')$

b) $\tau(k \circ h) = \tau(k) + k_* \tau(h)$

c) If $X = X_1 \cup_{X_0} X_2 \xrightarrow{h_1 \cup_{h_0} h_2} Y_1 \cup_{Y_0} Y_2$ and everything is a homotopy equivalence, then $\tau(h_1 \cup_{h_0} h_2) = i_{1*}\tau(h_1) + i_{2*}\tau(h_2) - i_{0*}\tau(h_0)$

d) If $X_1 \times X_2 \xrightarrow{h_1 \times h_2} Y_1 \times Y_2$ and everything is connected then $\tau(h_1 \times h_2) = \chi(Y_2) j_{1*}\tau(h_1) + \chi(Y_1) j_{2*}\tau(h_2)$

e) If $X \xrightarrow{f} Y$ is a homeomorphism, then $\tau(f) = 0$.

Assume that

$$\begin{tikzcd} M \arrow[rr, "\lambda", "\simeq"'] \arrow[dr, "f"] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where M is a compact CAT manifold (possibly with boundary), p is a CAT fiber bundle of compact manifolds, and λ is a homotopy equivalence.

Proposition 10.1. *If $\chi(B) = 0$ and f fibers, then $\tau(\lambda) = 0$.*

Proof. Write $B = B' \cup_{\partial e^n} e^n$ for some n-cell e^n so that

$$\begin{aligned}\tau(\lambda) &= \tau(\lambda\big|_{B'} \cup_{\lambda\big|_{\partial e^n}} \lambda\big|_{e^n}) \\ &= \tau(\lambda\big|_{B'}) + \tau(F \times e^n \longrightarrow F' \times e^n) - \tau(F \times \partial e^n \longrightarrow F' \times \partial e^n)\end{aligned}$$

and the last term is $(-1)^n \tau(F \longrightarrow F')$. Proceeding inductively over the cells of B, we get a factor of $\chi(B)$ which vanishes by assumption. $\square$

We really showed that

$$\tau(\lambda) \in \mathrm{Im}(Wh(\pi_1 F) \xrightarrow{i_* \chi(B)} Wh(\pi_1 M))$$

and using higher Whitehead torsion, we can do better!

Denote by $\underline{Wh}^{CAT}(F)$ the CAT Whitehead spectrum of F (depends only on the homotopy type of F, though on more than just $\pi_1 F$)

As this is a spectrum, we can write down a homology (with twisted coefficients, though), and there's a natural map

$$\begin{aligned}&\beta : H^0(B; \Omega\underline{Wh}^{CAT}(F)) \\ &\longrightarrow H^0(\mathrm{pt}; \Omega\underline{Wh}^{CAT}(F)) \cong \pi_1 \underline{Wh}^{CAT}(F) = Wh(\pi_1 F) \xrightarrow{i_* \chi(B)} Wh(\pi_1 M)\end{aligned}$$

Theorem 10.2. *Suppose we have*

(10.1)
$$\begin{tikzcd} M \arrow[rr, "\lambda", "\simeq"'] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

where p is a CAT fiber bundle of compact manifolds.
1) If f fibers, then $\tau_{fib}(f) := [\tau(\lambda)] \in \mathrm{cokernel}(\beta)$ vanishes.
2) $\tau_{fib}(f)$ does not depend on the choice of factorization (10.1).
3) If CAT=TOP and $\tau_{fib}(f) = 0$ then there exists $n \in \mathbb{N}$ such that $M \times \mathbb{D}^n \longrightarrow M \xrightarrow{f} B$ fibers (we say that 'f fibers stably')

Comparing with results of Dwyer-Weiss-Williams, can do better.

Theorem 10.3. *Let $f : M \longrightarrow B$ with M, B compact TOP mfds. Then f stably fibers iff*
i) hofib$(f) \cong$ finite CW complex
ii) Wall$(f) = 0 \in H^0(B; \underline{Wh}^{TOP}(F))$
iii) $\tau_{fib}(f) = 0$

Now a few words on higher Whitehead torsion:

Recall that classical Whitehead torsion is defined for a homotopy equivalence between compact manifolds $M \xrightarrow{f} N$. We can think of this as a factorization diagram

$$\begin{tikzcd} M \arrow[rr, "\lambda", "\simeq"'] \arrow[dr, "f"'] & & N \arrow[dl, "p"] \\ & \{\mathrm{pt}\} & \end{tikzcd}$$

and then $\tau(f) \in Wh(\pi_1 N) = H^0(\mathrm{pt}; \Omega \underline{Wh}^{CAT}(N))$.

For a higher version, start with

$$\begin{tikzcd} E \arrow[rr, "f", "\simeq"'] \arrow[dr, "p"'] & & E' \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

where f is a fiber homotopy equivalence between bundles of compact manifolds. Then we get $\tau(f) \in H^0(B; \Omega \underline{Wh}^{CAT}(F))$ (F is the fiber of p).

Note:
i) $\underline{Wh}^{CAT}(F)$ depends on more than just $\pi_1(F)$
ii) We get different invariants if CAT = TOP or CAT = Diff.

Consider the case CAT = Diff

$$B \xrightarrow{\text{transfer}(p)} \Omega^\infty \Sigma^\infty_+ E \quad H^*(E) \ni \omega \mapsto \int_{E/B} e(TE)\omega \in H^*(B)$$

Now $A(E)$ the algebraic A-theory of E,

$$A(E) \cong \Omega^\infty \Sigma^\infty_+ E \times \underline{Wh}^{Diff}(E)$$

and we have a map $\chi(p) : B \longrightarrow A(E)$ (A-theory Euler characteristic). There's a Riemann-Roch theorem of Dwyer-Weiss-Williams:

$$\begin{tikzcd} \Omega^\infty \Sigma^\infty_+ E \arrow[rr] & & A(E) \\ & B \arrow[ul, "\text{transfer}(p)"] \arrow[ur, "\chi(p)"'] & \end{tikzcd}$$

commutes up to a preferred homotopy.

Now if we bring in E', we get a diagram

$$\begin{tikzcd} \Omega^\infty \Sigma^\infty_+ E \arrow[rr] \arrow[dd] \arrow[dr, leftrightarrow] & & A(E) \arrow[dd] \\ & B \arrow[ur] \arrow[dl] \arrow[dr] & \\ \Omega^\infty \Sigma^\infty_+ E' \arrow[rr] & & A(E') \end{tikzcd}$$

which commutes up to homotopy, since both the transfer and the A-theory Euler characteristic are homotopy invariant. But the preferred homotopy depends on the smooth structure of the bundles E and E'. We obtain a map $B \longrightarrow \Omega \underline{Wh}^{Diff}(E)$, which is almost but not quite the parametrized torsion.

11. Kahle: T-duality and differential K-theory

joint with Valentino.

T stands originally for torus, but has been abstracted beyond that.

11.1. Topological T-Duality Cartoon.

T a torus, $\check{T}$ dual torus

Start with P and $\check{P}$ that fiber over B with actions of T and $\check{T}$ and suppose you have twists for $K(P)$ and $K(\check{P})$, H and $\check{H}$, you can use $P \times_X \check{P}$ to relate them via a push-pull reminiscent of the Fourier-Mukai transform in Algebraic Geometry.

Under certain conditions, you have $K^{H+\bullet}(P) \cong K^{\bullet+\check{H}-\dim T}(\check{P})$.

Most of the talk will be to set up a precise statement in differential K-theory.

11.2. Differential K-theory.

Mantra: For a given cohomology theory, the differential version describes the geometric enrichment.

Let E be a general cohomology theory, Let $V^\bullet = E^\bullet(\mathrm{pt}) \otimes \mathbb{R}$ the differential E theory sits in the commutative diagram

$$\begin{tikzcd}
\check{E}^\bullet(X) \arrow[r, "c"] \arrow[d, "\mathrm{curv}"] & E^\bullet(X) \arrow[d] \\
\Omega_{cl}(X;V)^\bullet \arrow[r, "\text{de Rham}"] & H(X;V)^\bullet
\end{tikzcd}$$

We have some exact sequences

$$0 \longrightarrow \Omega(X;V)^{\bullet-1} \longrightarrow \check{E}^\bullet(X) \longrightarrow E^\bullet(X) \longrightarrow 0$$

and

$$0 \longrightarrow E^{\bullet-1}(X, V/\mathbb{Z}) \longrightarrow \check{E}^\bullet(X) \xrightarrow{\mathrm{curv}} \Omega(X;V)^\bullet$$

where the kernel elements are called flat.

For instance if $E = H$ then $H^2(X)$ are classes of $U(1)$ bundles on X and $\check{H}^2(X)$ are these bundles with connections. Similarly, $H^1(X)$ corresponds to homotopy classes of maps $X \longrightarrow \mathbb{S}^1$ and $\check{H}^1(X)$ corresponds to smooth maps.

Another example is $E = K$. Morally speaking, in degree zero this is formal differences of vector bundles with connections, but the equivalence relation is a little more involved.

Often in physics, you're interested in more than just a cohomology class, but an actual cycle, because you can glue these together.

Choose a cochain model for $H^\bullet(X)$, then you can associate a p-groupoid $\mathscr{H}^p(X)$ to $H^p(X)$: The objects are $x, y \in Z^p(X)$ (so cocycles) and 1-morph: $\delta : x, y, \psi \in C^{p-1}(X)$ such that $x = y + \delta\psi$. Note that:

1) $\pi_0(\mathscr{H}^p(X)) = H^p(X)$
2) $Aut(0 \in \mathscr{H}^p(X)) = \mathscr{H}^{p-1}(X)$

When you play this game with differential cohomology, you get two groupoids depending on your choice of morphisms. E.g., for line bundles with connections: you could take isomorphisms that preserve the connections, or arbitrary isomorphisms. The first one satisfies property (1), and the second one satisfies property (2). We will care about the second version.

Now the setup we will care about: Start with a vector space with inner product V and Λ a full lattice in V, then let

$$T = V/\Lambda, \quad \check{T} = V^*/\check{\Lambda}$$

Let (P, ∇) be a principal T-bundle with connection over a space X, and $(\check{P}, \check{\nabla})$ a principal $\check{T}$-bundle over the same space.

Thus (P, ∇) is an object in $\check{H}^2(X; \Lambda)$ and similarly $(\check{P}, \check{\nabla})$

There is a product

$$\check{H}^p(X; \Lambda) \otimes \check{H}^q(X; \Lambda^v) \xrightarrow{\cdot} \check{H}^{p+q}(X; \mathbb{Z})$$

and we assume we have a trivialization $\sigma : 0 \longrightarrow P \cdot \check{P}$. We refer to the triple $(P, \check{P}, \sigma)$ as a differential T-duality pair.

Assume we have such a pair. If we pull-back $P \xrightarrow{\pi} X$ along π, we have a 'diagonal' section of $\pi^*(P) \longrightarrow P$ which we gives a morphism

$$\Delta_P : 0 \longrightarrow \pi^* P \in \check{H}^2(P; \Lambda)$$

This yields a morphism

$$\Delta_P \cdot \pi^* \check{P} : 0 \longrightarrow \pi^* P \cdot \pi^* \check{P}$$

which we can compare to $\pi^* \sigma$ to obtain a morphism

$$\tau = \Delta_P \cdot \pi^* \check{P} - \pi^* \sigma : 0 \to 0 \in \check{H}^4(P)$$

As an automorphism of $0 \in \check{H}^4(P; \mathbb{Z})$ we may canonically identify τ as an *object* in $\check{H}^3(P; \mathbb{Z})$.

In this way we obtain a differential 3-cohomology *co-cycle*, which should twist differential K-theory in the same way as 3-cohomology co-cycles twist topological K-theory. We can do the same thing starting with $\check{P} \longrightarrow X$ and so end up with

$$\begin{tikzcd} & P \times_X \check{P} \arrow[dl] \arrow[dr] & \\ P, \tau \arrow[dr] & & \check{P}, \check{\tau} \arrow[dl] \\ & X & \end{tikzcd}$$

A simple computation shows that on the correspondence space $P \times_X \check{P}$ the product $\check{\pi}^* \Delta_P \cdot \pi^* \Delta_{\check{P}}$ gives a morphism

$$\check{\pi}^* \Delta_P \cdot \pi^* \Delta_{\check{P}} : \check{\pi}^* \tau \to \pi^* \check{\tau}.$$

We may now we define a *canonical* T-duality homomorphism

$$T : \check{K}^{\bullet+\tau}(P) \xrightarrow{\check{\pi}^*} \check{K}^{\bullet+\check{\pi}^*\tau}(P \times_X \check{P}) \xrightarrow{\Delta_P \cdot \Delta_{\check{P}}} \check{K}^{\bullet+\pi^*\check{\tau}}(P \times_X \check{P}) \xrightarrow{\pi^*} \check{K}^{\bullet-\dim T+\check{\tau}}(\check{P})$$

and if you impose a notion of invariance you get a map

$$T^{\mathbb{T}} : (\check{K}^{\bullet+\tau}(P))^{\mathbb{T}} \longrightarrow (\check{K}^{\bullet-\dim T+\check{\tau}}(\check{P}))^{\check{\mathbb{T}}}$$

and the theorem is that this map is an isomorphism!

Notice that because of the twisting, it's not clear what is meant by the invariant part of the complex. Define

$$(\check{K}^{\bullet+\tau}(P))^{\mathbb{T}}$$

as those elements of $\check{K}^{\bullet+\tau}(P)$ whose image under the curvature map (in $\Omega^{\bullet+\tau}(P)$) is invariant under $\mathbb{T}$. Idea of the proof: Fives lemma.

11.3. **Relation with physics.** "D-branes charges are classified by (the appropriate) K-theory" where appropriate refers to the flavor of string theory you're working with, but also if there are B-fields then you need the appropriate B-twisted K-theory.

T-duality tells you that nice torus actions on spacetimes have a dual spacetime with a dual torus action, with the same physics. In particular you should have the same D-brane charges, and hence an isomorphisms of the appropriate B-twisted K-theory.

The fields associated to the charges are classified by the cocyles in the differential cohomology theory, so one should also expect to have the same differential K-theory. It seems that the physicists always assume invariance with respect to the torus actions implicitly.

The theorem is at the level of the differential K-theory, but physics suggests that it should be true at the level of cocycles!

12. van Erp:

joint with Paul Baum

Problem raised by Epstein and Melrose. Paul Baum asked his favorite question at the end of a talk, and this led to this project.

12.1. K-homology and index theory.

What's K-homology? $KK(C(X), \mathbb{C})$ Any sort of reasonable Fredholm operator that arises in a geometric context ("pseudolocal") determines an element in this KK-group. One way to understand these KK-cycles is that they pair with vector bundles to give integers.

This is based on ideas of Atiyah and formalized by Kasparov. An alternative picture was developed by Baum and Douglas: geometric K-homology (or topological), this has much more the flavor of ordinary homology theory.

Cycles are triples $K_0^{top}(X) = \{(M, E, \phi)\}$ where M is a spin-c manifold (closed), $\phi \in \mathcal{C}(M, X)$ and E is a complex vector bundle on M which is a suitable notion of coefficients here. There's an equivalence relation: if you have a boundary of such a gadget it's zero, since the complex vector bundle E is like the coefficients they add with direct sum. You want some way of being able to relate manifolds of different dimensions (Bott periodicity): you take even sphere bundles over M. Modding out you end up with an Abelian group.

The point at the end is that there is a map

$$\mu : K_j^{top}(X) \longrightarrow KK^j(C(X), \mathbb{C})$$

if X is a finite CW-complex then μ is an isomorphism. You take the Dirac operator on M twisted with the bundle E, this gives you an elliptic operator on M and you push forward via ϕ, this will in general no longer be a differential operator on X, but just some general Fredholm operator.

The "general index problem" according to Baum-Douglas: given Fredholm datum defining an element in $KK(C(X), \mathbb{C})$, you want to find the element of $K_0^{top}(X)$ associated to it. This is better because you somehow have the index with all possible twistings all at once. Indeed, if T is an element in $KK(C(X), \mathbb{C})$ with corresponding (M, E, ϕ) then

$$\operatorname{ind} T \otimes I_F = \int \operatorname{Ch}(E) \cup \operatorname{Ch}(\phi^* F) \cup Td(M)$$

Example: Atiyah-Singer

Start with (σ, F^0, F^1)

$$\begin{tikzcd} & K^0(T^*X) \arrow[dl, "\text{clutch}"'] \arrow[dr, "\text{quantize}"] & \\ K_0^{top}(X) \arrow[rr, "\mu"'] & & KK(C(X), \mathbb{C}) \end{tikzcd}$$

Think of the clutching map as a solution to the index problem in the sense of Baum-Douglas.

What I'm going to talk about is a problem raised by Epstein and Melrose (1997): Find the index of a hypoelliptic operator in the Heisenberg calculus on contact manifolds. For simplicity, we'll restrict to a particular example: Start with the Heisenberg group and mod out by a lattice.

Example:

$$G = \left\{ \begin{pmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, t \in \mathbb{R} \right\} \cong \mathbb{R}^3$$

is the Heisenberg group, Γ will be the lattice $x, y, t \in \mathbb{Z}$ and $X = G/\Gamma$. Define

$$W_1 = \partial_x, \quad W_2 = \partial_y + x\partial_t, \quad W_3 = \partial_t$$

notice that $[W_1, W_2] = W_3$.

We're interested in the operator

$$P = -W_1^2 - W_2^2 + i\gamma W_3$$

with $\gamma \in \mathcal{C}^\infty(X)$

Theorem 12.1 (Rockland)**.** *If for all $x \in X$ and all $\pi \in \widehat{G} \setminus \{0\}$,*

$$\pi(P_x) \textit{ is invertible}$$

then P has a parametrix (inverse modulo smoothing operators $\Psi^{-\infty}$).

Think of this as a generalization of ellipticity, with the representations of G replacing the Fourier transform.

This may not look very tractable at first, but it turns out to be. P_x is not an operator on the manifold, but on the group. Can represent $\widehat{G}$ as the union of the z-axis and the xy-plane, so the unit sphere (which are the only representations you really need to check) correspond to the unit circle on the plane and two rep's on the z-axis, $\pi_\pm$.

For the scalar representations you end up with $\xi^2 + \eta^2$, while for the representations $\pi_\pm : \pi_\pm(P_x)$ is the harmonic oscillator $\pm\gamma(x)$ and this will be invertible as long as $\pm\gamma(x)$ is never an odd integer.

Corollary 12.2. *P is hypoelliptic Fredholm if $\gamma(x) \notin 2\mathbb{Z} + 1$, for all $x \in X$.*

From P we pass to $P(\mathrm{Id} + P^*P)^{-1/2}$ to get a bounded operator (needs work, for example appeal to existence of Heisenberg calculus) hence an element of $KK(C(X), \mathbb{C})$ and the question is, what is the (M, E, ϕ)-cycle?

The index problem of such operators has been studied quite extensively by Epstein-Melrose. There is an unpublished manuscript, including a complicated Chern character.

The first step that deviates from Epstein-Melrose is that there is an analogue of the symbol in $K(T^*X)$.

Let $\sigma_H(P) = \{P_x : x \in X\}$. Each model operator P_x is a differential operator on the group, these fit together to a family of operators on a non-trivial bundle of groups. Define

$$T_H X = \bigsqcup_{x \in X} G_x$$

so that $\sigma_H(P)$ consists of a family of translation invariant operators on the fibers of $T_H X \longrightarrow X$.

Spent a lot of time when trying to write thesis, trying to define a K-theory element from this, but it turns out to be easy:

Fact:
If P is hypoelliptic, so that 'all' $\pi(P_x)$ invertible, then this operator $\sigma_H(P)$ is invertible modulo compactly supported smooth functions on $T_H X$ with respect to the natural convolution algebra. Moreover, $\sigma_H(P)$ is a two-sided multiplier of $\mathcal{C}_c^\infty(T_H X)$, so

$$\sigma_H(P) \in K_0(\mathcal{C}_c^\infty(T_H X))$$

and then you can map to $K_0(C^* T_H X)$ just passing to a suitable norm-closure of the algebra.

An indication that this is the right thing to do comes from the Connes-Thom isomorphism, which implies that for a nilpotent Lie group the K-theory is the same as for its Lie algebra, and then using a Mayer-Vietoris argument you get,

$$K_0(C^* T_H X) \xrightarrow{\cong} K^0(T^* X).$$

Thus we end up with a class in $K^0(T^*X)$ like we wanted.

Theorem 12.3. *The topological index of Atiyah-Singer applied to this class in $K^0(T^*X)$ gives the index of P.*

Charlie Epstein pointed out that you don't get a formula until you know how to compute what element of $K^0(T^*X)$ you get. Recently realized that the group $K^0(T^*X)$ is not really the right thing to look at, this is where K-homology comes in.

Recall the Atiyah-Singer diagram we had before

$$\begin{tikzcd}
 & K^0(T^*X) \arrow[dl, "c"', "\cong"] \arrow[dr, "\cong"] & \\
K_0^{top}(X) \arrow[rr, "\cong", "\mu"'] & & KK(C(X), \mathbb{C})
\end{tikzcd}$$

(note that the clutching map c is just Poincaré Duality) Now we construct maps

$$\begin{tikzcd} & K_0(C^*(T_H X)) \arrow[ddl, dashed, "?"' , "b"] \arrow[ddr, "\text{quantize}"] & \\ & K^0(T^*X) \arrow[u, "\cong"] \arrow[dl, "c"', "\cong"] \arrow[dr, "\cong"] & \\ K_0^{top}(X) \arrow[rr, "\cong", "\mu"'] & & KK(C(X), \mathbb{C}) \end{tikzcd}$$

(commutativity of the triangle on the right is shown through an adaptation of Connes' tangent groupoid approach)

The map b involves passing from non-commutative geometry to commutative geometry and this involves thinking about the decomposition of the C^* algebra into stably abelian factors.

Here recall that the Heisenberg group sits in between two Abelian groups

$$0 \longrightarrow \mathbb{R} \longrightarrow G \longrightarrow \mathbb{R}^2 \longrightarrow 0.$$

with $\mathbb{R}^2$ being the span of W_1, W_2.

This in turn leads to

$$0 \longrightarrow I \longrightarrow C^*(T_H X) \longrightarrow C^*(H) \longrightarrow 0$$

and I is Morita equivalent to the normal bundle of X, i.e., $X \times (\mathbb{R} \setminus \{0\})$.

This Morita equivalence yields

$$K_0(I) \cong K^1(X) \oplus K^1(X)$$

and now we can apply Poincaré Duality to K-homology. There are two PD's, related to the two spin-c structures on a contact manifold, one for each co-orientation of H.

$$\begin{tikzcd} K_0(I) \arrow[r, "\text{onto}"] \arrow[d, "\text{Morita}"', "\cong"] & K_0(C^*(T_H X)) \arrow[dd, "b"] \\ K^1(X) \oplus K^1(X) \arrow[d, "\cap[X^+] \oplus \cap[X^-]"] & \\ K_0^{top}(X) \oplus K_0^{top}(X) \arrow[r, "\text{addition}"] & K_0^{top}(X) \end{tikzcd}$$

This defined the map b that makes the diagram commute. In the end we get $b(\sigma_H(P)) = (M, E, \phi)$ with

$$M = X^+ \times \mathbb{S}^1 \bigsqcup X^- \times \mathbb{S}^1 \xrightarrow{\phi} X$$
$$E = E^+ \bigsqcup E^-$$

where

$$E^+ = \bigoplus_{j=0}^{N} \phi^* \mathrm{Sym}^j H^{1,0} \otimes L(\gamma - (1+2j))$$

$$E^- = \bigoplus_{j=0}^{N} \phi^* \mathrm{Sym}^j H^{0,1} \otimes L(\gamma + (1+2j))$$

for large enough N.

13. Moriyoshi: Eta cocycles and the Godbillon-Vey index theorem

joint with Paolo Piazza.

In this conference, the APS index theorem is the main subject, so let me start by reminding you of this theorem.

The setting is a compact even-dimensional manifold M with boundary. [Picture: surface with a truncated cylindrical end] Near the boundary the metric is a product. We have a Dirac-type operator

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

The APS index theorem says that

$$\operatorname{ind}(D^+ \text{ with APS boundary conditions }) = \int_M \widehat{A}(TM) - \frac{\eta + h}{2}(D_\partial)$$

where D_∂ is the Dirac operator on the boundary. We have seen this in many talks in this conference, indeed the opening talk was about the topological content of the eta invariant.

I'd like to add one more topological interpretation of the η-invariant. From our point of view

$$\eta\text{-invariant} = \text{ the evaluation by } b(\text{b-trace})$$

In the APS situation, there are two algebras invloved, A and B, and there is a surjective map

$$A \xrightarrow{\pi} B$$

We get a cocycle σ, the 'eta cocycle,'

$$\begin{tikzcd} & 0 \\ \cdot & \sigma \arrow[u, "b"] \arrow[l, "\pi^*"'] \\ \text{b-trace} \arrow[u, "b"] & \end{tikzcd}$$

Let V be the complete manifold obtained by attaching a cylindrical end to M. Let V_λ be the truncation M with a truncated cylindrical end, parametrized by λ ($V_0 = M$). Let W be the cylinder $\partial M \times \mathbb{R}$. [Picture: Manifold with cylindrical end]

First introduce the algebra $k(V)$,

$$k(V) = \{k : \text{ continuous kernel function on } V, \text{ with compact support}\}$$

and then

$$B(W) = \{\ell : \text{ continuous kernel function on } W, \\ \mathbb{R}\text{-invariant, with } \mathbb{R}\text{-compact support}\}$$

and the let $A(V)$ be the 'Toeplitz extension of B', with ρ a tempered step function (cut-off: equal to one on cylinder, zero on M)

$$A(V) = \{a = k + \rho\ell\rho : k \in k(V), \ell \in B(W)\}$$

In this situation we have three invariants.

1) Basic short exact sequence

$$0 \longrightarrow k(V) \longrightarrow A(V) \longrightarrow B(W) \longrightarrow 0$$

where the third map is $k + \rho\ell\rho \mapsto \ell$, and its C^*-completion,

$$0 \longrightarrow \mathcal{K} \longrightarrow A^* \longrightarrow B^* \longrightarrow 0.$$

2) Relative index class

From this exact sequence we get a relative index class $(e_0, e_1, p_t) \in K_0(A^*, B^*)$. Fix $s > 0$, then e_0 is defined as the graph projection of sD^+

$$e_0 = e_{sD^+}$$

explicitly, with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$e_0 = (sD + \varepsilon)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (sD + \varepsilon)$$

Then for e_1,

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and p_t is the family of projections

$$p_t = e_{tD_W}, \quad t \in [s, \infty)$$

defined in the same way as e_{sD^+}.

We assume that D_∂ invertible, which has the important property that p_t extends continuously to $t = \infty$, where we have

$$p_\infty = e_1.$$

Thus we have the two C^* algebras $A^* \xrightarrow{\pi} B^*$, two projections e_0 and e_1 'upstairs' (ie in A^*) and a one parameter family of projections p_t 'downstairs' (ie in B^*) connecting them. Hence (e_0, e_1, p_t) defines a class in $K_0(A^*, B^*)$ which we call the relative index.

3) On $B(W)$,

$$\sigma(\ell_0, \ell_1) = \mathrm{Tr}(\ell_0[\chi^0, \ell_1])$$

with χ^0 a step function. This is a 1-cocycle, which we call the eta cocycle.

Now we define τ, a pull-back of this sigma, on $a = k + \rho\ell\rho$,

$$\tau(a) = \lim_{\lambda \to \infty} \left(\int_V k(x, x)\, dx - \int_{V_{2\lambda} \setminus V_\lambda} \ell(x, x)\, dx \right)$$

Observe that $\pi^*\sigma = b\tau$ 'relative cocycle condition.'

Now I can present our interpretation of the eta-invariant. It is the b of b-trace.

The pair $(\tau,\sigma)\in HC^0(A,B)$ is a relative cyclic cocycle. Define $S^n(\tau,\sigma)=(\tau_{2n},\sigma_{2n+1})$, the 'Connes' S-operation'. We make the pairing

$$\langle(\tau_{2n},\sigma_{2n+1});\text{rel index class}\rangle$$
$$:=\frac{1}{n!}\left(\tau(e_i,\cdot,e_i)|_{i=1}^{i=0}-(2n+1)\int_s^\infty\sigma([p_t^\circ,p_t],p_t,\dots,p_t)\,dt\right)$$

Theorem 13.1. *As $s\to 0$, we get*

$$1^{st}\ \textit{term}\ \to\int_M\widehat{A}(TM),\ 2^{nd}\ \textit{term}\ \to\eta(D_\partial)$$

Piazza pointed out that in the definition of $\tau(a)$ subtracting off the second term is indispensable, and the limit is Melrose's b-trace.

Summarizing:
$({}^b\operatorname{Tr},\sigma)$ is a relative cyclic cocycle and

$$\langle({}^b\operatorname{Tr},\sigma),\ \text{rel index}\ \rangle=APS.$$

Also note that the pairing $\langle(\tau_{2n},\sigma_{2n+1});\text{rel index class}\rangle$ is equal to the pairing between $K(k)$ and $H(k)$, so that one recovers the index pairing.

Theorem 13.2.
1)We construct the Godbillon-Vey eta cocycle σ_{GV}, regularized GV-cochain τ^*_{GV} ($\pi^*\sigma=b\tau$) (b-trace extension of τ_{GV} due to Natsume-M.)
2) Relative pairing, get APS-type formula

Now let me explain the setup for foliations.

First, we choose a Gamma covering, $\widetilde{M}\longrightarrow M$, where Γ is a discrete group and this is a Galois covering.

Let Γ act on $\mathbb{S}^1$ by orientation preserving diffeo.

Define $X=\widetilde{M}\times_\Gamma\mathbb{S}^1$, this is known as a foliated $\mathbb{S}^1$ bundle. The leaves are

$$\mathcal{F}=\{\widetilde{M}\times\{*\}\}_{*\in\mathbb{S}^1}$$

We have a jet homomorphism, from the twisted product of Γ and $\mathbb{S}^1$ into the 2-jet group: Take $(\gamma,y)\in\Gamma\times\mathbb{S}^1$, look at the Taylor expansion of γ at y,

$$\gamma(t)=y+\alpha t+\tfrac{\beta}{2}t^2+\dots$$

and assign to it $\alpha t+\frac{\beta}{2}t^2$ in the 2-jet group. Note that this 2-jet group is isomorphic to the $ax+b$ group.

On this group, things are no longer unimodular. This group has some 'Heisenberg-type' 2-cocycles. That is, given the element $g=ax+b$ assign to it $\log a$ and you get a 1-cocycle of G, or assign to it b and you get a 1-cocycle with values in G-module $\Delta\cong\mathbb{R}$ (modular fucntion of G) Heisenberg 2-cocycle

$$c=\frac{1}{2}(\log a\cup b-b\cup\log a)\in H^2_{gp}(G;\Delta)$$

Definition 13.3. *Bott-Thurston cocycle* $BT = j^*(c) \in C^2(\Gamma, \Omega^1\mathbb{S}^1)$
Universal Godbillon-Vey class: $GV(\mathcal{F}) = [BT] \in H^3(E\Gamma \times_\Gamma \mathbb{S}^1; \mathbb{R})$

Main result with Paolo:
First, need to construct the 'basic exact sequence'. Let V be $(X, \mathcal{F})$ with cycilindrical end (assume you started out with a manifold with boundary and that the foliation on the boundary has the same codimension). Let W be $(\partial X \times \mathbb{R}, \mathcal{F}_\partial \times \mathbb{R})$. Get a short exact sequence

$$0 \longrightarrow C^*(V, \mathcal{F}) \longrightarrow A^* \longrightarrow B^* = C^*(\partial X, \mathcal{F}_\partial) \otimes C^*\mathbb{R} \longrightarrow 0$$

analogous to above.

Next, have a relative index class, given by taking D to be the longitudinal Dirac operator (assume leaves are even-dimensional).

Finally, we have a GV-eta cocycle given as follows: Use a regularized trace ω_Γ analogous to the b-trace

$$\sigma_{GV} = \frac{1}{3!} \sum_{\alpha \in S_3} \omega_\Gamma(\ell_0 \cdot \delta_{\alpha(1)}\ell_1 \cdot \delta_{\alpha(2)}\ell_2 \cdot \delta_{\alpha(3)}\ell_3)$$

where we use the derivations

$$\delta_1 = [\phi, \cdot], \quad \delta_2 = [\dot{\phi}, \cdot], \quad \delta_3 = [\chi^0, \cdot]$$

where ϕ is the log of the modular function of the holonomy groupoid.

Also have τ^r_{GV} on A using b-trace and it satisifes $\pi^*\sigma_{GV} = b\tau^r_{GV}$.

Theorem 13.4. *Assume that D_∂ is invertible. In the relative pairing*

$$\langle S^n(\tau^r_{GV}, \sigma_{GV}),\ \textit{rel index}\rangle = \int_X \widehat{A}(T\mathcal{F})\mathrm{GV}(\mathcal{F}) - (GV\ \textit{eta})\ \textit{as}\ s \to 0$$

where

$$GV\ \textit{eta} = C \int \sigma_{GV}([\dot{p}_t, p_t], p_t, \dots, p_t)\, dt.$$

One interesting question is: what is the spectral nature of this GV-eta invariant? Another, what about higher index theorem? Finally, what about when there are singularities? e.g., hyperbolic cusps? Hyperbolicity would allow you to isolate the zero spectrum, and hope to remove the invertibility assumption.

14. Ian Hambleton: Gauge theory and smooth group actions on 4-manifolds

An overview based on joint work with Ronnie Lee

.

This is an attempt to tell you about some problems that I think are interesting both for topology and for analysis.

The main focus is X a smooth four-manifold, and $\mathrm{Diff}(X)$. I think it's fair to say that not much is known about this object. One way to study the diffeomorphisms of a smoth four manifold is to study finite group actions on X. Why finite groups ? In dimension four, we are at a critical point between low-dimensional topology and high-dimensional topology. The quotient by a circle action gets you into a three manifold and so the techniques are very different.

We will assume that X is 1-connected, closed and oriented, and that the group acts by orientation preserving diffeomorphisms. Maybe you would like a concrete question ?

Question: Suppose X does not admit an effective circle action (action will always mean smooth and orientation preserving), does there exist a constant $C = C(X)$ such that $p > C$, p prime implies that

$$\mathbb{Z}_p \text{ does not act smoothly on } X?$$

We remark that if X is spin with non-zero signature, then the $\widehat{A}$-genus rules out any effective smooth circle actions.

Problem: Show that $\mathbb{Z}_p$ does not act smoothly on a $K3$ surface if $p > 24$.

The motivation for this kind of question is that the algebraic automorphisms of $K3$ are faithful in cohomology with rational coefficients, and there is a restriction on the automorphisms you could have on the cohomology lattice of a $K3$ surface.

Let me take a minute to explain why this is true. If you take $H_2(X, \mathbb{Z}) = \mathbb{Z}^{b_2}$ you have a $\mathbb{Z}[\mathbb{Z}_p]$ lattice, and there are three types, rationally equivalent to direct sums of $\mathbb{Z}$, $\mathbb{Z}[\zeta_p]$, or $\mathbb{Z}[\mathbb{Z}_p]$. so for $p > b_2$ you can eliminate all but the trivial ones.

Does there exist any $\mathbb{Z}_p$ action on a $K3$ which induces the identity on homology ? A theorem of Ruberman rules these out when $p = 2$. In the symplectic category, you can use a recent theorem of Chen-Kwasik to rule out homologically trivial actions.

Another general motivating question: do the equivariant smooth symmetries of algebraic surfaces resemble the algebraic ones?

The profusion of simply-connected four manifolds means that it is very hard to make reasonable conjectures. The idea that one might build them up out of algebraic pieces has been ruled out, for instance.

Another very nice result, due to Xiao (1990), says that if X is a minimal algebraic surface of general type, then the algebraic automorphism group is finite and its order is bounded by a multiple of K_X^2

In a sense, the question I was asking is about generalizing this to equivariant diffeomorphisms.

Tools: We are thinking primarily of the Yang-Mills moduli spaces, as developed by a number of mathematicians including Atiyah, Donaldson, Hitchin, Taubes, Uhlenbeck, etc.

W don't hear so much about Yang-Mills any more because, after the spectacular success of Donaldson in the 1980's, Seiberg-Witten introduced new invariants that swept the Yang-Mills theory away. This happened at an unfortunate time, in that the analytic foundations were not quite at a state where they could be picked up by non-experts.

You have a manifold with symmetry and you want to attach something and have the symmetries extend, is the sort of meta-math description. I'll need to describe the Yang-Mills equations, and the modifications needed to talk about the equivariant moduli space. Afterwards, I want to describe two concrete examples where this has been succesful and then indicate some work in progress on a new direction.

So now let X be a Riemannian 4-manifold as above, and let $P \longrightarrow X$ be a principal $SU(2)$ bundle, classified by the second Chern class. We want to study the space of connections modulo gauge

$$\mathcal{B}(P) = \mathcal{A}(P)/\mathcal{G}(P)$$

The great feature of 4-dimensional geometry is the star operator that allows you to look at

$$F_A^+ \in \Omega^2_+(\mathrm{ad}P)$$

where this bundle is $P \times_G \mathfrak{g} \longrightarrow X$. Say that A is anti-self-dual if $F_A^+ = 0$. The moduli space of anti-self-dual connections is

$$\mathcal{M}(P) \subseteq \mathcal{B}(P).$$

This space initially has no good strucuture. One of the methods of the early theory was to perturb the metric until you got the moduli space as smooth as possible. We have the space of irreducible connections

$$\mathcal{M}^*(P) \subseteq \mathcal{M}(P)$$

and the reducible ones where $c_2(P) = -c_1(L)^2$, and then the bundle $E = P \times_{SU(2)} \mathbb{C}^2 \longrightarrow X$ splits as $L \oplus L^{-1}$.

The space $\mathcal{M}^*(P)$ is smooth generically with dimension

$$8\ell - \frac{3}{2}(\chi(X) + \sigma(X)).$$

There were two questions that were very important in the early theory: Is this non-empty? (a big contribution of Taubes to the early theory) and how to compactify? (the foundational theory was developed by Uhlenbeck)

That's a brief sequence of bullet points about the Yang-Mills moduli space, now what about equivariance?

Let π be a finite group acting smoothly and preserving orientation on X, and let g be a π-invariant Riemannian metric. The first thing is very simple: there is an induced action on moduli space. Indeed, if you take an element $\gamma \in \pi$, and you pull-back P then you can cover γ by an 'extended Gauge transformation' and the indeterminacy consists of the usual gauge transformations, and so is absorbed in passage to the moduli space.

The second point I want to make is that a generic equivariant metric is not a generic metric. And we needed genericity to have a nice moduli spaces. Let me remind you that there were two ways of getting generic moduli spaces. one can perturb the metric, or one can perturb the local structure of the moduli space. We have the elliptic deformation complex

$$\Omega^0(\mathrm{ad}P) \xrightarrow{d_A} \Omega^1(\mathrm{ad}P) \xrightarrow{d_A^+} \Omega^2_+(\mathrm{ad}P)$$

and so we get local charts $H^1_A \xrightarrow{\phi} H^2_A$ describing the moduli space locally as

$$\phi^{-1}(0)/\Gamma_A$$

where $\Gamma_A = \pm 1$ or $\Gamma_A = \mathbb{S}^1$ depending on whether A is irreducible or reducible.

The second way to proceed is to take this map ϕ and perturb it chart by chart. What may or may not be well-known to this audience is:

Bierstone general position for smooth actions of compact Lie groups G

The simplest case is to look at $F : V \longrightarrow W$ a smooth Gi-equivariant map between two representation spaces. We want a nice generic structure for $F{-}1(0)$. If you try to make F equivariantly transverse, then the linearization would surject and so W would be an equivariant summand of V, so this is not always possible (equivariant transversality is open but not dense)

On the other hand, you could look at the stratification by orbit type, and try to have transversality at each stratum. This is a dense condition but not open.

Bierstone (1978) found a dense and open condition. The first installment in describing this, is to look at polynomial maps

$$F_1, \dots, F_k \in \mathcal{C}^\infty_G(V, W)$$

and there exist invariant functions $h_1, \dots, h_k \in \mathcal{C}^\infty_G(V, \mathbb{R})$ such that

$$F(x) = \sum_{i=1}^{k} h_i(x) F_i(x)$$

The general position condition is expressed by requiring that the intersection of the graph $\{(x, (h_1(x), \dots, h_k(x)) \subset V \times \mathbb{R}^k$ should be transverse to all strata of the algebraic zero set $U(\Phi, h) = 0$ of the polynomial map $\Phi \colon V \times \mathbb{R}^k \to W$ given by

$$\Phi(x, h) = \sum h_1 F_1(x) + \cdots + h_k F_k(x),$$

for $x \in V$ and $h \in \mathbb{R}^k$.

The pay-off is that $F^{-1}(0)$ is an equivariantly Whitney stratified set.

With Ronnie Lee, we managed to get the moduli space in general position using this notion of Bierstone.

What is this good for? Here's where the two examples come in. I should mention that the results so far have been few and very hard to get.

Let $X = \overline{\mathbb{CP}}^2$, so the algebraic automorphism group is

$$Aut(X) = PGL_3(\mathbb{C})$$

and one could ask if finite group acting effectively on X have to be a finite subgroup of this? Yes.

Theorem 14.1.
1) If π is finite and acts smoothly on X and is the identity of H_* then π is a subgroup of $PGL_3(\mathbb{C})$.

2) Same thing for connected sums (more than one) of X, you conclude that

$$\pi \subseteq \mathbb{S}^1 \times \mathbb{S}^1$$

Now consider $\pi = \mathbb{Z}_m$ with m odd, acting linearly on $\mathbb{CP}^2$ with 3 isolated fixed points. At a fixed point x we have the isotropy representation

$$T_x\mathbb{CP}^2 = \mathbb{C} \oplus \mathbb{C}$$

then the action of the generator is by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta^a & 0 \\ 0 & 0 & \zeta^b \end{pmatrix}$$

where $\zeta = e^{2\pi i/m}$ and you have the special weights (a, b), $(a - b, -b)$, and $(-a, b - a)$ at the three fixed points.

Algebraically you would have six rotation numbers, from two weights (a_i, b_i) at each fixed point. The relation is by the G-Signature formula

$$\sum \frac{(t^{a_i} + 1)(t^{b_i} + 1)}{(t^{a_i} - 1)(t^{b_i} - 1)} = 1 \text{ in } \mathbb{Q}[t]/(t^m - 1)$$

There's a nice picture on $\mathcal{M}_1(X)$ showing why any smooth action has the same weights as in some linear action (H-Lee, 1992).

The problem I have renewed hope in is another nice example: $X = \mathbb{S}^2 \times \mathbb{S}^2$, $\pi = C_m$ with m odd, so there are four isolated fixed points. You get a relation between the eight weights from the Atiyah-Singer equivariant index.

Conjecture: *rotation numbers have the form* (a, b), $(a, -b)$, (c, d), $(c, -d)$.

To construct the standard examples we can look at two copies of $\mathbb{S}^4$ with rotations of given weights and perform an equivariant connected sum on the free part.

The idea is to look at π-invariant surfaces (e.g., $\mathbb{T}^2$) that are non-zero in H_* with $[\mathbb{T}^2] \cdot [\mathbb{T}^2] = 0$. We want to study the compactification of the cylindrical end moduli space after putting it into equivariant general position.

15. Rochon: Pseudodifferential operators on manifolds with foliated boundaries

I'll start by first reviewing the situation where we have a fibered boundary: Let X be a manifold with boundary ∂X equipped with a fibration

$$\begin{tikzcd} Z \arrow[r, dash] & \partial X \arrow[d, "\phi"] \\ & Y \end{tikzcd}$$

Let $x \in \mathcal{C}^\infty(X)$ be a boundary defining function (bdf), i.e., $x^{-1}(0) = \partial X$, $x > 0$ on $X \setminus \partial X$, and $dx \neq 0$ on ∂X.

Consider a complete Riemannian metric, g_ϕ which near the boundary takes the form

$$g_\phi = \frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + g_Z$$

where g_Y is a metric on Y and g_Z induces a metric on each fiber.

First example: Euclidean metric on $\mathbb{R}^n$. Think of X as $\overline{\mathbb{B}}^n$, so $\partial X = \mathbb{S}^{n-1}$ and we think of $\mathbb{R}^n$ as the interior of the ball. We can take as a boundary defining function $x = 1/r$ where r is the distance to the origin.

An example that showed up yesterday, scattering metrics, is a generalization of this example. Take $Y = \partial X$, and $Z = \{\mathrm{pt}\}$, then the metric looks like an expanding cone at infinity over Y.

Another extreme example is to again have a trivial fibration, but now the other way around. So take $Y = \{\mathrm{pt}\}$ and $Z = \partial X$, now the metric looks like a cylindrical end near infinity.

A general ϕ metric looks like a cylinder in the fibers and an expanding cone in the base. These metrics come up for instance in the study of gravitational instantons.

In this setting, given a metric, one is interested in studying the geometric operators associated to it. For instance, the Laplacian, and the Dirac operator if the interior is spin. Since it is not compact, one needs to work to study these operators, and for this purpose, Mazzeo-Melrose introduced a pseudodifferential calculus. These operators show up naturally even if you are only interested in differential operators, e.g., if you want to study the resolvent.

To study this, Mazzeo-Melrose start by introducing a Lie algebra,

$$\begin{aligned} \mathcal{V}_\phi &= \{\xi \in \mathcal{C}^\infty(X, TX) : \exists c > 0 \text{ s.t. } g_\phi(\xi,\xi) < c\} \\ &= \{\xi \in \mathcal{C}^\infty(X, TX) : \xi x \in x^2 \mathcal{C}^\infty(X), \Phi_*(\xi\big|_{\partial X}) = 0\} \end{aligned}$$

in local coordinates (x, y, z), any such vector field ξ is of the form

$$\xi = a x^2 \partial_x + \sum b^i x \partial_{y_i} + \sum c^j \partial_{z_j}, \qquad \text{with } a, b^i, c^j \in \mathcal{C}^\infty(X).$$

One can easily check that these form a Lie algebra.

From this, once we have these vector fields what we can do is compose them to get a differential operator of higher order

$$\mathcal{V}_\phi \longrightarrow \mathrm{Diff}^m_\phi(X)$$

This will include operators naturally associated to the metric, such as the Laplacian. Then a microlocalization (which we will keep mysterious)

$$\mathrm{Diff}^m_\phi(X) \longrightarrow \Psi^m_\phi(X).$$

Question: What can we do if instead of the fibration we have a foliation $\mathcal{F}$ on ∂X?

The definitions of the Lie algebra and of $Diff^*_\phi$, have natural generalizations when the fibration is replaced by a foliation.

A natural example of complete metric involving a foliation at infinity is obtained by taking a certain Gibbons-Hawking metric, mod out by a finite group of rotations. On the quotient, the metric is no longer hyperKahler, but it is still Kahler and you end up with a foliation on the sphere at infinity.

To define pseudodifferential operators in that context, one natural approach is to use Lie groupoids. In fact you have a Lie algebroid which you can integrate to get a Lie groupoid and then there is an associated class of pseudodifferential operators.

An alternative approach is to use the ϕ-calculus of Mazzeo-Melrose. This required looking only at foliations that can be 'resolved' by a fibration. Then the idea is to use this fibration to define the operators.

More precisely, assume the foliation $\mathcal{F}$ arises as follows:
1) The boundary is a quotient of a possibly non-compact space $\partial\widetilde{X}$ by the action of a discrete group Γ acting freely and properly discontinuously on $\partial\widetilde{X}$.
2) Assume that there is a fibration

$$\begin{tikzcd} \widetilde{Z} \arrow[r, dash] & \partial\widetilde{X} \arrow[d, "\widetilde{\phi}"] \\ & Y \end{tikzcd}$$

where Y is compact (though $\widetilde{Z}$ and $\partial\widetilde{X}$ might not be).
3) Γ acts on Y in a locally free way (i.e., $\gamma \in \Gamma, \mathcal{U} \subseteq Y$ is open and $y \cdot \gamma = y$ for all $y \in \mathcal{U}$ implies $\gamma = \mathrm{Id}$) and $\widetilde{\phi}(p \cdot \gamma) = \widetilde{\phi}(p) \cdot \gamma$, for all $\gamma \in \Gamma$.
4) The leaves of $\mathcal{F}$ are the image of the fibers of $\widetilde{\phi}$ under the quotient map

$$q : \partial\widetilde{X} \longrightarrow \partial\widetilde{X}/\Gamma = \partial X.$$

This is a serious restriction, but still, there are many natural examples that arise in this way.

For instance, the Kronecker foliation:
Start with Π^2, and the orbits of a line with non-rational slope. We can construct this foliation by taking $\partial\widetilde{X}$ to be $\mathbb{R} \times (\mathbb{R}/\mathbb{Z})$ with the projection

onto the right factor, taking $\Gamma = \mathbb{Z}$ with the action

$$(x, y) \cdot k = (x + k, y - \theta k)$$

on $\widetilde{\partial} X$ and the induced action on Y. Then $\partial \widetilde{X}/\Gamma = \mathbb{T}^2$, in fact you can describe this explicitly

$$(\mathbb{R} \times (\mathbb{R}/\mathbb{Z}))/\mathbb{Z} \ni [x, [y]] \mapsto ([x], [y + \theta x]) \in \mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2.$$

Another example is givein by Seifert fibrations:
These are circle foliations on 3-manifolds. The space of leaves is generally an orbifold. Provided the orbifold is 'good', i.e., its universal cover is a smooth manifold, then you can obtain the foliation as above. The only cases where it is not possible are the teardrop and the 'American football' with two different angles.

Once you have such a fibration, which you can think of as a resolution to better understand the foliation, then you can do the following:

Take a collar neighborhood of ∂X

$$c : M = \partial X \times [0, \varepsilon)_x \longrightarrow \partial X$$

and a compatible $\widetilde{M} = \partial \widetilde{X} \times [0, \varepsilon)_x$ of $\partial \widetilde{X}$ with the trivial extension of the Γ action. On $\widetilde{M}$, we can consider $\Psi_{\phi,\Gamma}(\widetilde{M})$.

Definition 15.1. *A pseudodifferential operator associated to this boundary foliation*

$$P \in \Psi^m_{\mathcal{F}}(X)$$

is an operator of the form $P = q_*(P_1) + P_2$, where $P_1 \in \Psi_{\phi,\Gamma}(\widetilde{M})$ is properly supported and $P_2 \in \dot{\Psi}^m(X)$ (i.e., its integral kernel vanishes to infinite order at all boundary faces).

One can show that these operators compose, send smooth functions to smooth functions, and preserve the smooth functions that vanish to infinite order at the boundary.

There are two 'symbols' that determine compactness, Fredholmness, etc. The 'usual' symbol

$$0 \longrightarrow \Psi^{m-1}_{\mathcal{F}}(X) \longrightarrow \Psi^m_{\mathcal{F}}(X) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^{\mathcal{F}}TX \setminus \{0\}) \longrightarrow 0$$

and a normal operator at the boundary

$$0 \longrightarrow x\Psi^m_{\mathcal{F}}(X) \longrightarrow \Psi^m_{\mathcal{F}}(X) \xrightarrow{N_{\mathcal{F}}} \Psi^m_{\mathcal{F}-\mathrm{sus}}(X) \longrightarrow 0$$

coming from P_1 in the decomposition $q_*(P_1) + P_2$, but independent of the particular decomposition.

One can obtain an index theorem in this setting:

Assume that $\partial \widetilde{X}$ is compact, so $\widetilde{Z}$ is compact and Γ is finite (rules out the Kronecker fibration, but not the 'admissible' Seifert fibrations).

Assume that X, Z, and Y be spin in a compatible way, X is even dimensional, and let $g_{\mathcal{F}}$ be a metric such that

$$q^* c^* g_{\mathcal{F}} \text{ is a } \phi\text{-metric.}$$

With this data get a Dirac operator $\eth_{\mathcal{F}}$ and we assume that

$$q^*(\eth_{\mathcal{F}}\big|_{\partial X}) = \widetilde{\eth}_0$$

is invertible. (There's also a more general result, where we have a smoothing perturbation of a Dirac-type operator.)

Theorem 15.2 (R.)**.**

$$\operatorname{ind}(\eth_{\mathcal{F}}) = \int_X \widehat{A}(X, g_{\mathcal{F}}) - \frac{1}{|\Gamma|}\int_Y \widehat{A}(Y, g_Y)\eta(\widetilde{\eth}_0) + \frac{\rho}{2}$$

where ρ is given by

$$\rho = \frac{\eta(\widetilde{\eth}_\delta)}{|\Gamma|} - \eta(\eth_\delta)$$

with $\widetilde{\eth}_\delta$ the operator on $\partial\widetilde{X}$ associated to $\widetilde{g}_\delta = \frac{\widetilde{\phi}^* g_Y}{\delta} = g_{\widetilde{Z}}$, $\eth_\delta$ is the operator on $\widetilde{X}$ associated to $q_*(\widetilde{g}_\delta)$, for $\delta > 0$ sufficiently small (ρ is independent of the choice of δ).

A few words on the proof. The strategy is to take an adiabatic limit. In the more general setting where smoothing perturbations are allowed, a formula for the adiabatic limit can be deduced from an index theorem of R.-Albin.

16. Degeratu: Singular spin structure

This is joint work with Richard Melrose and Mark Stern. It is based on an idea that Mark Stern and I originally had. The idea is to have a way of thinking of spinors on non-spin manifolds. Why should one try to do this?

The motivation comes from the positive mass theorem:
Let (X^n, g) be an asymptotically flat n-manifold with one end. This means that outside of a compact set the manifold looks like a complement of a ball in $\mathbb{R}^n$, and the metric is asymptotic to the Euclidean metric. Assuming that $n \geq 3$, for decay of the metric of the form

$$g = \delta + \mathcal{O}(r^{-n+2}) \text{ and } \partial^l g = \mathcal{O}(r^{-n+2-l}) \quad n \geq 3,$$

one defines the ADM mass to be

$$\text{mass}_{ADM}(X) = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^{n-1}(r)} (\partial_j g_{ij} - \partial_i g_{jj})\, d\Omega^i.$$

The positive mass theorem says that, if the scalar curvature $R \geq 0$, then the ADM mass is positive.

In the late seventies, this was proven by Schoen-Yau for dimension 3, and then for dimensions less than or equal to 7 (by minimal submanifold methods).

In the early eighties, Witten gave another proof based on spinorial techniques. Noticing that in 3-dimensions X is spin, then you have a Dirac operator and a Lichnerowicz formula. Then, given a normalized asymptotically constant spinor on the Euclidean end, ϕ, you have

$$\int_X |\nabla\phi|^2 + \frac{R}{4}|\phi|^2 - |\eth\phi|^2 = \text{mass}_{ADM}(X).$$

If you can show that there exist a normalized asymptotically constant spinor in the null space of $\eth$, then you get the positive mass theorem. This existence part was made rigurous by Parker and Taubes.

Bartnik showed that you can get the positive mass theorem for spin manifolds in arbitrary dimension.

Nowdays, there are approaches to proving the positive mass theorem in higher dimensions by Schoen-Yau and Lokhamp, both using minimal surfaces techniques.

What we are interested in is trying to extend Witten's argument to non-spin manifolds.

Today, I will be telling you about an approach towards defining spinors and a Dirac operator on non-spin manifolds.

First let's talk about spin obstructions:

Probably every one knows that the obstruction to a spin structure on an oriented manifold is the second Stiefel-Whitney class, $w_2(X) \in H^2(X; \mathbb{Z}_2)$. The main idea is the following: on a non-spin manifold, find the Poincaré dual of this class, cut it out, and then do analysis on the complement.

In order to understand the Poincaré dual, it is better to think of the Stiefel-Whitney class as it was originally defined. Namely, $w_2(X)$ is the obstruction to extending $n-1$ linearly independent vector fields from the one-skeleton to the two-skeleton.

Thinking about it in these terms, you can use the Grassmanian. We know that we can think of these characteristic classes in terms of classifying spaces. For our purposes, it is enough to go to the oriented Grassmanian. Namely, consider a continuous map

$$X \xrightarrow{f} \mathrm{Gr}^+(n, \mathbb{R}^{n+N}), \quad \text{with } N \text{ very big.}$$

The class $w_2(X)$ is the pull-back under f of $w_2(\mathrm{Gr}^+(n, \mathbb{R}^{n+N}))$, the second Stiefel-Whitney class of the oriented Grassmannian. A result of Chern gives that the Poincaré dual of w_2 in Gr^+ is a Schubert variety, which we will call S_2.

In fact all cohomology classes of the (oriented) Grasmannian are represented by Schubert varieties. In general these are singular spaces.

To understand S_2 and its singularities, we decompose (with respect the canonical Euclidean inner-product)

$$\mathbb{R}^{n+N} = \mathbb{R}^{n-1} \oplus \mathbb{R}^{N+1}.$$

In this set-up, S_2 is

$$\begin{aligned} S_2 &= \{V \in \mathrm{Gr}^+(n, \mathbb{R}^{n+N}) \mid \dim(V \cap \mathbb{R}^{N+1}) \geq 2\} \\ &= \{V \in \mathrm{Gr}^+(n, \mathbb{R}^{n+N}) \mid \dim(\mathrm{Ker}\, p_V) \geq 2\}. \end{aligned}$$

Here $p_V : V \to \mathbb{R}^{n-1}$ is the orthogonal projection onto $\mathbb{R}^{n-1}$.

¿From this, you see that S_2 is a stratified space, whose strata are given by the size of $\dim(V \cap \mathbb{R}^{N+1})$.

Let $V \in S_2$. The normal directions in $\mathrm{Gr}^+(n, \mathbb{R}^{n+N})$ to the strata to which V belongs are

$$N_V = \mathrm{Hom}(\mathrm{Ker}\, p_V, \mathrm{Coker}\, p_V).$$

Hence the strata of S_2 have codimension $k(k-1)$ in $\mathrm{Gr}^+(n, \mathbb{R}^{n+N})$ with $2 \leq k \leq n$. In total we have $n-1$ strata in S_2, with the "smooth" stratum having codimension two in the oriented Grassmannian. The other $n-2$ strata form a stratification of the singular set of S_2.

Our first theorem is the following:

Theorem 16.1.
1) S_2 is an iterated conic stratified space.
2) There is a sequence of $(n-2)$ projective blow-ups in the Grassmanian (given by blowing-up the singular strata of S_2 in the appropriate order)

$$Gr^+ \longleftarrow Gr^+_{(1)} \longleftarrow \ldots \longleftarrow Gr^+_{(n-2)} = \widehat{G},$$

so that the proper transform $\widehat{S}$ of S_2 in the last blow-up $\widehat{G} = Gr^+_{(n-2)}$ is smooth and codimension 2.

3) There is also a sequence of $(n-1)$ radial blowups

$$Gr^+ \longleftarrow \widetilde{Gr}^+_{(1)} \longleftarrow \ldots \longleftarrow \widetilde{Gr}^+_{(n-2)} \longleftarrow \widetilde{Gr}^+_{(n-1)} = \widetilde{G},$$

producing $\widetilde{G}$ a manifold with corners.
4) The tautological bundle on the Grassmannian restricted to the complement of S_2 has a spin structure which extends smoothly to the manifold with corners $\widetilde{G}$.

So far, this is giving us a pretty good understanding of the Poincaré dual of the second Stiefel-Whitney class on the oriented Grassmannian. Using Thom's transversality, the map f can be deformed so that it is transverse to each of the strata of S_2. This allows us to pull-back S_2 to X under this deformed map. We obtain in this way a stratified set $Y \subset X$ which has the same structure as S_2. Thus

Corollary 16.2.
1) If X^n is a non-spin manifold of dimension $n \geq 4$, there exists $Y \subseteq X$, an iterated conic space with strata (possibly empty) in codimension $k(k-1)$ for $2 \leq k \leq n$, such that $X \setminus Y$ is spin.
2) Radially blowing up the singularities of Y produces a manifold with corners $\widetilde{X}$. The spin structure on $X \setminus Y$ extends smoothly over $\widetilde{X}$.

Why can't we choose Y to be a smooth submanifold? The first obstruction to having Y smooth is in codimension 6. This comes from the fact that the map f might hit the first singular stratum of $S_2 \subset \mathrm{Gr}^+(n, \mathbb{R}^{n+N})$. Avoiding this is obstructed by $w_2 w_4 + w_3^2$ (according to a result of Suzuki).

Moreover, note that in the case of dimension smaller than 6, Y can be chosen to be smooth. This is different from the Spin^c-condition, since not all 5-manifolds are Spin^c.

We now understand what set Y to cut from the manifold X so that the complement $X \setminus Y$ is spin. Next, we need to understand how to do analysis on this complement.

Assume for now that $Y^{n-2} \subseteq X^n$ is an oriented smooth submanifold of X, connected for simplicity. We see right away that the spin structure on $X \setminus Y$ has the following property:

Proposition 16.3. *If $X \setminus Y$ is spin, and X is non-spin, then the spin structure on $X \setminus Y$ has holonomy around Y in the normal fibers.*

Intuitively, this comes down to the fact that on $\mathbb{R}^2 \setminus \{0\}$ – which is topologically the circle – we have two spin structures: one for each double cover of the circle. It turns out that the trivial spin structure over the circle is the one that does not extend to the disk. The reason for this is that when you look at spinors for this spin structure on the circle, they are going to have holonomy as you go around the circle.

As a consequence, the Fourier decomposition for spinors on $X \setminus Y$ near Y is

$$\Psi(r,\theta,y) = \sum_{k\in\mathbb{Z}} e^{i(2k+1)\theta/2}\Psi_k(r,y)$$

for $y \in Y$ and (r,θ) coordinates in the normal fiber to $y \in Y$.

A result of Ammann-Bär (also of Fangyun Wang – MIT thesis) gives that in this context Y has an induced spin structure.

If you know this, then in the above Fourier decomposition, we can interpret $\Psi_k(r,y)$ as taking values in two copies of the spinor bundle $S(Y)$ on Y. With this, we have the following model for the Dirac operator on $X \setminus Y$ near Y

Lemma 16.4 (local models for the Dirac operator)**.** *The Dirac operator has the form*

$$\eth = \begin{pmatrix} \eth_Y & \partial_{\overline{z}} \\ -\partial_z & \eth_Y \end{pmatrix}$$

where, for simplicity, we are thinking of the normal variable as being complex.

Now we can talk about the minimal and maximal domain of this Dirac operator. The minimal domain is going to contain those Fourier modes which behave like $r^{(2k+1)/2}$ with $k \geq 0$ near Y, while the maximal domain is going to have in addition Fourier modes which behave like $r^{-1/2}$. We can then show that

Theorem 16.5.
(i) We have a well-defined boundary map $B : \mathcal{D}_{max} \longrightarrow L^2(Y, S(Y) \oplus S(Y))$ such that the sequence

$$0 \longrightarrow \mathcal{D}_{\min} \longrightarrow \mathcal{D}_{\max} \xrightarrow{B} L^2(Y, S(Y) \oplus S(Y))$$

is exact. B has closed range which includes the smooth sections.
(ii) $\eth$ has a self-adjoint extension, given by an APS boundary condition.
(iii) With the above APS boundary condition, $\eth$ is Fredholm.

17. Deeley: Index theory in geometric K-homology

Start by setting up some notation, then I'll tell you what the plan is for today.

X will be a CW-complex, A, B, and B_∂ will be unital C^*-algebras. $\phi : B_1 \longrightarrow B_2$ a unital $*$-homomorphism. (e.g., $\phi : \mathbb{C} \longrightarrow N$ a II_1-factor.)

Plan:
1)Geometric K-homology with coefficients in A ($K_*(X; A) \xrightarrow{\mu} KK^*(C(X), A)$)
2) Geometric K-homology with coefficients in $\phi : B_1 \longrightarrow B_2$, $K_*(X; \phi)$ which will again be a KK-group, but we'll leave it mysterious for now.
3)Example $\phi : \mathbb{C} \longrightarrow N$ and $\mathbb{R}/\mathbb{Z}$-index theory.

Idea is: relative theories involve objects with connecting morphism

Definition 17.1 ($K_*(X; A)$)**.** *A K-cycle (on X with coefficients in A) is a triple (M, E_A, f) where*
1) M a compact smooth spin-c manifold.
2) $f : M \longrightarrow X$ continuous
3) E_A finitely generated projective Hilbert A-module bundle

Remarks:
1)If $A = \mathbb{C}$, E_A is Hermitian vector bundle
2) In general, $E_A = M \times A^n$ is an example
3) $K_0(C(X) \otimes A)$ is the Grothendieck group of equivalence classes of such bundles.

Operations:
1) Disjoint union

$$(M, E_A, f) \cup (M', E'_A, f') = (M \cup M, E_A \cup E'_A, f \cup f')$$

2) Opposite spin-c structure

Relations:
1) $(M, E_A, f) \cup (M, E'_A, f) \sim (M, E_A \oplus E'_A, f)$
2) Vector bundle modification
3) Bordism

A bordism in $K_*(X, A)$ is a triple (W, E_A, f) where $\partial W \neq 0$

Now we can define groups $K_0(X; A)$ consists of even cycles modulo equivalence,
$K_1(X; A)$ consists of odd cycles modulo equivalence,
Given $\phi : B_1 \longrightarrow B_2$ unital get $\phi_* : K_*(X; B_1) \longrightarrow K_*(X; B_2)$ by sending (M, E_{B_1}, f) to $(M, E_{B_1} \otimes_\phi E_{B_2}, f)$

Relative construction:
By a ϕ-cycle we mean a triple $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ where
1) W is a compact spin-c manifold
2) $W \xrightarrow{f} X$
3) E_{B_2} over W, F_{B_1} over ∂W, $\alpha : E_{B_2}\big|_{\partial W} \cong F_{B_1} \otimes_\phi B_2$

Idea:
In the case of the map: $K_*(X) \longrightarrow K_*(X)$ defined by sending (M, E, f) to

$(M, E \oplus E \oplus E, f)$
Cycles in the relative theory: $(W, (E, F, \alpha), f)$ where $\alpha : E\big|_{\partial W} \xrightarrow{\cong} F \oplus F \oplus F$

Relations:
A) Disjoint unions / direct sum
B) Vector bundle modification
C) Bordism:

Definition 17.2. *A bordism in $K_*(X;\phi)$ is $(\overline{W}, W, (E_{B_2}, E_{B_1}, \alpha), f)$ where:*
1) $\overline{W}$, W compact spin-c manifold with boundary
2) $W \subseteq \partial\overline{W}$ "nicely"
3) E_{B_2} over $\overline{W}$, F_{B_1} over $\partial\overline{W} \setminus int(W)$
4) $f : W \longrightarrow X$

Now we can define groups:
$K_0(X;\phi)$ consists of even cycles modulo equivalence,
$K_1(X;\phi)$ consists of odd cycles modulo equivalence.

We have a map
$$r : K_*(X; B_2) \longrightarrow K_*(X;\phi)$$
given by $(M, E_{B_2}, f) \mapsto (M, (E_{B_2}, \emptyset), \phi), f)$. And also a map
$$\delta : K_*(X;\phi) \longrightarrow K_*(X; B_1)$$
given by $(W, (E_{B_2}, F_{B_1}, \alpha), f) \mapsto (\partial W, F_{B_1}, f\big|_{\partial W})$.

These fit into a six-term exact sequence

$$\begin{tikzcd}
K_0(X, B_1) \arrow[r, "\phi_*"] & K_0(X; B_2) \arrow[r, "r"] & K_0(X;\phi) \arrow[d, "\delta"] \\
K_1(X;\phi) \arrow[u, "\delta"] & K_1(X; B_2) \arrow[l, "r"] & K_1(X; B_1) \arrow[l, "\phi_*"]
\end{tikzcd}$$

Example:
$\phi : \mathbb{C} \longrightarrow N$, with N a II_1 factor. For $X = \{\text{pt}\}$, this sequence becomes

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "\phi_*"] & \mathbb{R} \arrow[r, "r"] & K_0(\text{pt};\phi) \arrow[d, "\delta"] \\
K_1(\text{pt};\phi) \arrow[u, "\delta"] & 0 \arrow[l, "r"] & 0 \arrow[l, "\phi_*"]
\end{tikzcd}$$

and the question is, can we define an 'index' map
$$K_0(X;\phi) \longrightarrow \mathbb{R}/\mathbb{Z}?$$

Following APS, we notice that
$$\mathbb{R}/\mathbb{Z} = \text{coker}(\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \oplus \mathbb{R})$$

Let $(W, (E_N, F_{\mathbb{C}}, \alpha), f)$ be a cycle.
Step 1: $\mathbb{Q}/\mathbb{Z}$-part
$(\partial W, F_{\mathbb{C}}, f\big|_{\partial W})$ is torsion in $K_1(X)$. Suppose that there exists k such that

$$k(\partial W, F_{\mathbb{C}}, f\big|_{\partial W}) = \partial(Q, \widetilde{F}, g)$$

then we can use the Freed-Melrose index theorem to extract a $\mathbb{Q}/\mathbb{Z}$ index.

Step 2: $\mathbb{R}$-part
First notice that this Q is a choice, it's playing the rôle that the trivialization plays in the APS flat bundle index theorem. So we need to use it in the $\mathbb{R}$-part. Start with

$$(-Q, \frac{1}{k}(\widetilde{F} \otimes_\phi N), g)$$

use the rest of the initial data to perform a gluing and clutching construction and end up with an element of $K_*(X; N)$ and then a $\mathbb{R}$-index (we have a von Neumann bundle).

Now just some remarks:
1) We've actually done a little bit more than what I just said, we've produced an isomorphism:

$$\mu : K_*(X; \phi) \xrightarrow{\cong} K_*(X; \mathbb{R}/\mathbb{Z})$$

2) We can answer the question I originally asked

$$\widetilde{\mu} : K_*(X; \phi) \longrightarrow KK(C(X), SC_\phi)$$

with SC_ϕ the suspension of the mapping cone of ϕ.
3) Is there a connection between higher APS and $K_*(X; \phi)$?