Singular foliations, holonomy and their use

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Cortona, June 2014

Foliations appear in many situations:

- Actions of Lie group(oid)s
- Poisson geometry...
- Stratified spaces...

Most foliations: singular

Aim: understand "space of leaves" $\mathcal{M}/\mathcal{F}$.
Best model: Holonomy groupoid $\mathrm{H}(\mathcal{F})$

- Desingularizes $\mathcal{F}$...
- No unnecessary isotropy...

Applications

1. NCG methods: Caclulate spectrum of Laplacian
2. Topology/DG: Normal form about a leaf, linearization

Noncommutative Geometry methods

Regular case: $H(\mathcal{F})$ smooth, attach $C^*(\mathcal{F})$.

- Leaves correspond to ideals.
- If all leaves are dense, $C^*(\mathcal{F})$ simple (Fack-Skandalis).

If $H(\mathcal{F})$ smooth, attach longitudinal pseudodifferential calculus.

- Replace leaves with operators...
- $C^*(\mathcal{F})$ carries all info about this calculus.

Particularly longitudinal Laplacian Δ: essentially self-adjoint, unbounded multiplier of $C^*(\mathcal{F})$.

Also Scroedinger-type operators $\Delta + f$...

Gaps in spectrum correspond to projections of $C^*(\mathcal{F})$. Calculations: K-theory, index theory, Baum-Connes map...

Motivation: Laplacian of Kronecker foliation

Kronecker foliation on $M = T^2$: $\mathcal{F} = \langle X = \frac{d}{dx} + \theta \frac{d}{dy} \rangle$. $L = \mathbb{R}$

Two Laplacians:

- $\Delta_L = -\frac{d^2}{dx^2}$ acting on $L^2(\mathbb{R})$
- $\Delta_M = -X^2$ acting on $L^2(M)$

By Fourier:

- $\Delta_L \rightsquigarrow$ mult. by ξ^2 on $L^2(\mathbb{R})$. Spectrum: $[0, +\infty)$.
- $\Delta_M \rightsquigarrow$ mult. by $(n + \theta k)^2$ on $L^2(\mathbb{Z}^2)$. Spectrum dense in $[0, +\infty)$.

Spectrum Calculation

Horocyclic foliation: Spectrum has no gaps

Consider the action of the "$ax + b$"-group on a compact manifold M.
e.g. $M = SL(2, \mathbb{R})/\Gamma$ where Γ discrete co-compact group.
Leaves = orbits of "$x + b$" subgroup (dense).

Spectrum of Laplacian is an interval $[m, +\infty)$

Proof: We show $C^*(M, F)$ has no projections.

- $\exists\ ax + b$-invariant measure of $M \Longrightarrow$ get trace of $C^*(M, F)$. Faithful because $C^*(M, F)$ simple (Fack-Skandalis).
- "ax" subgroup induces $\mathbb{R}_+^*$-action on $C^*(M, F)$ which scales the trace.
- Image of K_0 is a countable subgroup of $\mathbb{R}$, invariant with respect $\mathbb{R}_+^*$-action.

Singular foliations

Definition

A singular foliation $(M, \mathcal{F})$ is a $C^\infty(M)$-submodule of $\mathcal{X}_c(M)$ which is involutive and locally finitely generated.

$\mathcal{F}$ projective $\Rightarrow$ almost regular foliation.

Singular case: A-Skandalis constructions

For *any* singular foliation, we were able to construct:

- Holonomy groupoid $\mathrm{H}(\mathcal{F})$. Very singular...
- $\mathrm{C}^*(\mathcal{F})$, representations...
- The cotangent bundle $\mathcal{F}^*$: locally compact space.
- Pseudodifferential caclulus...
  1. $0 \to \mathrm{C}^*(M, \mathcal{F}) \to \Psi^*(M, \mathcal{F}) \to \mathrm{C}_0(\mathcal{F}^*) \to 0$
  2. Elliptic operators of order 0 are regular unbounded multipliers
- Analytic index (element of $\mathrm{KK}(\mathrm{C}_0(\mathcal{F}^*); \mathrm{C}^*(M, \mathcal{F}))$)

Holonomy groupoid: Examples

1. $\mathcal{F} =< X >$ s.t. X has non-periodic integral curves around $\partial\{X = 0\}$:

$$H(\mathcal{F}) = H(X)|_{\{X\neq 0\}} \cup \mathrm{Int}\{X = 0\} \cup (\mathbb{R} \times \partial\{X = 0\})$$

2. action of $SL(2, \mathbb{R})$ on $\mathbb{R}^2$:

$$H(\mathcal{F}) = (\mathbb{R}^2 \setminus \{0\})^2 \cup SL(2, \mathbb{R}) \times \{0\}$$

topology: Let $x \in \mathbb{R}^2 \setminus \{0\}$. Then $(\frac{x}{n}, \frac{x}{n}) \in H(\mathcal{F})$ converges to every g in stabilizer group of x... namely to every point of $\mathbb{R}$!

Debord

s-fibers of $H(\mathcal{F})$ is always smooth.

A-Zambon

$H(\mathcal{F})$ is a diffeological space (Souriau)

Results A-Skandalis

Theorem 1

M compact manifold, $X_1, \ldots, X_N \in C^\infty(M; TM)$ such that

$$[X_i, X_j] = \sum f_{ij}^k X_k$$

Then $\Delta = \sum X_i^* X_i$ is essentially self-adjoint (both in $L^2(M)$ and $L^2(L)$).

Proof

This operator is indeed a regular unbounded multiplier of our C^*-algebra.

What about calculating the spectrum?

Theorem 2

Assume that:

- the (dense open) set $\Omega \subset M$ where leaves have maximal dimension has Lebesgue measure 1.
- the restriction of all leaves to Ω are dense in Ω.
- the holonomy groupoid of the restriction of $\mathcal{F}$ to Ω is Hausdorff and amenable.

Then Δ_M and Δ_L have the same spectrum.

Calculation: Need to know the "shape" of $K_0(C^*(\mathcal{F}))$.

leaves of given dimension $\rightsquigarrow$ locally closed subsets $\rightsquigarrow$ filtration of $C^*(\mathcal{F})$

Now give a formula for the K-theory. Baum-Connes conjecture...

Holonomy revisited

Recall

Regular foliation = $\mathcal{F}$: projective module of vector fields.

- Choose path $\gamma : [0,1] \to L$ and $S_{\gamma(0)}, S_{\gamma(1)}$ small transversals of L.
- Path holonomy: (germ of) local diffeomorphism $S_{\gamma(0)} \to S_{\gamma(1)}$ by "sliding along γ in nearby leaves".
- Explicitly: Let $X \in \mathcal{F}$ whose flow at $\gamma(0)$ is γ.
  Now flow X at other points of $S_{\gamma(0)}$ until time 1.
  - $H(F) = \{\text{paths in leaves}\}/\{\text{path holonomy}\}$

Recall: Path holonomy depends only on the homotopy class of γ.
Get holonomy map

$$h : \pi_1(L, x) \to GermAut_{\mathcal{F}}(S_x; S_x)$$

Image H_x^x: holonomy group of F.

- Linearizes to representation

$$dh : \pi_1(L, x) \to GL(N_x L)$$

Path holonomy in the singular case fails!

Orbits of action by rotations in $\mathbb{R}^2$: $\mathcal{F} = span_{C^\infty(\mathbb{R}^2)} < x\partial_y - y\partial_x >$.

- Take γ: constant path at origin.
- Transversal S_0: open neighborhood of origin in $\mathbb{R}^2$.

Realize γ either by integrating the zero vector field or $x\partial_y - y\partial_x$ at the origin. Get completely different diffeomorphisms of S_0!

Here $\mathcal{F}$ is projective as well!

"Almost projective" (singular) case (Debord):

A projective foliation $\mathcal{F}$ always has a smooth holonomy groupoid.

Non-projective $\mathcal{F} = \text{span} < X >$: Take X with non-empty interior of $\{x \in M : X(x) = 0\}$

Singular case (projective or not):

$h : \pi_1(L, x) \to GermAut_{\mathcal{F}}(S_x; S_x)$ not defined!

Stability for regular foliations

Local Reeb stability theorem

If L is a compact embedded leaf and H_x^x is finite then nearby L the foliation F is isomorphic to its linearization.

Namely, around L the manifold looks like

$$\frac{\widetilde{L} \times \mathbb{R}^q}{\pi_1(L)}$$

$\pi_1(L)$ acts diagonally by deck transformations and linearized holonomy.

This is equal to

$$\frac{H_x \times N_x L}{H_x^x}$$

The action of H_x^x on $N_x L$ is the one that integrates the Bott connection

$$\nabla : F \to CDO(N), \quad (X, \langle Y \rangle) \to \langle [X, Y] \rangle$$

The holonomy map

Let $(M, \mathcal{F})$ a singular foliation, L a leaf, $x, y \in L$ and S_x, S_y slices of L at x, y respectively.

Theorem (A-Zambon)

There is an injective map

$$\Phi_x^y : H_x^y \to \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}$$

It defines a morphism of groupoids

$$\Phi : H \to \cup_{x,y} \frac{\mathrm{GermAut}_{\mathcal{F}}(S_x, S_y)}{\exp(I_x\mathcal{F}) \mid_{S_x}}$$

Regular case: then $\exp(I_x\mathcal{F}) \mid_{S_x} = \{\mathrm{Id}\}$.

Holonomy map and the Bott connection

1. Differentiating Φ gives

$$\Psi_L : H_L \to Iso(NL, NL)$$

   Lie groupoid representation of H_L on NL;

2. Differentiating Ψ_L gives

$$\nabla^{L,\perp} : A_L \to Der(NL)$$

   It is the Bott conection...

All this justifies the terminology "holonomy groupoid"!

Linearization

Vector field Y on M tangent to $L \rightsquigarrow$
Vector field Y_{lin} on NL, defined as follows:

$$Y_{lin} \text{ acts on the fibrewise constant functions as } Y|_L$$
$$Y_{lin} \text{ acts on } C^\infty_{lin}(NL) \equiv I_L/I_L^2 \text{ as } Y_{lin}[f] = [Y(f)].$$

The linearization of $\mathcal{F}$ at L is the foliation $\mathcal{F}_{lin}$ on NL generated by

$$\{Y_{lin} : Y \in \mathcal{F}\}$$

Lemma

Let L be a leaf. Then $\mathcal{F}_{lin}$ is the foliation induced by the Lie groupoid action Ψ_L of H_L on NL.

We say $\mathcal{F}$ is linearizable at L if there is a diffeomorphism mapping $\mathcal{F}$ to $\mathcal{F}_{lin}$.

For $\mathcal{F} = \langle X \rangle$ with X vanishing at $L = \{x\}$ linearizability means:

There is a diffeomorphism taking X to fX_{lin} for a non-vanishing function f.

This is a weaker condition than the linearizability of the vector field X!

Question: When is a singular foliation isomorphic to its linearization?

We don't know yet, but:

Proposition (A-Zambon)

Let L_x embedded leaf.

The following are equivalent:

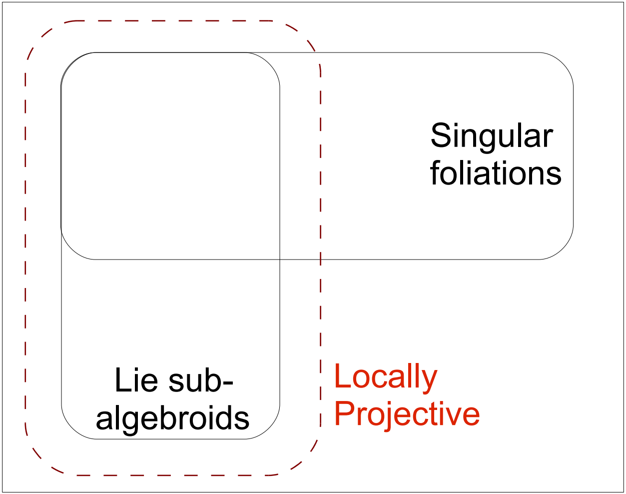
1. $\mathcal{F}$ is linearizable about L_x and H_x^x compact
2. there exists a tubular neighborhood U of L and a (Hausdorff) Lie groupoid $G \to U$, proper at x, inducing the foliation $\mathcal{F}|_U$.

In that case:

- G can be chosen to be the transformation groupoid of the action Ψ_L of H_L on NL.

- $(U, \mathcal{F}|_U)$ admits the structure of a singular Riemannian foliation.

The bigger picture



Singular subalgebroids

A-Zambon results

Log-symplectic manifolds, *e.g.* $(\mathbb{R}^2, \pi = x\mathrm{d}x \wedge \mathrm{d}y)$. Construct symplectic realizations?

Weinstein's programme: M. Gualtieri and S. Li used Melrose's blow-up construction to give a symplectic realization.

(Recall: B. Monthubert showed Melrose's b-calculus is really a *groupoid calculus*)

A-Zambon: Holonomy groupoid construction can be extended to any singular subalgebroid. Special case: symplectic groupoid of Gualtieri-Li. Can construct many other symplectic realizations this way...