Index theory for the Dirac operator on Lorentzian manifolds

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Index theory in Lorentzian signature?

Problem 1: Let D be a differential operator of order k over a closed manifold. Then $D : H^k \to L^2$ is Fredholm $\Leftrightarrow$ D is elliptic.

$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Problem 2: Closed Lorentzian manifolds violate causality conditions

$\Rightarrow$ useless as models in General Relativity

But: There is one for the Atiyah-Patodi-Singer index theorem!

Setup

- M Riemannian manifold, compact, with boundary ∂M
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\rightsquigarrow$ splitting $SM = S_R M \oplus S_L M$
- Hermitian vector bundle $E \to M$ with connection $\rightsquigarrow$ twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ where $V_{R/L} = S_{R/L} M \otimes E$

Need boundary conditions:
Let A_0 be the Dirac operator on ∂M.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projector

APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

Atiyah-Patodi-Singer index theorem

Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

Under APS-boundary conditions D is Fredholm and

$$\boxed{\begin{aligned}\mathrm{ind}(D_{\mathrm{APS}}) = &\int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) \\ &+ \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}\end{aligned}}$$

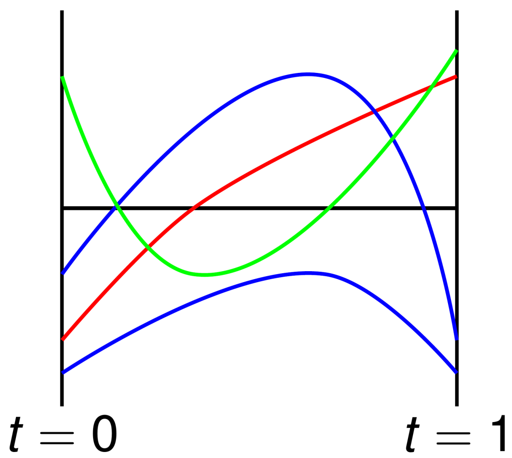
Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

Spectral flow

Special case: $M = \Sigma \times [0,1]$ and $g = dt^2 + g_t$. Then

$$\mathrm{sf}(A_{t\in[0,1]}) = \mathrm{ind}(D_{\mathrm{APS}}) + h(A_1)$$



Warning

APS-boundary conditions cannot be replaced by anti-Atiyah-Patodi-Singer boundary conditions,

$$P_-(f|_{\partial M}) = \chi_{(-\infty,0)}(A_0)(f|_{\partial M}) = 0$$

Example

- M = unit disk $\subset \mathbb{C}$
- $D = \overline{\partial} = \frac{\partial}{\partial \overline{z}}$
- Fourier expansion: $u|_{\partial M} = \sum_{n\in\mathbb{Z}} \alpha_n e^{in\theta}$
- $A_0 = i\frac{d}{d\theta}$
- Taylor expansion: $u = \sum_{n=0}^{\infty} \alpha_n z^n$

APS-boundary conditions:
$\alpha_n = 0$ for $n \geq 0 \Rightarrow \ker(D) = \{0\}$
aAPS-boundary conditions:
$\alpha_n = 0$ for $n < 0 \Rightarrow \ker(D) =$ infinite dimensional

More general boundary conditions

minimal extension D_{min} = closure of D with domain $C_{cc}(M, V_R)$
maximal extension D_{max} = distributional extension to L^2

$$\check{H}(A_0) := H^{\frac{1}{2}}_{(-\infty,0)}(A_0) \oplus H^{-\frac{1}{2}}_{[0,\infty)}(A_0)$$

Theorem(Ballmann-B. 2012)

1) the map $\Phi \mapsto \Phi|_{\partial M}$ on $C^\infty(M, V_R)$ extends uniquely to a continuous surjection $\mathcal{R} : \mathrm{dom}\, D_{\mathrm{max}} \to \check{H}(A_0)$.
2) $\mathrm{dom}\, D_{\mathrm{min}} = \{\Phi \in \mathrm{dom}\, D_{\mathrm{max}} \mid \mathcal{R}\Phi = 0\}$. In particular, $\check{H}(A_0) \cong \mathrm{dom}\, D_{\mathrm{max}} / \mathrm{dom}\, D_{\mathrm{min}}$.
3) for any closed subspace $B \subset \check{H}(A_0)$, the operator D_B with domain $\mathrm{dom}\, D_B = \{\Phi \in \mathrm{dom}\, D_{\mathrm{max}} \mid \mathcal{R}\Phi \in B\}$ is a closed extension of D between D_{min} and D_{max}, and any such extension is of this form.

Elliptic boundary conditions

Definition

A linear subspace $B \subset H^{\frac{1}{2}}(\partial M, V_R) \subset \check{H}(A_0)$ is said to be an **elliptic boundary condition** if there is an L^2-orthogonal decomposition

$$L^2(\partial M, V_R) = V_- \oplus W_- \oplus V_+ \oplus W_+$$

such that

$$B = W_+ \oplus \{v + gv \mid v \in V_- \cap H^{\frac{1}{2}}\}$$

where

1) $W_\pm \subset C^\infty(\partial M, V_R)$ finite-dimensional;
2) $V_- \oplus W_- \subset L^2_{(-\infty,a]}(A_0)$ and $V_+ \oplus W_+ \subset L^2_{[-a,\infty)}(A_0)$, for some $a \in \mathbb{R}$;
3) $g : V_- \to V_+$ and $g^* : V_+ \to V_-$ are operators of order 0.

Fredholm property and boundary regularity

Theorem (Ballmann-B. 2012)

Let B be an elliptic boundary condition. Then

$$D_B : \mathrm{dom}\, D_B \to L^2(M, V_L)$$

is Fredholm.

Theorem (Ballmann-B. 2012)

Let B be an elliptic boundary condition. Then

$$\Phi \in H^{k+1}(M, V_R) \Longleftrightarrow D_B\Phi \in H^k(M, V_L),$$

for all $\Phi \in \mathrm{dom}\, D_B$ and integers $k \geq 0$. In particular, $\Phi \in \mathrm{dom}\, D_B$ is smooth up to the boundary if and only if $D_B\Phi$ is smooth up to the boundary.

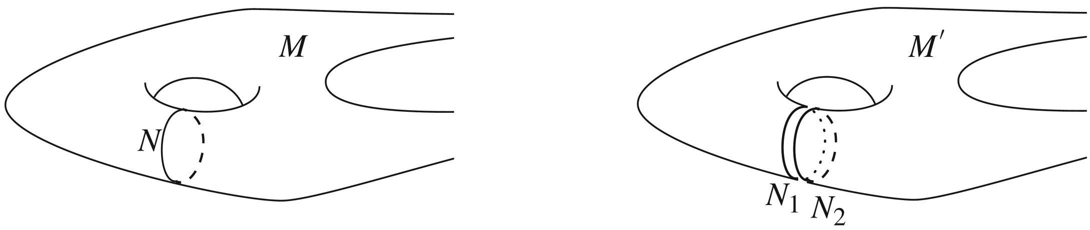
Examples

1) Generalized APS:
$V_- = L^2_{(-\infty,a)}(A_0)$, $V_+ = L^2_{[a,\infty)}(A_0)$, $W_- = W_+ = \{0\}$, $g = 0$.
Then
$$B = H^{\frac{1}{2}}_{(-\infty,a)}(A_0).$$

2) Classical local elliptic boundary conditions in the sense of Lopatinsky-Schapiro.

Examples

3) “Transmission” condition



$$V_+ = L^2_{(0,\infty)}(A_0 \oplus -A_0) = L^2_{(0,\infty)}(A_0) \oplus L^2_{(-\infty,0)}(A_0)$$

$$V_- = L^2_{(-\infty,0)}(A_0 \oplus -A_0) = L^2_{(-\infty,0)}(A_0) \oplus L^2_{(0,\infty)}(A_0)$$

$$W_+ = \{(\phi, \phi) \in \ker(A_0) \oplus \ker(A_0)\}$$

$$W_- = \{(\phi, -\phi) \in \ker(A_0) \oplus \ker(A_0)\}$$

$$g : V_-^{\frac{1}{2}} \to V_+^{\frac{1}{2}}, \quad g = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

Then

$$B = \left\{(\phi, \phi) \in H^{\frac{1}{2}}(N_1, V_R) \oplus H^{\frac{1}{2}}(N_2, V_R) \mid \phi \in H^{\frac{1}{2}}(N, V_R)\right\}$$

A deformation argument

Replace B by B_s where g is replaced by g_s with $g_s = s \cdot g$.
Then $B_1 =$ transmission condition and $B_0 =$ APS-condition.

Hence $\mathrm{ind}(D^M) = \mathrm{ind}(D^{M'}_{transm.}) = \mathrm{ind}(D^{M'}_{\mathrm{APS}})$.

Holds also if M is complete noncompact and D satisfies a coercivity condition at infinity.

Implies **relative index theorem** by Gromov and Lawson (1983).

Globally hyperbolic spacetimes

A subset $\Sigma \subset M$ is called Cauchy hypersurface if each inextendible timelike curve in M meets Σ exactly once.

If M has a Cauchy hypersurface then M is called globally hyperbolic.

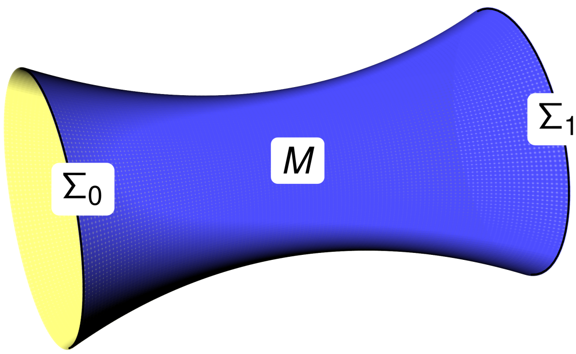
Examples:

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- $\cdots$

Globally hyperbolic spacetimes

Theorem (Bernal-Sánchez 2005)

Every globally hyperbolic Lorentzian manifold is isometric to $M = I \times \Sigma$ with metric $-N^2 dt^2 + g_t$ such that each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy hypersurface.



Let M be a globally hyperbolic Lorentzian manifold **with boundary** $\partial M = \Sigma_0 \sqcup \Sigma_1$

Σ_j compact smooth spacelike Cauchy hypersurfaces

The Cauchy problem

Well-posedness of Cauchy problem

The map $D \oplus \mathrm{res}_\Sigma : C^\infty(M; V_R) \to C^\infty(M; V_L) \oplus C^\infty(\Sigma; V_R)$ is an isomorphism of topological vector spaces.

Wave propagator U:

$$\begin{tikzcd} & \{v \in C^\infty(M; V_R) \mid Dv = 0\} \arrow[dl, "\mathrm{res}_{\Sigma_0}"', "\cong"] \arrow[dr, "\mathrm{res}_{\Sigma_1}", "\cong"'] & \\ C^\infty(\Sigma_0, V_R) \arrow[rr, "U"] & & C^\infty(\Sigma_1, V_R) \end{tikzcd}$$

U extends to **unitary** operator $L^2(\Sigma_0; V_R) \to L^2(\Sigma_1; V_R)$.

Fredholm pairs

Definition

Let H be a Hilbert space and $B_0, B_1 \subset H$ closed linear subspaces. Then (B_0, B_1) is called a Fredholm pair if $B_0 \cap B_1$ is finite dimensional and $B_0 + B_1$ is closed and has finite codimension. The number

$$\mathrm{ind}(B_0, B_1) = \dim(B_0 \cap B_1) - \dim(H/(B_0 + B_1))$$

is called the index of the pair (B_0, B_1).

Elementary properties:

1.) $\mathrm{ind}(B_0, B_1) = \mathrm{ind}(B_1, B_0)$

2.) $\mathrm{ind}(B_0, B_1) = -\,\mathrm{ind}(B_0^\perp, B_1^\perp)$

3.) Let $B_0 \subset B_0'$ with $\dim(B_0'/B_0) < \infty$. Then

$$\mathrm{ind}(B_0', B_1) = \mathrm{ind}(B_0, B_1) + \dim(B_0'/B_0).$$

Fredholm pairs and the Dirac operator

Let $B_0 \subset L^2(\Sigma_0, V_R)$ and $B_1 \subset L^2(\Sigma_1, V_R)$ be closed subspaces.

Observation (B.-Hannes 2017)

The following are equivalent:

(i) The pair $(B_0, U^{-1}B_1)$ is Fredholm of index k;

(ii) The pair (UB_0, B_1) is Fredholm of index k;

(iii) The restriction

$$D : \ker(\pi_{B_0^\perp} \circ \mathrm{res}_{\Sigma_0}) \cap \ker(\pi_{B_1^\perp} \circ \mathrm{res}_{\Sigma_1}) \to L^2(M_0, V_L)$$

is a Fredholm operator of index k.

Trivial example

Let $\dim(B_0) < \infty$ and $\operatorname{codim}(B_1) < \infty$.

Then D with these boundary conditions is Fredholm with index

$$\dim(B_0) - \operatorname{codim}(B_1)$$

The Lorentzian index theorem

Theorem (B.-Strohmaier 2015)
Under APS-boundary conditions D is a Fredholm operator.
The kernel consists of smooth spinor fields and

$$\boxed{\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}}$$

$$\begin{aligned}\mathrm{ind}(D_{\mathrm{APS}}) = {} & \dim\ker[D : C^\infty_{\mathrm{APS}}(M; V_R) \to C^\infty(M; V_L)] \\ & - \dim\ker[D : C^\infty_{\mathrm{aAPS}}(M; V_R) \to C^\infty(M; V_L)]\end{aligned}$$

aAPS conditions are as good as APS-boundary conditions.

Proof of the regularity statement

- If v is a distributional spinor solving $Dv = 0$ then $\mathrm{WF}(v) \subset \{\text{lightlike covectors}\}$
- v restricts to distributions along $\Sigma_{0/1}$
- APS conditions along $\Sigma_0 \Rightarrow$ $\mathrm{WF}(v) \subset \{\text{future-directed lightlike covectors}\}$ along Σ_0
- propagation of singularities $\Rightarrow$ $\mathrm{WF}(v) \subset \{\text{future-directed lightlike covectors}\}$ on all of M
- similarly, APS along $\Sigma_1 \Rightarrow$ $\mathrm{WF}(v) \subset \{\text{past-directed lightlike covectors}\}$
- $\Rightarrow \mathrm{WF}(v) = \emptyset$, i.e. v is smooth

Proof of the index theorem

Decompose the wave propagator

$$U = \begin{pmatrix} U_{++} & U_{+-} \\ U_{-+} & U_{--} \end{pmatrix}$$

w.r.t. decomposition

$$\begin{aligned} L^2(\Sigma_0; V_R) &= P_+ L^2(\Sigma_0; V_R) &&\oplus\quad (I - P_+) L^2(\Sigma_0; V_R) \\ &= L^2_{[0,\infty)}(\Sigma_0; V_R) &&\oplus\quad L^2_{(-\infty,0)}(\Sigma_0; V_R)\,, \\ L^2(\Sigma_1; V_R) &= (I - P_+) L^2(\Sigma_1; V_R) &&\oplus\quad P_+ L^2(\Sigma_1; V_R) \\ &= L^2_{(0,\infty)}(\Sigma_1; V_R) &&\oplus\quad L^2_{(-\infty,0]}(\Sigma_1; V_R) \end{aligned}$$

Then

$$\mathrm{ind}(D_{\mathrm{APS}}) = \dim\ker(U_{--}) - \dim\ker(U_{++})$$

Proof of the index theorem

Step 1: Show that D_{APS} is Fredholm
$\Leftrightarrow$ U_{++} and U_{--} are Fredholm
$\Leftarrow$ U_{+-} and U_{-+} are compact (uses microlocal analysis)

Step 2: Compute the index
Introduce auxiliary *Riemannian* metric $\hat{g}$ on M (“Wick rotation”)

$$\mathrm{sf}(A_t) = \mathrm{ind}(\hat{D}_{\mathrm{APS}}) = \text{geometric expression}(\hat{g}, \nabla^E)$$

$$\mathrm{sf}(A_t) = \mathrm{ind}(D_{\mathrm{APS}})$$

$$\text{geometric expression}(g, \nabla^E) = \text{geometric expression}(\hat{g}, \nabla^E)$$

Application to QFT

No natural physical interpretation of APS boundary conditions in the Riemannian case.
But the Lorentzian version allows to compute the chiral anomaly in QFT.

Boundary conditions in graph form

A pair (B_0, B_1) of closed subspaces $B_i \subset L^2(\Sigma_i, V^R)$ form elliptic boundary conditions if there are L^2-orthogonal decompositions

$$L^2(\Sigma_i, V^R) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+, \qquad i = 0, 1,$$

such that

(i) W_i^+, W_i^- are finite dimensional;

(ii) $W_i^- \oplus V_i^- = L^2_{(-\infty,a_i)}(A_i)$ and $W_i^+ \oplus V_i^+ = L^2_{[a_i,\infty)}(A_i)$ for some $a_i \in \mathbb{R}$;

(iii) There are bounded linear maps $g_0 : V_0^- \to V_0^+$ and $g_1 : V_1^+ \to V_1^-$ such that

$$B_0 = W_0^+ \oplus \Gamma(g_0),$$
$$B_1 = W_1^- \oplus \Gamma(g_1),$$

where $\Gamma(g_{0/1}) = \{v + g_{0/1} v \mid v \in V_{0/1}^{\mp}\}$.

Boundary conditions in graph form

Theorem (B.-Hannes 2017)

Let $a_0, a_1 \in \mathbb{R}$. Then the pair $(\Gamma(g_0), \Gamma(g_1))$ is Fredholm of the same index as $(\mathrm{APS}_0(a_0), \mathrm{APS}_1(a_1))$ provided

(A) g_0 or g_1 is compact **or**

(B) $\|g_0\| \cdot \|g_1\|$ is small enough.

1.) Applies if $g_0 = 0$ or $g_1 = 0$.

2.) Conditions (A) and (B) cannot both be dropped (counterexamples).

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