Torsion and the Dirac operator

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Cortona, 26th June 2017

The classical Dirac operator

- X compact oriented spin Riemannian manifold.
- $S^{TX}\left(TX, g^{TX}\right)$ spinors, $\nabla^{S^{TX}}$ LC connection.
- $\left(E, g^E, \nabla^E\right)$ Hermitian vector bundle with connection.
- D^X classical Dirac operator acts on $C^\infty\left(X, S^{TX} \otimes E\right)$.
- Δ^X Bochner Laplacian, K^X scalar curvature.
- Lichnerowicz formula

$$D^{X,2} = -\Delta^X + \frac{1}{4}K^X + \frac{1}{2}c\left(e_i\right)c\left(e_j\right)R^E\left(e_i, e_j\right).$$

The Dirac operator with torsion

- ∇_T^{TX} metric connection on TX with torsion T.
- $\langle T, \theta \rangle$ assumed to be antisymmetric.
- $\eta = \langle T \wedge \theta \rangle$ 3-form on X.
- $\nabla_T^{S^{TX}}$ connection on S^{TX} induced by ∇_T^{TX}.
- D_T^X self-adjoint Dirac operator associated with $\nabla_T^{S^{TX}}$.
- $D_T^X = D^X + \frac{1}{4}{}^c\eta$.
- Δ_T^H Bochner Laplacian associated with connection with torsion T.

Theorem B89

$$D_T^{X,2} = -\Delta_{3T}^H + \frac{K^X}{4} + \frac{1}{4}{}^c \left(d \left\langle T \wedge \theta \right\rangle\right) - \frac{1}{8} \left|T \wedge \theta\right|^2 .$$

The case where $\langle T \wedge \theta \rangle$ is closed.

• $D_T^{X,2} = -\Delta_{3T}^H + \frac{K^X}{4} - \frac{1}{8}\left|T \wedge \theta\right|^2.$

Theorem B89

The local index theorem holds, with $\widehat{A}(TX)$ calculated with ∇_{-3T}^{TX}.

The case of a compact Lie group G

- G compact Lie group with Lie algebra $\mathfrak{g}$.
- B a G-invariant scalar product on $\mathfrak{g}$.
- $TG \simeq \mathfrak{g}$ left-invariant vector fields.
- d trivial connection on TG, $T(U, V) = -[U, V]$.
- $\langle T \wedge \theta \rangle = -B(\theta^2, \theta)$ closed.

The Dirac operator of Kostant

- $\kappa^{\mathfrak{g}}(U,V,W) = B([U,V],W)$ closed, $\kappa^{\mathfrak{g}} = -\frac{1}{3}\langle T \wedge \theta\rangle$.
- $D^{\mathrm{Ko}} = D^G_{T/3}$.
- $D^{\mathrm{Ko}} = c(e_i)\nabla_{e_i} + \frac{1}{2}c(\kappa^{\mathfrak{g}})$.

Theorem (Kostant)

$$D^{K,2} = -\Delta^G + \frac{1}{24} f^2_{ijk}.$$

Proof.

The connection ∇^G_T is the canonical trivial connection on $TG \simeq \mathfrak{g}$. $\square$

A reductive group

- G connected reductive group, K maximal compact subgroup.
- θ Cartan involution, K fixed by θ.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ Cartan splitting.
- B a G invariant form on $\mathfrak{g}$, > 0 on $\mathfrak{p}$, < 0 on $\mathfrak{k}$.
- $X = G/K$ symmetric space.

The case where $G = \mathrm{SL}_2(\mathbf{R})$

- $G = \mathrm{SL}_2(\mathbf{R})$, $\theta g = \widetilde{g}^{-1}$.
- $K = S^1$, $\mathrm{sl}_2(\mathbf{R}) = \mathfrak{p} \oplus \mathfrak{k}$.
- $B(a, b) = 2\mathrm{Tr}[ab]$.
- X upper half-plane.

Casimir and Kostant

- $-\Delta^G$ is now the Casimir operator $C^{\mathfrak{g}}$ (not elliptic).
- We still have a Kostant operator D^{Ko} such that

$$D^{\mathrm{Ko},2} = C^{\mathfrak{g}} + c.$$

$\widehat{D}^{\mathrm{Ko}}$ and $C^\infty(G) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$

- $\widehat{c}(\mathfrak{g})$ acts on $\Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\widehat{D}^{\mathrm{Ko}} = \widehat{c}(e_i)\, e_i + \frac{1}{2}\widehat{c}(-\kappa^{\mathfrak{g}})$ acts on $C^\infty(G) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\widehat{D}^{\mathrm{Ko},2} = -C^{\mathfrak{g}} - c \ldots$
- …analogue of $(-d_x + d_x^*)^2 = \frac{\partial^2}{\partial x^2}$.
- $Z = \Gamma \setminus X$ compact quotient.
- $\mathrm{Tr}\left[\exp\left(t\left(\Delta^Z - c\right)\right)\right] = \mathrm{Tr}\left[\exp\left(t\widehat{D}^{\mathrm{Ko},2}\right)\right] \ldots$
- …looks like a McKean-Singer formula …
- …but it is not, because $\Lambda^{\cdot}(\mathfrak{g}^*)$ appears in the right hand-side.

How to make $\Lambda^{\cdot}(V^*)$ great again

- V vector space.
- $\mathcal{A}(V^*) = \Lambda^{\cdot}(V^*) \otimes S^{\cdot}(V^*)$.
- In representation theory, $\mathcal{A}(V^*) \simeq \mathbf{R}$.
- $\left(\mathcal{A}(V^*), d^V\right)$ de Rham complex of polynomial forms.
- This complex is acyclic, and $H^0 = \mathbf{R}$.
- $\left[d^V, i_Y\right] = L_Y$, and $L_Y = N^{\mathcal{A}(V^*)}$.
- If h^V scalar product (positivity can be dropped!), i_Y is the adjoint of d^V.
- The above is a Hodge theoretic proof of acyclicity.

Action of $\mathfrak{D}_b$ on $C^\infty(G) \otimes S(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$

- $d^{\mathfrak{g}} + i_Y$ acts on $S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\widehat{D}^{\mathrm{Ko}}$ acts on $C^\infty(G) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- $\mathfrak{D}_b = \widehat{D}^{\mathrm{Ko}} + \frac{1}{b}(d^{\mathfrak{g}} + i_Y)$.
- $\mathfrak{D}_b$ acts on $C^\infty(G) \otimes S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$.
- A spectral sequence argument shows that as $b \to 0$, $S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)$ is replaced by $\mathbf{R}$.
- If P projection $S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*) \to \mathbf{R}$, $P\widehat{D}^{\mathrm{Ko}}P = 0$.
- $\mathfrak{D}_b$ deforms the operator 0.

Quotienting by K

- The above construction is invariant by K.
- $\mathfrak{D}_b^X$ acts on $\left[C^{\infty}(G) \otimes S^{\cdot}(\mathfrak{g}^*) \otimes \Lambda^{\cdot}(\mathfrak{g}^*)\right]^K$.
- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ descends to $TX \oplus N$.
- $\mathfrak{D}_b^X$ acts on $C^{\infty}(X, S^{\cdot}(T^*X \oplus N^*) \otimes \Lambda^{\cdot}(T^*X \oplus N^*))$.

The Bargmann isomorphism

- V Euclidean vector space.
- $B : \overline{S}(V^*) \simeq L_2(V)$ isomorphisms of Hilbert spaces.
- B depends explicitly on the metric.

The geometric action of $\mathfrak{D}_b^X$

- $\mathfrak{D}_b^X$ acts on $C^\infty\left(X, S^{\cdot}\left(T^*X \oplus N^*\right) \otimes \Lambda^{\cdot}\left(T^*X \oplus N^*\right)\right)$.
- $\mathfrak{D}_b^X$ on $C^\infty\left(X, L_2\left(TX \oplus N\right) \otimes \Lambda^{\cdot}\left(T^*X \oplus N^*\right)\right)$.
- $\widehat{\mathcal{X}}$ total space of $TX \oplus N$.
- $\mathfrak{D}_b^X$ acts on $C^\infty\left(\widehat{\mathcal{X}}, \widehat{\pi}^*\left(\Lambda^{\cdot}\left(T^*X \oplus N^*\right)\right)\right)$.

The explicit form of $\mathfrak{D}_b^X$

$$\mathfrak{D}_b^X = \widehat{D}^{\mathrm{Ko}} + \underbrace{ic\left(\left[Y^N, Y^{TX}\right]\right)}_{\text{mystery}} + \frac{1}{b}\underbrace{\left(d^{\mathfrak{p}} + Y^{\mathfrak{p}} \wedge + d^{\mathfrak{p}*} + i_{Y^{\mathfrak{p}}}\right)}_{\text{Witten}}$$
$$+ \frac{\sqrt{-1}}{b}\left(-d^{\mathfrak{k}} - Y^{\mathfrak{k}} \wedge + d^{\mathfrak{k}*} + i_{Y^{\mathfrak{k}}}\right).$$

- P orthogonal projection on $\ker\left(d^{\mathfrak{p}} + \ldots\right)$.
- $P\left(\widehat{D}^{\mathrm{Ko}} + ic\left(\left[Y^N, Y^{TX}\right]\right)\right)P = 0$.
- $\mathfrak{D}_b^X$ deforms the operator 0.

The operator $\mathcal{L}_b^X$

- $\mathcal{L}_b^X = \frac{1}{2}\left(-\widehat{D}^{\mathrm{Ko},2} + \mathfrak{D}_b^{X,2}\right)$ acts on
$$C^{\infty}\left(\widehat{\mathcal{X}}, \widehat{\pi}^{*}\Lambda^{\cdot}\left(T^{*}X \oplus N^{*}\right)\right).$$
- $\mathcal{L}_b^X$ deforms $\frac{1}{2}\left(-\Delta^X + c\right)$.

A formula for $\mathcal{L}_b^X$

θ Cartan involution $= \pm 1$ on N, TX.

$$\mathcal{L}_b^X = \underbrace{\frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2}_{\text{quartic term}} + \underbrace{\frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right)}_{\text{Harmonic oscillator of } TX\oplus N} + \frac{N^{\Lambda^{\cdot}(T^*X\oplus N^*)}}{b^2}$$

$$+\frac{1}{b}\Bigg(\underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}} + \widehat{c}\left(\mathrm{ad}\left(Y^{TX}\right)\right) - c\left(\mathrm{ad}\left(Y^{TX}\right) + i\theta\mathrm{ad}\left(Y^N\right)\right)\Bigg).$$

Remark

By Hörmander, $\frac{\partial}{\partial t} + \mathcal{L}_b^X$ is hypoelliptic.

A fundamental identity

Theorem B11

If $Z = \Gamma \setminus X$ compact quotient, for $t > 0, b > 0$,

$$\mathrm{Tr}^{C^\infty(Z,\mathbf{R})}\left[\exp\left(t\left(\Delta^Z - c\right)/2\right)\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right].$$

Proof

Limit as $b \to 0$, Bianchi identity
$\left[\mathfrak{D}_b^Z, \mathcal{L}_b^Z\right] = \left[\mathfrak{D}_b^Z, \left(\mathfrak{D}_b^{Z,2} + C^{\mathfrak{g}}\right)/2\right] = 0$, combined with

$$\frac{\partial}{\partial b}\mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^Z\right)\right] = -\frac{t}{2}\mathrm{Tr_s}\left[\left[\mathfrak{D}_b^Z, \frac{\partial}{\partial b}\mathfrak{D}_b^Z \exp\left(-t\mathcal{L}_b^Z\right)\right]\right] = 0.$$

Splitting the identity

1. The identity splits as identity of orbital integrals...
2. ...which are contributions of the conjugacy classes of $\Gamma = \pi_1(Z)$.

Semisimple orbital integrals

- $\gamma \in G$ semisimple, $[\gamma]$ conjugacy class.
- For $t > 0$, $I([\gamma]) = \mathrm{Tr}^{[\gamma]}\left[\exp\left(t\left(\Delta^X - c\right)/2\right)\right]$ orbital integral of heat kernel on orbit of γ:

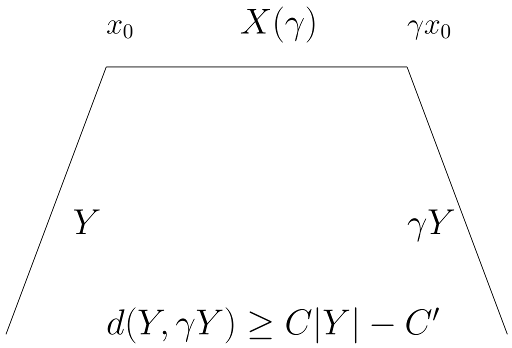
$$I([\gamma]) = \int_{Z(\gamma)\backslash G} p_t^X\left(g^{-1}\gamma g\right) dg.$$

The minimizing set

•$X(\gamma) \subset X$ minimizing set for the convex displacement function $d(x, \gamma x)$.
•$X(\gamma) \subset X$ totally geodesic symmetric space for the centralizer $Z(\gamma)$.

Geometric form of the orbital integral

$$I(\gamma) = \int_{N_{X(\gamma)/X}} \operatorname{Tr}\left[p_t^X(Y, \gamma Y)\right] \underbrace{r(Y)}_{\text{Jacobian}} dY.$$



$$p_t^X(x, x') \leq C \exp(-C' d^2(x, x')).$$

The heat kernel for $\mathcal{L}_b^X$

$\exp\left(-t\mathcal{L}_b^X\right)$ has a heat kernel $q_{b,t}^X\left(\left(x,Y\right),\left(x',Y'\right)\right)$.

Theorem (B2011)

- For $b \in]0, M]$, $t > 0$ fixed,

$$\begin{aligned}
&\left|q_{b,t}^X\left(\left(x,Y\right),\left(x',Y'\right)\right)\right| \\
&\leq C\exp\left(-C'\left(d^2\left(x,x'\right)+\left|Y\right|^2+\left|Y'\right|^2\right)\right),
\end{aligned}$$

$$\begin{aligned}
&q_{b,t}^X\left(\left(x,Y\right),\left(x',Y'\right)\right) \xrightarrow{b\to 0} \\
&\mathbf{P}p_t^X\left(x,x'\right)\pi^{-\dim\mathfrak{g}/2}\exp\left(\frac{1}{2}\left(\left|Y\right|^2+\left|Y'\right|^2\right)\right)\mathbf{P}.
\end{aligned}$$

A second fundamental identity

Theorem B2011

For $b > 0, t > 0$,

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(t\left(\Delta^X - c\right)/2\right)\right] = \mathrm{Tr_s}^{[\gamma]}\left[\exp\left(-t\mathcal{L}_b^X\right)\right].$$

Remark

The proof uses the fact that $\mathrm{Tr}^{[\gamma]}$ is a trace on the algebra of G-invariants smooth kernels on X with Gaussian decay.

The limit as $b \to +\infty$

- After rescaling of Y^{TX}, Y^N, as $b \to +\infty$, $\mathcal{L}_b \simeq \frac{b^4}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \frac{1}{2}\left|Y\right|^2 - \underbrace{\nabla_{Y^{TX}}}_{\text{geodesic flow}}$.
- As $b \to +\infty$, the orbital integral localizes near $X(\gamma)$ exactly like in Lefschetz formulas.
- $\gamma = e^a k^{-1}, a \in \mathfrak{p}, k \in K, \mathrm{Ad}(k)\, a = a$.
- $Z(\gamma)$ centralizer of γ, $\mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$ Lie algebra of $Z(\gamma)$.

Semisimple orbital integrals

Theorem (B. 2011)

There is an explicit function $J_\gamma\left(Y_0^{\mathfrak{k}}\right), Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$, such that

$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(t\left(\Delta^Z - c\right)/2\right)\right] = \frac{\exp\left(-|a|^2/2t\right)}{(2\pi t)^{p/2}} \int_{\mathfrak{k}(\gamma)} J_\gamma\left(Y_0^{\mathfrak{k}}\right) \mathrm{Tr}^E\left[\rho^E\left(k^{-1}\right)\exp\left(-i\rho^E\left(Y_0^{\mathfrak{k}}\right)\right)\right] \exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{(2\pi t)^{q/2}}.$$

The function $J_\gamma(Y_0), Y_0^{\mathfrak{k}} \in \mathfrak{k}(\gamma)$

Definition

$$J_\gamma\left(Y_0^{\mathfrak{k}}\right) = \frac{1}{\left|\det\left(1 - \mathrm{Ad}(\gamma)\right)|_{\mathfrak{z}_0^\perp}\right|^{1/2}} \frac{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)|_{\mathfrak{p}(\gamma)}\right)}{\widehat{A}\left(i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)_{\mathfrak{k}(\gamma)}\right)}$$

$$\left[\frac{1}{\det\left(1 - \mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{z}_0^\perp(\gamma)}} \right.$$

$$\left.\frac{\det\left(1 - \exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1 - \exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

The formula of Atiyah-Bott

• Compare with fixed point formulas by Atiyah-Bott

$$L(g) = \int_{X_g} \widehat{A}_g(TX) \operatorname{ch}_g(E).$$

• We ultimately compute the trace of any heat kernel, and not 'only' the index of a Dirac operator, and this by a 'local' formula.

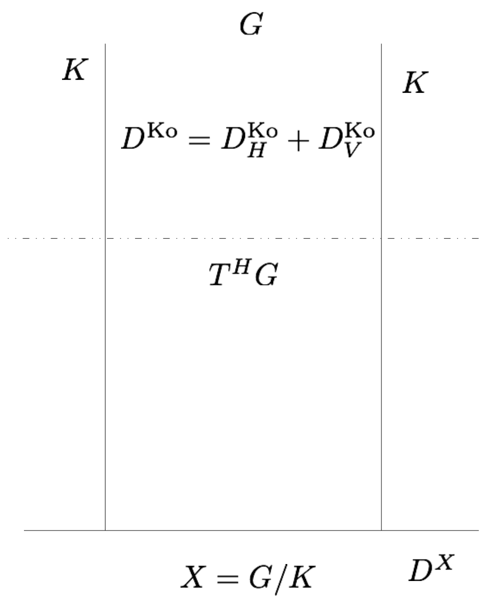
D^{Ko} and D^X

- Recall that $D^{\mathrm{Ko},2} = C^{\mathfrak{g}} + c$.
- D^{Ko} splits as the sum of two commuting pieces

$$D_H^{\mathrm{Ko}} = \sum_1^m c(e_i) e_i,$$

$$D_V^{\mathrm{Ko}} = -\sum_{m+1}^{m+n} c(e_i)(e_i + c(\mathrm{ad}(e_i)|_{\mathfrak{p}})) + \frac{1}{2} c(\kappa^{\mathfrak{k}}).$$

The fibration $G \to X$



The descent of D^{Ko} to D^X

- For eta invariant of D^X, Casimir not enough.
- Assume K to be simply connected.
- Then D^{Ko} descends to $D^X + \frac{1}{2} c\left(\kappa^{\mathfrak{k}}\right)$ acting on $C^{\infty}\left(X, S^{TX} \otimes \Lambda^{\cdot}(N^*)\right)$.
- Before, D^{Ko} was acting on $C^{\infty}(X, S^{\cdot}(T^*X \oplus N^*) \otimes \Lambda^{\cdot}(T^*X \oplus N^*))$.
- We have to combine the action of $\widehat{c}(\mathfrak{g}^*)$ on $\Lambda^{\cdot}(\mathfrak{g}^*)$ and on $S^{\mathfrak{p}}$.

An action of SO (2)

- $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$.
- Introduce another copy $\overline{\mathfrak{p}}$ of $\mathfrak{p}$, so that $\mathfrak{g} \oplus \overline{\mathfrak{p}} = \mathfrak{p} \oplus \overline{\mathfrak{p}} \oplus \mathfrak{k}$.
- SO (2) acts by rotations on $\mathfrak{p} \oplus \overline{\mathfrak{p}}$.
- If $e \in \mathfrak{p}$, $R_\vartheta \widehat{c}(e) R_\vartheta^{-1} = \cos(\vartheta) \widehat{c}(e) + \sin(\vartheta) \widehat{c}(\overline{e})$.
- $\widehat{D}_\vartheta^{\mathrm{Ko}} = R_\vartheta \widehat{D}^{\mathrm{Ko}} R_\vartheta^{-1}$.
- $\widehat{D}_\vartheta^{\mathrm{Ko}} = \sin(\vartheta) \widehat{D}^X + \ldots$
- Ultimately, we can recover results of Moscovici-Stanton on geometric evaluation of eta invariants on locally symmetric spaces.

The results of Shu Shen on analytic torsion

Using the above geometric formulas for orbital integrals, Shu Shen was able to complete the results of Moscovici-Stanton on the Fried conjecture for analytic torsion on locally symmetric spaces.

Geodesic flow and Fourier transform

- $Z = \sum Y^i \frac{\partial}{\partial x^i}$.
- $\sigma(Z) = \sqrt{-1}\langle Y, \xi \rangle =$ Fourier.

Exterior algebra and symmetric algebra

- Exterior algebra $\Lambda^{\cdot}(T^*X)$ in de Rham $\left(\Omega^{\cdot}(X), d^X\right)$.
- Symmetric (polynomial) algebra $S^{\cdot}(T^*X)$ is less popular.
- Introducing $S^{\cdot}(T^*X)$ restores supersymmetry.
- If g^{TX} Riemannian metric, $\overline{S}^{\cdot}(T^*X)$ is L_2 space for fibrewise Gaussian measure.
- If $a_i = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial Y^i} + Y^i\right), a_i^* = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial Y^i} + Y^i\right)$ annihilation, creation operators, geodesic flow

$$Z = Y^i \frac{\partial}{\partial x^i} = \frac{1}{\sqrt{2}}\left(a_i + a_i^*\right)\frac{\partial}{\partial x^i}$$

- Z Bosonic Dirac operator.

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