L^2-ACYCLIC MANIFOLDS

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L^2-ACYCLIC

X finite complex (or compact manifold)

$$\begin{array}{c} \overline{X} \\ \big\downarrow \\ X \end{array} \quad \text{regular } G\text{-cover (e.g. Universal cover)}$$

Hilbert space $l^2G = \{f : G \to \mathbb{C} \mid \sum |f(g)|^2 < \infty\}$

$$\cdots \to C_{p+1}(\overline{X}) \otimes_{\mathbb{Z}G} l^2G \xrightarrow{\partial_{p+1}} C_p(\overline{X}) \otimes_{\mathbb{Z}G} l^2G \xrightarrow{\partial_p} C_{p-1}(\overline{X}) \otimes_{\mathbb{Z}G} l^2G \to \dots$$

$X \to BG$ is L^2-*acyclic* if $\ker \partial_p = \overline{im\ \partial_{p+1}}$ for all p

Exercise: S^1 is L^2-acyclic.
Remark: $X \to BG$ L^2-acyclic $\iff b_*^{(2)}(\overline{X}; G) = 0 \iff L^2$-Laplacian on $\overline{X}$ is injective.

Special case $G = \mathbb{Z}$, $BG = S^1$

Theorem (J. Cohen)

TFAE

- $X \to S^1$ *is* L^2*-acyclic*
- $H_*\overline{X}$ *is torsion over* $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$
- $0 = H_*(X; \mathbb{Q}(t)) := H_*(C(\overline{X}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Q}(t))$ *"twisted coefficients"*

Idea of proof: In general, $C(\overline{X}) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Q}(t)$ is sum of an acyclic complex and a complex with 0 differential.

Special case $G = \mathbb{Z}$, $BG = S^1$, KM Surgery!

Let $X \to S^1$ be a k-manifold (with L^2-acyclic boundary).
Question: Is $X \to S^1$ bordant (rel ∂) to an L^2-acyclic manifold?

Theorem (CDW)

- *k odd. Answer is yes.*
- *k even. $X \to S^1$ is bordant (rel ∂) to a "highly connected" manifold, i.e. $H_{<k/2}(\overline{X})$ is a f.g $\mathbb{Z}$-module.*
  ($\Longrightarrow H_{<k/2}(X; \mathbb{Q}(t)) = 0$.)

Framing issues are dealt with using:

Embedding Lemma

Let M be a connected k-manifold, $1 < p < k-2$, and $t \in \pi_1 M$. Suppose $\alpha \in \pi_p M$ is represented by an embedded sphere. Then $(t-1)\alpha \in \pi_p M$ is represented by an embedding $S^p \times D^{k-p} \hookrightarrow M$.

Special case $G = \mathbb{Z}$, $BG = S^1$ symmetric signature

Let $k = 2j$, let $X \to S^1$ be a compact k-manifold with L^2-acyclic boundary.

Definition

Symmetric signature $\sigma(X \to S^1) \in L_k(\mathbb{Q}(t))$ is the Witt class of the intersection form

$$I_X : H_j(X; \mathbb{Q}(t)) \times H_j(X; \mathbb{Q}(t)) \to \mathbb{Q}(t)$$

Theorem (CDW)

For even $k > 4$, $X \to S^1$ is bordant to an L^2-acyclic manifold iff $\sigma(X \to S^1) = 0$.

Special case $G = \mathbb{Z}$, $BG = S^1$, Repackaging

Definition

$\Omega_k^2(BG)$ is the bordism group of closed L^2-acyclic k-manifolds

Theorem (CDW)

$$\sigma : \Omega_k^{2 \to SO}(B\mathbb{Z}) \to L_k(\mathbb{Q}(t))$$

is an isomorphism for $k > 4$ and onto for $k = 4$.

Theorem (CDW)

LES: $\cdots \to \Omega_k^2(B\mathbb{Z}) \to \Omega_k^{SO}(B\mathbb{Z}) \to L_k(\mathbb{Q}(t)) \to \cdots \to L_4(\mathbb{Q}(t))$

$$L_k(\mathbb{Q}(t)) = \begin{cases} 0 & k \text{ odd} \\ \mathbb{Z}^\infty \oplus (\mathbb{Z}/4)^\infty \oplus (\mathbb{Z}/2)^\infty & k \text{ even} \end{cases}$$

For every odd dimension, there an (infinitely generated) group of L^2-acyclic manifolds (with secondary invariants).

L^2-ACYCLIC MANIFOLD GROUPS

Question (Weinberger): What are the fundamental groups of L^2-acyclic manifolds?

• Method 1 (Surgery) : (CDW) If G is polycyclic-by-finite, there for any $n > 4$, there is an (null bordant$/G$) L^2-acyclic manifold with fundamental group G.
• Method 2 (Handlebody and analysis): (D-Schick) If a finitely presented G satisfies

- $b_2^{(2)}G = b_1^{(2)}G = b_0^{(2)}G = 0$
- "Quantization condition" $\exists \varepsilon > 0$ so that if M is a finitely presented $\mathbb{Z}G$-module so that $\dim \mathcal{N}G \otimes M < \varepsilon$, then $\dim \mathcal{N}G \otimes M = 0$

The Quantization condition holds, for example, if G is a torsionfree group satisfying the Atiyah Conjecture, e.g the free group times $\mathbb{Z}^2$ (take $\varepsilon = 1$).

L^2-acyclicity and algebra

Question: For which groups G is there a homological criterion for acyclicity?

Answer: If $\mathbb{Z}G$ has a semisimple (Ore) ring of quotients.

Definition

Let $\Delta \subset R$ be the nonzero divisors. R has a *classical ring of quotients* if there is a ring hom $\phi : R \to K$ so that

- $\phi(\Delta) \subset K^\times$
- $K = \phi(\Delta)^{-1}\phi(R)$.

Write $K = \Delta^{-1}R$.

Lemma

Suppose $\Delta^{-1}\mathbb{Z}G$ exists and is semisimple. $X \to BG$ is L^2-acyclic iff $H_*(X; \Delta^{-1}\mathbb{Z}G) = 0$.

RING OF QUOTIENTS

Question: What groups G have a (semisimple) ring of quotients?

Answer: No for G the free group, yes for EAB-groups (elementary amenable with a bound on the order of finite subgroups)

MAIN THEOREM

Theorem (CDW)

For G polycyclic-by-finite,

$$\sigma : \Omega_k^{2\to SO}(BG) \to L_k(\Delta^{-1}\mathbb{Z}G)$$

is an isomorphism for $k > 4$ and is surjective for $k = 4$.

Algebraic L-theory

- L is the letter after K
- L_n : Rings with involution $\to$ Abelian groups
- 4-periodic: $L_n(R,-) = L_{n+4}(R,-)$
- If $1/2 \in R$, L_0R (resp. L_2R) is the Witt group of Hermitian (resp. Skew-Hermitian) forms.
- $L_0\mathbb{R} = L_0(\mathbb{C},-) \xrightarrow{\cong} \mathbb{Z}$ signature
- $L_0\mathbb{C} = \mathbb{Z}/2$ rank
- $L_0(R,-) = L_2(R,-)$ if $\exists \alpha \in R^\times$ s.t. $\overline{\alpha} = -\alpha$.
- intersection forms $\in L$-group

L-GROUPS

Question: What is the computation of $L_k(\Delta^{-1}\mathbb{Z}G)$?
Question: What is the torsion?
Question: What should we conjecture about $L_k(\Delta^{-1}\mathbb{Z}G) \otimes \mathbb{Q}$ for G torsionfree?

- $L_0(\mathbb{Q}(t)) \cong L_2(\mathbb{Q}(t)) \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty$
- $L_{odd}(\Delta^{-1}\mathbb{Z}G) = 0$ for G torsionfree.
- $L_{odd}(\Delta^{-1}\mathbb{Z}G)$ detected by semicharacteristics.

Related to Pfister theory, Hilbert's 17-th problem, Milnor conjecture in algebraic K-theory, etc.

L^2-acyclicity and low-dimensional topology

There is a homomorphism from knot concordance group

$$\begin{aligned} \mathcal{C} &\to \Omega_3^2(S^1) \\ K &\mapsto M_K \text{ 0-surgery on } K \end{aligned}$$

Question: Is $\sigma : \Omega_4^{2\to SO}(BG) \to L_4(\Delta^{-1}\mathbb{Z}G)$ an isomorphism?

Theorem (Jae Choon Cha)

There are algebraically slice knots K so that M_K (0-surgery on K) is nontrivial in $\Omega_3^{(2)}(B\mathbb{Z})$?

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Conclusion

There are a lot of questions:

- What are the fundamental groups of L^2-acyclic manifolds?
- For which G does $\mathbb{Z}G$ have a semisimple localization?
- What is the conjectural picture for $L_*(\Delta^{-1}\mathbb{Z}G)$? For the torsion?
- How to compute (or study) $\Omega_3^{(2)}(BG)$?
- Extension questions
- Connections with bordism of diffeomorphisms (a la Kreck)