PSC metrics via end-periodic manifolds

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References

Joint work with my Masters degree student, Michael Hallam (University of Adelaide)

[HM17]

Michael Hallam and V. M. ,
PSC metrics via end-periodic manifolds,
31 pages, [arXiv:math/1706.09354].

Abstract

Recall from basic differential geometry:
(X, g) Riemannian manifold $\Rightarrow$ Riemannian curvature tensor $\overset{\text{tr}}{\Rightarrow}$ scalar curvature κ_g.

Question 1: Which manifolds have a metric g with $\kappa_g > 0$?
We say that g is a metric of positive scalar curvature (PSC).

Question 2: If a manifold has a metric g with $\kappa_g > 0$, what is the topology of the space of all such metrics (mod diffeos)?

Abstract

Not all manifolds admit metrics with PSC, as can be shown by many techniques:

Dirac operators (Lichnerowicz; Gromov-Lawson-Rosenberg) and minimal surfaces (Schoen-Yau) in all dimensions;

gauge theory (Seiberg-Witten) special to dimensions 3 and 4.

I will a describe a technique to address these questions for even dimensional manifolds, based on the analysis of the Dirac operator on end-periodic, non-compact manifolds.

Abstract

We obtain two types of results on PSC metrics for compact spin manifolds that are even dimensional.

The **first** type of result are **obstructions** to the existence of PSC metrics on such manifolds, expressed in terms of **end-periodic eta invariants** that were defined by Mrowka-Ruberman-Saveliev (MRS), and are the **even dimensional analogs** of the results by Higson-Roe.
Also Keswani, Weinberger and myself, Piazza, Schick, Benameur, Deeley, Goffeng,....

Abstract

The **second** type of result studies the **size of the group of components** of the space of PSC metrics for compact spin manifolds that are even dimensional, whenever this space is non-empty.These are the **even dimensional analogs** of the results by Botvinnik-Gilkey and refine certain results in MRS. Hitchin, Piazza-Schick, ...

End-periodic analogs of K-homology, structure groups and spin bordism theory are defined and are utilised to prove many of our results.

End-periodic manifolds

Let Z be a compact spin manifold with boundary Y and that Y is an oriented, connected submanifold of a compact spin X that is Poincaré dual to $\gamma \in H^1(X, \mathbb{Z})$. Let W be the fundamental segment obtained by cutting X open along Y,

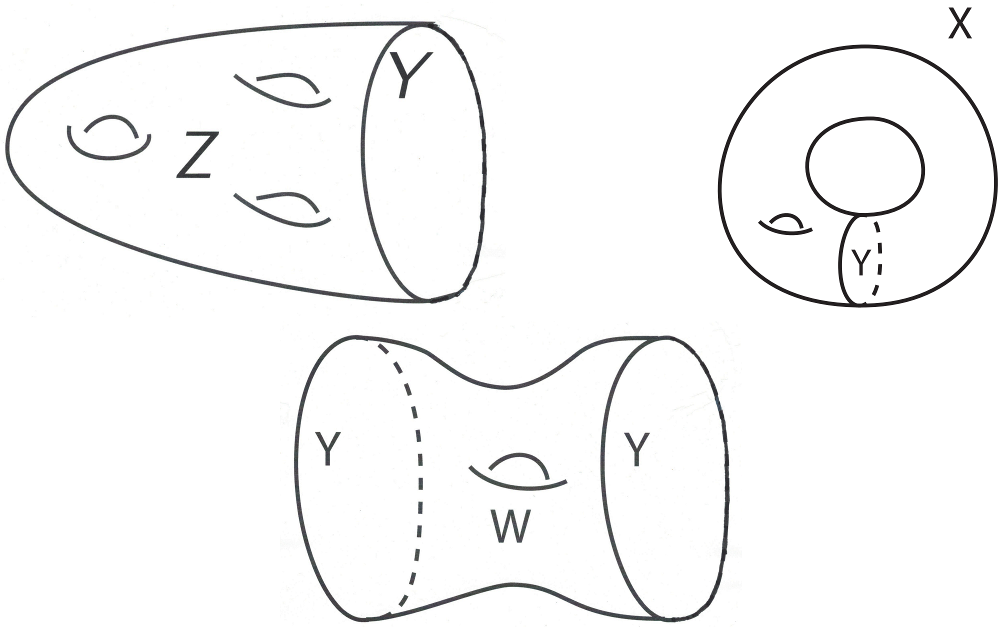


Figure: Pieces of an end-periodic manifold

End-periodic manifolds

If W_k are isometric copies of W, then we can attach $X_1 = \cup_{k \geq 0} W_k$ to the boundary component Y of Z, forming the end-periodic manifold Z_∞.

$$Z_\infty = Z \cup_Y W_0 \cup_Y W_1 \cup_Y W_2 \cdots$$

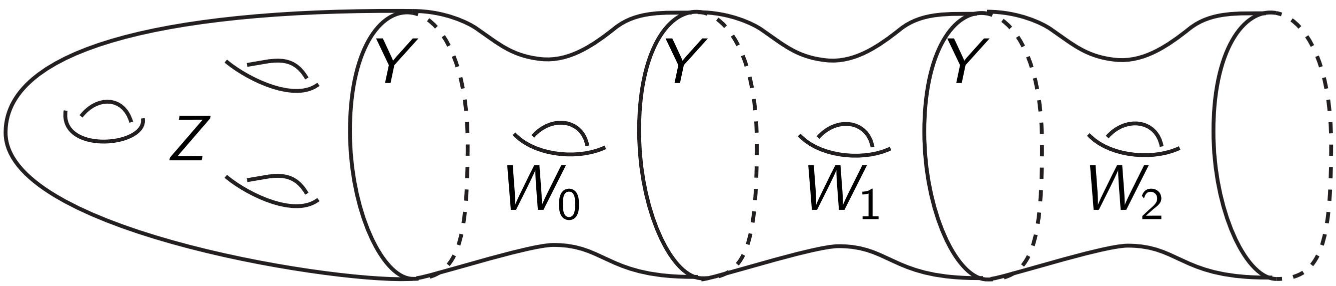


Figure: End-periodic manifold

End-periodic manifolds

Often in the talk, we also deal with manifolds with more than one periodic end. End-periodic elliptic operators on end-periodic manifolds were studied by Taubes, who first established conditions under which the L^2-closure of such operators is Fredholm, and also calculated the index of the end-periodic analog of the anti-self-dual operator occurring in Yang-Mills theory.

Index theorem for end-periodic manifolds

A key step is to introduce the **weighted Sobolev spaces** on Z_∞ as follows. First recall that the Sobolev space $L^2_k(Z_\infty, S)$ for an integer $k \geq 0$, is defined as the completion of $C_0^\infty(Z_\infty, S)$ in the norm

$$\|u\|^2_{L^2_k(Z_\infty,S)} = \sum_{j\leq k} \int_{Z_\infty} |\nabla^j u|^2$$

for a fixed choice of end-periodic metric and compatible end-periodic Clifford connection on Z_∞.

Index theorem for end-periodic manifolds

Now, restrict the upstairs covering map $F : \tilde{X} \to \mathbb{R}$ to the half-cover $X_1 = \bigcup_{k \geq 0} W_k$, and choose an extension of this map to Z_∞, which we continue to denote F.For a weight $\delta \in \mathbb{R}$ and an integer $k \geq 0$, then $u \in L^2_{k,\delta}(Z_\infty, S)$ if $e^{\delta F} u \in L^2_k(Z_\infty, S)$. Define the weighted Sobolev $L^2_{k,\delta}$-norm by

$$\|u\|_{L^2_{k,\delta}(Z_\infty, S)} = \| e^{\delta F} u\|_{L^2_k(Z_\infty, S)}.$$

Index theorem for end-periodic manifolds

It is easy to check that up to equivalence of norms, this is independent of the choice of extension of F to Z_∞, since the region over which we are choosing an extension is compact. The spaces $L^2_{k,\delta}(Z_\infty, S)$ are all complete in this norm, and the operator $D^+(Z_\infty)$ extends to a bounded operator

$$D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-) \tag{0.1}$$

for every k and δ.

Index theorem for end-periodic manifolds

The following theorem of Taubes characterises Fredholmness of the operator $D^+(Z_\infty)$ in terms of invertibility of the family $D^+_z(X) = D^+(X) - \ln(z)\, c(dF)$.

Lemma (Taubes)

The operator $D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-)$ is Fredholm if and only if the operators $D^+_z(X)$ are invertible for all z on the circle $|z| = e^\delta$.

Index theorem for end-periodic manifolds

Corollary

A necessary condition for the operator $D^+(Z_\infty)$ to be Fredholm is that $\operatorname{index} D^+(X) = 0$.

It also follows that the operator $D^+(Z_\infty)$ acting on the Sobolev spaces of weight δ is Fredholm for all but a discrete (and finite on bounded intervals) set of $\delta \in \mathbb{R}$.

Index theorem for end-periodic manifolds

Theorem (MRS Index Theorem A)

If the L^2–closure of the operator $D^+(Z_\infty)$ is Fredholm, and choose a form ω on X such that $d\omega = \mathbf{I}(D^+(X))$, then

$$ind_{L^2}(D^+(Z_\infty)) = \int_Z \mathbf{I}(D^+(Z)) - \int_Y \omega + \int_X \gamma \wedge \omega - \frac{1}{2}\eta^{ep}(D^+(X)).$$

MRS Theorem C relaxes the Fredholm assumption, when their theorem is harder to state. Their theorem reduces to the APS index theorem when the end is a cylinder.

End-periodic eta invariant

$$\eta^{ep}(D^+(X)) = \frac{1}{\pi i}\int_0^\infty \oint_{|z|=1} \operatorname{Tr}\left(c(\gamma)\cdot D_z^+ \exp(-t(D_z^+)^* D_z^+)\right)\,\frac{dz}{z}\,dt,$$
$$= 2\int_0^\infty \tau(c(\gamma)D^+ e^{-tD^-D^+})\,dt$$

where τ is the von Neumann trace (cf. [Atiyah76]) on $\tilde{X}$.

The definition extends to flat Hermitian bundles, or equivalently unitary representations of the fundamental group.

End-periodic eta invariant - asymmetry of the spectral set interpretation

It turns out that the family $\mathbb{C}^* \ni z \mapsto D_z^+(X)$ is meromorphic, invertible when $|z| = 1$ (by Fredholmness - Taubes).

The **poles** of this family is called the **spectral set** of $D^+(X)$.

Then $\eta^{\mathrm{ep}}(D^+(X))$ can be interpreted as the **asymmetry** of the **spectral set** wrt the circle $|z| = 1$. More precisely, it is a regularization of the number of spectral points with $|z| > 1$ minus the number of spectral points with $|z| < 1$.

End-periodic eta invariant

η^{ep} changes sign when either the orientation of X changes or γ goes to $-\gamma$.

η^{ep} can be twisted by flat Hermitian bundles/unitary reps of the fundamental group. Also when $X = S^1 \times Y$, then $\eta^{\mathrm{ep}}(D^+(X) \otimes \alpha) = \eta(D(Y) \otimes \alpha)$ for a unitary rep α of $\pi_1(Y)$.

End-periodic rho invariant

Define the **end-periodic rho invariant**, ρ^{ep} as follows:

$$\rho^{\mathrm{ep}}(D^+(X), \alpha_1, \alpha_2) = \eta^{\mathrm{ep}}(D^+(X) \otimes \alpha_1) - \eta^{\mathrm{ep}}(D^+(X) \otimes \alpha_2)$$

Then follows from MRS that

$$\rho^{\mathrm{ep}}(D^+(X), \alpha_1, \alpha_2) \mod \mathbb{Z}$$

is metric independent. More generally,
ρ^{ep} *has the analogous properties of the usual rho invariant.*

End-periodic K-homology

Definition

An (odd) **end-periodic K-cycle**, or simply a K^{ep}*-cycle* for a discrete group π is a quadruple (X, S, γ, f), where X is a compact oriented **even**-dimensional Riemannian manifold, $S = S^+ \oplus S^-$ is a $\mathbb{Z}_2$-graded Dirac bundle over X, $\gamma \in H^1(X, \mathbb{Z})$ is a cohomology class whose restriction to each connected component of X is primitive, and a cts map $f : X \to B\pi$.

End-periodic K-homology

The $\mathbb{Z}_2$-graded structure of S includes a Clifford multiplication by tangent vectors to X which swaps the positive and negative sub-bundles. As in K-homology, the manifold X is allowed to be disconnected, with the connected components possibly having different even dimensions. NB the definition of a K^{ep}-cycle imposes topological restrictions on X, namely each connected component of X must have non-trivial first cohomology in order for the class γ to be primitive on each component.

End-periodic K-homology

Definition

Two K^{ep}-cycles (X, S, γ, f) and (X', S', γ', f') are **isomorphic** if there exists an orientation preserving diffeomorphism $\varphi : X \to X'$ which is covered by a $\mathbb{Z}_2$-graded isometric bundle isomorphism $\psi : S \to S'$ such that

$$\psi \circ c_X(v) = c_{X'}(\varphi_* v) \circ \psi$$

for all $v \in TX$. The diffeomorphism φ must additionally satisfy $\varphi^*(\gamma') = \gamma$, and $f' \circ \varphi = f$.

End-periodic K-homology

We now define what it means for a K^{ep}-cycle (X, S, γ, f) to be a boundary. First, let $Y \subset X$ be a codimension-1 submanifold that is Poincaré dual to γ. The orientation of Y is such that for all closed forms α of codimension 1 (over each component of X),

$$\int_Y \iota^*(\alpha) = \int_X \gamma \wedge \alpha,$$

where $\iota : Y \to X$ is the inclusion. In other words, the orientation of Y is such that the signs of the above two integrals always agree.

End-periodic K-homology

Now, cut X open along Y to obtain a compact manifold W with boundary $\partial W = Y \amalg -Y$, with our boundary orientation conventions. Glue infinitely many isometric copies W_k of W end to end along Y to obtain the complete oriented Riemannian manifold $X_1 = \bigcup_{k \geq 0} W_k$ with boundary $\partial X_1 = -Y$. Pull back the Dirac bundle S on X to get a $\mathbb{Z}_2$-graded Dirac bundle on X_1, also denoted S, and pull back f to get a map $f : X_1 \to B\pi$.

End-periodic K-homology

Definition

The K^{ep}-cycle (X, S, γ, g) is a **boundary** if there exists a compact oriented Riemannian manifold Z with boundary $\partial Z = +Y$, which can be attached to X_1 along Y to form a complete oriented Riemannian manifold $Z_\infty = Z \cup_Y X_1$, such that the bundle S extends to a $\mathbb{Z}_2$-graded Dirac bundle on Z_∞ and the map f extends to a continuous map $f : Z_\infty \to B\pi$.

The manifold Z_∞ is an *end-periodic* manifold, with end modelled on (X, γ).

End-periodic K-homology

The **negative** of a K^{ep}-cycle (X, S, γ, f) is simply $(X, S, -\gamma, f)$. This is so that the disjoint union of a K^{ep}-cycle with its negative is a boundary – it is clear that the $\mathbb{Z}$-cover $\tilde{X}$ of X corresponding to γ is an end-periodic manifold with end modelled on $(X \amalg X, \gamma \amalg -\gamma)$. The definitions of bordism and direct sum – disjoint union are exactly the same as in K-homology, with the class γ left unchanged.

End-periodic K-homology

In the case of bundle modification, the class $\hat{\gamma}$ on $\hat{X} = X \times_\rho S^{2k}$ is the pullback of γ by the projection $p : \hat{X} \to X$. There is also one more relation we define which relates the orientation on X to the one-form γ:

$$(X, S, -\gamma, f) \sim (-X, \Pi(S), \gamma, f)$$

where $-X$ is X with the reversed orientation and $\Pi(S)$ is S with its $\mathbb{Z}_2$-grading reversed. We call this relation *orientation – sign*, as it links the orientation on X to the sign of γ.

End-periodic K-homology

Definition

The **end-periodic K-homology group**, $K_1^{\mathrm{ep}}(B\pi)$, is the abelian group consisting of K^{ep}-cycles up to the equivalence relation generated by isomorphism of K^{ep}-cycles, bordism, direct sum – disjoint union, bundle modification, and orientation – sign. Addition is given by disjoint union of cycles

$$(X, S, \gamma, f) \amalg (X', S', \gamma', f') = (X \amalg X', S \amalg S', \gamma \amalg \gamma', f \amalg f').$$

As for K-homology, we could also define the group $K_0^{\mathrm{ep}}(B\pi)$ using odd-dimensional manifolds.

End-periodic K-homology - the isomorphism

We now ST there is a natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

First we describe the map $K_1(B\pi) \to K_1^{\mathrm{ep}}(B\pi)$. Let (M, S, f) be a K-cycle for $B\pi$. Define $X = S^1 \times M$ an even dimensional manifold with the product orientation and Riemannian metric, the Dirac bundle $S \oplus S \to X$ with Clifford multiplication as before, $\gamma = d\theta \in H^1(X, \mathbb{Z})$, and $f : X \to B\pi$ the extension of $f : M \to B\pi$. Map the equivalence class of (M, S, f) in $K_1(B\pi)$ to the equivalence class of $(S^1 \times M, S \oplus S, d\theta, f)$ in $K_1^{\mathrm{ep}}(B\pi)$.

End-periodic K-homology - the isomorphism

Now for the inverse map, $K_1^{\mathrm{ep}}(B\pi) \to K_1(B\pi)$.

Let (X, S, γ, f) be an end-periodic cycle. Choose a submanifold $Y \subset X$ Poincaré dual to γ, oriented as before. We map the cycle (X, S, γ, f) to (Y, S^+, f), where S^+ and f are restricted to Y.

End-periodic K-homology

If ω is an oriented volume form for Y then we let ∂_t be the unit normal to Y such that $\partial_t \wedge \omega$ is the orientation on X. The Clifford multiplication on S^+ is then defined to be

$$c_Y(v) = c_X(\partial_t)c_X(v)$$

for $v \in TY$. One easily verifies that this indeed defines a Clifford multiplication on S^+.

Theorem (HM)

The above maps between K-homologies define an isomorphism of groups $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

End-periodic K-homology

Proof.

We must check that the above maps on K-homologies are inverse to each other. If we begin with a cycle (M, S, f), this maps to $(S^1 \times M, S \oplus S, d\theta, f)$. Mapping this again, we get (M, S, f) back, so this direction is easy.

Now suppose we begin with a cycle (X, S, γ, f). This maps to (Y, S^+, f) which then maps to $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$. We will show this cycle is bordant to the original cycle (X, S, γ, f).

Consider the half cover X_1 of X obtained using $-\gamma$. Near the boundary, this is diffeomorphic to a product $(-\delta, 0] \times Y$. The half cover of $S^1 \times Y$ obtained from $d\theta$ is $\mathbb{R}_{\leq 0} \times Y$. $\square$

End-periodic K-homology

Proof.

The two half covers clearly glue together to produce and end-periodic manifold with two ends as in the figure.

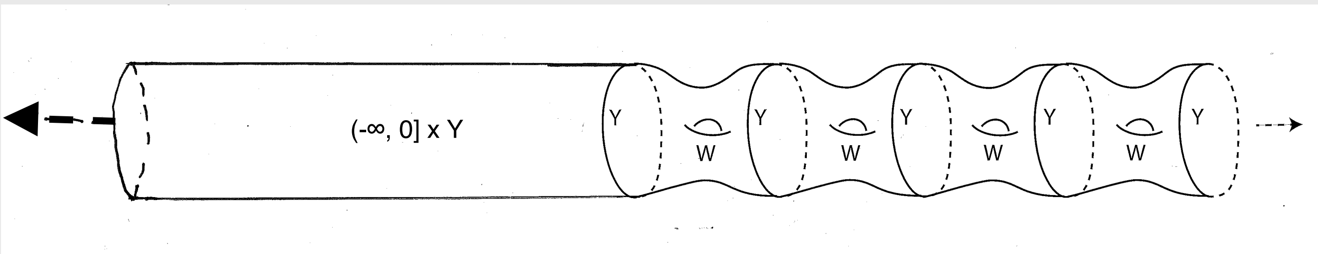


Figure: End-periodic manifold with two ends

The Dirac bundles and maps to $B\pi$ extend over this manifold, and hence the two cycles are bordant. □

End-periodic K-homology & end-periodic rho invariants

We use the end-periodic eta invariant of MRS to define homomorphisms from the end-periodic K-homology group $K_1^{\mathrm{ep}}(B\pi)$ to $\mathbb{R}/\mathbb{Z}$. Any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$ will determine such a homomorphism, and we see that this homomorphism agrees with that constructed in Higson-Roe under the natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$ constructed earlier.

End-periodic K-homology & $\mathbb{R}/\mathbb{Z}$-index theorem

Theorem (HM)

The mod $\mathbb{Z}$ reduction of the end-periodic rho invariant $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2)$ associated to $\sigma_1, \sigma_2 : \pi \to U(N)$ depends only on the equivalence class of (X, S, γ, f) in $K_1^{\mathrm{ep}}(B\pi)$. Hence there is a well-defined group homomorphism

$$\rho^{\mathrm{ep}} : K_1^{\mathrm{ep}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Furthermore, the following diagram commutes:

$$\begin{tikzcd} K_1^{\mathrm{ep}}(B\pi) \arrow[rr, "\cong"] \arrow[dr, "\rho^{\mathrm{ep}}"'] & & K_1(B\pi) \arrow[dl, "\rho"] \\ & \mathbb{R}/\mathbb{Z} & \end{tikzcd} \tag{0.2}$$

End-periodic K-homology & $\mathbb{R}/\mathbb{Z}$-index theorem

Proof.

Key is the invariance of ρ^{ep} under **bordism**. First suppose that (X, S, γ, f) is a boundary, with a Dirac operator $D^+(X)$ whose associated family $D_z^+(X)$ has discrete spectrum. Then the families associated to the twisted operators $D_1^+(X)$ and $D_2^+(X)$ have discrete spectrum, and we apply the MRS index theorem to each operator separately to get

$$ind_{\mathrm{MRS}} D_i^+(Z_\infty) \;=\; \int_Z \mathbf{I}(D_i^+(Z)) - \int_Y \omega_i + \int_X df \wedge \omega_i - \frac{h_i + \eta^{\mathrm{ep}}(D_i^+(X))}{2}$$

for $i = 1, 2$. Now, since we are twisting by flat vector bundles, both the index form and the transgression classes for the twisted operators are constant multiplies of the index form and transgression class of the original operator. $\square$

End-periodic K-homology & $\mathbb{R}/\mathbb{Z}$-index theorem

Proof.

Hence when we subtract the two equations, the terms involving these vanish and we are left with

$$\rho^{\mathrm{ep}} = ind_{\mathsf{MRS}} D_2^+(Z_\infty) - ind_{\mathsf{MRS}} D_1^+(Z_\infty)$$

which is an integer. Now the end-periodic rho invariant behaves additively under disjoint unions of cycles and changes sign when the negative of a cycle is taken. This proves bordism invariance mod $\mathbb{Z}$ for cycles having Dirac operators whose families have discrete spectra....

$\square$

End-periodic K-homology & $\mathbb{R}/\mathbb{Z}$-index theorem

Proof.

Now the K^{ep}-cycle (X, S, γ, f) is bordant to $(S^1 \times Y, S, d\theta, f)$, where Y is Poincaré dual to γ. By Section 6.3 of MRS, the end-periodic rho invariant of $(S^1 \times Y, S, d\theta, f)$ is equal to the rho invariant of the K-cycle (Y, S, f). Hence

$$\rho^{\mathrm{ep}}(X, S, \gamma, f; \sigma_1, \sigma_2) = \rho(Y, S, f; \sigma_1, \sigma_2) \mod \mathbb{Z}.$$

The isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$ then immediately implies the theorem. $\square$

End-periodic analogs of other functors

We also define End-periodic analogs of other functors and prove results about these in our paper.

These include end-periodic spin bordism groups $\Omega_*^{\mathrm{ep},\mathit{spin}}(B\pi)$, end-periodic psc spin bordism groups $\Omega_*^{\mathrm{ep},\mathit{spin},+}(B\pi)$, end periodic structure groups $S_*^{\mathrm{ep}}(B\pi)$ etc.

PSC obstructions & end-periodic rho invariant

Using the above isomorphisms of K-homologies (and cobordism theories), we can immediately transfer results on positive scalar curvature from the odd-dimensional case to the even-dimensional case in which a primitive 1-form is given.

Odd-dimensional results in the literature

First we will state the odd-dimensional results that we will be generalising to the even-dimensional case using our isomorphisms. The first ones are obstructions to positive scalar curvature.

PSC obstructions & rho invariant

Theorem (Weinberger, Higson-Roe Theorem 6.9)

Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(M, S, f, \sigma_1, \sigma_2)$ is a rational number.

PSC obstructions & rho invariant

Theorem (Higson-Roe Remark 6.10)

Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If π is torsion-free, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(M, S, f, \sigma_1, \sigma_2)$ is an integer.

This uses, the result that the maximal Baum-Connes map for π is injective whenever for instance π is a torsion-free linear discrete group, [Guentner-Higson-Weinberger].

PSC obstructions & rho invariant

Theorem (Higson-Roe Theorem 1.1, Keswani)

Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If the maximal Baum-Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(M, S, f, \sigma_1, \sigma_2)$ is zero.

NB. The maximal Baum-Connes conjecture holds for π whenever π is K-amenable.

Our even dimensional results

The following is our even dimensional analog of Theorem HRW.

Theorem

Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold M whose scalar curvature in the induced metric is positive. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2)$ is a rational number.

PSC obstructions & end-periodic rho invariant

Proof.

The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle for $B\pi$, $(M, S\big|, f\big|)$ where M is a Poincaré dual submanifold for γ having positive scalar curvature. Clearly M has an induced spin structure. By Theorem HRW, $\rho(M, S\big|, f\big|, \sigma_1, \sigma_2) \in \mathrm{Q}$. By our $RE/\mathbb{Z}$ index Theorem it follows that $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2) \in \mathrm{Q}$ as claimed. $\square$

PSC obstructions & end-periodic rho invariant

Theorem

Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold M whose scalar curvature in the induced metric is positive. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2)$ is an integer.

PSC obstructions & end-periodic rho invariant

Proof.

The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle for $B\pi$, $(M, S\big|, f\big|)$ where M is a Poincaré dual submanifold for γ having positive scalar curvature. Clearly M has an induced spin structure. By Theorem 11, $\rho(M, S\big|, f\big|, \sigma_1, \sigma_2) \in \mathbb{Z}$. By our $RE/\mathbb{Z}$ index Theorem it follows that $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2) \in \mathbb{Z}$ is an integer. $\square$

PSC obstructions & end-periodic rho invariant

Theorem

Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold M whose scalar curvature in the induced metric is positive. If the maximal Baum-Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2)$ is zero.

PSC obstructions & end-periodic rho invariant

Proof.

The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle for $B\pi$, $(M, S\big|, f\big|)$ where M is a Poincaré dual submanifold for γ having positive scalar curvature. Clearly M has an induced spin structure. By Theorem 12, $\rho(M, S\big|, f\big|, \sigma_1, \sigma_2) = 0$. By Theorem 8.5, [MRS] it follows that $\rho^{\mathrm{ep}}(S^1 \times M, S, \gamma, f, \sigma_1, \sigma_2) = 0$. By the bordism invariance of ρ^{ep}, we deduce that $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2) = 0$.
This can also be proved via results on the end periodic structure group. $\square$

Size of the space of components of positive scalar curvature metrics

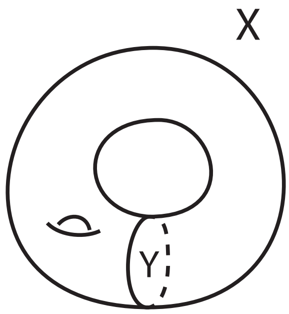
Hitchin proved the first results on the size of the space of components of the space of Riemannian metrics of positive scalar curvature metrics on a compact spin manifold, when non-empty. Botvinnik-Gilkey, Piazza-Schick and others

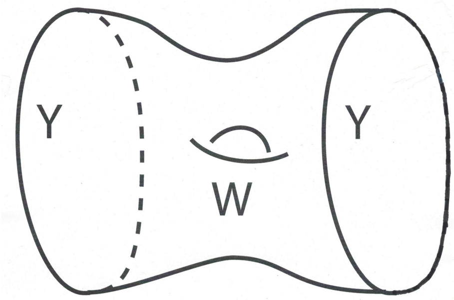
π_0(PSC metrics) & end-periodic rho invariant

Theorem (HM)

Let Y be a compact spin manifold of dimension $(4n-1), n>1$, admitting a metric of positive scalar curvature such that $\pi_1(Y)$ is finite and nontrivial, with $Y \hookrightarrow X$ is a smooth submanifold, X is a **psc-adaptable** *compact spin manifold of dimension $4n$ wrt Y, with a map $f : X \to B\pi_1(Y)$, then $\pi_0(\mathfrak{M}^+(X))$ is infinite, where $\mathfrak{M}^+(X)$ denotes the quotient of the space of positive scalar curvature metrics by the diffeomorphism group.*

PSC obstructions & psc-adaptable

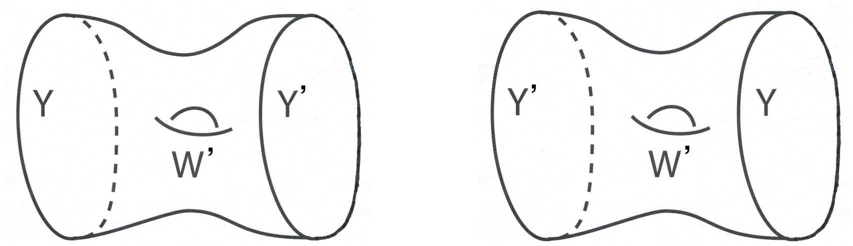




Cut X along Y. Since Y has psc, a theorem of Miyazaki and Rosenberg enables one to *push* the psc metric on Y across the bordism (pictured on the right) to get a possibly different psc metric on Y. The problem is that one doesn't know whether the new psc metric on Y is isotopic to the original (this would be true if the general *concordance = isotopy conjecture* were true). Hence the concept **psc-adaptable** which hypothesizes this.
It is the case when the bordism is *symmetric* for instance.

PSC obstructions & psc-adaptable

That is starting with a bordism W' from Y to Y', we get a bordism from Y to itself by thinking of W' as a bordism from Y' to Y and gluing to the original bordism, see Figure below.



Then one can use the Miyazaki-Rosenberg construction starting with the psc metric Y to get another another psc metric on Y' halfway through, and then reverse the M-R construction from the psc metric on the halfway Y' to get a psc metric on Y on the other end. In this case, we end up with the original psc metric on Y.

π_0(PSC metrics) & end-periodic rho invariant

Mrowka, Ruberman and Saveliev (Theorem 9.20 also note a class of psc-adaptable manifolds – those of the form $(S^1 \times Y)\#M$ where Y and M are manifolds of positive scalar curvature.

π_0(PSC metrics) & end-periodic rho invariant

Proof.

Start with the product metrics $d\theta^2 + g_j, j = 1, \ldots$ on $S^1 \times Y$, where g_j is an infinite family of PSC metrics on Y that lie in different components of $\mathfrak{M}^+(Y)$, as constructed in Botvinnik-Gilkey (B-G) Theorem 0.3, and having the property that $\rho(Y, D_Y, g_i, \alpha) \neq \rho(Y, D_Y, g_j, \alpha)$ if $i \neq j$, for the explicitly constructed $\alpha \in R_0(\pi_1(Y))$. The condition in B-G that $r_m(\pi_1(Y)) > 0$ is automatically satisfied since $m = 4n - 1$, see the remark following Theorem 0.1 in B-G.

Since X is assumed to be psc-adaptable, each psc metric g_j on Y determines a psc metric h_j on X. By MRS Theorem 8.5, $\rho^{\mathrm{ep}}(X, D, h_j, \alpha) = \rho(Y, D_Y, g_j, \alpha)$.

Therefore, $\rho^{\mathrm{ep}}(X, D, h_i, \alpha) \neq \rho^{\mathrm{ep}}(X, D, h_j, \alpha)$ if $i \neq j$.

Conclude that $\pi_0(\mathfrak{M}^+(X))$ is infinite. $\square$

Vanishing of end-periodic rho using the representation variety

Next, we give a proof of the vanishing of the end-periodic rho invariant of the twisted Dirac operator with coefficients in a flat Hermitian vector bundle on a compact even dimensional Riemannian spin manifold X of positive scalar curvature using the representation variety of $\pi_1(X)$ instead.

Let $\iota\colon Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a generator $\gamma \in H^1(X, \mathbb{Z})$. Given a representation $\alpha\colon \pi_1(Y) \to U(N)$, define a representation $\tilde{\alpha}\colon \pi_1(X) \to U(N)$ using the commutative diagram,

$$\begin{tikzcd} \pi_1(X) \arrow[rr, "\tilde{\alpha}"] & & U(N) \\ & \pi_1(Y) \arrow[ul, "\iota_*"] \arrow[ur, "\alpha"'] & \end{tikzcd} \tag{0.3}$$

Vanishing of end-periodic rho using the representation variety

Let $\mathfrak{R} = \mathrm{Hom}(\pi, U(N))$ denote the representation variety of $\pi = \pi_1(Y)$, and $\tilde{\mathfrak{R}}$ denote the representation variety of $\pi_1(X)$. We now construct a generalization of the Poincaré vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$. Let $E\pi \to B\pi$ be a principal π-bundle over the space $B\pi$ with contractible total space $E\pi$. Let $h\colon Y \to B\pi$ be a continuous map classifying the universal π-covering of Y. We construct a tautological rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$ as follows: consider the action of π on $E\pi \times \mathfrak{R} \times \mathrm{C}^N$ given by

$$\begin{aligned} E\pi \times \mathfrak{R} \times \mathrm{C}^N \times \pi &\longrightarrow E\pi \times \mathfrak{R} \times \mathrm{C}^N \\ ((q, \sigma, v), \tau) &\longrightarrow (q\tau, \sigma, \sigma(\tau^{-1})v). \end{aligned}$$

Vanishing of end-periodic rho using the representation variety

Define the universal rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$ to be the quotient $(E\pi \times \mathfrak{R} \times \mathrm{C}^N)/\pi$. Then $\mathcal{P}$ has the property that the restriction $\mathcal{P}\big|_{B\pi\times\sigma}$ is the flat Hermitian vector bundle over $B\pi$ defined by σ. Let I denote the closed unit interval $[0,1]$ and $\beta : I \to \mathfrak{R}$ be a smooth path in $\mathfrak{R}$ joining the unitary representation α to the trivial representation. Define $E = (f \times \beta)^*\mathcal{P} \to X \times I$ to be the Hermitian vector bundle over $X \times I$. By the Kunneth Theorem in cohomology, we have $\mathrm{ch}(F) = \sum_i x_i\xi_i$, where $\mathrm{ch}(F)$ is the Chern character of F, for some $x_i \in H^*(B\pi, \mathbb{R})$ and $\xi_i \in H^*(\mathfrak{R}, \mathbb{R})$, by the Kunneth theorem. It follows that if $y_i = f^*(x_i)$ and $\mu_i = \beta^*(\xi_i)$,then $\mathrm{ch}(E) = \sum_i y_i\mu_i$. Note that the pullback connection makes E into a Hermitian vector bundle over $Y \times I$.

Vanishing of end-periodic rho using the representation variety

Theorem (PSC and vanishing of end-periodic rho)

Let (X, g) be a compact spin manifold of even dimension, and let $\iota\colon Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$. Suppose that

1. *g is a Riemannian metric of positive scalar curvature;*
2. *the restriction $g\big|_Y$ is also a metric of positive scalar curvature.*

Let π denote the fundamental group of Y and $\alpha\colon \pi \to U(N)$ a unitary representation that can be connected by a smooth path $\beta : I \to \mathfrak{R}$ to the trivial representation in the representation space $\mathfrak{R}$, and the induced unitary representation $\tilde{\alpha}\colon \tilde{\pi} \to U(N)$, where $\tilde{\pi} = \pi_1(X)$. Then $\rho^{\mathrm{ep}}(X, S, \gamma, g; \tilde{\alpha}, 1) = 0$, where the flat hermitian bundle $E_{\tilde{\alpha}}$ is determined by $\tilde{\alpha}$.