Obstructions to positive scalar curvature via submanifolds

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Intro: Coarse index theory

Let \((X, g_X)\) be a complete spin manifold endowed with a free and proper \(\Lambda\)-action.

- **Coarse index** \(\text{Ind}_\Lambda(\mathcal{D}_X) \in KO_*(C^*_\Lambda(X))\).
- If \(X = \tilde{N}, \Lambda = \pi_1N\) with \(N\) closed spin, we recover **Rosenberg index** \(\text{Ind}_\Lambda(\mathcal{D}_{\tilde{N}}) = \alpha(N) \in KO_*(C^*_\Lambda(\tilde{N})) \cong KO_*(C^*_\Gamma\Lambda)\).
- \(\text{Ind}_\Lambda(\mathcal{D}_X) \neq 0 \Rightarrow X\) does not admit a \(\Lambda\)-invariant metric of uniform psc in the same quasi-isometry class.
- Assume that \(X\) is \(\Lambda\)-invariantly partitioned by \(\tilde{N} \subset X\). Then there is a map
  \[
  \partial_{MV} : KO_*(C^*_\Lambda(X)) \to KO_{* - 1}(C^*_\Gamma\Lambda)
  \]
  with \(\text{Ind}_\Lambda(\mathcal{D}_X) \mapsto \alpha(N)\). (**Roe’s part. mfd. index thm.**)
- If \(\text{Ind}_\Lambda(\mathcal{D}_X) \neq 0\), then \(X_- \cup \tilde{N} X_+\) does not admit a complete \(\Lambda\)-invariant metric of uniform psc **outside** \(X_\pm\) in the same quasi-isometry class.
Intro II: Consequences of partitioned mfd. index thm.

Let $\Lambda = \pi_1 N$ with $N$ closed spin.

- Consider $N \times \mathbb{R}$ endowed with some complete metric. Then:
  - $KO_*(C_\Lambda^* (\tilde{N} \times \mathbb{R})) \to KO_{*-1}(C_\Lambda^* \Lambda)$, $\text{Ind}^\Lambda(\mathcal{D}_{\tilde{N} \times \mathbb{R}}) \mapsto \alpha(N)$.

- Consider $N \times \mathbb{R}^2$.
  - Let $Z$ be the double of $N \times (\mathbb{R}^2 \setminus B(0))$, so $Z \cong N \times S^1 \times \mathbb{R}$.
  - If $N \times \mathbb{R}^2$ admits uniform psc, then $Z$ admits a complete metric of uniform psc outside $N \times S^1 \times \mathbb{R}_+$.
  - If $\alpha(N) \neq 0$, then $\alpha(N \times S^1) \neq 0$, so $N \times \mathbb{R}^2$ does not admit a metric of uniform positive scalar curvature.
Low codimensions

**Theorem (Hanke–Pape–Schick)**

Let $M$ be closed spin, $N \hookrightarrow M$ a codimension 2 submanifold with trivial normal bundle. Suppose that $\pi_1(N) \hookrightarrow \pi_1(M)$ and $\pi_2(N) \twoheadrightarrow \pi_2(M)$. Then, if $\alpha(N) \neq 0$, $M$ does not admit psc.

**Theorem (Z.)**

Let $N \hookrightarrow M$ a codimension 1 submanifold with trivial normal bundle and $\Lambda := \pi_1(N) \hookrightarrow \pi_1(M) =: \Gamma$, then there exists a homomorphism

$$\tau^{(a)}_{\Gamma,\Lambda} : \text{KO}_*(C^*_\Gamma \Gamma) \to \text{KO}_{*-1}(C^*_\Gamma \Lambda)$$

**mapping** $\alpha(M)$ **to** $\alpha(N)$. 
**Conjecture (Gromov–Lawson)**

An **aspherical** manifold does **not** admit a metric of **positive scalar curvature (psc)**.

Goal: Study a “partial version” of this conjecture.

**Definition**

Let $M$ be a closed spin manifold, $\pi_1 M =: \Gamma$, $c: M \to B \Gamma$. $M$ is called

- **(rationally) essential** if $c_*[M] \neq 0 \in H_*(B\Gamma) \otimes \mathbb{Q}$.
- **K-essential** if $c_*[M]_K \neq 0 \in K_*(B\Gamma)$.

- Chern character: Rationally K-essential $\iff$ rationally essential
- Aspherical manifolds are rationally essential.
- If $\Gamma$ satisfies the strong Novikov conjecture,

\[ \nu: K_*(B\Gamma) \otimes \mathbb{Q} \hookrightarrow K_*(C^*_\Gamma) \otimes \mathbb{Q} \]

then a rationally essential spin manifold $M$ with $\pi_1 M = \Gamma$ does not admit psc.
High codimensions

Theorem (Engel, Schick–Z.)

Under hypotheses \((\ast)\), there is a commutative diagram:

\[
\begin{array}{ccc}
H_*(M) & \overset{c_M}{\longrightarrow} & H_*(B\Gamma) \\
\downarrow^{\tau_{M,N}} & & \downarrow^{\tau_{\Gamma,\Lambda}} \\
H_{*-k}(N) & \overset{c_N}{\longrightarrow} & H_{*-k}(B\Lambda).
\end{array}
\]

(Enough to assume that the normal bundle is orientable and the Euler class of the normal bundle vanishes on \(\text{im}(\pi_k(N) \rightarrow H_k(N))\).)

Corollary

If \(N\) is (rationally) essential, then so is \(M\).
Theorem (Schick–Z.)

Assume $(\ast)$ and let $k = 2$. Let $E$ be a multiplicative generalized homology theory. Then there is a commutative diagram:

\[
\begin{array}{ccc}
E_\ast(M) & \xrightarrow{c_M} & E_\ast(B\Gamma) \\
\downarrow^{\tau_{M,N}} & & \downarrow^{\tau_{\Gamma,\Lambda}} \\
E_{\ast-2}(N) & \xrightarrow{c_N} & E_{\ast-2}(B\Lambda).
\end{array}
\]

Corollary

If $N$ is $K$- or $KO$-essential, then so is $M$. 

Codimension two and generalized homology
Proof strategy

• Let \( p : \tilde{X} \to X \) some covering and \( \theta \in H^k_{vc}(\tilde{X}) \).
  
  \[ \to \text{obtain map } \tau_\theta : H_*(X) \to H_{*-k}(\tilde{X}), \quad x \mapsto \theta \cap p^!(x). \]

\( \tilde{M} \to M \) covering with \( \pi_1 \tilde{M} = \Lambda \), set \( \theta := \text{pd}(N \hookrightarrow \tilde{M}) \in H^k_c(\tilde{M}) \).

\[
\begin{array}{ccc}
H_*(\tilde{M}) & \longrightarrow & H_*(B\Gamma) \\
\downarrow_{\tau_{M,N}} & & \downarrow_{\tau_\theta} \\
H_*(N) & \longrightarrow & H_{*-k}(M) \longrightarrow H_{*-k}(B\Lambda).
\end{array}
\]

\( \to \) Need to extend \( \theta \) to \( \vartheta \in H^k_{vc}(B\Lambda \to B\Gamma) \).

Construct \( B\Gamma \) out of \( M \) by attaching cells:

• Start with \( (k + 1) \)-cells attached to \( M \) along \( N \) to make \( \pi_k \) trivial.

• Inductively attach \( (l + 1) \)-cells to the resulting space to make \( \pi_l \) trivial for all \( l > k \) and get models for \( B\Gamma \) and \( B\Lambda \).

• Represent \( \theta \) by a map \( (\tilde{M}, \tilde{M} \setminus D\nu(N)) \to (K(\mathbb{Z}, k), *) \) and work out conditions that allow to extend it over \( B\Lambda \).
Let $M$ be a closed spin $n$-manifold.

The assembly map takes the KO-homology fundamental class to the Rosenberg index.

It fits into the Higson–Roe l.e.s., where the third term is the analytic structure group.

For each $g \in \mathcal{R}^+(M)$, there is $\rho(g) \in S^\Gamma_n(\tilde{M})$.

For $g_0, g_1 \in \mathcal{R}^+(M)$, there is $\alpha_{\text{diff}}(g_0, g_1) \in \text{KO}_{n+1}(C^*_r\Gamma)$ with $\partial \alpha_{\text{diff}}(g_0, g_1) = \rho(g_0) - \rho(g_1)$.

There is a version for non-(co)compact manifolds and secondary partitioned manifold index theorems (Piazza–Schick, Z.)
The secondary codimension one obstruction

Theorem (Z.)

Let \( M \) be a closed connected spin manifold and \( N \subset M \) a closed submanifold of codimension 1 with trivial normal bundle such that \( \Lambda := \pi_1(N) \leftarrow \pi_1(M) =: \Gamma \). Then there exists a diagram,

\[
\begin{array}{cccc}
S^\Gamma_* (\tilde{M}) & \longrightarrow & KO_* (M) & \longrightarrow & KO_* (C^*_\Gamma) \\
\downarrow_{\tau_s} & & \downarrow_{\tau_t} & & \downarrow_{\tau_a} \\
S^\Lambda_{*+1} (\tilde{N}) & \longrightarrow & KO_{*-1} (N) & \longrightarrow & KO_{*-1} (C^*_\Gamma \Lambda),
\end{array}
\]

with the following properties:

(i) \( \tau_t ([M]) = [N] \) and \( \tau_a (\alpha (M)) = \alpha (N) \),

(ii) \( \tau_s (\rho (g)) = \rho (g_N) \) for all \( g \in \mathcal{R}^+ (M) \) of producture structure \( g_N \oplus dt^2 \) near \( N \),

(iii) \( \tau_a (\alpha_{\text{diff}} (g_0, g_1)) = \alpha_{\text{diff}} (g_{N,0}, g_{N,1}) \) for all \( g_0, g_1 \in \mathcal{R}^+ (M) \) of product structure \( g_{N,i} \oplus dt^2, i = 0, 1, \) near \( N \).
The secondary codimension two obstruction

**Theorem (Z.)**

Let $M$ be a closed connected spin manifold and $N \subset M$ a closed submanifold of codimension 2 with trivial normal bundle and $g_0, g_1 \in \mathcal{R}^+(M)$ such that

- $\Lambda := \pi_1(N) \hookrightarrow \pi_1(M) =: \Gamma$,
- $\pi_2(N) \twoheadrightarrow \pi_2(M)$,
- On $N \times D^2 \subseteq M$, $g_i$ is of the form $g_{N,i} \oplus g_{D,i}$, where $g_{D,i}$ is cylindrical near $\partial D^2$.
- $(c_N)_* \rho(g_{N,0}) \neq (c_N)_* \rho(g_{N,1}) \in S^n_{n-2}(E\Lambda)$.

Then $g_0$ and $g_1$ are not concordant on $M$. 