Obstructions to positive scalar curvature via submanifolds

Rudolf Zeidler
WWU Münster

Cortona, June 2017

Intro: Coarse index theory

Let (X, g_X) be a complete spin manifold endowed with a free and proper Λ-action.

- **Coarse index** $\mathrm{Ind}_\Lambda(\not{\mathfrak{D}}_X) \in \mathrm{KO}_*(\mathrm{C}^*_\Lambda(X))$.
- If $X = \tilde{N}$, $\Lambda = \pi_1 N$ with N closed spin, we recover **Rosenberg index** $\mathrm{Ind}^\Lambda(\not{\mathfrak{D}}_{\tilde{N}}) = \alpha(N) \in \mathrm{KO}_*(\mathrm{C}^*_\Lambda\left(\tilde{N}\right)) \cong \mathrm{KO}_*(\mathrm{C}^*_\mathrm{r}\Lambda)$.
- $\mathrm{Ind}_\Lambda(\not{\mathfrak{D}}_X) \neq 0 \Rightarrow X$ does not admit a Λ-invariant metric of uniform psc in the same quasi-isometry class.
- Assume that X is Λ-invariantly partitioned by $\tilde{N} \subseteq X$. Then there is a map

  $$\partial_{\mathrm{MV}} \colon \mathrm{KO}_*(\mathrm{C}^*_\Lambda(X)) \to \mathrm{KO}_{*-1}(\mathrm{C}^*_\mathrm{r}\Lambda)$$

  with $\mathrm{Ind}^\Lambda(\not{\mathfrak{D}}_X) \mapsto \alpha(N)$. (**Roe's part. mfd. index thm.**)
- If $\mathrm{Ind}^\Lambda(\not{\mathfrak{D}}_X) \neq 0$, then $X_- \cup_{\tilde{N}} X_+$ does not admit a complete Λ-invariant metric of uniform psc **outside** $X_\pm$ in the same quasi-isometry class.

Intro II: Consequences of partitioned mfd. index thm.

Let $\Lambda = \pi_1 N$ with N closed spin.

- Consider $N \times \mathbb{R}$ endowed with some complete metric. Then:
  - $\mathrm{KO}_*(\mathrm{C}^*_\Lambda(\tilde{N} \times \mathbb{R})) \to \mathrm{KO}_{*-1}(\mathrm{C}^*_\mathrm{r}\Lambda)$, $\mathrm{Ind}^\Lambda(\not{\mathfrak{D}}_{\tilde{N}\times\mathbb{R}}) \mapsto \alpha(N)$.
- Consider $N \times \mathbb{R}^2$.
  - Let Z be the double of $N \times (\mathbb{R}^2 \setminus \mathrm{B}(0))$, so $Z \cong N \times \mathrm{S}^1 \times \mathbb{R}$.
  - If $N \times \mathbb{R}^2$ admits uniform psc, then Z admits a complete metric of uniform psc outside $N \times \mathrm{S}^1 \times \mathbb{R}_+$.
  - If $\alpha(N) \neq 0$, then $\alpha(N \times \mathrm{S}^1) \neq 0$, so $N \times \mathbb{R}^2$ does not admit a metric of uniform positive scalar curvature.

Low codimensions

Theorem (Hanke–Pape–Schick)

Let M be closed spin, $N \hookrightarrow M$ a codimension 2 submanifold with trivial normal bundle. Suppose that $\pi_1(N) \hookrightarrow \pi_1(M)$ and $\pi_2(N) \twoheadrightarrow \pi_2(M)$. Then, if $\alpha(N) \neq 0$, M does not admit psc.

Theorem (Z.)

Let $N \hookrightarrow M$ a codimension 1 submanifold with trivial normal bundle and $\Lambda := \pi_1(N) \hookrightarrow \pi_1(M) =: \Gamma$, then there exists a homomorphism

$$\tau^{(a)}_{\Gamma,\Lambda} \colon \mathrm{KO}_*(\mathrm{C}^*_\mathrm{r}\Gamma) \to \mathrm{KO}_{*-1}(\mathrm{C}^*_\mathrm{r}\Lambda)$$

mapping $\alpha(M)$ to $\alpha(N)$.

Conjecture (Gromov–Lawson)

An **aspherical** manifold does **not** admit a metric of **positive scalar curvature (psc)**.

Goal: Study a "partial version" of this conjecture.

Definition

Let M be a closed spin manifold, $\pi_1 M =: \Gamma$, $c\colon M \to \mathrm{B}\Gamma$. M is called

- *(rationally) essential* if $c_*[M] \neq 0 \in \mathrm{H}_*(\mathrm{B}\Gamma)$ $(\otimes\mathbb{Q})$.
- K-*essential* if $c_*[M]_{\mathrm{K}} \neq 0 \in \mathrm{K}_*(\mathrm{B}\Gamma)$.

- Chern character: Rationally K-essential $\Leftrightarrow$ rationally essential
- Aspherical manifolds are rationally essential.
- If Γ satisfies the strong Novikov conjecture,

  $$\nu\colon \mathrm{K}_*(\mathrm{B}\Gamma) \otimes \mathbb{Q} \hookrightarrow \mathrm{K}_*(\mathrm{C}^*_{\mathrm{r}}\Gamma) \otimes \mathbb{Q}$$

  then a rationally essential spin manifold M with $\pi_1 M = \Gamma$ does not admit psc.

High codimensions

Theorem (Engel, Schick–Z.)

Under hypotheses $(\star)$*, there is a commutative diagram:*

$$\begin{array}{ccc} \mathrm{H}_*(M) & \xrightarrow{c_M} & \mathrm{H}_*(\mathrm{B}\Gamma) \\ \downarrow{\tau_{M,N}} & & \downarrow{\tau_{\Gamma,\Lambda}} \\ \mathrm{H}_{*-k}(N) & \xrightarrow{c_N} & \mathrm{H}_{*-k}(\mathrm{B}\Lambda). \end{array}$$

(Enough to assume that the normal bundle is orientable and the Euler class of the normal bundle vanishes on $\mathrm{im}(\pi_k(N) \to H_k(N))$*.)*

Corollary

If N *is (rationally) essential, then so is* M*.*

Codimension two and generalized homology

Theorem (Schick–Z.)

Assume $(\star)$ *and let* $k = 2$*. Let* E *be a multiplicative generalized homology theory. Then there is a commutative diagram:*

$$\begin{array}{ccc} E_*(M) & \xrightarrow{c_M} & E_*(\mathrm{B}\Gamma) \\ \downarrow{\scriptstyle \tau_{M,N}} & & \downarrow{\scriptstyle \tau_{\Gamma,\Lambda}} \\ E_{*-2}(N) & \xrightarrow{c_N} & E_{*-2}(\mathrm{B}\Lambda). \end{array}$$

Corollary

If N *is* K*- or* KO*-essential, then so is* M*.*

Proof strategy

- Let $p\colon \bar{X} \to X$ some covering and $\theta \in \mathrm{H}^k_{\mathrm{vc}}(\bar{X})$.
  $\to$ obtain map $\tau_\theta\colon \mathrm{H}_*(X) \to \mathrm{H}_{*-k}(\bar{X})$, $x \mapsto \theta \cap p^!(x)$.

$\bar{M} \to M$ covering with $\pi_1 \bar{M} = \Lambda$, set $\theta := \mathrm{pd}(N \hookrightarrow \bar{M}) \in \mathrm{H}^k_{\mathrm{c}}(\bar{M})$.

$$\begin{tikzcd}
 & \mathrm{H}_*(\bar{M}) \arrow[r] \arrow[d, "\tau_\theta"] & \mathrm{H}_*(\mathrm{B}\Gamma) \arrow[d, dashed, "\tau_\vartheta"] \\
\mathrm{H}_*(N) \arrow[ur] \arrow[r, "\tau_{M,N}"] & \mathrm{H}_{*-k}(M) \arrow[r] & \mathrm{H}_{*-k}(\mathrm{B}\Lambda).
\end{tikzcd}$$

$\to$ Need to extend θ to $\vartheta \in \mathrm{H}^k_{\mathrm{vc}}(\mathrm{B}\Lambda \to \mathrm{B}\Gamma)$.

Construct $\mathrm{B}\Gamma$ out of M by attaching cells:

- Start with $(k+1)$-cells attached to M along N to make π_k trivial.
- Inductively attach $(l+1)$-cells to the resulting space to make π_l trivial for all $l > k$ and get models for $\mathrm{B}\Gamma$ and $\mathrm{B}\Lambda$.
- Represent θ by a map $(\bar{M}, \bar{M} \setminus \mathrm{D}\nu(N)) \to (\mathrm{K}(\mathbb{Z}, k), *)$ and work out conditions that allow to extend it over $\mathrm{B}\Lambda$.

The Higson–Roe sequence & secondary indices

$$
\begin{array}{ccccccc}
 & & \rho(g) & \longmapsto & [M]_{\mathrm{KO}} & \longmapsto & \alpha(M) \\
\mathrm{KO}_{n+1}(\mathrm{C}^*_{\mathrm{r}}\Gamma) & \longrightarrow & \mathrm{S}^{\Gamma}_n(\tilde{M}) & \longrightarrow & \mathrm{KO}_n(M) & \xrightarrow{\nu} & \mathrm{KO}_n(\mathrm{C}^*_{\mathrm{r}}\Gamma)
\end{array}
$$

- Let M be a closed spin n-manifold.
- The assembly map takes the KO-homology fundamental class to the Rosenberg index.
- It fits into the **Higson–Roe l.e.s.**, where the third term is the **analytic structure group**.
- For each $g \in \mathcal{R}^+(M)$, there is $\rho(g) \in \mathrm{S}^{\Gamma}_n(\tilde{M})$.
- For $g_0, g_1 \in \mathcal{R}^+(M)$, there is $\alpha_{\mathrm{diff}}(g_0, g_1) \in \mathrm{KO}_{n+1}(\mathrm{C}^*_{\mathrm{r}}\Gamma)$ with $\partial\alpha_{\mathrm{diff}}(g_0, g_1) = \rho(g_0) - \rho(g_1)$.
- There is a version for non-(co)compact manifolds and secondary partitioned manifold index theorems (Piazza–Schick, Z.)

The secondary codimension one obstruction

Theorem (Z.)

Let M be a closed connected spin manifold and $N \subset M$ a closed submanifold of codimension 1 with trivial normal bundle such that $\Lambda := \pi_1(N) \hookrightarrow \pi_1(M) =: \Gamma$. Then there exists a diagram,

$$\begin{array}{ccccc} \mathrm{S}^{\Gamma}_{*}\left(\tilde{M}\right) & \longrightarrow & \mathrm{KO}_{*}\left(M\right) & \longrightarrow & \mathrm{KO}_{*}(\mathrm{C}^{*}_{\mathrm{r}}\Gamma) \\ \downarrow{\scriptstyle \tau_{\mathrm{s}}} & & \downarrow{\scriptstyle \tau_{\mathrm{t}}} & & \downarrow{\scriptstyle \tau_{\mathrm{a}}} \\ \mathrm{S}^{\Lambda}_{*-1}\left(\tilde{N}\right) & \longrightarrow & \mathrm{KO}_{*-1}\left(N\right) & \longrightarrow & \mathrm{KO}_{*-1}(\mathrm{C}^{*}_{\mathrm{r}}\Lambda), \end{array}$$

with the following properties:

(i) $\tau_{\mathrm{t}}([M]) = [N]$ *and* $\tau_{\mathrm{a}}(\alpha\,(M)) = \alpha\,(N)$,

(ii) $\tau_{\mathrm{s}}(\rho(g)) = \rho(g_N)$ *for all* $g \in \mathcal{R}^{+}(M)$ *of producture structure* $g_N \oplus \mathrm{d}t^2$ *near* N,

(iii) $\tau_{\mathrm{a}}(\alpha_{\mathrm{diff}}\,(g_0, g_1)) = \alpha_{\mathrm{diff}}\,(g_{N,0}, g_{N,1})$ *for all* $g_0, g_1 \in \mathcal{R}^{+}\,(M)$ *of product structure* $g_{N,i} \oplus \mathrm{d}t^2$, $i = 0, 1$, *near* N.

The secondary codimension two obstruction

Theorem (Z.)

Let M be a closed connected spin manifold and $N \subset M$ a closed submanifold of codimension 2 with trivial normal bundle and $g_0, g_1 \in \mathcal{R}^+(M)$ such that

- $\Lambda := \pi_1(N) \hookrightarrow \pi_1(M) =: \Gamma$,
- $\pi_2(N) \twoheadrightarrow \pi_2(M)$,
- *On $N \times D^2 \subseteq M$, g_i is of the form $g_{N,i} \oplus g_{D,i}$, where $g_{D,i}$ is cylindrical near ∂D^2.*
- $(c_N)_* \rho(g_{N,0}) \neq (c_N)_* \rho(g_{N,1}) \in \mathrm{S}^{\Lambda}_{n-2}(\mathrm{E}\Lambda)$.

Then g_0 and g_1 are not concordant on M.