Deformation of constant curvature conical metrics

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Outline

Constant curvature metric with conical singularities

Consider a compact Riemann surface M, with the following data:

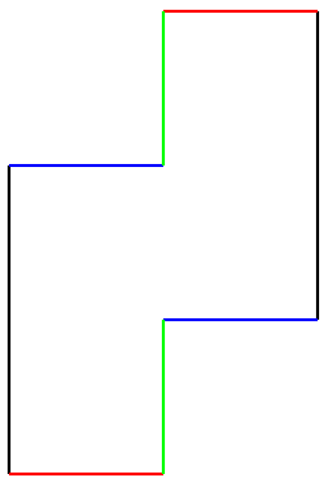
- k distinct points $\mathfrak{p} = (p_1, \dots, p_k)$
- Angle data $\vec{\beta} = (\beta_1, \dots, \beta_k) \in (0, \infty)^k$
- Curvature constant $K \in \{-1, 0, 1\}$
- Area A
- Conformal structure $\mathfrak{c}$ given by M

A constant curvature metric with prescribed conical singularities is a smooth metric with constant curvature, except near p_j the metric is asymptotic to a cone with angle $2\pi\beta_j$.

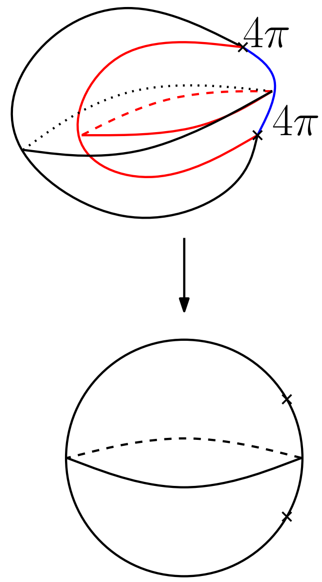
$$\text{(Gauss–Bonnet) } \chi(M, \vec{\beta}) := \chi(M) + \sum_{j=1}^{k} (\beta_j - 1) = \frac{1}{2\pi} KA.$$

Some examples of constant curvature conical metrics

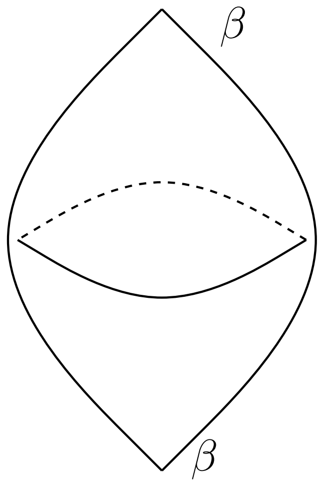
Translation surfaces



Covers of surfaces of constant curvature



Spherical footballs



Existence and Uniqueness

Theorem (88' McOwen, 91' Troyanov, 92' Luo–Tian)

For any compact surface M and conical data $(\mathfrak{p}, \vec{\beta})$ satisfying one of the following constraints:

- $\chi(M, \vec{\beta}) \leq 0$; *or*
- $\chi(M, \vec{\beta}) > 0, \vec{\beta} \in (0, 1)^k$
    - $k = 2,\ \beta_1 = \beta_2$; *or*
    - $k \geq 3,\ \beta_j + k - \chi(M) > \sum_{i \neq j} \beta_i, \forall j.$

there is a unique constant curvature metric with the prescribed singularities.

The moduli space for $\vec{\beta} \in (0, 1)^k$

Theorem (Mazzeo–Weiss, 2015)

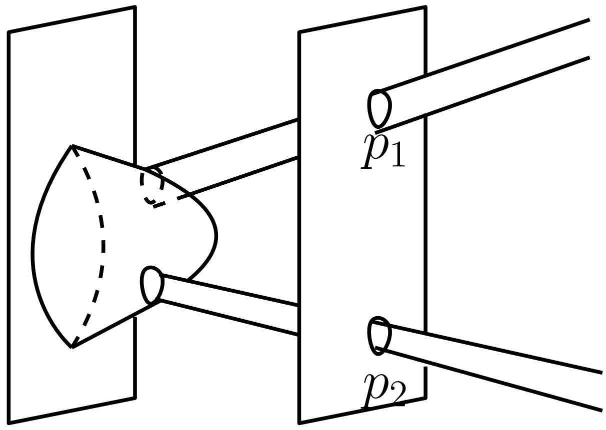
- *["Teichmüller space"] The space of constant curvature conical metrics* $\mathcal{CM}_{cc}(M, \mathfrak{p})$ *are Banach manifolds.*
- *[Moduli space] There is an embedded* $(6\gamma - 6 + 3k)$*-dimensional submanifold* $S \subset \mathcal{CM}_{cc}(M, \mathfrak{p})$ *as the quotient by the action of diffeomorphism group.*

Question

1. What happens when cone points collide?
2. Compactification of the moduli space?

When two points collide

- Scale back the distance between two cone points (Blow up)
- Half sphere at the collision point, with two cone points over the half sphere:



- Flat metric on the half sphere, and curvature K metric on the original surface

Iterative structure

- “bubble over bubble” structure
- Higher codimensional faces from deeper scaling
- Flat conical metrics on all the new faces

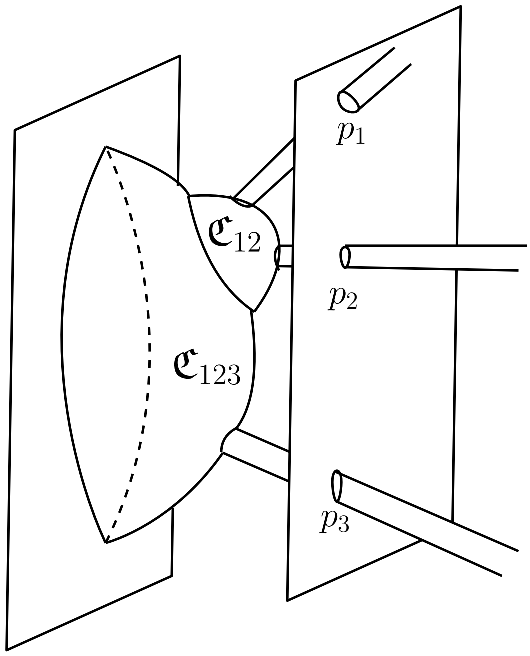


Figure: One of the singular fibers in $\mathcal{C}_3$, where two of the points collide faster than the third one

Resolution of the configuration family

Resolve the product fiberation $M^{k+1} \to M^k$

- Step 1: In the base, blow up the diagonals iteratively according to the partial order on the index ("extended configuration space")

$$\tilde{\mathcal{H}}_k = [M^k; \cup_{\mathcal{I}} \Delta_{\mathcal{I}}].$$

- Step 2: Lift the fibration to $\tilde{\mathcal{H}}_k \times M \to \tilde{\mathcal{H}}_k$, blow up all the partial diagonals ("compactified configuration family")

$$\tilde{\mathcal{C}}_k = [\tilde{\mathcal{H}}_k \times M; \cup \Delta^{\mathcal{C}}_{\mathcal{I}}]$$

- The above two steps resolve the compact group action by Σ_k. Take the quotient to get the unordered version: $\mathcal{H}_k = \tilde{\mathcal{H}}_k / \Sigma_k,\ \mathcal{C}_k = \tilde{\mathcal{C}}_k / \Sigma_k$. [Albin-Melrose, 2010]
- We obtain a b-fibration

$$\pi_k : \mathcal{C}_k \to \mathcal{H}_k$$

Example: $\mathcal{C}_2 \to \mathcal{H}_2$

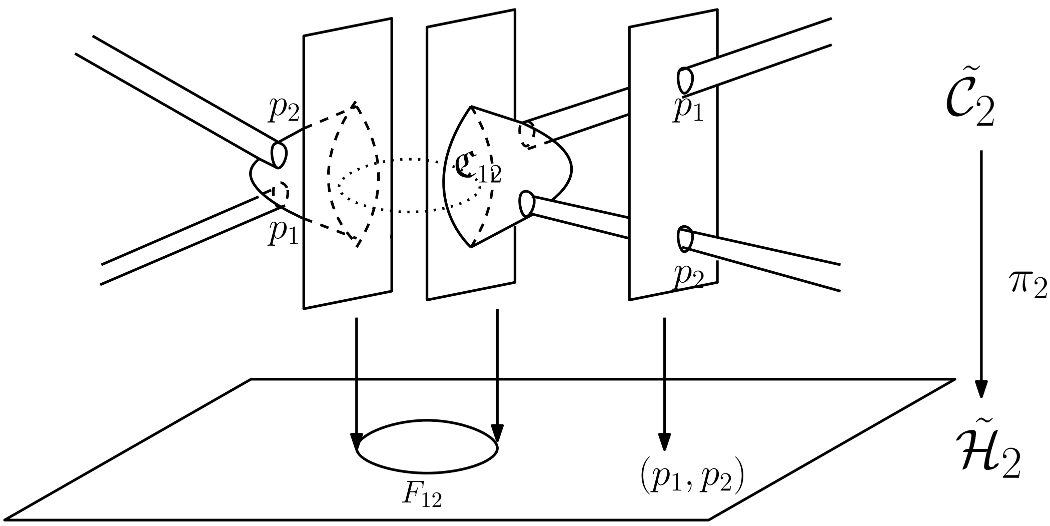


Figure: "Centered" projection of $\tilde{\mathcal{C}}_2 \to \tilde{\mathcal{H}}_2$

Results about fiber metrics on $\mathcal{C}_k$

Theorem (Mazzeo-Z, 2017)

For any given $\vec{\beta}$, the family of constant curvature metrics with conic singularities is polyhomogeneous on $\mathcal{C}_k$.*

- *: The metric family can be hyperbolic / flat (with any cone angles), or spherical (with angles less than 2π)
- Solve the curvature equation $\Delta_{g_0} f - Ke^{2f} + K_{g_0} = 0$ uniformly
- The leading term of the metric is given by the flat metric
- When $K = \pm 1$, the difference from the flat metric is bounded by $O(\rho^2)$
- This matches the blow up limit

Motivation: positive curvature with big cone angles

- $M = \mathbb{S}^2$, $k = 2, 3, 4$: [Troyanov, 1991] [Umehara–Yamada, 2000] [Eremenko, 2000] [Eremenko–Gabrielov–Tarasov, 2014] [Eremenko–Gabrielov, 2015]
- Genus ≥ 1, minimax theory [Carlotto–Malchiodi, 2011] [Malchiodi, 2016].
- [Mondello–Panov, 2016] $M = \mathbb{S}^2$, necessary condition & construction in the interior

$$d_1(\vec{\beta} - \vec{1}, \mathbb{Z}^k_{odd}) \geq 1$$

- The boundary of the Mondello-Panov region [Dey, 2017] [Kapovich, 2017] [Eremenko, 2017]

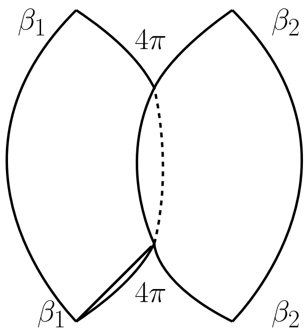
Questions to answer

Goal

- Find out the structure of the moduli space
- The full solution space: for given admissible $(\vec{\beta}, \mathfrak{p}, \mathfrak{c})$, how many solutions are there?
- Deformation theory: is there a manifold structure?

The eigenvalue 2

- The linearized operator $\Delta_{g_0} - 2K$ acting on the Friedrichs domain
- When $\vec{\beta} \in (0,1)^k$, $\lambda_1 \geq 2K$; when angles increase, eigenvalues decrease
- Example: two footballs glued together



- The surjectivity (and injectivity) of linearized operator is lost, hence implicit function theorem fails
- Indicial roots of $\Delta_g - 2$ given by $\{\pm \frac{k}{\beta}\}$

Geometric realization of the indicial roots

We discover that one key step to make it unobstructed is the following:

Proposition (Mazzeo-Z, in progress)

The linearized space generated by the splitting of cone angles are spanned by $\{r^{-\frac{k_i}{\beta}}, 1 < k_i \leq \beta\}$.

- Proof by computing the Jacobi field generated by the geometric motion
- The linearized operator is surjective after adding those parameters
- It provides additional coordinates for the moduli space

Summary

- We constructed the compactification of configuration space
  - characterized geometrically the process of merging cone points
  - developed new regularity results regarding the constant curvature metric family
- We hope to apply this construction
  - to capture the analytic behavior of the curvature operator
  - to understand the moduli space of spherical conical metrics with no angle constraints, including its compactification

Thank you for your attention!