Moduli spaces of Riemannian and Lorentzian manifolds

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Overview

This overview talk consists of two parts which connect spaces of special Riemannian and Lorentzian metrics.

- From a curve in the moduli space of Ricci-flat Riemannian metrics with parallel spinor on an n-dimensional manifold to a Lorentzian manifold of dimension $n+2$ with a (lightlike) parallel spinor and special holonomy
  Work by B. Ammann, K. Kröncke, O. Müller,
  Connected work in collaboration with H. Weiß, F. Witt
  Arxiv 1903.02064 and 1512.07390

- From homotopy groups of the space $\mathcal{R}^+(M)$ of metrics of positive scalar curvature to
  to homotopy groups of the space $\mathcal{I}^+(M)$ of Lorentz initial data on M satisfying the dominant energy condition strictly

$$\pi_k\Big(\mathcal{R}^+(M)\Big) \longrightarrow \pi_{k+1}\Big(\mathcal{I}^+(M)\Big)$$

$$\alpha_{\mathrm{Riem}} : \pi_k\Big(\mathcal{R}^+(M)\Big) \to KO^{-n-k-1}(\{*\}), \qquad \alpha_{\mathrm{Lor}} : \pi_{k+1}\Big(\mathcal{I}^+(M)\Big) \to KO^{-n-k-1}(\{*\})$$

$n = \dim M$

Work by Jonathan Glöckle, Regensburg

Arxiv 1906.00099

Spin geometry

Let (N, h) be a time- and space-oriented semi-Riemannian manifold.
We assume that we have a fixed spin structure, i.e. a choice of a complex vector bundle $\$^h N$, called the spinor bundle, with

$$\$^h N \otimes_{\mathbb{C}} \$^h N = \bigwedge^{\bullet/\mathrm{even}} T^*N \otimes_{\mathbb{R}} \mathbb{C}.$$

This bundle carries (fiberwise over $p \in M$)

- a non-degenerate hermitian product (positive definit in the Riemannian case)
- a compatible connection
- a compatible Clifford multiplication $\mathrm{cl} : TN \otimes \$^h N \to \$^h N$, $\mathrm{cl}(X \otimes \varphi) =: X \cdot \varphi$ such that

$$X \cdot Y \cdot \varphi + Y \cdot X \cdot \varphi + 2h(X, Y)\,\varphi = 0.$$

Spinors and holonomy

Let (N,h) be a Riemannian or Lorentzian spin manifold. Parallel transport along a loop $c:[0,1]\to N$, $p=c(0)=c(1)$ gives a map

$$\begin{array}{ccc} \mathcal{P}^{\$^h N}(c) \in & \mathrm{Spin}(\$^h_p N) \subset \mathrm{U}(\$^h_p N) \\ & \big\downarrow 2:1 \\ \mathcal{P}^{TN}(c) \in & \mathrm{SO}(T_pN,h) \end{array}$$

Assume that $\varphi \neq 0$ is a parallel spinor, i.e. a parallel section of $\Gamma(\$^h N)$. Then the holonomy group

$$\mathrm{Hol}(N,h,p) := \{\mathcal{P}^{TN}(c) \mid c \text{ loop based in } p\} \subsetneq \mathrm{SO}(T_pN,h).$$

is special, i.e. $\dim \mathrm{Hol}(N,h,p) < \dim \mathrm{SO}(T_pN,h)$.

Parallel spinors

Let (N, h) be a Riemannian or Lorentzian spin manifold.
Assume that $\varphi \neq 0$ is a parallel spinor,

$\Rightarrow \quad R_{X,Y}\varphi = 0$

$\Rightarrow \quad 0 = \sum \pm e_i \cdot R_{e_i,Y}\varphi \stackrel{!}{=} -\frac{1}{2}\operatorname{Ric}(Y)\cdot\varphi$

$\Rightarrow \quad h(\operatorname{Ric}(Y), \operatorname{Ric}(Y))\varphi = -\operatorname{Ric}(Y)\cdot\operatorname{Ric}(Y)\cdot\varphi = 0$

In the Riemannian case: $\operatorname{Ric} = 0$

In the Lorentzian case:

$\operatorname{Ric}(Y)$ is lightlike for all Y

$\Rightarrow \quad \operatorname{Ric} = f\alpha \otimes \alpha$ for a lightlike 1-form α.

The Dirac current of (N, h) is the vector field V_φ with

$$h(X, V_\varphi) = -\langle X \cdot \varphi, \varphi\rangle$$

If φ is parallel, then V_φ is a parallel vector field and $V_\varphi \parallel \alpha^\#$.

Spacelike hypersurfaces

Work by H. Baum, T. Leistner, A. Lischewski

If (N, h) is a Lorentzian manifold with a parallel spinor φ.

Then $h(V_\varphi, V_\varphi) \leq 0$, i.e. V_φ is causal.

We assume V_φ is lightlike.

Let M be a spacelike hypersurface of N with induced metric g and Weingarten map W.

On M we write

$$V_\varphi|_M = U_\varphi + u_\varphi \nu$$

U_φ tangential to M

ν future unit normal of M

If we “restrict” φ to M it satisfies the constraint equation

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TM, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

The Cauchy problem for parallel spinors

Conversely, if we have a Riemannian manifold (M, g) with a non-trivial solution of

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TM, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

then it extends to a Lorentzian metric on $M \times (-\epsilon, \epsilon)$ with a parallel spinor φ with V_φ lightlike.

Again: work by H. Baum, T. Leistner, A. Lischewski
Simplified by Julian Seipel (Master thesis, Regensburg), following ideas by P. Chrusciel

Our Goal: Find solutions to (CE).

Machinery for solutions of the constraint equations

Let Q be a closed spin manifold.
$\mathcal{M}(Q) := \{\text{Riemannian metrics } g \text{ on } Q\}$
$\mathcal{M}_{\|}(Q) := \{g \in \mathcal{M}(Q) \mid (Q, g) \text{ has a parallel spinor}\}$
$\mathrm{Diff}(Q) := \{\text{Diffeomorphisms } Q \to Q\}$
$\mathrm{Diff}_{\mathrm{Id}}(Q)$ is the identity component of $\mathrm{Diff}(Q)$.

Last time in Cortona, I sketched how to prove the following:

Theorem (Ammann, Kröncke, Weiß, Witt 2015)

The premoduli space

$$\mathcal{M}od_{\|}(Q) := \mathcal{M}_{\|}(Q) / \mathrm{Diff}_{\mathrm{Id}}(Q)$$

is a smooth manifold, and the map $\mathcal{M}od_{\|}(Q) \to \mathbb{N}$,
$[g] \mapsto \dim \Gamma_{\|}(\$^g Q)$ is locally constant.

The parallel spinors from a vector bundle over

$$\Gamma_{\|} \to \mathcal{M}od_{\|}(Q)$$

of locally constant rank.
The bundle has

- a connection: given by work of Bourguignon—Gauduchon, Bär—Gauduchon—Moroianu and Müller—Nowaczyk.
- and a compatible metric: the L^2-metric

The connection in fact comes from a natural connection on the bundle

$$\coprod_{g \in \mathcal{M}(Q)} \Gamma(\$^g Q) \to \mathcal{M}(Q),$$

using the following (for us amazing) proposition:

Proposition (AKWW/AKM)

Along a divergence-free path of Ricci-flat metrics $(g_t)_{a \leq t \leq b}$ *the parallel transport of a parallel spinor remains parallel.*

We say that $(g_t)_{a \leq t \leq b}$ is divergence-free if

$$\mathrm{div}^{g_t}\left(\frac{d}{dt} g_t\right) = 0.$$

This means that this path of metrics is orthogonal to the orbits of $\mathrm{Diff}_{\mathrm{Id}}(Q)$.

Some comments on the proof

The following argument provides an infinitesimal version of the proposition.
Unfortunately it requires some work to obtain the full version out of it. In particular, this step requires the previous theorem.

McKenzie Wang's argument

The McKenzie Wang map $\mathcal{W}$ for $\varphi \in \Gamma_{\|}(Q, g)$ is given by

$$\bigodot^2 T^*Q \hookrightarrow \bigotimes^2 T^*Q \xrightarrow{\mathrm{cl}^g(\,\cdot\,,\varphi)\otimes\mathrm{id}} \$^g Q \otimes T^*Q$$

where $\mathrm{cl}(\alpha, \varphi) := \alpha^\# \cdot \varphi$ is the Clifford multiplication of 1-forms with spinors. If $\nabla\varphi = 0$, then

$$\begin{array}{ccc} \Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad\mathcal{W}\quad} & \Gamma(\$^g Q \otimes T^*Q) \\ \Big\downarrow \Delta^E & & \Big\downarrow (D^{T^*Q})^2 \\ \Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad\mathcal{W}\quad} & \Gamma(\$^g Q \otimes T^*Q) \end{array}$$

commutes.
Here: $\Delta^E h = \nabla^*\nabla h - 2\mathring{R}h$. $\qquad \mathring{R}h(X, Y) := h(R_{e_i,X}Y, e_i)$.

Consequences of McKenzie Wang

Δ^E is the linearization of $g \mapsto 2\,\mathrm{Ric}^g$ in divergence, trace-free directions.

- $\mathrm{spec}(\Delta^E) \subset [0, \infty)$. Thus $g \in \mathcal{M}_{\|}(Q)$ cannot be deformed to a metric of positive scalar curvature.
- $\ker \Delta^E \subset \ker \left(D^{T^*Q}\right)^2$.
- Elliptic theory on compact manifolds: $\ker \left(D^{T^*Q}\right)^2 = \ker D^{T^*Q}$.

Proof of the infinitesimal version of the proposition

$$
\begin{array}{ccc}
\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad \mathcal{W} \quad} & \Gamma(\$^g Q \otimes T^*Q) \\
 & \color{red}{\searrow^{4\frac{d}{dg}\nabla\varphi}} & \big\downarrow D^{T^*Q} \\
\Delta^E \big\downarrow & & \Gamma(\$^g Q \otimes T^*Q) \\
 & & \big\downarrow D^{T^*Q} \\
\Gamma(\bigodot^2 T^*Q) & \xrightarrow{\quad \mathcal{W} \quad} & \Gamma(\$^g Q \otimes T^*Q)
\end{array}
$$

commutes.

Application to the Lorentzian problem

This structure provides solutions to the constraint equations:

$$\begin{aligned} \nabla^M_X \varphi &= \frac{i}{2} W(X) \cdot \varphi, \qquad \forall X \in TM, \\ U_\varphi \cdot \varphi &= i u_\varphi \varphi, \end{aligned} \tag{CE}$$

Theorem (Ammann, Kröncke, Müller 2019)

For any smooth curve $G : (a, b) \to \mathcal{M}od_{||}(Q)$ *and any smooth function* $F : (a, b) \to (0, \infty)$ *we obtain a solution of the constraint equation on* $M = Q \times (a, b)$ *with* $\ldots$

Theorem (Ammann, Kröncke, Müller 2019)

For any smooth curve $G : s \in (a,b) \to \mathcal{M}od_{\|}(Q) \ni G_s$ and any smooth function $F : (a,b) \to (0,\infty)$ we obtain a solution of the constraint equation on $M = Q \times (a,b) \ni (x,s)$. The data on M are given as follows:

- *The metric Γ on M is $\Gamma = g_s + ds^2$ for a divergence-free family of metrics g_s with $[g_s] = G_s$*
- *The spinor is obtained as follows:*
  - *Choose a parallel spinor ψ_r on (Q, g_r) for some $r \in (a,b)$.*
  - *By parallel transport along $s \mapsto g_s$ choose a family $(\psi_s)_{s\in(a,b)}$ of spinors which are parallel on $(Q, g_s)_{s\in(a,b)}$.*
  - *View $\varphi := (F(s)\psi_s)_{s\in(a,b)}$ as a spinor over $M = Q \times (a,b)$.*
  - *This spinor φ satisfies the constraint equations.*

Summary of this part

For an interval I we get a map

$$C^\infty(I, \mathcal{M}od_{\|}(Q)) \times C^\infty(I, \mathbb{R}_+) \to \mathcal{M}od_{\|}^{\mathrm{Lor}}(Q \times I \times (-\epsilon, \epsilon))$$

Similarly closed curves $\mathcal{M}od_{\|}(Q)$ yield Lorentzian metrics on $Q \times S^1 \times (-\epsilon, \epsilon)$ if a “closing” condition holds

Slogan: Curves of Riemannian special holononmy metrics on Q yield a Lorentzian special holonomy metric on a manifold N with $\dim N = \dim Q + 2$.

Topology of the space of Lorentzian initial data satisfying the dominant energy condition strictly

The dominant energy condition

Let h be a Lorentzian metric on N
Energy-momentum tensor or Einstein tensor

$$T^h := \mathrm{Ric}^h - \frac{1}{2}\mathrm{scal}^h h$$

We say that h satisfies the dominant energy condition in $x \in N$ if for all causal future oriented vectors $X, Y \in T_x N$:

$$T(X, Y) \geq 0. \qquad \text{(DEC)}$$

DEC on spacelike hypersurfaces

If M is a space-like hypersurface with induced metric g, and future-oriented unit normal, then we define:
Energy density $\rho := T^h(\nu, \nu) = \frac{1}{2}\left(\mathrm{scal}^g + (\mathrm{tr} W)^2 - \mathrm{tr}(W^2)\right)$
Momentum density $j := T^h(\nu, \cdot)|_{T_x M} = \mathrm{div} W - \mathrm{d}\, \mathrm{tr} W$
DEC for h implies $\rho \geq |j|$.

Definition

Let g be a Riemannian metric and W a g-symmetric endomorphism section. We say that (g, W) satisfies

- the dominant energy condition if $\rho \geq |j|$ (DEC)
- the strict dominant energy condition if $\rho > |j|$ (DEC$_+$)

$$\mathcal{I}^+(M) := \{(g, W) \text{ satisfying } (\mathrm{DEC}_+)\}.$$

The inclusion $\mathcal{R}^+(M) \to \mathcal{I}^+(M)$

$\mathcal{R}(M) \hookrightarrow \mathcal{I}(M)$, $g \mapsto (g, 0)$
$\mathcal{R}^+(M) = \mathcal{R}(M) \cap \mathcal{I}^+(M)$

Lemma

If $g \in \mathcal{R}^+(M)$, then $(g, \lambda \,\mathrm{Id}) \in \mathcal{I}^+(M)$ for all $\lambda \in \mathbb{R}$.

Lemma

If K is a compact set and $K \to \mathcal{I}(M)$, $k \mapsto (g_k, W_k) \in \mathcal{I}(M)$, then there is a $\lambda_\pm \in \mathbb{R}$ with $\pm\lambda_\pm \gg 0$ and such that for all $k \in K$

$$(g_k, W_k + \lambda_\pm \,\mathrm{Id}) \in \mathcal{I}^+(M).$$

With such arguments it follows that the inclusion $\mathcal{R}^+(M) \hookrightarrow \mathcal{I}^+(M)$ is homotopic to a constant map. This leads to maps

$$\mathrm{Cone}\big(\mathcal{R}^+(M)\big) \to \mathcal{I}^+(M)$$

one map for $\lambda_+ \gg 0$ and one for $\lambda_- \ll 0$.
Glued together we get a map $\Sigma\big(\mathcal{R}^+(M)\big) \to \mathcal{I}^+(M)$.

The Dirac-Witten operator

Restrict the spinor $\$N$ from (N, h) to (M, g).
As Clifford module $\$N|_M$ is one or two copies of $\$M$.
However: scalar product on $\$N$ is indefinit, scalar product on $\$M$ positive definit.
The connections differ:

$$\nabla^N_X \varphi = \nabla^M_X \varphi - \frac{1}{2}\nu \cdot W(X) \cdot \varphi$$

Dirac-Witten-Operator

$$D^{(g,W)}\varphi = \sum_{j=1}^{n} e_j \cdot \nabla^N_{e_j}\varphi$$

where $(e_1, \ldots, e_n)$ is a locally defined orthonormal frame of TM.

Theorem (Witten 1981, Parker-Taubes, Hijazi-Zhang, ...)

$D^{(g,W)}$ is self-adjoint and invertible if $(g, W) \in \mathcal{I}^+(M)$.

The Lorentzian α-index

Attention: $\mathrm{Cl}_{n,1}$-linear spinors instead of complex spinors

For every $(g, W) \in \mathcal{I}(M)$ we get an odd $\mathrm{Cl}_{n,1}$-linear self-adjoint Fredholm operator $D^{(g,W)}$.

$D^{(g,W)}$ is invertible if $(g, W) \in \mathcal{I}^+(M)$.

For any $\Psi : S^{k+1} \to \mathcal{I}^+(M)$ J. Glöckle constructs $\alpha_{\mathrm{Lor}}(\Psi) \in \mathrm{KO}^{-n-(k+1)-1,1}(\{*\}) \cong \mathrm{KO}^{-n-k-1}(\{*\})$.

Theorem (J. Glöckle 2019)

The diagram

$$
\begin{array}{ccccc}
\pi_k\Big(\mathcal{R}^+(M)\Big) & \xrightarrow{\ \Sigma\ } & \pi_{k+1}\Big(\Sigma(\mathcal{R}^+(M)\Big) & \longrightarrow & \pi_{k+1}\Big(\mathcal{I}^+(M)\Big) \\
& \searrow^{\alpha_{\mathrm{Riem}}} & & \swarrow_{\alpha_{\mathrm{Lor}}} & \\
& & KO^{-n-k-1}(\{*\}) & &
\end{array}
$$

commutes.

Application

Non-triviality of many $\pi_k(\mathcal{R}^+(M))$ was shown by Crowley, Hanke, Steimle, Schick and Botvinnik, Ebert, Randal-Williams.
Strategy:

- Description of some $\Psi : S^k \to \mathcal{R}^+(M)$.
- Show $\alpha_{\mathrm{Riem}}(\Psi) \neq 0$.

Corollary (Glöckle)

For each such non-trivial $\pi_k(\mathcal{R}^+(M))$ we get a non-trivial $\pi_{k+1}(\mathcal{I}^+(M))$.