Lecture notes on "Character Rigidity"*

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1 Introduction

Given a discrete group Γ, a **character** on Γ is a function $\tau : \Gamma \to \mathbb{C}$ which satisfies:

1. $\tau(e) = 1$;
2. τ is positive definite, i.e., for each $\gamma_1, \ldots, \gamma_n \in \Gamma$ the matrix $(\tau(\gamma_j^{-1}\gamma_i))_{i,j}$ is non-negative definite;
3. $\tau(\gamma_1\gamma_2) = \tau(\gamma_2\gamma_1)$ for all $\gamma_1, \gamma_2 \in \Gamma$.

The space of characters $\mathrm{Char}(\Gamma)$ forms a weak*-closed convex subset of $\ell^\infty\Gamma$. A character $\tau \in \mathrm{Char}(\Gamma)$ is **almost periodic** if the set of translates $\{x \mapsto \tau(\gamma x)\}_{\gamma\in\Gamma}$ forms a uniformly pre-compact set in $\ell^\infty\Gamma$.

The purpose of these notes is to discuss the following result and its corollaries:

Theorem A (P [Pet15a]). *Let Γ be a lattice in a simple real Lie group G which has trivial center, and real rank at least 2. If $\tau \in \mathrm{Char}(\Gamma)$ is an extreme point then either*

1. *τ is almost periodic, or else*
2. *$\tau = \delta_e$.*

Corollary B (Stuck-Zimmer [SZ94]). *Let Γ be as in Theorem A, then any probability measure-preserving ergodic action of Γ on a standard Lebesgue space is essentially free.*

Corollary C (Margulis [Mar79]). *Let Γ be as in Theorem A, then any non-trival normal subgroup of Γ is of finite index.*

Characters on groups naturally correspond to representations of the group in a tracial von Neumann algebra. In the latter setting Theorem A is equivalent (by Theorem 5.7 below) to the following:

Theorem D (P [Pet15a]). *Let Γ be as in Theorem A, if M is a finite factor, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation such that $\pi(\Gamma)'' = M$, then either*

1. *M is finite dimensional, or else*

2. π extends to an isomorphism $L\Gamma \xrightarrow{\sim} M$.

An example where the above theorem applies is the case $G = PSL_n(\mathbb{R})$ (and e.g., $\Gamma = PSL_n(\mathbb{Z})$) for $n \geq 3$. For simplicity we will consider this case only in these notes. For the general case one must adapt the proofs below using the structure theory of simple real Lie groups.

1.1 Previous results

We'll say that a discrete group Γ is **character rigid** (or **operator algebraic superrigid**) if the consequences of Theorem A hold for Γ. That lattices in higher-rank simple Lie groups are character rigid was conjectured by Connes in the early 80's as a von Neumann algebraic analogue of Margulis' superrigidity theorem (see [Jon00]). At the time no such lattes where known to be character rigid, the first examples being found by Bekka [Bek07] who showed that this holds for the groups $PSL_n(\mathbb{Z})$ for $n \geq 3$. Bekka's techniques were generalized by the author and Thom in [PT13] where the same results were obtained for the groups $SL_2(A)$, where $A = \mathcal{O}$ is a ring of integers (or, more generally, $A = \mathcal{O}S^{-1}$ a localization) with infinitely many units. Unfortunately though, Bekka's techniques do not appear to generalize to arbitrary lattices.

The first examples of higher-rank groups such that *arbitrary* irreducible lattices are character rigid were found by the author and Creutz in [CP12, CP13], where this was shown to be the case for irreducible lattices in certain product groups where one of the product factors is an algebraic group over a non-archimedian local field. These groups however are not simple real Lie groups and so are disjoint from the groups considered in Theorem A.

1.2 General strategy

Suppose $G = PSL_n(\mathbb{R})$ for $n \geq 3$, $\Gamma < G$ is a lattice, M is a finite factor and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation such that $\pi(\Gamma)'' = M$. We let $P < G$ be the minimal parabolic subgroup consisting of upper triangular matrices in $PSL_n(\mathbb{R})$.

The general strategy to prove Theorem D is to mimic Margulis [Mar79] (in the setting of groups) and Stuck-Zimmer [SZ94] (in the setting of discrete measured equivalence relations) by splitting the result into a "property (T) half" and an "amenability half". The strategy to prove each "half" is then roughly as follows:

Property (T) half:

1. (Kazhdan [Kaž67]) G and Γ have property (T).
2. (Connes-Jones [CJ85], Popa [Pop86]) M has property (T).

Amenability half:

1. (Zimmer [Zim77]) The von Neumann algebra $(L^\infty(G/P; \mathcal{B}(L^2M))^\Gamma$ is amenable.
2. (Margulis [Mar79]) The action $\Gamma \curvearrowright G/P$ satisfies a "Lebesgue density" type property.

3. (P [Pet15a], Creutz-P [CP13]) If π does not extend to an isomorphism $L\Gamma \xrightarrow{\sim} M$, then we have $M = (L^\infty(G/P; \mathcal{B}(L^2M))^\Gamma$.

4. (Connes [Con76]) If M is amenable, then it satisfies a "Reiter type" property.

Once the results above are established the proof of Theorem D then follows from the easy observation that if M is a finite factor with property (T) and satisfying a "Reiter type" property then M must be finite dimensional.

1.3 Background

We assume that the reader is familiar with basic concepts in Banach spaces/algebras, von Neumann algebras, standard Borel/probability spaces, and locally compact groups. Familiarity with [Zhu93] and the first four chapters of [Fol95] will suffice for most of these notes. Familiarity with Chapters 1, 2, 3, and 5 of [Pet15b] will also suffice. Another good resource is [Zim84], specifically Chapters 2, 4, 7, and 8, which contain a detailed proof of Corollary C above.

2 The Howe-Moore property for $SL_n(\mathbb{R})$

A continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is **mixing** if G is not compact and in the weak operator topology we have $\lim_{g\to\infty} \pi(g) = 0$, i.e., for each $\xi, \eta \in \mathcal{H}$, and $\varepsilon > 0$, there exists a compact set $L \subset G$ so that $|\langle \pi_x \xi, \eta \rangle| < \varepsilon$ for all $x \notin L$. A locally compact group G has the **Howe-Moore property** if every continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$ without non-trivial invariant vectors is mixing. In this section we'll show that for $n \geq 2$, $SL_n(\mathbb{R})$ has the Howe-Moore property [HM79]. We'll present here the proof of Veech [Vee79] which works even for uniformly bounded representations on Banach spaces.[1]

Lemma 2.1. *Suppose G is a Polish group, X is a Banach space, and $\pi : G \to \mathrm{GL}(X)$ is a uniformly bounded continuous representation. If $k_n, \tilde{k}_n, g_n \in G$ such that $k_n \to k$, $\tilde{k}_n \to \tilde{k}$, and WOT-$\lim_{n\to\infty} \pi(g_n) = T$, then WOT-$\lim_{n\to\infty} \pi(k_n g_n \tilde{k}_n) = \pi(k) T \pi(\tilde{k})$.*

Proof. We first note that as G is separable, it is enough to consider the case when X is separable. Next, note that it is easy to see that WOT-$\lim_{n\to\infty} \pi(g_n \tilde{k}_n) = T\pi(\tilde{k})$. Thus, replacing g_n with $g_n \tilde{k}_n$, and T with $T\pi(\tilde{k})$ we may assume that $\tilde{k}_n = \tilde{k} = e$.

Since X is separable, $\mathcal{B}(X)$ with the weak operator topology is metrizable on bounded subsets. Moreover, the action of G on $\mathcal{B}(X)$ given by $(x, T) \mapsto \pi(x)T$, is separately continuous. Hence, by Fort's joint continuity theorem[2] this action is jointly continuous on bounded subsets, and the result then follows. □

Lemma 2.2 (Mautner [Mau57])**.** *Suppose G is a Polish group, X is a Banach space, $\pi : G \to \mathrm{GL}(X)$ is a uniformly bounded continuous representation, and $x, y \in X$. If $b, a_n \in G$, such that we have the weak limit $\lim_{n\to\infty} \pi(a_n)x = y$, and such that $a_n^{-1} b a_n \to e$, then $\pi(b)x = y$.*

[1] Veech's interest was to show that every weakly almost periodic function on G has a limit at infinity, this follows from Theorem 2.3, together with Ryll-Nardzewski's result that the space of weakly almost periodic functions has a (unique) G-invariant mean.

[2] See Theorem 8.51 in [Kec95] for a simple proof of Fort's theorem.

Proof. Taking limits in the weak topology we have

$$\pi(b)x = \lim_{n\to\infty} \pi(ba_n)x = \lim_{n\to\infty} \pi(a_n(a_n^{-1}ba_n))x = \lim_{n\to\infty} \pi(a_n)x = y.$$

□

Fix $n \geq 2$, and set $G = SL_n(\mathbb{R})$, and $K = SO(n) < G$. We let A_+ denote the set of matrices $g \in G$ such that g is a diagonal matrix whose diagonal entries are positive and non-increasing, and we let A be the subgroup generated by A_+. Note that if $g \in G$, then if we consider the polar decomposition of g we may write $g = k_0 h$, where h is positive-definite and $k_0 \in K$. Since positive-definite matrices can be diagonalized there then exists $k_1 \in K$ so that $h = k_2^{-1} a k_2$, for $a \in A_+$. If we set $k_1 = k_0 k_2^{-1}$, then we have $g = k_1 a k_2$, with $k_1, k_2 \in K$, and $a \in A_+$. Hence, we have established the **Cartan decomposition** $G = KA_+K$.

Theorem 2.3 (Veech)**.** *Fix $m \geq 2$ and set $G = SL_m(\mathbb{R})$. Suppose X is a Banach space, and $\pi : G \to \mathrm{GL}(X)$ is a uniformly bounded continuous representation, then any weak operator topology cluster point of $\pi(G)$ is a projection onto X^G.*

Proof. We first consider the case $G = SL_2(\mathbb{R})$. Suppose $\pi : G \to \mathrm{GL}(X)$ is a uniformly bounded continuous representation on a Banach space X.

Let T be a WOT-cluster point of some sequence $\{\pi(g_n)\}_n$, where $g_n \to \infty$. Using the Cartan decomposition we may write $g_n = k_n a_n k_n'$ where $k_n, k_n' \in K$, and $a_n \in A_+$, and taking a subsequence we will assume that for $k, \tilde{k} \in K$ we have $k_n \to k$, $\tilde{k}_n \to \tilde{k}$, and WOT-$\lim_{n\to\infty} \pi(g_n) = T$. By Lemma 2.1 we then have $S = \text{WOT-}\lim_{n\to\infty} \pi(a_n) = \pi(k)T\pi(\tilde{k})$.

Write $a_n = \begin{pmatrix} r_n & 0 \\ 0 & r_n^{-1} \end{pmatrix}$, where $r_n \to \infty$, and consider the subgroup $N \subset G$ consisting of upper triangular matrices with entries 1 on the diagonal. Note that the conjugation action of $A = \langle A_+ \rangle$ on N is given by

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & r^2 s \\ 0 & 1 \end{pmatrix},$$

thus, for $x \in N$ we have $a_n^{-1} x a_n \to e \in G$. Hence, by Mautner's lemma it follows that $\pi(x)S = S$, for all $x \in N$.

Since N is non-compact we may again use the Cartan decomposition to conclude that there is a sequence $b_n \in A_+$, $h_n, \tilde{h}_n \in K$, and $h, \tilde{h} \in K$, such that $b_n \to \infty$, $h_n \to h$, $\tilde{h}_n \to \tilde{h}$, and $\pi(h_n b_n \tilde{h}_n)S = S$ for all $n \in \mathbb{N}$. Therefore we have SOT-$\lim_{n\to\infty} \pi(b_n)\pi(\tilde{h})S = \pi(h^{-1})S$.

Now fix $x \in X$ in the range of $\pi(\tilde{h})S = \pi(\tilde{h}k)T\pi(\tilde{k})$. Since $b_n \to \infty$, and $\pi(b_n)x$ converges it follows that for each $\varepsilon > 0$, $\{b \in A_+ \mid \|\pi(b)x - x\| < \varepsilon\}$ is non-compact. Thus, there exists a sequence $c_n \in A$ such that $c_n \to \infty$, and $\lim_{n\to\infty} \pi(c_n)x = x$. Since the representation is uniformly bounded we then have also that $\lim_{n\to\infty} \pi(c_n^{-1})x = x$, and hence taking a subsequence and replacing c_n with c_n^{-1} if necessary we may assume that $c_n \in A_+$. Since $\lim_{n\to\infty} \pi(c_n)x = x$ we then have from Mautner's lemma that $\pi(g)x = x$ for all $g \in N$. Since we also have $\lim_{n\to\infty} \pi(c_n^{-1})x = x$, Mautner's lemma also shows that $\pi(g)x = x$ for all $g \in N^T$.

Since $\langle N, N^T \rangle = G$ we then conclude that x is G-invariant. Since x was an arbitrary vector in the range of $\pi(k\tilde{h})T\pi(\tilde{k})$, and since $\pi(k\tilde{h}), \pi(\tilde{k}) \in \mathrm{Isom}(X)$ it then follows that every vector in the range of T is G-invariant. Thus, T is a projection onto the space of G-invariant vectors, finishing the proof for $G = SL_2(\mathbb{R})$.

We now suppose that $G = SL_m(\mathbb{R})$, with $m > 2$. We first note that the same argument as above shows that if $\{g_n\}_n$ is any sequence in G converging to infinity, and if $T \in \mathcal{B}(X)$ is a weak operator topology cluster point of $\{\pi(g_n)\}_n$, then we may find $S \in \mathcal{B}(X)$, $k, \tilde{k}, \tilde{h} \in K$, such that for any x in the range of $\pi(\tilde{h}k)T\pi(\tilde{k})$, there is a sequence $\{c_n\}_n \subset A$, with $c_n \to \infty$, such that $S = \text{WOT-lim}_{n\to\infty} \pi(a_n) = \pi(k)T\pi(\tilde{k})$, and $\lim_{n\to\infty} \pi(c_n)x = x$.

For $i \neq j$, let $N_{i,j} \subset SL_m(\mathbb{R})$ denote the subgroup consisting of matrices with diagonal entries equal to 1, and all other entries zero except possibly the (i, j)-th entry. For $I \subset \{1, 2, \ldots, m\}$ we let A_I denote the subgroup of A consisting of all diagonal matrices whose diagonal entries are 1 except possibly those diagonal entries corresponding to indices in I.

As $c_n \to \infty$, taking a subsequence if necessary, there must be $1 \leq i, j \leq m$, $i \neq j$ so that the ith diagonal entry of c_n converges to infinity, while the jth diagonal entry of c_n converges to 0. It then follows from Mautner's lemma that x is fixed by the subgroup $N_{i,j}$, and applying Mautner's lemma to the inverse sequence shows that x is also fixed by $N_{j,i}$, and hence is fixed by $A_{\{i,j\}} \subset \langle N_{i,j}, N_{j,i}\rangle$.

We let Y denote the set of $A_{\{i,j\}}$-invariant vectors, then to finish the proof it is enough to show that Y is G-invariant. Indeed, if this is the case then $A_{\{i,j\}}$ is contained in the kernel of the representation restricted to Y, and since G is simple we then have $Y = X^G$.

To see that Y is G-invariant note that $N_{i',j'}$ commutes with $A_{\{i,j\}}$ whenever $\{i, j\} \cap \{i', j'\} = \emptyset$, in which case $N_{i',j'}$ leaves A invariant. On the other hand, if $\{i', j'\} \cap \{i, j\} \neq \emptyset$ then $A_{i,j}$ acts on $N_{i',j'}$ by conjugation, and the closure of any orbit contains the identity matrix. Thus, again by Mautner's lemma we have that any vector which is fixed by $A_{i,j}$ is also fixed by $N_{i',j'}$ and, in particular, we have that $N_{i',j'}$ leaves Y invariant in this case as well.

Since G is generated by $N_{i',j'}$, for $1 \leq i', j' \leq m$, $i' \neq j'$, this then shows that Y is indeed G-invariant. □

Note that it may be the case that $\pi(G)$ has no weak operator topology cluster point, e.g., consider the translation action of G on L^1G.

If X is reflexive, e.g., if X is a Hilbert space, then it follows easily from the Banach-Alaoglu, that $\text{Isom}(X) \subset \mathcal{B}(X)$ is compact in the weak operator topology, thus a consequence of the previous theorem is the Howe-Moore property.

Corollary 2.4 (Howe-Moore)**.** *$SL_n(\mathbb{R})$ has the Howe-Moore property for $n \geq 2$.*

Corollary 2.5 (Moore's ergodicity theorem)**.** *Set $G = SL_n(\mathbb{R})$. Let $\Gamma < G$ be a lattice and suppose $H < G$ is a non-compact closed subgroup. Then the action $H \curvearrowright G/\Gamma$ is ergodic, as is the action $\Gamma \curvearrowright G/H$.*

Proof. Let μ be the G-invariant probability measure on G/Γ. Suppose a measurable subset $E \subset G/\Gamma$ is H-invariant and consider the Koopman representation $\pi : G \to \mathcal{U}(L^2(G/\Gamma, \mu))$ given by $\pi(g)(f)(x) = f(g^{-1}x)$. Then we have that 1_E is H-invariant and since H is non-compact it then follows from the Howe-Moore property that 1_E is contained in the space of G-invariant functions. As the G action is transitive it is clearly ergodic and hence 1_E is essentially constant, i.e., $\mu(E) \in \{0, 1\}$.

We have canonical identifications

$$L^\infty(G/\Gamma)^H \cong L^\infty(G)^{\Gamma\times H} \cong L^\infty(G/H)^\Gamma,$$

and hence ergodicity of the action $\Gamma \curvearrowright G/H$ follows as well. □

3 Property (T)

Let G be a locally compact group and $\pi : G \to \mathcal{U}(\mathcal{H})$ a continuous unitary representation. A vector $\xi \in \mathcal{H}$ is an **invariant vector** if $\pi_x \xi = \xi$ for all $x \in G$. The representation has **almost invariant vectors** if there is a sequence of unit vectors $\xi_n \in \mathcal{H}$ such that $\|\pi(x)\xi_n - \xi_n\| \to 0$ uniformly on compact subsets of G.

Proposition 3.1. *Let G be a group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c > 0$ such that $\mathrm{Re}(\langle \pi_x \xi, \xi \rangle) \geq c\|\xi\|^2$ for all $x \in G$, then π contains an invariant vector ξ_0 such that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$.*

Proof. Let K be the closed convex hull of the orbit $\pi(G)\xi$. Then K is G-invariant and $\mathrm{Re}(\langle \eta, \xi \rangle) \geq c\|\xi\|^2$ for every $\eta \in K$. Let $\xi_0 \in K$ be the unique element of minimal norm, then since G acts isometrically we have that for each $x \in G$, $\pi_x \xi_0$ is the unique element of minimal norm for $\pi_x K = K$, and hence $\pi_x \xi_0 = \xi_0$ for each $x \in G$. Since $\xi_0 \in K$ we have that $\mathrm{Re}(\langle \xi_0, \xi \rangle) \geq c\|\xi\|^2$. □

Corollary 3.2. *Let G be a group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c < \sqrt{2}$ such that $\|\pi_x \xi - \xi\| \leq c\|\xi\|$ for all $x \in G$, then π contains a non-zero invariant vector.*

Proof. For each $x \in G$ we have

$$2\mathrm{Re}(\langle \pi_x \xi, \xi \rangle) = 2\|\xi\|^2 - \|\pi_x \xi - \xi\|^2 \geq (2 - c^2)\|\xi\|^2.$$

Hence, we may apply Proposition 3.1. □

Let G be a locally compact group, and $H < G$ a closed subgroup. The pair (G, H) has **relative property (T)** if every representation of G which has almost invariant vectors, has a non-zero H-invariant vector. G has **property (T)** if the pair (G, G) has relative property (T).

Proposition 3.3. *Let G be a locally compact group and $H \lhd G$ a closed normal subgroup, then the following are equivalent:*

1. *The pair (G, H) has relative property (T).*
2. *For any sequence of positive definite functions $\varphi_n : G \to \mathbb{C}$ such that $\varphi_n \to 1$ uniformly on compact subsets we have that $\sup_{x \in H} |1 - \varphi_n(x)| \to 0$.*

Proof. First suppose that the pair (G, H) has relative property (T), and $\varphi_n : G \to \mathbb{C}$ is a sequence of positive definite functions such that $\varphi_n \to 1$ uniformly on compact subsets. Normalizing φ_n we assume that $\varphi_n(e) = 1$. By the GNS-construction there exists a sequence of representations $\pi_n : G \to \mathcal{U}(\mathcal{H}_n)$ and unit vectors $\xi_n \in \mathcal{H}_n$ so that $\varphi_n(x) = \langle \pi_n(x)\xi_n, \xi_n \rangle$. We let $\mathcal{H}_n^0$ denote the space of H-invariant vectors in $\mathcal{H}_n$ and note that since H is normal we have that G preserves $\mathcal{H}_n^0$ and hence also preserves its orthogonal compliment ${\mathcal{H}_n^0}^\perp$. We let η_n denote the projection of ξ_n to ${\mathcal{H}_n^0}^\perp$ and ζ_n denote the projection of ξ_n to $\mathcal{H}_n^0$, so that ζ_n is H-invariant. Then $\oplus_n {\mathcal{H}_n^0}^\perp$ is a representation without H-invariant vectors, and η_n are almost invariant. Since (G, H) has relative property (T) we then conclude that $\|\eta_n\| \to 0$, and hence $\xi_n - \zeta_n \to 0$. Thus by Cauchy-Schwarz we have

$$\sup_{x\in H} |1 - \varphi_n(x)| = \sup_{x\in H} |\langle \xi_n - \pi_n(x)\xi_n, \xi_n\rangle| \leq 2\|\xi_n - \zeta_n\| \to 0.$$

Conversely, if (G, H) does not have relative property (T) then there exists a representation $\pi : G \to \mathcal{U}(\mathcal{H})$ without H-invariant vectors, but having almost invariant unit vectors $\xi_n \in \mathcal{H}$. If we set $\varphi_n(x) = \langle \pi(x)\xi_n, \xi_n\rangle$, then φ_n is a sequence of positive definite functions which converges uniformly on compact subsets to 1. However φ_n cannot converge uniformly on H by Proposition 3.1. □

We remark that the previous lemma also holds in the case when H is not normal, although the proof is quite a bit more complicated [Jol05].

Suppose that G and A are locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism, and let $\hat{\alpha} : G \to \mathrm{Aut}(\hat{A})$ denote the dual homomorphism given by $\hat{\alpha}_x(\chi) = \chi \circ \alpha_{x^{-1}}$, for $x \in G$, and $\chi \in \hat{A}$. Note that for $f \in L^1H$, and $x \in G$ we have $\widehat{f \circ \alpha_x}(\chi) = \int f \circ \alpha_x(y)\overline{\chi}(y)\,\mathrm{d}y = \int f(y)\overline{\hat{\alpha}_x(\chi)(y)}\frac{\mathrm{d}\alpha_{x^{-1}}y}{\mathrm{d}y}\,\mathrm{d}y = \left(\widehat{f\frac{\mathrm{d}\alpha_{x^{-1}}y}{\mathrm{d}y}}\right) \circ \hat{\alpha}_x(\chi)$.

Lemma 3.4. *Let G and A be as above. Suppose $\{\varphi_i\}_{i\in I} \subset \mathcal{P}_1(G \ltimes A)$ is a net of positive definite functions, and let $\nu_i \in \mathrm{Prob}(\hat{A})$ denote the net of probability measures which corresponds by Bochner's theorem to the functions φ_i restricted to A. If $\varphi_i \to 1$ uniformly on compact subsets of G then $\|\hat{\alpha}_x\nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.*

Proof. We let $\pi_i : G \ltimes A \to \mathcal{U}(\mathcal{H}_i)$ be the cyclic representations associated to φ_i with cyclic vector $\xi_i \in \mathcal{H}_i$. For $f \in L^1H$ we have

$$\begin{aligned}\int \hat{f}\,\mathrm{d}\hat{\alpha}_x\nu_i &= \int \hat{f} \circ \hat{\alpha}_x\,\mathrm{d}\nu_i \\ &= \int f \circ \alpha_x(y)\langle \pi_i(y)\xi_i, \xi_i\rangle\,\mathrm{d}\alpha_x(y) \\ &= \int f(y)\langle \pi_i(y)\pi(x^{-1})\xi_i, \pi(x^{-1})\xi_i\rangle\,\mathrm{d}y \\ &= \langle \pi_i(f)\pi_i(x^{-1})\xi_i, \pi(x^{-1})\xi_i\rangle.\end{aligned}$$

Thus, we have $|\int \hat{f}\,\mathrm{d}\hat{\alpha}_x\nu_i - \int \hat{f}\,\mathrm{d}\nu_i| \leq 2\|\pi_i(f)\|\|\xi_i - \pi_i(x)\xi_i\| = 2\|\hat{f}\|_\infty\|\xi_i - \pi(x)\xi_i\|$.

Since $\mathcal{F}(L^1H)$ is dense in $C_0\hat{A}$ it then follows that $\|\hat{\alpha}_x\nu_i - \nu_i\| \leq 2\|\xi_i - \pi_i(x)\xi_i\| \to 0$ uniformly on compact subsets of G. □

Lemma 3.5. *Let G and A be second countable locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism. Then the following conditions are equivalent:*

(i) *$(G \ltimes A, A)$ does not have relative property (T).*

(ii) *There exists a net $\{\nu_i\}_{i\in I} \subset \mathrm{Prob}(\hat{A})$, such that $\nu_i(\{e\}) = 0$, $\nu_i \to \delta_{\{e\}}$ weak*, and $\|\hat{\alpha}_y\nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.*

Proof. For (i) $\implies$ (ii), suppose that $(G \ltimes A, A)$ does not have relative property (T). Thus, there exists a continuous representation $\pi : G \ltimes A \to \mathcal{U}(\mathcal{H})$ without A-invariant vectors, and a net of unit vectors $\{\xi_i\}_i \subset \mathcal{H}$ such that $\|\pi(x)\xi_i - \xi_i\| \to 0$ uniformly on compact subsets of $G \ltimes H$.

We let $\varphi_i : G \ltimes A \to \mathbb{C}$ denote the function of positive definite given by $\varphi_i(x) = \langle \pi(x)\xi_i, \xi_i \rangle$, and we let $\nu_i \in \mathrm{Prob}(\hat{A})$ denote the probability measure corresponding to φ_i restricted to A, given by Bochner's theorem. Since π does not have A-invariant vectors we have that $\nu_i(\{e\}) = 0$. As $\varphi_i \to 1$ uniformly on compact sets of A we have $\nu_i \to \delta_{\{e\}}$ weak*, and since $\varphi_i \to 1$ uniformly on compact subsets of G by the previous lemma we have that $\|\hat{\alpha}_x \nu_i - \nu_i\| \to 0$ uniformly on compact subsets of G.

Conversely, for (ii) $\implies$ (i), let $\{\nu_i\}_{i \in I} \subset \mathrm{Prob}(A)$ be the net given by (ii). Fix $\mu_0 \in \mathrm{Prob}(G)$ in the same measure class as Haar measure, and set $\tilde{\nu}_i = \hat{\alpha}(\mu_0) * \nu_i = \int \hat{\alpha}_x \nu_i \, \mathrm{d}\mu_0(x)$. Then we again have that $\tilde{\nu}_i \to \delta_{\{e\}}$ weak*, and $\|\hat{\alpha}_x \tilde{\nu}_i - \tilde{\nu}_i\| \to 0$ uniformly on compact subsets, and moreover we have that $\tilde{\nu}_i$ is quasi-invariant for the G action on $\hat{A}$.

We define $\pi_i : G \ltimes H \to \mathcal{U}(L^2(\hat{A}, \tilde{\nu}_i))$ by $\pi_i(xh) = U_x h$ where U is the Koopman representation corresponding to the action of G on $(\hat{A}, \tilde{\nu}_i)$. Then π_i gives a unitary representation, and as $\tilde{\nu}_i \to \delta_{\{e\}}$ weak* we see that $\{\xi_i\}_{i \in I}$ forms a net of almost invariant vectors for H. Moreover, for $x \in G$ we have

$$
\begin{aligned}
|\langle \pi_i(x)\xi_i - \xi_i, \xi_i \rangle| &\leq \int \left| \left(\frac{\mathrm{d}\hat{\alpha}_x \nu_i}{\mathrm{d}\nu_i} \right)^{1/2} - 1 \right| \mathrm{d}\nu_i \\
&\leq \left\| \frac{\mathrm{d}\hat{\alpha}_x \nu_i}{\mathrm{d}\nu_i} - 1 \right\|_1^{1/2} \\
&\leq \|\hat{\alpha}_x \nu_i - \nu_i\|^{1/2}.
\end{aligned}
$$

Hence, $\{\xi_i\}_{i \in I}$ also forms a net of almost invariant vectors for G. We let $\mathcal{K}_i \subset L^2(\hat{A}, \tilde{\nu}_i)$ denote the space of A-invariant vectors. Then as $A \lhd G \ltimes A$ is normal it follows that $\mathcal{K}_i$ is $A \lhd G$-invariant and hence so is $\mathcal{K}_i^\perp$. If $(G \ltimes A, A)$ had relative property (T) then since $\{P_{\mathcal{K}_i}^\perp \xi_i\}_{i \in I}$ is almost invariant for $G \ltimes A$ in $\oplus_{i \in I} \pi_i$ it then follows that we must have $P_{\mathcal{K}_i}^\perp \xi_i \to 0$. Hence, it follows that $\tilde{\nu}_i(\{e\}) \to 1$. However, $\tilde{\nu}_i(\{e\}) = \tilde{\nu}_i(\{e\}) - \nu_i(\{e\}) \to 0$ and we would then have a contradiction. ☐

Corollary 3.6. *Let G and A be second countable locally compact groups such that A is abelian. Suppose $\alpha : G \to \mathrm{Aut}(A)$ is a continuous homomorphism. If $(G \ltimes A, A)$ does not have relative property (T) then there exists a state $\varphi \in B_\infty(\hat{A})^*$ such that $\varphi(1_{\{e\}}) = 0$, $\varphi(1_O) = 1$ for every neighborhood O of e, and $\varphi(f \circ \hat{\alpha}_x) = \varphi(f)$ for all $f \in B_\infty(\hat{A})$, and $x \in G$.*

Proof. Suppose $(G \ltimes A, A)$ does not have relative property (T), and let $\{\nu_i\}_{i \in I} \subset \mathrm{Prob}(\hat{A})$ be as in the previous lemma. Then each ν_i gives a state on $B_\infty(\hat{A})$. If we let φ be a weak*-cluster point of $\{\nu_i\}_{i \in I}$ then we have $\varphi(1_{\{e\}}) = 0$ since $\nu_i(\{e\}) = 0$ for all $i \in I$. We also have $\varphi(1_O) = 1$ for every neighborhood O of e since $\nu_i \to \delta_{\{e\}}$ weak*. And we have $\varphi(f \circ \hat{\alpha}_x) = \varphi(f)$ for all $f \in B_\infty(\hat{A})$, and $x \in G$, since $\|\hat{\alpha}_x \nu_i - \nu_i\| \to 0$ for all $x \in G$. ☐

We consider the natural action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$ given by matrix multiplication, and let $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$ be the semi-direct product.

Theorem 3.7. *The pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative property (T).*

Proof. Under the identification $\hat{\mathbb{R}}^2 = \mathbb{R}^2$ given by the pairing $\langle a, \xi \rangle = e^{\pi i a \cdot \xi}$ we have that the dual action of $SL_2(\mathbb{R})$ is given by matrix multiplication with the inverse transpose.

Suppose that $\varphi \in B_\infty(\mathbb{R}^2)^*$ is a $SL_2(\mathbb{R})$-invariant state such that $\varphi(1_O) = 1$ for any neighborhood O of $(0,0)$.

We set

$$A = \{(x,y) \in \mathbb{R}^2 \mid x > 0, -x < y \leq x\};$$
$$B = \{(x,y) \in \mathbb{R}^2 \mid y > 0, -y \leq x < y\};$$
$$C = \{(x,y) \in \mathbb{R}^2 \mid x < 0, x \leq y < -x\};$$
$$D = \{(x,y) \in \mathbb{R}^2 \mid y < 0, y < x \leq -y\}.$$

A simple calculation shows that for $k \geq 0$ the sets $A_k = \left(\begin{smallmatrix} 1 & 0 \\ 2^k & 1 \end{smallmatrix}\right) A$ are pairwise disjoint. Thus, we must have that $\varphi(1_A) = 0$. A similar argument also shows that $\varphi(1_B) = \varphi(1_C) = \varphi(1_D) = 0$. Hence we conclude that $\varphi(1_{\{(0,0)\}}) = 1 - \varphi(1_{A \cup B \cup C \cup D}) = 1$. By the previous corollary it then follows that $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative property (T). □

Theorem 3.8 ([Kaž67])**.** *$SL_m(\mathbb{R})$ has property (T) for $m \geq 3$.*

Proof. We consider the group $SL_2(\mathbb{R}) < SL_m(\mathbb{R})$ embedded as matrices in the upper left corner. We also consider the group $\mathbb{R}^2 < SL_m(\mathbb{R})$ embedded as those matrices with 1's on the diagonal, and all other entries zero except possibly the $(1,n)$th, and $(2,n)$th entries. Note that the embedding of $SL_2(\mathbb{R})$ normalizes the embedding of $\mathbb{R}^2$, and these groups generate a copy of $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

If $\pi : SL_m(\mathbb{R})$ is a representation which has almost invariant vectors, then by Theorem 3.7 we have that the copy of $\mathbb{R}^2$ has a non-zero invariant vector. By the Howe-Moore property it then follows that π has an $SL_m(\mathbb{R})$-invariant vector. □

Suppose that G is a locally compact group and $\Gamma < G$ is a lattice. If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, we denote by $L^2(G, \mathcal{H})^\Gamma$ the set of measurable function $f : G \to \mathcal{H}$ which satisfy $\pi(\gamma^{-1}) f(g\gamma) = f(g)$ for all $g \in G$, and $\gamma \in \Gamma$, and $\int_{G/\Gamma} \|f(g)\|^2 \mathrm{d}g < \infty$, where we identify two functions if they agree almost everywhere. We define an inner-product on $L^2(G, \mathcal{H})^\Gamma$ by $\langle f_1, f_2 \rangle = \int_{G/\Gamma} f_1(g) \overline{f_2(g)} \, \mathrm{d}g$. With this inner-product it is not hard to see that $L^2(G, \mathcal{H})^\Gamma$ forms a Hilbert space.

The **induced representation** $\tilde{\pi} : G \to \mathcal{U}(L^2(G, \mathcal{H})^\Gamma)$ is given by $(\tilde{\pi}(x) f)(y) = f(x^{-1} y)$. It is easy to see that $\tilde{\pi}$ gives a continuous unitary representation of G.

Theorem 3.9. *[Kaž67] Let G be a second countable locally compact group, and $\Gamma < G$ a lattice, if G has property (T) then Γ has property (T).*

Proof. We fix a Borel fundamental domain Σ for Γ so that the map $\Sigma \times \Gamma \ni (\sigma, \gamma) \to \sigma\gamma \in G$ is a Borel isomorphism, and we choose a Haar measure on G so that Σ has measure 1. We let $\alpha : G \times \Sigma \to \Gamma$ be defined so that $\alpha(g, \sigma)$ is the unique element in Γ which satisfies $g\sigma\alpha(g, \sigma) \in \Sigma$.

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation with almost invariant unit vectors $\{\xi_n\}_{n \in \mathbb{N}}$. We consider the vectors $\tilde{\xi}_n \in L^2(G, \mathcal{H})^\Gamma$ given by $\tilde{\xi}_n(\sigma\gamma) = \pi(\gamma)\xi_n$, for $\sigma \in \Sigma$, $\gamma \in \Gamma$. Then for $g \in G$ we have

$$\|\tilde{\pi}(g)\tilde{\xi}_n - \tilde{\xi}_n\|^2 = \int_{G/\Gamma} \|\tilde{\xi}_n(g^{-1}x) - \tilde{\xi}_n(x)\|^2 \, \mathrm{d}x = \int_\Sigma \|\xi_n - \pi(\alpha(g^{-1}, \sigma))\xi_n\|^2 \, \mathrm{d}\sigma \to 0.$$

Thus, $\{\tilde{\xi}_n\}_{n\in\mathbb{N}}$ forms a sequence of almost invariant vectors for $\tilde{\pi}$. Since G has property (T), it follows that there exists a non-zero invariant vector $\tilde{\xi}_0 \in L^2(G, \mathcal{H})^\Gamma$, i.e., $\tilde{\xi}_0(gx) = \tilde{\xi}_0(x)$ for almost all $g, x \in G$. It then follows that $\tilde{\xi}_0$ is essentially constant. We let $\xi_0 \neq 0$ denote the essential range of $\tilde{\xi}_0$. Since $\tilde{\xi}_0 \in L^2(G, \mathcal{H})^\Gamma$ we have that $\pi(\gamma)\xi_0 = \xi_0$ for all $\gamma \in \Gamma$. Thus, $\xi_0 \in \mathcal{H}$ is a non-zero invariant vector and hence Γ has property (T). □

We remark that the converse of the previous theorem is also true.

4 Amenability

A (left) **invariant mean** m on a locally compact group G is a finitely additive Borel probability measure on G, which is absolutely continuous with respect to Haar measure, and which is invariant under the action of left multiplication, i.e., $m : \text{Borel}(G) \to [0, 1]$ such that $m(G) = 1$, $m(E) = 0$ if $\lambda(E) = 0$, where λ is the Haar measure on G, and if $A_1, \ldots, A_n \in \text{Borel}(G)$ are disjoint then $m(\cup_{j=1}^n A_n) = \sum_{j=1}^n m(A_n)$, and if $A \in \text{Borel}(G)$, then $m(xA) = m(A)$ for all $x \in G$. If G possesses an invariant mean then G is **amenable**. We can similarly define right invariant means, and in fact if m is a left invariant mean then $m^*(A) = m(A^{-1})$ defines a right invariant mean. Amenable groups were first introduced by von Neumann in his investigations of the Banach-Tarski paradox.

Given a right invariant mean m on G it is possible to define an integral over G just as in the case if m were a countably additive measure. We therefore obtain a state $\phi_m \in (L^\infty G)^*$ by the formula $\phi_m(f) = \int f\, dm$, and this state is left invariant, i.e., $\phi_m(L_x(f)) = \phi_m(f)$ for all $x \in G$, $f \in L^\infty G$. Conversely, if $\phi \in (L^\infty G)^*$ is a left invariant state, then restricting ϕ to characteristic functions defines a right invariant mean.

Example 4.1. Let $\mathbb{F}_2$ be the free group on two generators a, and b. Let A^+ be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a, let A^- be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a^{-1}, let B^+, and B^- be defined analogously, and consider $C = \{e, b, b^2, \ldots\}$. Then we have that

$$\mathbb{F}_2 = A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C)$$
$$= A^+ \sqcup aA^-$$
$$= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C).$$

If m were a left-invariant mean on $\mathbb{F}_2$ then we would have

$$m(\mathbb{F}_2) = m(A^+) + m(A^-) + m(B^+ \setminus C) + m(B^- \cup C)$$
$$= m(A^+) + m(aA^-) + m(b^{-1}(B^+ \setminus C)) + m(B^- \cup C)$$
$$= m(A^+ \sqcup aA^-) + m(b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)) = 2m(\mathbb{F}_2).$$

Hence, $\mathbb{F}_2$ is non-amenable.

An **approximately invariant mean** on G is a net $f_i \in L^1(G)_+$ such that $\int f_i = 1$, and $\|L_x(f_i) - f_i\|_1 \to 0$, uniformly on compact subsets of G.

A **Følner net** is a net of non-null finite measure Borel subsets $F_i \subset G$ such that $\lambda(F_i \Delta x F_i)/\lambda(F_i) \to 0$, uniformly on compact subsets of G. Note that we do not require that $\Gamma = \cup_i F_i$, nor do we require that F_i are increasing, however, if G is not compact then it is easy to see that any Følner net $\{F_i\}_i$ must satisfy $\lambda(F_i) \to \infty$.

Theorem 4.2. *Let G be a locally compact group, then the following conditions are equivalent.*

(i) *G is amenable.*

(ii) *$C_{\rm b}G$ admits a left invariant state.*

(iii) *$C_{\rm b}^{\rm lu}(G)$ admits a left invariant state.*

(iv) *$L^\infty G$ has an $L^1 G$-invariant state.*

(v) *G has an approximate invariant mean.*

(vi) *The left regular representation $\lambda : G \to \mathcal{U}(L^2 G)$ has almost invariant vectors.*

(vii) *The representation $\lambda : G \to \mathcal{U}(L^2 G)$ has almost invariant vectors when G is viewed as a discrete group.*

(viii) *The (discontinuous) action of G on its Stone-Čech compactification βG which is induced by left-multiplication admits an invariant Radon probability measure.*

(ix) *Any continuous action $G \curvearrowright K$ on a compact metric space K admits an invariant Radon probability measure.*

Proof. First note that (i) $\implies$ (ii), and (ii) $\implies$ (iii) are obvious.

To see (iii) $\implies$ (iv) suppose m is a left invariant state on $C_{\rm b}^{\rm lu}(G)$. Note that since G acts continuously on $C_{\rm b}^{\rm lu}(G)$ it follows that for all $f \in L^1 G$, and $g \in C_{\rm b}^{\rm lu}(G)$ we have that the integral $f * g = \int f(y) \delta_y * g \, d\lambda(y)$ converges in norm. Hence we have $m(f * g) = m(g)$, for all $f \in L^1(G)_{1,+} = \{f \in L^1 G \mid f \geq 0; \int f = 1\}$ and $g \in C_{\rm b}^{\rm lu}(G)$.

Fix $f_0 \in L^1(G)_{1,+}$ and define a state on $L^\infty G$ by $\tilde{m}(g) = m(f_0 * g)$ (recall that $L^1 G * L^\infty G \subset C_{\rm b}^{\rm lu}(G)$). If $\{f_n\}_{n \in \mathbb{N}} \subset L^1(G)_{1,+}$ is an approximate identity then $m(f_0 * g) = \lim_{n \to \infty} m(f_0 * f_n * g) = \lim_{n \to \infty} m(f_n * g)$, and thus $\tilde{m}$ is independent of f_0. Thus, for $f \in L^1(G)_{1,+}$ and $g \in L^\infty G$ we have $\tilde{m}(f * g) = m(f_0 * f * g) = \tilde{m}(g)$.

We show (iv) $\implies$ (v) using the method of Day: Since $L^\infty G = (L^1 G)^*$, the unit ball in $L^1 G$ is weak*-dense in the unit ball of $(L^\infty G)^* = (L^1 G)^{**}$. It follows that $L^1(G)_{1,+}$ is weak*-dense in the state space of $L^\infty G$.

Let $S \subset L^1 G_{+,1}$, be finite and let $K \subset \prod_{f \in S} L^1 G$ be the weak-closure of the set $\{S \ni g \mapsto (g * f - f) \mid f \in L^1(G)_{1,+}\}$. Since G has an $L^1 G$-invariant state on $L^\infty G$, and since $L^1(G)_{1,+}$ is weak*-dense in the state space of $L^\infty G$, we have that $0 \in K$. However, K is convex and so by the Hahn-Banach separation theorem the weak-closure coincides with the norm closure. Thus, there exists a net $\{f_i\} \subset L^1(G)_{1,+}$ such that for all $g \in L^1 G_{+,1}$ we have

$$\|g * f_i - f_i\|_1 \to 0.$$

If $S \subset L^1G_{+,1}$ is compact then the above convergence may be taken uniformly for $g \in S$. Indeed, if $\varepsilon > 0$ then let $S_0 \subset S$ be finite such that $\inf_{g_0 \in S_0} \|g - g_0\|_1 < \varepsilon$, for all $g \in S$. Then there exists f_i such that $\|g_0 * f_i - f_i\|_1 < \varepsilon$ for all $g_0 \in S_0$, and we then have $\|g * f_i - f_i\|_1 < 2\varepsilon$ for all $g \in S$.

If $K \subset G$ is compact and $g_0 \in L^1G_{+,1}$, then since the action of G on L^1G is continuous it follows that $\{L_k(g_0) \mid k \in K\} \subset L^1G_{+,1}$ is compact, and hence we have

$$\lim_{i\to\infty} \sup_{k \in K} \|L_k(g_0 * f_i) - (g_0 * f_i)\|_1 \leq \lim_{i\to\infty} \sup_{k \in K} \|L_k(g_0) * f_i - f_i\|_1 + \|f_i - g_0 * f_i\|_1 = 0.$$

For (v) $\implies$ (vi) just notice that if $\{f_i\}_i \subset L^1(G)_!$ is an approximately invariant mean, then $\{\sqrt{f_i}\}_i \subset L^2G$ is a net of almost invariant vectors.

(vi) $\implies$ (vii) is obvious. For (vii) $\implies$ (i) let $\xi_i \in L^2G$ be a net of almost invariant vectors for G as a discrete group. We define states φ_i on $\mathcal{B}(L^2G)$ by $\varphi_i(T) = \langle T\xi_i, \xi_i\rangle$. By weak* compactness of the state space, we may take a subnet and assume that this converges in the weak* topology to $\varphi \in \mathcal{B}(L^2G)^*$. We then have that for all $T \in \mathcal{B}(L^2G)$ and $x \in G$,

$$|\varphi(\lambda(x)T - T\lambda(x))| = \lim_i |\langle(\lambda(x)T - T\lambda(x))\xi_i, \xi_i\rangle|$$

$$= \lim_i |\langle T\xi_i, \lambda(x^{-1})\xi_i\rangle - \langle T\lambda(x)\xi_i, \xi_i\rangle|$$

$$\leq \lim_i \|T\|(\|\lambda(x^{-1})\xi_i - \xi_i\| + \|\lambda(x)\xi_i - \xi_i\|) = 0.$$

We consider the usual embedding $M : L^\infty G \to \mathcal{B}(L^2G)$ given by point-wise multiplication. For $f \in L^\infty G$ and $x \in G$ we have $\lambda(x)M_f\lambda(x^{-1}) = M_{L_x(f)}$. Thus, restricting φ to $L^\infty G$ gives a state on $L^\infty G$ which is G-invariant.

(ii) $\iff$ (viii), follows from the G-equivariant identification $C_bG \cong C(\beta G)$, together with the Riesz representation theorem.

For (viii) $\implies$ (ix), suppose G acts continuously on a compact Hausdorff space K, and fix a point $x_0 \in K$. Then the map $f(g) = gx_0$ on G extends uniquely to a continuous map $\beta f : \beta G \to K$, moreover since f is G-equivariant, so is βf. If μ is an invariant Radon probability measure for the action on βG then we obtain the invariant Radon probability measure $f_*\mu$ on K.

For (ix) $\implies$ (iii), fix $F \subset C_b^{\mathrm{lu}}G$ any finite subset. Then since G acts continuously on $C_b^{\mathrm{lu}}G$ it follows that the G-invariant unital C^*-subalgebra $A \subset C_b^{\mathrm{lu}}G$ generated by F is separable, and hence $\sigma(A)$ is a compact metrizable space and the natural action of G on $\sigma(A)$ is continuous. By hypothesis there then exists a G-invariant probabiliy measure on $\sigma(A)$ which corresponds to a G-invariant state φ_F on A. By the Hahn-Banach theorem we may extend φ_F to a (possibly no longer G-invariant) state $\tilde{\varphi}_F$ on $C_b^{\mathrm{lu}}G$. Considering the finite subsets of $C_b^{\mathrm{lu}}G$ as a partially ordered set by inclusion we then have a net of states $\{\tilde{\varphi}_F\}$ on $C_b^{\mathrm{lu}}G$, and we may let φ_∞ be a cluster point of this set. It is then easy to see that φ_∞ is G-invariant. □

The previous theorem is the combined work of many mathematicians, including von Neumann, Følner, Day, Namioka, Hulanicki, Reiter, and Kesten.

Example 4.3. Any compact group is amenable, and any group which is locally amenable (each compactly generated subgroup is amenable) is also amenable. The group $\mathbb{Z}^n$ is amenable (consider the Følner sequence $F_k = \{1, \ldots, k\}^n$ for example). From this it then follows easily that all discrete abelian groups are amenable. Moreover, from part (ix) we see that if a locally compact group is amenable as a discrete group then it is also amenable as a locally compact group, thus all abelian locally compact groups are amenable.

Closed subgroups of amenable groups are also amenable (hence any locally compact group containing $\mathbb{F}_2$ as a closed subgroup is non-amenable). This follows from the fact that if $H < G$ is a closed subgroup, then there exists a Borel set $\Sigma \subset G$ such that the map $H \times \Sigma \ni (h, \sigma) \mapsto h\sigma \in G$ gives a Borel bijection. This then gives an H-equivariant homomorphism θ from $L^\infty H \to L^\infty G$, given by $\theta(f)(h\sigma) = f(h)$. Restricting a G-invariant state to the image of $L^\infty H$ then gives an H-invariant state on $L^\infty H$.

If G is amenable and $H \lhd G$ is a closed subgroup then G/H is again amenable. Indeed, we may view $L^\infty(G/H)$ as the space of right H-invariant functions in $L^\infty G$, and hence we may restrict a G-invariant mean to $L^\infty(G/H)$.

From part (ix) in Theorem 4.2 it follows that if $H \lhd G$ is closed, such that H and G/H are amenable then G is also amenable. Indeed, if $G \curvearrowright K$ is a continuous action on a compact Hausdorff space, then if we consider $\mathrm{Prob}(K)^H$ the set of H-invariant probability measures, then $\mathrm{Prob}(K)^H$ is a non-empty compact set on which G/H acts continuously. Thus there is a G/H-invariant probability measure $\tilde{\mu} \in \mathrm{Prob}(\mathrm{Prob}(K)^H)$ and if we consider the barycenter $\mu = \int \nu \, d\tilde{\mu}(\nu)$, then μ is a G invariant probability measure on K.

It then follows that all solvable groups amenable, e.g., the group of upper triangular real invertible matrices.

5 Finite von Neumann algebras

Fix a separable Hilbert space $\mathcal{H}$. A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is **finite** if there exists a linear functional $\tau : M \to \mathbb{C}$ (called the trace) which satisfies the following properties:

1. τ is a state, i.e., $\tau(1) = 1$ and $\tau(x^*x) \geq 0$ for all $x \in M$;

2. τ is faithful, i.e., $\tau(x^*x) = 0$ if and only if $x = 0$;

3. τ is normal, i.e., τ is continuous with respect to the strong operator topology (or equivalently, τ is continuous with respect to the weak operator topology).

4. τ is tracial, i.e., $\tau(xy) = \tau(yx)$ for all $x, y \in M$.

By a tracial von Neumann algebra we mean a pair (M, τ), where M is a finite von Neumann algebra and τ is a trace on M. We remark that by a result of Murray and von Neumann, if M is a finite factor then the trace is unique.

Given a finite von Neumann algebra M with a normal faithful tracial state τ, we obtain an inner-product on M by

$$\langle x, y \rangle = \tau(y^*x).$$

We denote by $\| \cdot \|_2$ the resulting norm from this inner-product, and we denote by $L^2(M, \tau)$ the resulting Hilbert space completion. We then have a normal representation of M acting on $L^2(M, \tau)$ given by left multiplication. (This is the GNS-construction). Thus, we may view M both as an algebra of bounded operators on $\mathcal{B}(L^2(M,\tau))$ and as a dense subspace of the Hilbert space $L^2(M,\tau)$. When we wish to emphasize the latter perspective we will write $\hat{x}$ for an element $x \in M$ when it is viewed as an element in $L^2(M,\tau)$. We will also write $P_{\hat{x}}$ to denote the rank 1 projection onto $\mathbb{C}\hat{x} \subset L^2(M,\tau)$. When M is a finite factor, the trace is unique, and so, in this case, we use the notation L^2M, for $L^2(M,\tau)$.

The tracial property of τ entails that the conjugation map $M \ni \hat{x} \mapsto \hat{x^*}$ extends to a conjugate linear isometry $J : L^2(M,\tau) \to L^2(M,\tau)$. We than have $JyJ\hat{x} = \widehat{xy^*}$, and from this we see that $JMJ \subset M'$, and hence $M \subset JMJ'$.

Lemma 5.1. *Suppose (M,τ) is a tracial von Neumann algebra. Then on bounded sets the strong operator topology is equivalent to the topology induced from $\langle \cdot, \cdot \rangle$, and the weak operator topology is equivalent to the weak topology induced from $\langle \cdot, \cdot \rangle$.*

Proof. If $\{x_i\}_i$ is a uniformly bounded net such that $x_i \to x$ in the strong operator topology, then we have $\|x_i - x\|_2 = \|(x_i - x)\hat{1}\|_2 \to 0$. Conversely, if we have $\|x_i - x\|_2 \to 0$, then for any $y \in M$ we have $\|(x_i - x)\hat{y}\| = \|Jy^*J(x_i - x)\hat{1}\| \leq \|y\|\|x_i - x\|_2 \to 0$. Since $\hat{M}$ is dense in $L^2(M,\tau)$ and since $\{x_i\}_i$ is uniformly bounded it then follows that $x_i \to x$ in the strong operator topology.

Similarly, if $\{x_i\}_i$ is a uniformly bounded net such that $x_i \to x$ weakly, then for all $\xi \in L^2(M,\tau)$, we have $\langle (x_i - x)\hat{1}, \xi \rangle \to 0$. Conversely, if $\langle (x_i - x)\hat{1}, \xi \rangle \to 0$ for every $\xi \in L^2(M,\tau)$, then for each $y \in M$ and $\xi \in L^2(M,\tau)$ we have $\langle (x_i - x)\hat{y}, \xi \rangle = \langle (x_i - x)\hat{1}, (JyJ)\xi \rangle \to 0$. Just as above it then follows that $x_i \to x$ in the weak operator topology. □

Corollary 5.2. *The unit ball $(M)_1$ is a closed subset of $L^2(M,\tau)$.*

Proof. Any limit point in $(M)_1 \subset L^2(M,\tau)$ must also be a weak limit point. Since $(M)_1$ is compact in the weak operator topology it then follows from the previous lemma that any weak limit point in $(M)_1 \subset L^2(M,\tau)$ must be contained in $(M)_1$. □

Proposition 5.3. *If (M,τ) is a tracial von Neumann algebra then we have the equality $JMJ = M'$ and $M = JM'J$.*

Proof. If $x, y \in M$ then $\langle J\hat{x}, J\hat{y} \rangle = \tau(yx^*) = \langle \hat{y}, \hat{x} \rangle$. Thus, for all $\xi, \eta \in L^2(M,\tau)$ we have $\langle J\xi, J\eta \rangle = \langle \eta, \xi \rangle$.

If $a \in M'$ and $z \in M$ we have

$$\langle JaJ\hat{1}, \hat{z} \rangle = \langle J\hat{z}, aJ\hat{1} \rangle = \langle a^* z^* \hat{1}, \hat{1} \rangle = \langle a^* \hat{1}, \hat{z} \rangle$$

Since M is dense in $L^2(M,\tau)$ we then have $JaJ\hat{1} = a^*\hat{1}$.

If we also have $b \in M'$ then

$$\langle bJaJ\hat{1}, \hat{z} \rangle = \langle a^* \hat{z^*}, b^* \hat{1} \rangle = \langle \hat{z^*}, aJbJ\hat{1} \rangle = \langle JaJb\hat{1}, \hat{z} \rangle,$$

Since $z \in M$ is arbitrary we then have $bJaJ\hat{1} = JaJb\hat{1}$. If $a, b, c \in M'$ then applying this last equation with either ac or c in place of a gives

$$(bJaJ)Jc\hat{1} = JacJb\hat{1} = (JaJb)Jc\hat{1}.$$

Since $JM'\hat{1}$ is dense in $L^2(M,\tau)$ it then follows that $b(JaJ) = (JaJ)b$. Thus, $JaJ \in M'' = M$ and so $JM'J \subset M \subset JM'J$.

Therefore $JM'J = M$ and taking commutants give $JMJ = M'$. □

If Γ is a countable group, the group von Neumann algebra is the von Neumann algebra $L\Gamma \subset \mathcal{B}(\ell^2\Gamma)$ generated by the left-regular representation of Γ. This is a finite von Neumann algebra as $L\Gamma \ni x \mapsto \langle x\delta_e, \delta_e \rangle$ is easily seen to give a normal faithful tracial state. Since this gives a trace on $L\Gamma$ to each operator $x \in L\Gamma$ we may associate the vector $x\delta_e \in \ell^2\Gamma$. If $x\delta_e = \sum_{\gamma\in\Gamma} \alpha_\gamma \delta_\gamma$ then we call the coefficients α_γ the **Fourier coefficients** of x and we sometimes write $x = \sum_{\gamma\in\Gamma} \alpha_\gamma u_\gamma$ for notational convenience.

We also denote by $R\Gamma$ the von Neumann algebra generated by the right-regular representation. Note that $L\Gamma$ is standardly represented on $\mathcal{B}(\ell^2\Gamma)$ and we have $R\Gamma = JL\Gamma J$. In particular, it follows from Proposition 5.3 that we have $R\Gamma = L\Gamma'$.

Exercise 5.4. Let Γ be a countable group. Given vectors $\xi = \sum_{\gamma\in\Gamma} \alpha_\gamma \delta_\gamma, \eta = \sum_{\gamma\in\Gamma} \beta_\gamma \delta_\gamma \in \ell^2\Gamma$ the convolution of ξ with η is given by

$$\xi * \eta = \sum_{\gamma\in\Gamma} \left(\sum_{x\in\Gamma} \alpha_x \beta_{x^{-1}\gamma} \right) \delta_\gamma \in \ell^1\Gamma.$$

A vector $\xi \in \ell^2\Gamma$ is a **left-convolver** if $\xi * \eta \in \ell^2\Gamma$ for all $\eta \in \ell^2\Gamma$. By the closed graph theorem to each left-convolver ξ we obtain a bounded operator L_ξ given by $L_\xi \eta = \xi * \eta$.

Prove that $L\Gamma = \left\{ L_\xi \mid \xi \in \ell^2\Gamma \text{ is a left} - \text{convolver.} \right\}$.

Example 5.5. If Γ is abelian then we may consider the dual group $\hat{\Gamma} = \mathrm{Hom}(\Gamma, \mathbb{T})$ which is a compact group when endowed with the topology of pointwise convergence. We consider this group endowed with a Haar measure μ normalized so that $\mu(\hat{\Gamma}) = 1$. The Fourier transform $\mathcal{F} : \ell^2\Gamma \to L^2\hat{\Gamma}$ is defined as $(\mathcal{F}\xi)(\chi) = \sum_{g\in\Gamma} \xi(g)\langle \chi, g\rangle$. The Fourier transform implements a unitary between $\ell^2\Gamma$ and $\ell^2\hat{\Gamma}$.

If $\xi \in \ell^2\Gamma$ is a (left) convolver, then we have $L_\xi = \mathcal{F}^{-1} M_{\mathcal{F}(\xi)} \mathcal{F}$, and hence we obtain a canonical isomorphism $L\Gamma \cong L^\infty\hat{\Gamma}$. Moreover, we have $\tau(L_\xi) = \int \mathcal{F}(\xi)\, d\mu$, for each $L_\xi \in L\Gamma$.

A discrete group Γ is **i.c.c.** if every non-trival conjugacy class is infinite.

Theorem 5.6 (Murray-von Neumann [MvN43])**.** *Let Γ be a discrete group. Then $L\Gamma$ is a factor if and only if Γ is i.c.c.*

Proof. First suppose that $h \in \Gamma \setminus \{e\}$, such that $h^\Gamma = \{ghg^{-1} \mid g \in G\}$ is finite. Let $x = \sum_{k\in h^\Gamma} u_k$. Then $x \neq \mathbb{C}$, and for all $g \in G$ we have $u_g x u_g^* = \sum_{k\in h^\Gamma} u_{gkg^{-1}} = x$, hence $x \in \{u_g\}'_{g\in\Gamma} \cap L\Gamma = \mathcal{Z}(L\Gamma)$.

Conversely, suppose that Γ is i.c.c. and $x = \sum_{g\in\Gamma} \alpha_g u_g \in \mathcal{Z}(L\Gamma) \setminus \mathbb{C}$, then for all $h \in \Gamma$ we have $x = u_h x u_h^* = \sum_{g\in\Gamma} \alpha_g u_{hgh^{-1}} = \sum_{g\in\Gamma} \alpha_{h^{-1}gh} u_g$. Thus the Fourier coefficients for x are constant on conjugacy classes, and since $x \in L\Gamma \subset \ell^2\Gamma$ we have $\alpha_g = 0$ for all $g \neq e$, hence $x = \tau(x) \in \mathbb{C}$. □

5.1 Group characters and tracial von Neumann algebras

Let Γ be a discrete group. Recall that a function $\varphi : \Gamma \to \mathbb{C}$ is a character if it is of positive definite, is constant on conjugacy classes, and is normalized so that $\varphi(e) = 1$. Characters arise from representations into finite von Neumann algebras. Indeed, if M is a finite von Neumann algebra with a normal faithful trace τ, and if $\pi : \Gamma \to \mathcal{U}(M) \subset \mathcal{U}(L^2(M,\tau))$ is a representation then $\varphi(g) = \tau(\pi(g)) = \langle \pi(g)1_\tau, 1_\tau \rangle$ defines a character on Γ. Conversely, if $\varphi : \Gamma \to \mathbb{C}$ is a character then the cyclic vector ξ in the corresponding GNS-representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ satisfies $\langle \pi(gh)\xi, \xi \rangle = \varphi(gh) = \varphi(hg) = \langle \pi(hg)\xi, \xi \rangle$ for all $g, h \in \Gamma$. Since the linear functional $T \mapsto \langle T\xi, \xi \rangle$ is normal we may then extend this to a normal faithful trace $\tau : \pi(\Gamma)'' \to \mathbb{C}$ by the formula $\tau(x) = \langle x\xi, \xi \rangle$. In particular this shows that $\pi(\Gamma)''$ is finite since it has a normal faithful trace.

If (M_i, τ_i) are finite von Neumann algebras with normal faithful traces τ_i, $i \in \{1, 2\}$, and $\pi_i : \Gamma \to \mathcal{U}(M_i)$ then we will consider π_1 and π_2 to be equivalent if there is a trace preserving automorphism $\alpha : M_1 \to M_2$, such that $\alpha(\pi_1(g)) = \pi_2(g)$ for all $g \in \Gamma$. Clearly, this is equivalent to requiring that there exist a unitary $U : L^2(M_1, \tau_1) \to L^2(M_2, \tau_2)$ such that $U1_{\tau_1} = 1_{\tau_2}$, and $U\pi_1(g) = \pi_2(g)U$ for all $g \in G$.

Note that the space of characters is a convex set, which is closed in the topology of pointwise convergence.

Theorem 5.7 (Thoma). *Let Γ be a discrete group. There is a one to one correspondence between:*

1. *Equivalence classes of embeddings $\pi : \Gamma \to \mathcal{U}(M)$ where M is a finite von Neumann algebra with a given normal faithful trace τ, and such that $\pi(\Gamma)'' = M$, and*

2. *Characters $\varphi : \Gamma \to \mathbb{C}$,*

which is given by $\varphi(g) = \tau(\pi(g))$. Moreover, M is a factor if and only if φ is an extreme point in the space of characters.

Proof. The one to one correspondence follows from the discussion preceding the theorem, thus we only need to show the correspondence between factors and extreme points. If $p \in \mathcal{P}(\mathcal{Z}(M))$, is a non-trivial projection then we obtain characters φ_1, and φ_2 by the formulas $\varphi_1(g) = \frac{1}{\tau(p)}\tau(\pi(g)p)$, and $\varphi_2(g) = \frac{1}{\tau(1-p)}\tau(\pi(g)(1-p))$, and we have $\varphi = \tau(p)\varphi_1 + \tau(1-p)\varphi_2$. Since $p \in M = \pi(\Gamma)''$, there exists a sequence $x_n \in \mathbb{C}\Gamma$ such that $\frac{1}{\tau(p)}\tau(\pi(x_n)p) \to 1$, and $\frac{1}{\tau(1-p)}\tau(\pi(x_n)(1-p)) \to 0$, it then follows that $\varphi_1 \neq \varphi_2$ and hence φ is not an extreme point.

Conversely, if $\varphi = \frac{1}{2}(\varphi_1 + \varphi_2)$ with $\varphi_1 \neq \varphi_2$ then if we consider the corresponding representations $\pi_i : \Gamma \to \mathcal{U}(N_i)$, we obtain a trace preserving embedding $\alpha : N \to N_1 \oplus N_2$, which satisfies $\alpha(\pi(g)) = \pi_1(g) \oplus \pi_2(g)$. If we denote by p the projection $1 \oplus 0$ then it need not be the case that $p \in \alpha(N)$, however by considering p we may then define a new trace τ' on N by $\tau'(x) = \frac{1}{4}\tau_1(\alpha(x)p) + \frac{3}{4}\tau_2(\alpha(x)(1-p))$. Since $\varphi_1 \neq \varphi_2$ we must have that $\tau'(\pi(g)) \neq \tau(\pi(g))$ for some $g \in \Gamma$. Thus, N does not have unique trace and so is not a factor by a result of Murray and von Neumann [MvN37]. $\square$

6 Completely positive maps

Given a C^*-algebra A we denote by $\mathbb{M}_n(A)$ the space of $n \times n$ matrices over A.

If $\phi : A \to B$ is a linear map between C^*-algeras, then we denote by $\phi^{(n)} : \mathbb{M}_n(A) \to \mathbb{M}_n(B)$ the map defined by $\phi^{(n)}([a_{i,j}]) = [\phi(a_{i,j})]$. We say that ϕ is **positive** if $\phi(a) \geq 0$, whenever $a \geq 0$. If $\phi^{(n)}$ is positive then we say that ϕ is ***n*-positive** and if ϕ is n-positive for every $n \in \mathbb{N}$ then we say that ϕ is **completely positive**. If A and B are unital and $\phi : A \to B$ such that $\phi(1) = 1$ then we say that ϕ is unital.

Example 6.1. Let A be a C^*-algebra and K a compact Hausdorff space. If $\pi : K \to S(A)$ is a continuous map, then we obtain a unital (completely) positive map $\phi : A \to C(K)$ given by

$$\phi(a)(x) = \pi(x)(a), \qquad a \in A, x \in K.$$

Conversely, this formula also defines a continuous map $\pi : K \to S(A)$ whenever $\phi : A \to C(K)$ is unital (completely) positive.

Similarly, if (X, μ) is σ-finite measure space and $\pi : X \to S(A)$ is a measurable map, then we obtain a unital (completely) positive map $\phi : A \to L^\infty(X, \mu)$ given by the same formula. Less obvious is that if A is separable then every unital (completely) positive map from A into $L^\infty(X, \mu)$ arrises in this way.

If $\pi : A \to \mathcal{B}(\mathcal{K})$ is a representation of a C^*-algebra A and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the operator $\phi : A \to \mathcal{B}(\mathcal{H})$ given by $\phi(x) = V^*\pi(x)V$ is completely positive. Indeed, if we consider the operator $V^{(n)} \in \mathcal{B}(\mathcal{H}^{\oplus n}, \mathcal{K}^{\oplus n})$ given by $V^{(n)}((\xi_i)_i) = (V\xi_i)_i$ then for all $x \in \mathbb{M}_n(A)$ we have

$$\begin{aligned}\phi^{(n)}(x^*x) &= {V^{(n)}}^*\pi^{(n)}(x^*x)V^{(n)} \\ &= (\pi^{(n)}(x)V^{(n)})^*(\pi^{(n)}(x)V^{(n)}) \geq 0.\end{aligned}$$

Generalizing the GNS construction Stinespring showed that every completely positive map from A to $\mathcal{B}(\mathcal{H})$ arises in this way.

Theorem 6.2 (Stinespring's dilation theorem)**.** *Let A be a unital C^*-algebra, and suppose $\phi : A \to \mathcal{B}(\mathcal{H})$, then ϕ is completely positive if and only if there exists a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x) = V^*\pi(x)V$. We also have $\|\phi\| = \|V\|^2$, and if ϕ is unital then V is an isometry. Moreover, if A is a von Neumann algebra and ϕ is a normal completely positive map, then π is a normal representation.*

Proof. Consider the sesquilinear form on $A \otimes \mathcal{H}$ given by $\langle a \otimes \xi, b \otimes \eta\rangle_\phi = \langle \phi(b^*a)\xi, \eta\rangle$, for $a, b \in A$, $\xi, \eta \in \mathcal{H}$. If $(a_i)_i \in A^{\oplus n}$, and $(\xi_i)_i \in \mathcal{H}^{\oplus n}$, then we have

$$\begin{aligned}\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j\rangle_\phi &= \sum_{i,j} \langle \phi(a_j^* a_i)\xi_i, \xi_j\rangle \\ &= \langle \phi((a_i)_i^*(a_i)_i)(\xi_i)_i, (\xi_i)_i\rangle \geq 0.\end{aligned}$$

Thus, this form is non-negative definite and we can consider N_ϕ the kernel of this form so that $\langle \cdot, \cdot \rangle_\phi$ is positive definite on $\mathcal{K}_0 = (A \otimes \mathcal{H})/N_\phi$. Hence, we can take the Hilbert space completion $\mathcal{K} = \overline{\mathcal{K}_0}$.

As in the case of the GNS construction, we define a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ by first setting $\pi_0(x)(a \otimes \xi) = (xa) \otimes \xi$ for $a \otimes \xi \in A \otimes \mathcal{H}$. Note that since ϕ is positive we have $\phi(a^*x^*xa) \leq \|x\|^2\phi(a^*a)$, applying this to $\phi^{(n)}$ we see that $\|\pi_0(x) \sum_i a_i \otimes \xi_i\|_\phi^2 \leq \|x\|^2 \| \sum_i a_i \otimes \xi_i\|_\phi^2$. Thus, $\pi_0(x)$ descends to a well defined bounded map on $\mathcal{K}_0$ and then extends to a bounded operator $\pi(x) \in \mathcal{B}(\mathcal{K})$.

If we define $V_0 : \mathcal{H} \to \mathcal{K}_0$ by $V_0(\xi) = 1 \otimes \xi$, then we see that V_0 is bounded by $\|\phi(1)\|$ and hence extends to a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $x \in A$, $\xi, \eta \in \mathcal{H}$ we then check that

$$\begin{aligned}\langle V^*\pi(x)V\xi, \eta\rangle &= \langle \pi(x)(1 \otimes \xi), 1 \otimes \eta\rangle_\phi \\ &= \langle x \otimes \xi, 1 \otimes \eta\rangle_\phi = \langle \phi(x)\xi, \eta\rangle.\end{aligned}$$

Thus, $\phi(x) = V^*\pi(x)V$ as claimed. □

Corollary 6.3 (Kadison's inequality)**.** *If A and B are unital C^*-algebras, and $\phi : A \to B$ is unital compleley positive then for all $x \in A$ we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$*

Proof. We assume that $B \subset \mathcal{B}(\mathcal{H})$. If we consider the Stinespring dilation $\phi(x) = V^*\pi(x)V$, then since ϕ is unital we have that V is an isometry. Hence $1 - VV^* \geq 0$ and so we have

$$\begin{aligned} \phi(x^*x) - \phi(x)^*\phi(x) &= V^*\pi(x^*x)V - V^*\pi(x)^*VV^*\pi(x)V \\ &= V^*\pi(x^*)(1 - VV^*)\pi(x)V \geq 0. \end{aligned}$$ □

Lemma 6.4. *Let A be a C^*-algebra, if $\begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \in \mathbb{M}_2(A)$ is positive, then $x = 0$, and $y \geq 0$.*

Proof. We may assume A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, hence if $\xi, \eta \in \mathcal{H}$ we have

$$2\mathrm{Re}(\langle x^*\eta, \xi\rangle) + \langle y\eta, \eta\rangle = \left\langle \begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \geq 0.$$

The result then follows easily. □

Theorem 6.5 (Choi)**.** *If $\phi : A \to B$ is a unital 2-positive map between C^*-algebras, then for $a \in A$ we have $\phi(a^*a) = \phi(a^*)\phi(a)$ if and only if $\phi(xa) = \phi(x)\phi(a)$, and $\phi(a^*x) = \phi(a^*)\phi(x)$, for all $x \in A$.*

Proof. Applying Kadison's inequality to $\phi^{(2)}$ it follows that for all $x \in A$ we have

$$\begin{aligned} \begin{pmatrix} \phi(a^*a) & \phi(a^*x) \\ \phi(x^*a) & \phi(aa^* + x^*x) \end{pmatrix} &= \phi^{(2)}\left(\left| \begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right|^2 \right) \geq \left| \phi^{(2)}\left(\begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right) \right|^2 \\ &= \begin{pmatrix} \phi(a^*)\phi(a) & \phi(a^*)\phi(x) \\ \phi(x^*)\phi(a) & \phi(a)\phi(a^*) + \phi(x^*)\phi(x) \end{pmatrix}. \end{aligned}$$

Since $\phi(a^*a) = \phi(a)^*\phi(a)$ it follows from the previous lemma that $\phi(x^*a) = \phi(x^*)\phi(a)$, and $\phi(a^*x) = \phi(a)^*\phi(x)$. □

6.1 Conditional expectations

If A is a unital C^*-algebra, and $B \subset A$ is a unital C^*-subalgebra, then a **conditional expectation** from A to B is a unital completely positive $E : A \to B$ such that $E_{|B} = \mathrm{id}$. Note that by Choi's theorem we have $E(axb) = aE(x)b$ for all $a, b \in B$, $x \in A$.

Theorem 6.6 (Umegaki)**.** *Let M be a finite von Neumann algebra with normal faithful trace τ, and let $N \subset M$ be a von Neumann subalgebra, then there exists a unique normal conditional expectation $E : M \to N$ such that $\tau \circ E = \tau$.*

Proof. Let $e_N \in \mathcal{B}(L^2(M,\tau))$ be the projection onto $L^2(N,\tau) \subset L^2(M,\tau)$, and let J be the conjugation operator on $L^2(M,\tau)$ which we also view as the conjugation operator on $L^2(N,\tau)$. Note that $N' \cap \mathcal{B}(L^2(N,\tau)) = JNJ$ by Proposition 5.3. Since $L^2(N,\tau)$ is invariant under JNJ we have $e_N(JyJ) = (JyJ)e_N$ for all $y \in N$.

If $x \in M$, $y \in N$, then

$$\begin{aligned} e_N x e_N JyJ = e_N xJyJ &= e_N JyJxe_N \\ &= JyJe_N xe_N. \end{aligned}$$

Thus, $e_N x e_N \in (JNJ)' = N$ and we denote this operator by $E(x)$. Clearly, $E : M \to N$ is normal unital completely positive, and $E_{|N} = \mathrm{id}$, thus E is a normal conditional expectation. Also, for $x \in M$ we have

$$\tau(E(x)) = \langle e_N x e_N 1_\tau, 1_\tau\rangle = \langle x1_\tau, 1_\tau\rangle = \tau(x).$$

If $\tilde{E}$ were another trace preserving conditional expectation, then for $x \in M$, and $y \in N$ we would have

$$\begin{aligned} \tau(\tilde{E}(x)y) = \tau(\tilde{E}(xy)) &= \tau(xy) \\ &= \tau(E(xy)) = \tau(E(x)y), \end{aligned}$$

from which it follows that $\tilde{E} = E$. □

7 Amenable von Neumann algebras

A von Neumann algebra $M \subset \mathcal{B}(\mathcal{H})$ is **amenable** if there exists a (not in general normal) conditional expectation $E : \mathcal{B}(\mathcal{H}) \to M$.

If X and Y are Banach spaces and $x \in X$, and $y \in Y$ then we define the linear map $x \otimes y : \mathcal{B}(X, Y^*) \to \mathbb{C}$ by $(x \otimes y)(L) = L(x)(y)$. Note that $|(x \otimes y)(L)| \leq \|L\|\|x\|\|y\|$ and hence $x \otimes y$ is bounded and indeed $\|x \otimes y\| \leq \|x\|\|y\|$. Let Z be the norm closed linear span in $\mathcal{B}(X, Y^*)^*$ of all $x \otimes y$.

Lemma 7.1. *The pairing $\langle L, x \otimes y\rangle = (x \otimes y)(L)$ extends to an isometric identification between Z^* and $\mathcal{B}(X, Y^*)$.*

Proof. It is easy to see that this pairing gives an isometric embedding of $\mathcal{B}(X, Y^*)$ into Z^*. To see that this is onto consider $\varphi \in Z^*$, and for each $x \in X$ define $L_x : Y \to \mathbb{C}$ by $L_x(y) = \varphi(x \otimes y)$. Then we have $|L_x(y)| \leq \|\varphi\|\|x\|\|y\|$ and hence $L_x \in Y^*$. The mapping $x \mapsto L_x$ is easily seen to be linear and thus we have $L \in \mathcal{B}(X, Y^*)$, and clearly L is mapped to φ under this pairing. □

Lemma 7.2. *If X and Y are Banach spaces and L_i is a bounded net in $\mathcal{B}(X, Y^*)$ then $L_i \to L$ in the weak*-topology described above if and only if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$.*

Proof. If L_i converges to L in the weak*-topology then for all $x \in X$ and $y \in Y$ we have $L_i(x)(y) = (x \otimes y)(L_i) \to (x \otimes y)(L) = L(x)(y)$ showing that $L_i(x) \to L(x)$ is the weak*-topology for all $x \in X$. Conversely, if $L_i(x) \to L(x)$ in the weak*-topology for all $x \in X$ then in particular we have $(x \otimes y)(L_i) \to (x \otimes y)(L)$ for each $x \in X$ and $y \in Y$, thus this also holds for the linear span of all $x \otimes y$ and since the net is bounded we then have convergence on the closed linear span. □

Corollary 7.3. *Let A be a unital C^*-algebra, then the set of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is compact in the topology of point-wise weak operator topology convergence.*

Proof. First note that it is easy to see that the space of contractive completely positive maps from E to $\mathcal{B}(\mathcal{H})$ is closed in this topology. Since $\mathcal{B}(\mathcal{H})$ is a dual space, and on bounded sets the weak operator topology is the same as the weak*-topology, the result then follows from Alaoglu's theorem, together with the previous two lemmas. $\square$

Theorem 7.4 (Schwartz [Sch63]). *Let G be a locally compact amenable group, M a von Neumann algebra, and suppose that $\alpha : G \to \mathrm{Aut}(M)$ is a continuous action of G on M. Then there exists a conditional expectation $E : M \to M^G$.*

Proof. First suppose that G is amenable and let $\{\mu_i\}_i \subset \mathrm{Prob}(G)$ be a net such that $\|\mu_i - g_* \mu_i\|_1 \to 0$ for each $g \in G$. We then consider the completely positive maps $\phi_i : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ given by $\phi_i(T) = \int \alpha_g(T)\, \mathrm{d}\mu_i$.

By the previous corollary we may take a subsequence and assume that $\{\phi_i\}_i$ converges to a completely positive map E in the topology of pointwise weak operator topology convergence. Note that for $x \in M^G$ we have $\phi_i(x) = x$ and hence it follows that $E(x) = x$ for each $x \in M^G$. Moreover, if $T \in M$ then for $g \in G$ we have $\|\alpha_g(\phi_i(T)) - \phi_i(T)\rho(\gamma)\| \leq \|T\| \|\mu_i - g_*\mu_i\|_1 \to 0$. Hence it follows that $E(T) \in M^G$ and thus $E : M \to M^G$ is a conditional expectation. $\square$

Corollary 7.5. *Let Γ be a discrete group, then Γ is amenable if and only if $L\Gamma$ is amenable.*

Proof. If Γ is amenable then we may consider the action of Γ on $\mathcal{B}(\ell^2\Gamma)$ given by $\alpha_\gamma(T) = \rho(\gamma)T\rho(\gamma^{-1})$. From the previous theorem there then exists a conditional expectation from $\mathcal{B}(\ell^2\Gamma)$ onto $\rho(\Gamma)' = L\Gamma$ and hence $L\Gamma$ is amenable.

Conversely, if $L\Gamma$ is amenable and $E : \mathcal{B}(\ell^2\Gamma) \to L\Gamma$ is a conditional expectation, then we may consider the state $\varphi = \tau \circ E_{|\ell^\infty\Gamma}$ and a simple calculation shows that φ is Γ-invariant showing that Γ is amenable. $\square$

7.1 Amenable finite von Neumann algebras

Proposition 7.6 (Powers-Stormer inequality). *If $S, T \in \mathcal{B}(\mathcal{H})$ are positive Hilbert-Schmidt operators, then*

$$\|S - T\|_{\mathrm{HS}}^2 \leq \|S^2 - T^2\|_{\mathrm{TC}} \leq \|S + T\|_{\mathrm{HS}} \|S - T\|_{\mathrm{HS}}.$$

Proof. Note that if $x, y \in \mathcal{B}(\mathcal{H})$ positive, at least one of which is a trace class operator, then $\mathrm{Tr}(xy) \geq 0$. We consider the spectral projection $p = \chi_{[0,\infty)}(S - T)$, so that $p(S - T) \geq 0$ and $p^\perp(T - S) \geq 0$. Then

$$\begin{aligned}
\mathrm{Tr}((S-T)^2) &= \mathrm{Tr}(p(S-T)^2 + p^\perp(T-S)^2) \\
&\leq \mathrm{Tr}(p(S+T)(S-T) + p^\perp(T-S)(T+S)) \\
&= \mathrm{Tr}(p(S^2 - T^2) + p^\perp(T^2 - S^2)) \\
&\leq \mathrm{Tr}(p|S^2 - T^2| + p^\perp|T^2 - S^2|) = \mathrm{Tr}(|S^2 - T^2|),
\end{aligned}$$

where the third line follows from the fact that $\mathrm{Tr}(p(S+T)(S-T)) = \mathrm{Tr}((S+T)(S-T)p)$.

The second inequality follows form the formula $S^2 - T^2 = \frac{1}{2}((S+T)(S-T) + (S-T)(S+T))$, the Cauchy-Schwarz inequality, and the fact that $\|x\|_{\rm HS} = \||x|\|_{\rm HS}$ for any Hilbert-Schmidt operator x. □

The following theorem is the von Neumann algebra analogue of Theorem 4.2.

Theorem 7.7 (Connes [Con76]). *Let (M, τ) be a tracial von Neumann algebra, then the following conditions are equivalent.*

1. *M is amenable.*
2. *There exists a state φ on $\mathcal{B}(L^2(M,\tau))$ such that $\varphi_{|M} = \tau$, and $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and $T \in \mathcal{B}(L^2(M,\tau))$.*
3. *There exists a sequence of positive trace class operators $\{T_i\}_i \subset \mathcal{B}(L^2(M,\tau))$ so that $\tau(x) = {\rm Tr}(xT_i)$ for all $x \in M$ and $i \in \mathbb{N}$, and such that $\|xT_i - T_i x\|_{\rm Tr} \to 0$ for all $x \in M$.*
4. *There exists a sequence of Hilbert-Schmidt operators $\xi_i \in \mathcal{B}(L^2(M,\tau))$ such that $\langle x\xi_i, \xi_i\rangle = \langle \xi_i x, \xi_i\rangle = \tau(x)$ for every $x \in M$ and $i \in \mathbb{N}$, and such that $\|x\xi_i - \xi_i x\|_{\rm HS} \to 0$ for every $x \in M$.*

Proof. (1) $\implies$ (2): Suppose that $E : \mathcal{B}(L^2(M,\tau)) \to M$ is a conditional expectation and set $\varphi(T) = \tau(E(T))$. We then have that $\varphi_{|M} = \tau$ and for $T \in \mathcal{B}(L^2(M,\tau))$ and $x \in M$ we have

$$\varphi(xT) = \tau(xE(T)) = \tau(E(T)x) = \varphi(Tx).$$

(2) $\implies$ (1): Suppose now that φ is a state on $\mathcal{B}(L^2(M,\tau))$ such that $\varphi_{|M} = \tau$, and $\varphi(xT) = \varphi(Tx)$ for all $x \in M$ and $T \in \mathcal{B}(L^2(M,\tau))$. Performing the GNS-construction gives a representation $\pi : \mathcal{B}(L^2(M,\tau)) \to \mathcal{B}(\mathcal{K})$ and a unit vector $\xi_0 \in \mathcal{K}$ so that for all $T \in \mathcal{B}(L^2(M,\tau))$ we have $\varphi(T) = \langle \pi(T)\xi_0, \xi_0\rangle$.

Since $\varphi_{|M} = \tau$ we see that the map $\hat{x} \mapsto \pi(x)\xi_0$ extends to an isometry $V : L^2(M,\tau) \to \mathcal{K}$. It's easy to see that for $x \in M$ we have $\pi(x)V = Vx$.

We then obtain a unital completely positive map $E : \mathcal{B}(L^2(M,\tau)) \to \mathcal{B}(L^2(M,\tau))$ by $E(T) = V^*\pi(T)V$. Since $\pi(x)V = Vx$ we have that $E(x) = x$ for all $x \in M$.

Also, if $T \in \mathcal{B}(L^2(M,\tau))$, and $x, a, b \in M$ we have

$$\begin{aligned}\langle JxJE(T)\hat{a}, \hat{b}\rangle &= \langle V^*\pi(T)V\hat{a}, Jx^*J\hat{b}\rangle = \langle \pi(T)\pi(a)\xi_0, \pi(bx)\xi_0\rangle \\ &= \varphi(x^*b^*Ta) = \varphi(b^*Tax^*) \\ &= \langle V^*\pi(T)V\hat{ax^*}, \hat{b^*}\rangle = \langle E(T)JxJ\hat{a}, \hat{b}\rangle.\end{aligned}$$

We therefore conclude that $E(T) \in JMJ' = M$ and it follows that $E : \mathcal{B}(L^2(M,\tau)) \to M$ is a conditional expectation.

(2) $\implies$ (3): We let X denote the space of trace class operators T such that ${\rm Tr}(xT) = \tau(x){\rm Tr}(T)$. Then X is a closed subspace of the trace class operators and the annihilator of X in $\mathcal{B}(L^2(M,\tau))$ is $\{x \in M \mid \tau(x) = 0\}$. It then follows that $\varphi \in X^{**}$.

Fix a sequence $\{x_n\}_n \subset M$ which is dense in the strong operator topology. For each $n \in \mathbb{N}$ we let C_n denote the convex set $\{\oplus_{i=1}^n (x_iT - Tx_i) \mid T \in X, T \geq 0, {\rm Tr}(T) = 1\} \subset X \subset X^{**}$. Since $\varphi \in X^{**}$ it follows that 0 is in the weak* closure of C_n in X^{**}, and since C_n is convex we have that

0 must also be in the strong closure of C_n. Therefore there exist positive trace class operators T_n such that $\mathrm{Tr}(xT_n) = \tau(x)$ for all $x \in M$, and $\|x_i T_n - T_n x_i\|_{\mathrm{TC}} < 1/n$.

Since $\{x_n\}_n$ is dense in the strong operator topology we then have that $\|xT_n - T_n x\|_{\mathrm{TC}} \to 0$ for each $x \in M$.

(3) $\implies$ (4): If we set $S_n = T_n^{1/2}$ then the Powers-Stormer inequality gives $\|xS_n - S_n x\|_{\mathrm{HS}} \to 0$, and for $x \in M$ we have $\langle xS_n, S_n\rangle = \langle S_n x, S_n\rangle = \mathrm{Tr}(xT_n) = \tau(x)$.

(4) $\implies$ (2): We consider the states on $\mathcal{B}(L^2(M,\tau))$ given by $\varphi_n(T) = \langle TS_n, S_n\rangle$. We let φ be a weak* cluster point of $\{\varphi_n\}_{n\in\mathbb{N}}$. Then $\varphi_{|M} = \tau$ and if $x \in M$ and $T \in \mathcal{B}(L^2(M,\tau))$ we have that $\varphi(xT - Tx)$ is a cluster point of $\varphi_n(xT - Tx)$. However, from the Cauchy-Schwarz inequality we have

$$\begin{aligned} |\varphi_n(xT - Tx)| &= |\langle (xT - Tx)S_n, S_n\rangle| \\ &\leq |\langle xTS_n - TS_n x, S_n\rangle| + \|T\|\|xS_n - S_n x\|_{\mathrm{HS}} \\ &= |\langle xTS_n - TS_n, S_n x^*\rangle| + \|T\|\|xS_n - S_n x\|_{\mathrm{HS}} \\ &\leq 2\|T\|\|xS_n - S_n x\|_{\mathrm{HS}} \to 0. \end{aligned}$$

□

While we will not use this below, we also mention the following result of Connes:

Theorem 7.8 (Connes [Con76]). *There is a unique (up to isomorphism) amenable separable II_1 factor R.*

8 Property (T) for finite von Neumann algebras

A tracial von Neumann algebra (M,τ) has **property (T)** if for any normal M-bimodule $\mathcal{H}$, and any sequence of vectors $\xi_i \in \mathcal{H}$ satisfying $\|x\xi_i\|^2, \|\xi_i x\|^2 \leq \|x\|_2^2$ for all $x \in M$, and $i \in \mathbb{N}$, and $\|x\xi_i - \xi_i x\| \to 0$ for all $x \in M$, there exists a sequence of vectors $\eta_i \in \mathcal{H}$, such that $x\eta_i = \eta_i x$ for all $x \in M$ and $i \in \mathbb{N}$, and such that $\|\eta_i - \xi_i\| \to 0$.

Theorem 8.1 (Connes-Jones [CJ85]). *Suppose Γ is a discrete group with property (T), (M,τ) is a tracial von Neumann algebra, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation so that $M = \pi(\Gamma)''$. Then (M,τ) has property (T).*

Proof. Let $\mathcal{H}$ be a normal Hilbert M-bimodule, and $\{\xi_i\}_i \subset \mathcal{H}$ a sequence of vectors satisfying $\|x\xi_i\|^2, \|\xi_i x\|^2 \leq \|x\|_2^2$ for all $x \in M$, and $i \in \mathbb{N}$, and $\|x\xi_i - \xi_i x\| \to 0$ for all $x \in M$.

Consider the representation of Γ on $\mathcal{H}$ given by conjugation $\xi \mapsto \pi(\gamma)\xi\pi(\gamma^{-1})$. Then we have $\|\xi_i - \pi(\gamma)\xi_i\pi(\gamma^{-1})\| \to 0$ for all $\gamma \in \Gamma$ and since Γ has property (T) there then exists a sequence of Γ-invariant vectors $\eta_i \in \mathcal{H}$ so that $\|\xi_i - \eta_i\| \to 0$.

Since η_i are Γ-invariant we have that $\pi(\gamma)\eta_i = \eta_i\pi(\gamma)$ for all $\gamma \in \Gamma$. Taking linear combinations we then have that $x\eta_i = \eta_i x$ for any x in the linear span of $\pi(\Gamma)$. Since this linear span is weak operator topology dense in M it then follows that $x\eta_i = \eta_i x$ for all $x \in M$. □

In the case of the group-von Neumann algebra the converse of the previous result also holds, although we will not need that here.

Theorem 8.2 (Connes-Jones [CJ85]). *Let Γ be a discrete group. then Γ has property (T) only if $L\Gamma$ has property (T).*

Just as the case for groups, amenability and property (T) intersect only in very elementary situations.

Proposition 8.3. *Let M be a finite factor, if M is amenable and has property (T) then M is finite dimensional.*

Proof. Suppose M is a finite factor which is amenable and has property (T). By Theorem 7.7 there then exists a sequence of Hilbert-Schmidt operators $\xi_n \in \mathcal{H}(L^2(M))$ so that $\langle x\xi_i, \xi_i\rangle = \langle \xi_i x, \xi_i\rangle = \tau(x)$ for each $x \in M$ and $\|x\xi_i - \xi_i x\|_{\mathrm{HS}} \to 0$ for each $x \in M$.

Since left and right multiplication on the space of Hilbert-Schmidt operators gives a normal bimodule it then follows from property (T) that there exists a sequence of Hilbert-Schmidt operators $\eta_i \in \mathcal{B}(L^2(M))$ so that $x\eta_i = \eta_i x$ for all $x \in M$ and $\|\xi_i - \eta_i\|_{\mathrm{HS}} \to 0$. In particular, there must exist some non-zero Hilbert-Schmidt operator η which commutes with M. We then have that $J\eta J \in JMJ' = M$ and is Hilbert-Schmidt. Taking spectral projections then shows that M contains a projection onto a finite dimensional subspace and hence must be a type I factor. As M is a finite type I factor it must then be finite dimensional. □

9 Induced actions and ergodicity

Let G be a locally compact group, and $\Gamma < G$ a closed subgroup. Suppose M is a von Neumann algebra and $\theta : \Gamma \to \mathrm{Aut}(M)$ is a continuous action of Γ on M. We let $\alpha : G \times G/\Gamma \to \Gamma$ be a cocycle corresponding to the identity map $\Gamma \to \Gamma$, e.g., $\alpha(g, x) = \psi(gx)^{-1} g\psi(x)$, where $\psi : G/\Gamma \to G$ is a Borel section. We may then induce the action of Γ on M to an action of G on $L^\infty(G/\Gamma)\overline{\otimes} M$ as $\tilde{\theta}_g(f)(x) = \theta_{\alpha(g,g^{-1}x)}(f(g^{-1}x))$.

We consider the action $L : G \to \mathrm{Aut}(L^\infty G)$ (resp. $R : G \to \mathrm{Aut}(L^\infty G)$) induced by left (resp. right) multiplication. An alternate way to view the induced action of G on $L^\infty(G/\Gamma)\overline{\otimes} M$ is to consider the action of $\Gamma \curvearrowright^{R\otimes\theta} L^\infty(G)\overline{\otimes} M$. Fixing a Borel section $\psi : G/\Gamma \to G$ then gives an isomorphism $\Psi : L^\infty(G/\Gamma)\overline{\otimes} M \to (L^\infty(G)\overline{\otimes} M)^{(R\otimes\theta)(\Gamma)}$ by $\Psi(f)(g) = \theta_{\psi(\Gamma)\alpha(g^{-1},g\Gamma)}(f(g\Gamma))$. Under this isomorphism, the induced action of G on $L^\infty(G/\Gamma)\overline{\otimes} M$ then translates to the action $G \curvearrowright^{L\otimes \mathrm{id}} (L^\infty(G)\overline{\otimes} M)^{(R\otimes\theta)(\Gamma)}$.

Theorem 9.1 (Zimmer [Zim77]). *Let G be a locally compact group, $H < G$ a closed amenable subgroup, and $\Gamma < G$ a discrete subgroup. For any unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, we may consider the corresponding action of Γ on $\mathcal{B}(\mathcal{H})$ given by $\theta_\gamma(T) = \pi_\gamma T \pi_\gamma^*$. Then the von Neumann algebra $(L^\infty(G/H)\overline{\otimes}\mathcal{B}(\mathcal{H}))^\Gamma$ is amenable.*

Proof. As described above, we may identify the von Neumann algebras $(L^\infty(G/H)\overline{\otimes}\mathcal{B}(\mathcal{H}))^\Gamma$ and $(L^\infty(G)\overline{\otimes}\mathcal{B}(\mathcal{H}))^{H\times\Gamma}$. We similarly have an identification between this von Neumann algebra and $(L^\infty(G/\Gamma)\overline{\otimes}\mathcal{B}(\mathcal{H}))^H$ (where the action of H here is the one induced from the action of Γ). Since H and $L^\infty(G/\Gamma)\overline{\otimes}\mathcal{B}(\mathcal{H})$ are amenable the result then follows from Theorem 7.4. □

An action of a group on a von Neumann algebra is **ergodic** if the fixed point subalgebra is $\mathbb{C}$. Note that from the isomorphism given by Ψ above, we have that the action $\Gamma \curvearrowright^\theta M$ is ergodic if and only if the induced action $G \curvearrowright^{\tilde{\theta}} L^\infty(G/\Gamma)\overline{\otimes} M$ is ergodic.

Suppose now that $\Gamma < G$ is a lattice and M is a finite von Neumann algebra with a normal faithful trace τ, and assume also that θ_γ preserves the trace on M for each $\gamma \in \Gamma$. We then have that $\tilde{\theta}$ preserves the trace on $L^\infty(G/\Gamma)\overline{\otimes}M$ given by $\int \otimes \tau$ (where the integral corresponds to the unique G-invariant probability measure on G/Γ).

Suppose $H < G$ is a closed subgroup such that ΓH is dense in G. For $O \subset G$ a non-empty open subset we set $\Gamma_O = \Gamma \cap OH$, which is non-empty since ΓH is dense in G. For $x \in M$ we set $\mathcal{K}_x(O) = \overline{\text{co}}\{\theta_\gamma(x) \mid \gamma \in \Gamma_O\}$, and $\tilde{\mathcal{K}}_x(O) = \overline{\text{co}}\{\theta_{\gamma^{-1}}(x) \mid \gamma \in \Gamma_O\}$. For each $g \in G$ we let $\kappa_x(g)$ (resp. $\tilde{\kappa}_x(g)$) be the element of minimal $\|\cdot\|_2$ in $\cap_{O\in\mathcal{N}(g)}\mathcal{K}_x(O)$ (resp. $\cap_{O\in\mathcal{N}(g)}\tilde{\mathcal{K}}_x(O)$), where $\mathcal{N}(g)$ denotes the set of open neighborhoods of g. Note that for each $g \in G$, $\kappa_x(g), \tilde{\kappa}_x(g) \in M \subset L^2(M, \tau)$ by Corollary 5.2.

Proposition 9.2. *Using the notation above, if the induced action $H \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic, then $\kappa_x(g) = \tau(x)$ for all $g \in G$.*

Proof. Using the isomorphism $(L^\infty(G/\Gamma)\overline{\otimes}M)^H \cong (L^\infty(G)\overline{\otimes}M)^{H\times\Gamma}$ as described above, ergodicity of the H action is equivalent to ergodicity of the $H \times \Gamma$ action on $L^\infty(G)\overline{\otimes}M$, where the action of H is given by $h \mapsto L_h \otimes \text{id}$, and the action of Γ is given by $\gamma \mapsto R_\gamma \otimes \theta_\gamma$.

Note that $\kappa_x : G \to M$ is a bounded Borel map. Indeed, we have $\|\kappa_x(g)\| \leq \|x\|$ for all $g \in G$, hence κ_x is bounded. If $\{g_n\}_{n\in\mathbb{N}}$ is a countable dense subset of G, then for each $O \in \mathcal{N}(e)$ we may consider the simple function $\kappa_{x,O} : G \to M$ given by setting $\kappa_{x,O}(g)$ to be the unique element of minimal norm in $\mathcal{K}_x(g_jO)$, where j is the smallest natural number such that $g \in g_jO$. Taking a sequence $O_n \in \mathcal{N}(e)$ such that $\cap O_n = \{e\}$ we then have that κ_{x,O_n} converges pointwise in the strong operator topology to κ_x, hence κ_x is Borel.

For $g \in G$, $h \in H$, $\gamma \in \Gamma$, and $O \in \mathcal{N}(g)$, we have $\theta_\gamma(\mathcal{K}_x(O)) = \mathcal{K}_x(\gamma Oh)$, hence it follows that $\theta_\gamma(\kappa_x(g)) = \kappa_x(\gamma gh)$.

Thus, $\kappa_x \in (L^\infty(G)\overline{\otimes}M)^{H\times\Gamma} = \mathbb{C}$, and so $\kappa_x(g) = \tau(x)$ for almost every $g \in G$. As $\{g \in G \mid \kappa_x(g) = \tau(x)\}$ is closed, it then follows that $\kappa_x(g) = \tau(x)$ for all $g \in G$. □

In the sequel we will need to consider convex combinations of the form $\theta_{\gamma^{-1}}(x)$ for $\gamma \in \Gamma_O$. In the case when $H < G$ is normal we have $\kappa_x(g) = \tilde{\kappa}_x(g)$, and the above proposition suffices. For the general case we have the following argument:

Proposition 9.3. *Using the notation above, if the induced action $H \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic, then $\tilde{\kappa}_x(g) = \tau(x)$ for all $g \in G$.*

Proof. Let $\varepsilon > 0$ be given. We claim that there exists $U \in \mathcal{N}(e)$ such that for all $\gamma \in \Gamma_U$ we have $\|\theta_\gamma(\tilde{\kappa}_x(e)) - \tilde{\kappa}_x(e)\|_2 = \|\tilde{\kappa}_x(e) - \theta_{\gamma^{-1}}(\tilde{\kappa}_x(e))\|_2 < \varepsilon$.

Indeed, from the definition of $\tilde{\kappa}_x(e)$ there exists $O \in \mathcal{N}(e)$ such that if y is the element of minimal norm in $\tilde{\mathcal{K}}_x(O)$ then $\|\tilde{\kappa}_x(e) - y\|_2 < \varepsilon/2$. We may write y as a convex combination $y = \sum_i \alpha_i \theta_{\gamma_i^{-1}}(x)$ where $\gamma_i = g_i h_i$ with $g_i \in O$, and $h_i \in H$. If no such U existed, then there would exist a sequence $\tilde{\gamma}_k = \tilde{g}_k \tilde{h}_k$ with $\tilde{h}_k \in H$, and $\tilde{g}_k \to e$ such that $\|y - \theta_{\tilde{\gamma}_k^{-1}}(y)\|_2 \geq \varepsilon/4$.

However, $\theta_{\tilde{\gamma}_k^{-1}}(y) = \sum_i \alpha_i \theta_{\tilde{\gamma}_k^{-1}\gamma_i^{-1}}(x)$, and since $\tilde{g}_k \to e$, for each i there exists large enough k so that $\gamma_i\tilde{\gamma}_k = g_i(h_i\tilde{g}_k h_i^{-1})h_i\tilde{h}_k \in OH$. It then follows from uniqueness of y that $\|y - \theta_{\tilde{\gamma}_k^{-1}}(y)\|_2 \to 0$, proving the claim.

From the claim it then follows that $\tilde{\kappa}_x(e)$ is the element of minimal $\|\cdot\|_2$ in $\cap_{O\in\mathcal{N}(e)}\mathcal{K}_{\tilde{\kappa}_x(e)}(O)$, and it then follows from Proposition 9.2 that $\tilde{\kappa}_x(e) = \tau(\tilde{\kappa}_x(e)) = \tau(x)$.

For $\gamma \in \Gamma$, $h \in H$, and $O \subset G$ a non-empty open set, we have $\tilde{\mathcal{K}}_x(\gamma O h) = \tilde{\mathcal{K}}_{\theta_{\gamma^{-1}}(x)}(O)$, hence it follows that $\tilde{\kappa}_x(\gamma h) = \tilde{\kappa}_{\theta_{\gamma^{-1}}(x)}(e) = \tau(\theta_{\gamma^{-1}}(x)) = \tau(x)$. Since ΓH is dense in G and since $\{g \in G \mid \tilde{\kappa}_x(g) = \tau(x)\}$ is closed, it then follows that $\tilde{\kappa}_x(g) = \tau(x)$ for all $g \in G$. $\square$

10 Contracting automorphisms

Let H be a locally compact, compactly generated group. An automorphism $\alpha \in \mathrm{Aut}(H)$ is **contracting** if for any neighborhood U of e and any compact set $K \subset H$, there is an integer N so that $\alpha^n(K) \subset U$ for all $n \geq N$.

Theorem 10.1 (Lebesgue's density theorem)**.** *Suppose X is a locally compact seperable Hausdorff space and μ is a σ-finite measure on X which is positive on open sets and finite on compact sets. Suppose that for each $x \in X$ we have a decreasing sequence $B_{x,n}$ of precompact open neighborhoods of x, forming a basis for the open sets at x, and such that*

1. *$\mu(B_{x,n-1})/\mu(B_{x,n})$ is constant independent of x, n.*
2. *If $n \leq p$ and $B_{x,n} \cap B_{y,p} \neq \emptyset$, then $B_{y,p} \subset B_{x,n-2}$.*

   If $E \subset X$ is a measurable set, then for almost all $x \in E$,

$$\lim_{n\to\infty} \mu(B_{x,n} \cap E)/\mu(B_{x,n}) = 1.$$

Proposition 10.2. *Suppose that H is a compactly generated locally compact group and $\alpha \in \mathrm{Aut}(H)$ is a contracting automorphism of H. Suppose that $U \subset H$ is a precompact open neighborhood of the identity with $U^{-1} = U$ and such that $\alpha(U^2) \subset U$. For $x \in H$ let $B_{x,n} = x\alpha^n(U)$. Then these sets satisfy the hypothesis of Theorem 10.1, where μ is the left Haar measure.*

Proof. First note that since μ and $\alpha_*\mu$ are both left Haar measures it follows that there is some constant $c > 0$ so that $\alpha_*\mu = c\mu$. Thus, for $x \in H$ and $n > 1$ we have

$$\frac{\mu(B_{x,n-1})}{\mu(B_{x,n})} = \frac{\mu(x\alpha^{n-1}(U))}{\mu(x\alpha^n(U))} = c.$$

To verify (2) suppose there exists $z \in y\alpha^p(U) \cap x\alpha^n(U)$, with $n \leq p$. Let $w \in y\alpha^p(U)$. Then

$$z^{-1}w \in \alpha^p(U)y^{-1}y\alpha^p(U) = \alpha^p(U^2) \subset \alpha^{p-1}(U) \subset \alpha^{n-1}(U).$$

Thus $w \in z\alpha^{n-1}(U)$ and since $z \in x\alpha^n(U)$ we have

$$w \in x\alpha^n(U)\alpha^{n-1}(U) \subset x\alpha^{n-1}(U^2) \subset x\alpha^{n-2}(U).$$

Since $w \in y\alpha^p(U)$ was arbitrary we then have $y\alpha^p(U) \subset x\alpha^{n-2}(U)$. $\square$

Proposition 10.3 (Margulis)**.** *Suppose that H is a compactly generated locally compact group and $\alpha \in \mathrm{Aut}(H)$ so that α^{-1} is contracting. Then replacing α by some power of α if necessary, it follows that for any measurable set $E \subset H$ and almost every $h \in E$ we have that $\alpha^n(h^{-1}E)$ converges in measure to H.*

Proof. Fix a precompact open neighborhood U of the identity such that $U^{-1} = U$, and $\cup_{n \geq 1} U^n = H$. Since α^{-1} is contracting we may replace α with a fixed power and assume that $\alpha^{-1}(U^2) \subset U$. Note that we then have $\alpha^{-1}((U^k)^2) \subset U^k$ for all $k > 1$. We again have $\alpha_* \mu = c\mu$, for some $c > 0$.

Let $E \subset H$ be a measurable set and fix $k \geq 1$. Then by the previous proposition and Lebesgue's density theorem we have that for almost every $h \in E$

$$1 = \lim_{n\to\infty} \mu(h\alpha^{-n}(U^k) \cap E)/\mu(h\alpha^{-n}(U^k)) = \lim_{n\to\infty} \frac{c^n}{c^n} \mu(U^k \cap \alpha^n(h^{-1}E))/\mu(U^k).$$

Hence $\lim_{n\to\infty} \mu(U^k \cap \alpha^n(h^{-1}E)) = \mu(U^k)$. Since $H = \cup_{k\geq 1} U^k$ it then follows that $\alpha^n(h^{-1}E)$ converges to H in measure. □

Notation 10.4. For the remaining of this section, and through all of the next section we use the following notation:

Fix $G = \mathrm{SL}_n(\mathbb{R})$ for $n \geq 3$. We denote by P the subgroup of upper triangular matrices, and V the subgroup of upper triangular matrices whose diagonal entries are 1. Suppose also that $1 \leq j_1 < j_2 < \cdots < j_k = n$, with $k > 1$. We denote by $P_0 < G$ the subgroup of block triangular matrices of determinant 1 of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ 0 & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix},$$

where A_{hh} is a square matrix of order $j_h - j_{h-1}$ (here we assume $j_0 = 0$). We also denote by $V_0 < V$ the subgroup block triangular matrices of the form

$$\begin{pmatrix} E_1 & A_{12} & \cdots & A_{1k} \\ 0 & E_2 & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & E_k \end{pmatrix},$$

where E_h denotes the $j_h - j_{h-1}$ identity matrix, and R_0 consists of the block diagonal matrices of the form

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}.$$

We let L_0 be the subgroup of R_0 whose block matrices A_{hh} are upper triangular with diagonal entries equal to 1.

We also denote by $\overline{P}, \overline{V}$, etc. the corresponding transpose subgroups.

Example 10.5. Choose real numbers $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ so that $\prod_{j=1}^k \lambda_j = 1$ and let s be the block diagonal matrix

$$s = \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix},$$

where A_{ii} is the diagonal matrix with diagonal entries equal to λ_i. Then $s \in \mathcal{Z}(R_0)$ and $\mathrm{Int}(s^{-1})_{|\overline{V_0}}$ and $\mathrm{Int}(s)_{|V_0}$ are contracting.

Lemma 10.6. *The mapping $\overline{V} \to G/P$ given by $v \mapsto vP$ is an injective Borel map onto a conull subset, hence gives rise to a measure space isomorphism.*

Proof. If $A \in SL_n(\mathbb{R})$, we let A_j be the upper left-hand corner $j \times j$ submatrix. The lemma can easily be reduced to the following statement in linear algebra which is readily verified by induction on n: For $A \in SL_n(\mathbb{R})$ we can write $A = BC$ where $B \in \overline{V}$ and $C \in P$ if and only if for all $1 \leq j \leq n$, $\det(A_j) \neq 0$, and if this is the case then the representation of A is unique. □

Under the above isomorphism, the action of $\overline{V}$ on G/P transforms to left multiplication on $\overline{V}$, while the action of R on G/P transforms to the action induced by conjugation on $\overline{V}$. As $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$ we then have a measure space isomorphism between G/P and $\overline{V_0} \times \overline{L_0}$, and hence we obtain a corresponding isomorphism between von Neumann algebras $L^\infty(G/P) \cong L^\infty(\overline{V_0}) \overline{\otimes} L^\infty(\overline{L_0})$. Moreover, the natural projectiion map G/P to G/P_0 gives rise to an embedding $L^\infty(G/P_0) \subset L^\infty(G/P)$ which corresponds under the above isomorphism to the natural embedding $L^\infty(\overline{V_0}) \subset L^\infty(\overline{V_0}) \overline{\otimes} L^\infty(\overline{L_0})$. Note that the functions in $L^\infty(\overline{L_0})$ are fixed by any element in $\overline{V_0} \rtimes \mathcal{Z}(R_0)$.

Lemma 10.7. *Suppose X is a second countable locally compact Hausdorff space equipped with a probability $\mu \in$ Prob(X) such that μ gives positive measure to any non-empty open set. Suppose also that $\mathbb{Z} \curvearrowright X$ by homeomorphisms, preserving the measure μ, such that the action $\mathbb{Z} \curvearrowright (X, \mu)$ is ergodic. Then for almost every $x \in X$ the set $\mathbb{N} \cdot x$ is dense.*

Proof. If $W \subset X$ is a non-empty open set, then $E = (-\mathbb{N}) \cdot W \subset X$ is a positive measure subset of X such that $(-1) \cdot E \subset E$. Since the action is measure-preserving it then follows that E is invariant and hence must be conull in (X, μ). Thus, if $\{W_i\}_i$ is a countable basis of non-empty open sets then we have that $F = \cap_i (\cup_{j \leq -1} j \cdot W_i)$ is conull in (X, μ), and for each $x \in F$ we have that $\mathbb{N} \cdot x$ is dense since it intersects each basis set W_i. □

Lemma 10.8. *If $s \in \mathcal{Z}(R_0)$ is as in Example 10.5, then for almost every $v \in \overline{V_0}$ the set $\{\Gamma v s^{-j} \mid j \in \mathbb{N}\}$ is dense in G.*

Proof. Set $W = \{g \in G \mid \{\Gamma g s^{-j} \mid j \in \mathbb{N}\}$ is dense in G$\}$. By Moore's ergodicity theorem $\{s^{-n} \mid n \in \mathbb{Z}\}$ acts ergodically on G/Γ and hence by Lemma 10.7 W is conull.

Recall from Lemma 10.6 that the multiplication map $(\overline{V_0} \rtimes \overline{L_0}) \times P \to G$ is an injective map onto a conull subset of G. For each $v \in \overline{V_0}$ we set $Y_v = \{(g, h) \mid g \in \overline{L_0}, h \in P, vgh \in W\}$, and set $U = \{v \in \overline{V_0} \mid Y_v$ is conull in $\overline{L_0} \times P\}$. Then by Fubini, U is conull in $\overline{V_0}$. Thus, to prove the lemma it suffices to show that $\{\Gamma v s^{-j} \mid j \in \mathbb{N}\}$ is dense in G for each $v \in U$.

Fix $v \in U$, and choose $g \in \overline{L_0}$ and $h \in P$ so that $(g, h) \in Y_v$, so that $\{\Gamma vghs^{-j} \mid j \in \mathbb{N}\}$ is dense in G. Since $gh \in P_0 = R_0 \ltimes V_0$, we may write $gh = ur$ where $u \in V_0$, and $r \in R_0$. Since $[s, r] = e$ it then follows that $\{\Gamma vus^{-j} \mid j \in \mathbb{N}\}$ is dense in G. Note also that since Γ is discrete we must then have that $\{\Gamma vus^{-j} \mid j \geq N\}$ is dense for any $N \geq 1$.

Suppose now that $O \subset G$ is a non-empty open set. We let $O_1 \subset O$ be a non-empty precompact open set with $\overline{O_1} \subset O$, and we let $C \subset G$ be an open neighborhood of the identity so that $\overline{O_1}C \subset O$. Since $\mathrm{Int}(s)_{|V_0}$ is contracting there exists $N > 1$ so that $s^j u^{-1} s^{-j} \in C$ for all $j \geq N$. Since $\{\Gamma vus^{-j} \mid j \geq N\}$ is dense, there exists $j \geq N$ and $\gamma \in \Gamma$ so that $\gamma vus^{-j} \in O_1$. We then have

$$\gamma vs^{-j} = \gamma vus^{-j}(s^j u^{-1} s^{-j}) \in (\gamma vus^{-j})C \subset O_1 C \subset O.$$

Since $O \subset G$ was arbitrary it follows that $\{\Gamma vs^{-j} \mid j \in \mathbb{N}\}$ is dense in G. □

Theorem 10.9 (Margulis)**.** *Let $\nu \in \mathrm{Prob}(\overline{V_0})$ be in the same measure class as Haar measure. If $E \subset \overline{V_0}$ has positive measure and $g \in G$, then there exist sequences $\{\gamma_j\} \subset \Gamma$ and $\{h_j\} \subset \overline{V_0} \rtimes \mathcal{Z}(R_0)$ such that $\gamma_j h_j^{-1} \to g$, and $\nu(\gamma_j E) \to 1$.*

Proof. From Example 10.5 there exists $s \in \mathcal{Z}(R_0)$ such that $\mathrm{Int}s^{-1}{}_{|\overline{V_0}}$ and $\mathrm{Int}s_{|V_0}$ are contracting, and by Lemma 10.8, for almost every $v \in \overline{V_0}$ we have that $\{\Gamma vs^{-j} \mid j \in \mathbb{N}\}$ is dense in G. We denote by F_1 the set of such $v \in \overline{V_0}$.

As $\mathrm{Int}s^{-1}{}_{|\overline{V_0}}$ is contracting, Proposition 10.3 shows that for almost every $v \in \overline{V_0}$ we have $\nu(s^j v^{-1} E) \to 1$. We denote by F_2 the set of such $v \in \overline{V_0}$.

Fix $v_0 \in F_1 \cap F_2$. Since $v_0 \in F_1$ there exists a sequence $\{\gamma_i\} \subset \Gamma$, and a subsequence $\{s^{-j_i}\}$ such that $\gamma_i v_0 s^{-j_i} \to g$. Hence

$$\lim_{i\to\infty} \nu(\gamma_i E) = \lim_{i\to\infty} \nu(g s^{j_i} v_0^{-1} E) = 1.$$

Since $h_j = v_0 s^{-j_i} \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$, this finishes the proof. □

Margulis' normal subgroup theorem (Corollary C) follows from the following ergodicity result.

Theorem 10.10 (Margulis)**.** *Using the notation above with $G = SL_n(\mathbb{R})$ and $\Gamma < G$ a lattice, let $\mu \in \mathrm{Prob}(G/P)$ be a quasi-invariant probability measure. If $N \lhd \Gamma$ is a normal subgroup such that $N \not< \mathcal{Z}(G)$, then the action $N \curvearrowright (G/P, \mu)$ is ergodic.*

Proof. Using the notation in 10.4 we measurably identify G/P with $\overline{V} = \overline{V_0} \rtimes \overline{L_0}$ as above. Suppose that $E \subset \overline{V}$ is invariant under the action of N. Write $E = \cup_{h \in \overline{V_0}} \{h\} \times E_h$. We consider the measure algebra $\mathcal{M}(\overline{L_0})$ consisting of measurable subsets of $\overline{L_0}$ where we identify two subsets if their symmetric difference is null. Then $\mathcal{M}(\overline{L_0})$ is a Polish space when equipped with the topology of convergence in measure and the map $\psi : \overline{V_0} \to \mathcal{M}(\overline{L_0})$ is a measurable map. We fix a probability measures $\zeta \in \mathrm{Prob}(\overline{V_0})$, and $\nu \in \mathrm{Prob}(\overline{L_0})$ in the class of Haar measure.

We fix $g \in G$ and let $E_0 \subset \mathcal{M}(\overline{L_0})$ be in the essential range of ψ, and we fix a countable dense collection $\{F_i\}_i \subset \mathcal{M}(\overline{L_0})$. Then for any $j > 1$ and, there exists a positive measure set $A_j \subset \overline{L_0}$ so that $|\nu(E_h \Delta F_i) - \nu(E_0 \Delta F_i)| < 1/j$ for all $1 \leq i \leq j$, and all $h \in A_j$. By the previous theorem there then exists sequences $\{\gamma_j\}_j \subset \Gamma$ and $\{h_j\}_j \subset \overline{V_0} \rtimes \mathcal{Z}(R_0)$ so that $\zeta(\gamma_j A_j) \to 1$, and $\gamma_j h_j^{-1} \to g$.

Since the action of $\overline{V_0} \rtimes \mathcal{Z}(R_0)$ on $\overline{L_0}$ is trivial we then have that $\gamma_j E$ converges in measure to $gE_0 \in \mathcal{M}(\overline{L_0})$, and since $N \lhd \Gamma$ is normal we have that each $\gamma_j E$ is N-invariant and hence so is gE_0. Equivalently, we have shown that E_0 is invariant under the action of the group $g^{-1}Ng$. Since this was true for an arbitrary element $g \in G$ we then have that E_0 is invariant under the action of the closed normal subgroup in G which is generated by N. However, as $N \not< \mathcal{Z}(G)$ it then follows that E_0 is invariant under all of G and so must be either null or conull.

Since E_0 was an arbitrary set in the essential range of ψ it follows that there exists $A \subset \overline{L_0}$ measurable so that $E = A \times \overline{L_0}$ up to measure zero, i.e., the characteristic function $1_E \in L^\infty(G/P_0) \subset L^\infty(G/P)$. Since $P < P_0 \lneq G$ was arbitrary and since G is generated by such subgroups (here is where we use $n \geq 3$) it then follows that $1_E \in \mathbb{C}$, i.e., E is either null or conull and hence the action $N \curvearrowright (G/P, \mu)$ is ergodic. □

11 Character rigidity for lattices

We continue to use the notation from 10.4.

Suppose that M is a finite factor with normal faithful trace τ, and $\pi : \Gamma \to \mathcal{U}(M)$ is a homomorphism such that $\pi(\Gamma)'' = M$. We let Γ act on $\mathcal{B}(L^2(M))$ by conjugation by $J\pi(\gamma)J$ and we let

$$\mathcal{B} = (L^\infty(G/P) \overline{\otimes} \mathcal{B}(L^2 M))^\Gamma,$$

denote the von Neumann algebra of essentially bounded Γ-equivariant functions from G/P to $\mathcal{B}(L^2 M)$. We denote by $\sigma : \Gamma \to \mathrm{Aut}(L^\infty(G/P))$ the action given by $\sigma_\gamma(f)(x) = f(\gamma^{-1}x)$. We also fix a quasi-invariant probability measure $\mu \in \mathrm{Prob}(G/P)$ and let $\sigma^0 : \Gamma \to L^2(G/P, \mu)$ denote the Koopman representation $\sigma^0_\gamma(f)(x) = f(\gamma^{-1}x)(\frac{d\gamma_*\mu}{d\mu})^{1/2}(x)$. Note that viewing $L^\infty(G/P)$ as bounded operators on $L^2(G/P, \mu)$ we have $\sigma_\gamma(f) = \sigma^0_\gamma f \sigma^0_{\gamma^{-1}}$.

We have the following operator algebraic consequence of Lemma 10.9:

Lemma 11.1. *Using the notation from 10.4, suppose $x = x^* \in L^\infty(\overline{V_0}) \overline{\otimes} L^\infty(\overline{L_0}) \overline{\otimes} \mathcal{B}(L^2 M)$ and suppose $x_0 \in L^\infty(\overline{L_0}) \overline{\otimes} \mathcal{B}(L^2 M)$ is in the strong operator topology $\overline{V_0}$-essential range of x, then there exists $y = y^* \in \mathcal{B}$ such that $yP_{\hat{1}} \in L^\infty(\overline{L_0}) \overline{\otimes} \mathcal{B}(L^2 M)$, and $P_{\hat{1}} y P_{\hat{1}} = P_{\hat{1}} x_0 P_{\hat{1}}$.*

Proof. For each $j \in \mathbb{N}$ we let $E_j \subset \overline{V_0}$ be a positive measure subset such that $x(v_j) - x_0 \to 0$ strongly whenever $v_j \in E_j$. From Lemma 10.9 there exists sequences $\{\gamma_j\} \subset \Gamma$ and $\{h_j\} \subset \overline{V_0} \rtimes \mathcal{Z}(R_0)$ such that $\gamma_j h_j^{-1} \to e$, and $\nu(\gamma_j E_j) \to 1$.

Hence, for all $\xi \in L^2(\overline{V_0} \times \overline{L_0})$ and $\eta \in L^2 M$ we have

$$\begin{aligned}\|1_{\gamma_j E_j}(\sigma_{\gamma_j}(x) - x_0)\xi \otimes \eta\| &\leq \|1_{\gamma_j E_j}(\sigma_{\gamma_j}(x - x_0))\xi \otimes \eta\| + \|1_{\gamma_j E_j}(\sigma_{\gamma_j h_j^{-1}}(x_0) - x_0)\xi \otimes \eta\| \\ &= \left(\int_{\gamma_j E_j} \|(\sigma_{\gamma_j}(x - x_0))\eta\|_2^2 |\xi|^2 \right)^{1/2} \\ &\quad + \|1_{\gamma_j E_j}(\sigma_{\gamma_j h_j^{-1}}(x_0) - x_0)\xi \otimes \eta\| \to 0.\end{aligned}$$

Therefore, $1_{\gamma_j E_j}(\sigma_{\gamma_j}(x) - x_0) \to 0$ strongly. Since $\nu(\gamma_j E_j) \to 1$, we then have that $1_{\gamma_j E_j} \to 1$ strongly, and hence $\sigma_{\gamma_j}(x) \to x_0$ strongly.

We let $y \in \mathcal{B}$ be a weak operator topology cluster point of the set $\{\pi(\gamma_j)x\pi(\gamma_j^{-1})\}$. It then follows that $yP_{\hat{1}}$ is a weak operator topology cluster point of

$$\begin{aligned}\{\pi(\gamma_j)x\pi(\gamma_j^{-1})P_{\hat{1}}\} &= \{(\pi(\gamma_j)(J\pi(\gamma_j)J))(J\pi(\gamma_j^{-1})J)x(J\pi(\gamma_j)J)P_{\hat{1}}\} \\ &= \{(\pi(\gamma_j)(J\pi(\gamma_j)J))\sigma_{\gamma_j}(x)P_{\hat{1}}\}.\end{aligned}$$

Since $\sigma_{\gamma_j}(x) \to x_0$ strongly, we then have that $yP_{\hat{1}}$ is a weak operator topology cluster point of $\{(\pi(\gamma_j)(J\pi(\gamma_j)J))x_0P_{\hat{1}}\} \subset L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, thus $y \in \mathcal{B}$, and $yP_{\hat{1}} \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$.

Similarly, $P_{\hat{1}}yP_{\hat{1}}$ is a weak operator topology cluster point of

$$\{P_{\hat{1}}\pi(\gamma_j)x\pi(\gamma_j^{-1})P_{\hat{1}}\} = \{P_{\hat{1}}\sigma_{\gamma_j}(x)P_{\hat{1}}\}$$

and hence $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$. □

We say the representation $\pi : \Gamma \to \mathcal{U}(M)$ is induced from a character on $\mathcal{Z}(\Gamma)$ if $\tau(\pi(\gamma)) = 0$ for all $\gamma \notin \mathcal{Z}(\Gamma)$.

Theorem 11.2. *Using the notation above, either π is induced from a character on $\mathcal{Z}(\Gamma)$, or else $M = \mathcal{B}$.*

Proof. Suppose that $x = x^* \in \mathcal{B}$. Using the notation above consider $x \in L^\infty(\overline{V_0})\overline{\otimes}L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$. Let $x_0 \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$ be in the $\overline{V_0}$-essential range of x. By Lemma 11.1 there exists $y = y^* \in \mathcal{B}$ such that $yP_{\hat{1}} \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, and $P_{\hat{1}}yP_{\hat{1}} = P_{\hat{1}}x_0P_{\hat{1}}$.

Since the action $G \curvearrowright L^\infty(G/\Gamma)\overline{\otimes}M$ is ergodic Moore's ergodicity theorem shows that the action restricted to $\overline{V_0} \rtimes \mathcal{Z}(R_0)$ is ergodic (this group is non-compact since $P_0 \neq G$). By Proposition 9.3, for every open neighborhood $O \subset G$ of e, setting $\Gamma_O = \Gamma \cap O(\overline{V_0} \rtimes \mathcal{Z}(R_0))$ we have $\tau(x) \in \overline{\mathrm{co}}\{\pi(\gamma^{-1})x\pi(\gamma) \mid \gamma \in \Gamma_O\}$.

If π is not induced from a character on $\mathcal{Z}(\Gamma)$ there exists $\gamma_0 \in \Gamma \setminus \mathcal{Z}(\Gamma)$ such that $\alpha_0 = \tau(\pi(\gamma_0)) \neq 0$. Thus, for each open neighborhood $O \subset G$ of e there exists $0 \leq \alpha_i \leq 1$, $\sum_{i=1}^j \alpha_i = 1$, and $\gamma_i \in \Gamma_O$ such that

$$\begin{aligned}[\sigma_{\gamma_0}^0 \otimes \alpha_0, P_{\hat{1}}x_0P_{\hat{1}}] &= [\sigma_{\gamma_0}^0 \otimes \alpha_0, P_{\hat{1}}yP_{\hat{1}}] \\ &\sim \sum_{i=1}^j \alpha_i P_{\hat{1}}[\sigma_{\gamma_0}^0 \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y]P_{\hat{1}},\end{aligned}$$

where the approximation above is in the strong operator topology. We may write $\gamma_i = g_ih_i$ where $g_i \in O$, and $h_i \in \overline{V_0} \rtimes \mathcal{Z}(R_0)$, and since $yP_{\hat{1}}, P_{\hat{1}}y \in L^\infty(\overline{L_0})\overline{\otimes}\mathcal{B}(L^2M)$, we have that $\sigma_{h_i}(yP_{\hat{1}}) = yP_{\hat{1}}$, and $\sigma_{h_i}(P_{\hat{1}}y) = P_{\hat{1}}y$. Taking O to be a small enough neighborhood, we then have the strong operator topology approximations $\sigma_{\gamma_i}(yP_{\hat{1}}) \sim yP_{\hat{1}}$, and $\sigma_{\gamma_i}(P_{\hat{1}}y) \sim P_{\hat{1}}y$.

It follows that we then have the weak operator topology approximation

$$\begin{aligned}[\sigma_{\gamma_0}^0 \otimes \alpha_0, P_{\hat{1}}x_0P_{\hat{1}}] &\sim \sum_{i=1}^j \alpha_i P_{\hat{1}}[\sigma_{\gamma_0}^0 \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, (\sigma_{\gamma_i}^0 \otimes 1)y(\sigma_{\gamma_i^{-1}}^0 \otimes 1)]P_{\hat{1}} \\ &= \sum_{i=1}^j \alpha_i P_{\hat{1}}(\sigma_{\gamma_i}^0 \otimes 1)[\sigma_{\gamma_i^{-1}\gamma_0\gamma_i}^0 \otimes J\pi(\gamma_i^{-1}\gamma_0\gamma_i)J, y](\sigma_{\gamma_i^{-1}}^0 \otimes 1)P_{\hat{1}} = 0.\end{aligned}$$

Hence, we conclude that $[\sigma^0_{\gamma_0} \otimes \alpha_0, P_{\hat{1}} x_0 P_{\hat{1}}] = 0$, and since $\alpha_0 \neq 0$ we then have that $[\sigma^0_{\gamma_0} \otimes 1, P_{\hat{1}} x_0 P_{\hat{1}}] = 0$.

Hence $\sigma_{\gamma_0}(P_{\hat{1}} x_0 P_{\hat{1}}) = P_{\hat{1}} x_0 P_{\hat{1}}$, and since $\tau(\pi(\gamma\gamma_0\gamma^{-1})) = \tau(\pi(\gamma_0))$ the same argument shows that $\sigma_\gamma(P_{\hat{1}} x_0 P_{\hat{1}}) = P_{\hat{1}} x_0 P_{\hat{1}}$ for all $\gamma \in \langle\langle \gamma_0 \rangle\rangle$, the normal closure of γ_0. Since $\gamma_0 \not\in \mathcal{Z}(\Gamma)$, Theorem 10.10 then shows that $\langle\langle \gamma_0 \rangle\rangle$ acts ergodically on $L^\infty(G/P)$. Thus, we have $P_{\hat{1}} x_0 P_{\hat{1}} \in P_{\hat{1}} \mathcal{B}(L^2 M) P_{\hat{1}} = \mathbb{C} P_{\hat{1}}$.

As x_0 was an arbitrary element in the essential range of x, we conclude that $P_{\hat{1}} x P_{\hat{1}} \in L^\infty(\overline{V_0}) \overline{\otimes} \mathbb{C} P_{\hat{1}}$, and, as x was an arbitrary self-adjoint element, we then have $P_{\hat{1}} \mathcal{B} P_{\hat{1}} \subset L^\infty(\overline{V_0}) \overline{\otimes} \mathbb{C} P_{\hat{1}}$. If $a, b \in M$ and $z \in \mathcal{B}$, we then have $\langle z\hat{a}, \hat{b} \rangle = \langle (b^* z a)\hat{1}, \hat{1} \rangle \in L^\infty(\overline{V_0})$, and hence it follows that $\mathcal{B} \subset L^\infty(\overline{V_0}) \overline{\otimes} \mathcal{B}(L^2 M) = L^\infty(G/P_0) \overline{\otimes} \mathcal{B}(L^2 M)$.

However, P_0 was an arbitrary subgroup such that $P < P_0 \lneq G$. Since G is generated by all such subgroups (note that this is were we need $n \geq 3$) we therefore have $\mathcal{B} \subset \mathcal{B}(L^2 M)$. Thus, $\mathcal{B} = \mathcal{B}(L^2 M) \cap \{J\pi(\gamma)J \mid \gamma \in \Gamma\}' = M$. □

Proof of Theorem D. From Theorem 3.8 we see that G has property (T), and hence so does Γ by Theorem 3.9. Thus, M has property (T) by Theorem 8.1.

Since $\mathcal{Z}(G) = \{e\}$, we have that $\mathcal{Z}(\Gamma) = \{e\}$. Thus, if π does not extend to an isomorphism $L\Gamma \xrightarrow{\sim} M$, then by Theorem 11.2 we have that $M = \mathcal{B}$ and hence is amenable by Theorem 9.1. Proposition 8.3 would then show that M is finite dimensional. □

12 Appendix: Background material

In this section we collect the basic results we need along with references to proofs.

12.1 Bounded linear operators

If $\mathcal{H}$ is a Hilbert space then $\mathcal{B}(\mathcal{H})$, the algebra of all bounded linear operators, is a C^*-algebra with norm

$$\|x\| = \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \|x\xi\| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, \eta \rangle|,$$

and involution given by the adjoint, i.e., x^* is the unique bounded linear operator such that

$$\langle \xi, x^* \eta \rangle = \langle x\xi, \eta \rangle,$$

for all $\xi, \eta \in \mathcal{H}$. Note that $\|x^*\| = \|x\|$ since

$$\|x\| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, \eta \rangle| = \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle \xi, x^* \eta \rangle| = \|x^*\|.$$

To see the C^*-identity note that we clearly have $\|x^* x\| \leq \|x^*\| \|x\|$, and for the reverse inequality we have

$$\begin{aligned} \|x\|^2 &= \sup_{\xi \in \mathcal{H}, \|\xi\| \leq 1} \langle x\xi, x\xi \rangle \\ &\leq \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x\xi, x\eta \rangle| \\ &= \sup_{\xi, \eta \in \mathcal{H}, \|\xi\|, \|\eta\| \leq 1} |\langle x^* x\xi, \eta \rangle| = \|x^* x\|. \end{aligned}$$

Proposition 12.1 (Polar decomposition)**.** *Let $\mathcal{H}$ be a Hilbert space, and $x \in \mathcal{B}(\mathcal{H})$, then there exists a partial isometry v such that $x = v|x|$, and $\ker(v) = \ker(|x|) = \ker(x)$. Moreover, this decomposition is unique, in that if $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$ then $y = |x|$, and $w = v$.*

Proof. We define a linear operator $v_0 : R(|x|) \to R(x)$ by $v_0(|x|\xi) = x\xi$, for $\xi \in \mathcal{H}$. Since $\||x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$ it follows that v_0 is well defined and extends to a partial isometry v from $\overline{R(|x|)}$ to $\overline{R(x)}$, and we have $v|x| = x$. We also have $\ker(v) = R(|x|)^\perp = \ker(|x|) = \ker(x)$.

To see the uniqueness of this decomposition suppose $x = wy$ where $y \geq 0$, and w is a partial isometry with $\ker(w) = \ker(y)$. Then $|x|^2 = x^*x = yw^*wy = y^2$, and hence $|x| = (|x|^2)^{1/2} = (y^2)^{1/2} = y$. We then have $\ker(w) = \overline{R(|x|)}^\perp$, and $\|w|x|\xi\| = \|x\xi\|$, for all $\xi \in \mathcal{H}$, hence $w = v$. □

12.2 Locally convex topologies on the space of operators

Let $\mathcal{H}$ be a Hilbert space. On $\mathcal{B}(\mathcal{H})$ we define the following locally convex topologies:

- The **weak operator topology** (WOT) is defined by the family of semi-norms $T \mapsto |\langle T\xi, \eta\rangle|$, for $\xi, \eta \in \mathcal{H}$.
- The **strong operator topology** (SOT) is defined by the family of semi-norms $T \mapsto \|T\xi\|$, for $\xi \in \mathcal{H}$.

Note that the from coarsest to finest topologies we have

$$\text{WOT} \prec \text{SOT} \prec \text{Uniform}.$$

Also note that since an operator T is normal if and only if $\|T\xi\| = \|T^*\xi\|$ for all $\xi \in \mathcal{H}$, it follows that the adjoint is SOT continuous on the set of normal operators.

Lemma 12.2. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H}$ such that $\varphi(T) = \sum_{i=1}^n \langle T\xi_i, \eta_i\rangle$, for all $T \in \mathcal{B}(\mathcal{H})$.

(ii) φ is WOT continuous.

(iii) φ is SOT continuous.

Proof. The implications (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear and so we will only show (iii) $\implies$ (i). Suppose φ is SOT continuous. Thus, the inverse image of the open ball in $\mathbb{C}$ is open in the SOT and hence by considering the semi-norms which define the topology we have that there exists a constant $K > 0$, and $\xi_1, \ldots, \xi_n \in \mathcal{H}$ such that

$$|\varphi(T)|^2 \leq K \sum_{i=1}^n \|T\xi_i\|^2.$$

If we then consider $\{\oplus_{i=1}^n T\xi_i \mid T \in \mathcal{B}(\mathcal{H})\} \subset \mathcal{H}^{\oplus n}$, and let $\mathcal{H}_0$ be its closure, we have that

$$\oplus_{i=1}^n T\xi_i \mapsto \varphi(T)$$

extends to a well defined, continuous linear functional on $\mathcal{H}_0$ and hence by the Riesz representation theorem there exists $\eta_1, \ldots, \eta_n \in \mathcal{H}$ such that

$$\varphi(T) = \sum_{i=1}^{n} \langle T\xi_i, \eta_i \rangle,$$

for all $T \in \mathcal{B}(\mathcal{H})$. □

Corollary 12.3. *Let $K \subset \mathcal{B}(\mathcal{H})$ be a convex set, then the WOT, SOT, and closures of K coincide.*

Proof. By Lemma 12.2 the three topologies above give rise to the same dual space, hence this follows from the the Hahn-Banach separation theorem. □

If $\mathcal{H}$ is a Hilbert space then the map $\mathrm{id} \otimes 1 : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ defined by $(\mathrm{id} \otimes 1)(x) = x \otimes 1$ need not be continuous in either of the locally convex topologies defined above even though it is an isometric C^*-homomorphism with respect to the uniform topology. Thus, on $\mathcal{B}(\mathcal{H})$ we define the following additional locally convex topologies:

- The **σ-weak operator topology** (σ-WOT) is defined by pulling back the WOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.
- The **σ-strong operator topology** (σ-SOT) is defined by pulling back the SOT of $\mathcal{B}(\mathcal{H} \overline{\otimes} \ell^2\mathbb{N})$ under the map $\mathrm{id} \otimes 1$.

Note that the σ-weak operator topology can alternately be defined by the family of semi-norms $T \mapsto |\operatorname{Tr}(Ta)|$, for $a \in L^1(\mathcal{B}(\mathcal{H}))$. Hence, under the identification $\mathcal{B}(\mathcal{H}) = L^1(\mathcal{B}(\mathcal{H}))^*$, we have that the weak*-topology on $\mathcal{B}(\mathcal{H})$ agrees with the σ-WOT.

Lemma 12.4. *Let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be a linear functional, then the following are equivalent:*

(i) There exists a trace class operator $a \in L^1(\mathcal{B}(\mathcal{H}))$ such that $\varphi(x) = \operatorname{Tr}(xa)$ for all $x \in \mathcal{B}(\mathcal{H})$

(ii) φ is σ-WOT continuous.

(iii) φ is σ-SOT continuous.

Proof. Again, we need only show the implication (iii) $\implies$ (i), so suppose φ is σ-SOT continuous. Then by the Hahn-Banach theorem, considering $\mathcal{B}(\mathcal{H})$ as a subspace of $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$ through the map $\mathrm{id} \otimes 1$, we may extend φ to a SOT continuous linear functional on $\mathcal{B}(\mathcal{H} \otimes \ell^2\mathbb{N})$. Hence by Lemma 12.2 there exists $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n \in \mathcal{H} \overline{\otimes} \ell^2\mathbb{N}$ such that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\varphi(x) = \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle.$$

For each $1 \leq i \leq n$ we may define $a_i, b_i \in \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$ as the operators corresponding to ξ_i, η_i in the Hilbert space isomorphism $\mathcal{H} \otimes \ell^2\mathbb{N} \cong \mathrm{HS}(\mathcal{H}, \ell^2\mathbb{N})$. By considering $a = \sum_{i=1}^{n} b_i^* a_i \in L^1(\mathcal{B}(\mathcal{H}))$, it then follows that for all $x \in \mathcal{B}(\mathcal{H})$ we have

$$\begin{aligned}\mathrm{Tr}(xa) &= \sum_{i=1}^{n} \langle a_i x, b_i \rangle_2 \\ &= \sum_{i=1}^{n} \langle (\mathrm{id} \otimes 1)(x)\xi_i, \eta_i \rangle = \varphi(x).\end{aligned}$$ □

Corollary 12.5. *The unit ball in $\mathcal{B}(\mathcal{H})$ is compact in the σ-WOT.*

Corollary 12.6. *The WOT and the σ-WOT agree on bounded sets.*

Proof. The identity map is clearly continuous from the σ-WOT to the WOT. Since both spaces are Hausdorff it follows that this is a homeomorphism from the σ-WOT compact unit ball in $\mathcal{B}(\mathcal{H})$. By scaling we therefore have that this is a homeomorphism on any bounded set. □

Exercise 12.7. Show that the adjoint $T \mapsto T^*$ is continuous in the WOT, and when restricted to the space of normal operators is continuous in the SOT, but is not continuous in the SOT on the space of all bounded operators.

Exercise 12.8. Show that operator composition is jointly continuous in the SOT on bounded subsets.

Exercise 12.9. Show that the SOT agrees with the σ-SOT on bounded subsets of $\mathcal{B}(\mathcal{H})$.

Exercise 12.10. Show that pairing $\langle x, a \rangle = \mathrm{Tr}(a^* x)$ gives an identification between $\mathcal{K}(\mathcal{H})^*$ and $(L^1(\mathcal{B}(\mathcal{H})), \| \cdot \|_1)$.

12.3 Von Neumann algebras and the double commutant theorem

A **von Neumann algebra** (over a Hilbert space $\mathcal{H}$) is a $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which contains 1 and is closed in the weak operator topology.

Note that since subalgebras are of course convex, it follows from Corollary 12.3 that von Neumann algebras are also closed in the strong operator topology.

If $A \subset \mathcal{B}(\mathcal{H})$ then we denote by $W^*(A)$ the von Neumann subalgebra which is generated by A, i.e., $W^*(A)$ is the smallest von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ which contains A.

Lemma 12.11. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. Then $(A)_1$ is compact in the WOT.*

Proof. This follows directly from Corollary 12.5. □

Corollary 12.12. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, then $(A)_1$ and $A_{\mathrm{s.a.}}$ are closed in the weak and strong operator topologies.*

Proof. Since taking adjoints is continuous in the weak operator topology it follows that $A_{\mathrm{s.a.}}$ is closed in the weak operator topology, and by the previous result this is also the case for $(A)_1$. □

If $B \subset \mathcal{B}(\mathcal{H})$, the **commutant** of B is

$$B' = \{T \in \mathcal{B}(\mathcal{H}) \mid TS = ST, \text{ for all } S \in B\}.$$

We also use the notation $B'' = (B')'$ for the **double commutant**.

Theorem 12.13. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint set, then A' is a von Neumann algebra.*

Proof. It is easy to see that A' is a self-adjoint algebra containing 1. To see that it is closed in the weak operator topology just notice that if $x_\alpha \in A'$ is a net such that $x_\alpha \to x \in \mathcal{B}(\mathcal{H})$ then for any $a \in A$, and $\xi, \eta \in \mathcal{H}$, we have

$$\langle [x,a]\xi, \eta\rangle = \langle xa\xi, \eta\rangle - \langle x\xi, a^*\eta\rangle$$

$$= \lim_{\alpha\to\infty} \langle x_\alpha a\xi, \eta\rangle - \langle x_\alpha \xi, a^*\eta\rangle = \lim_{\alpha\to\infty} \langle [x_\alpha, a]\xi, \eta\rangle = 0. \qquad \square$$

Corollary 12.14. *A self-adjoint maximal abelian subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra.*

Proof. Since A is maximal abelian we have $A = A'$. $\square$

Lemma 12.15. *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then for all $\xi \in \mathcal{H}$, and $x \in A''$ there exists $x_\alpha \in A$ such that $\lim_{\alpha\to\infty} \|(x - x_\alpha)\xi\| = 0$.*

Proof. Consider the closed subspace $\mathcal{K} = \overline{A\xi} \subset \mathcal{H}$, and denote by p the projection onto this subspace. Since for all $a \in A$ we have $a\mathcal{K} \subset \mathcal{K}$, it follows that $ap = pap$. But since A is self-adjoint it then also follows that for all $a \in A$ we have $pa = (a^*p)^* = (pa^*p)^* = pap = ap$, and hence $p \in A'$.

We therefore have that $xp = xp^2 = pxp$ and hence $x\mathcal{K} \subset \mathcal{K}$. Since $1 \in A$ it follows that $\xi \in \mathcal{K}$ and hence also $x\xi \in \overline{A\xi}$. $\square$

Theorem 12.16 (Von Neumann's double commutant theorem)**.** *Suppose $A \subset \mathcal{B}(\mathcal{H})$ is a self-adjoint algebra containing 1. Then A'' is equal to the weak operator topology closure of A.*

Proof. By Theorem 12.13 we have that A'' is closed in the weak operator topology, and we clearly have $A \subset A''$, so we just need to show that $A \subset A''$ is dense in the weak operator topology. For this we use the previous lemma together with a matrix trick.

Let $\xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$ and consider the subalgebra $\tilde{A}$ of $\mathcal{B}(\mathcal{H}^n) \cong \mathbb{M}_n(\mathcal{B}(\mathcal{H}))$ consisting of diagonal matrices with constant diagonal coefficients contained in A. Then the diagonal matrix whose diagonal entries are all x is easily seen to be contained in $\tilde{A}''$, hence the previous lemma applies and so there exists a net $a_\alpha \in A$ such that $\lim_{\alpha\to\infty} \|(x - a_\alpha)\xi_k\| = 0$, for all $1 \le k \le n$. This shows that $A \subset A''$ is dense in the strong operator topology. $\square$

We also have the following formulation which is easily seen to be equivalent.

Corollary 12.17. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra. Then A is a von Neumann algebra if and only if $A = A''$.*

Corollary 12.18. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $x \in A$, and consider the polar decomposition $x = v|x|$. Then $v \in A$.*

Proof. Note that $\ker(v) = \ker(|x|)$, and if $a \in A'$ then we have $a \ker(|x|) \subset \ker(|x|)$. Also, we have

$$\|(av - va)|x|\xi\| = \|ax\xi - xa\xi\| = 0,$$

for all $\xi \in \mathcal{H}$. Hence av and va agree on $\ker(|x|) + \overline{R(|x|)} = \mathcal{H}$, and so $v \in A'' = A$. $\square$

Proposition 12.19. *Let (X, μ) be a σ-finite measure space. Consider the Hilbert space $L^2(X, \mu)$, and the map $M : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ defined by $(M_g\xi)(x) = g(x)\xi(x)$, for all $\xi \in L^2(X, \mu)$. Then M is an isometric $*$-isomorphism from $L^\infty(X, \mu)$ onto a maximal abelian von Neumann subalgebra of $\mathcal{B}(L^2(X, \mu))$.*

Proof. The fact that M is a $*$-isomorphism onto its image is clear. If $g \in L^\infty(X, \mu)$ then by definition of $\|g\|_\infty$ we can find a sequence E_n of measurable subsets of X such that $0 < \mu(E_n) < \infty$, and $|g|_{|E_n} \geq \|g\|_\infty - 1/n$, for all $n \in \mathbb{N}$. We then have

$$\|M_g\| \geq \|M_g 1_{E_n}\|_2 / \|1_{E_n}\|_2 \geq \|g\|_\infty - 1/n.$$

The inequality $\|g\|_\infty \leq \|M_g\|$ is also clear and hence M is isometric.

To see that $M(L^\infty(X, \mu))$ is maximal abelian let's suppose $T \in \mathcal{B}(L^2(X, \mu))$ commutes with M_f for all $f \in L^\infty(X, \mu)$. We take $E_n \subset X$ measurable sets such that $0 < \mu(E_n) < \infty$, $E_n \subset E_{n+1}$, and $X = \cup_{n \in \mathbb{N}} E_n$. Define $f_n \in L^2(X, \mu)$ by $f_n = T(1_{E_n})$.

For each $g, h \in L^\infty(X, \mu) \cap L^2(X, \mu)$, we have

$$|\int f_n g \overline{h} d\mu| = |\langle M_g T(1_{E_n}), h\rangle| = |\langle T(g_{|E_n}), h\rangle| \leq \|T\| \|g\|_2 \|h\|_2.$$

Since $L^\infty(X, \mu) \cap L^2(X, \mu)$ is dense in $L^2(X, \mu)$, it then follows from Hölder's inequality that $f_n \in L^\infty(X, \mu)$ with $\|f_n\|_\infty \leq \|T\|$, and that $M_{1_{E_n}} T = M_{f_n}$. Note that for $m \geq n$, $1_{E_m} f_n = 1_{E_m} T(1_{E_n}) = T(1_{E_n}) = f_n$. Hence, $\{f_n\}$ converges almost every where to a measurable function f. Since $\|f_n\|_\infty \leq \|T\|$ for each n, we have $\|f\|_\infty \leq \|T\|$. Moreover, if $g, h \in L^2(X, \mu)$ then we have

$$\int f g \overline{h} d\mu = \lim_{n \to \infty} \int f_n g \overline{h} d\mu = \lim_{n \to \infty} \langle 1_{E_n} T(g), h\rangle = \langle T(g), h\rangle.$$

Thus, $T = M_f$. $\square$

Because of the previous result we will often identify $L^\infty(X, \mu)$ with the subalgebra of $\mathcal{B}(L^2(X, \mu))$ as described above. Conversely, every abelian von Neumann algebra A is isomorphic to $L^\infty(X, \mu)$ for some measure space (X, μ), if A is separable in the strong operator topology then (X, μ) may be taken be taken to be a standard probability space.

Exercise 12.20. Let X be an uncountable set, $\mathcal{B}_1$ the set of all subsets of X, $\mathcal{B}_2 \subset B_1$ the set consisting of all sets which are either countable or have countable complement, and μ the counting measure on X. Show that the identity map implements a unitary operator $\mathrm{id} : L^2(X, \mathcal{B}_1, \mu) \to L^2(X, \mathcal{B}_2, \mu)$, and we have $L^\infty(X, \mathcal{B}_2, \mu) \subsetneq L^\infty(X, \mathcal{B}_2, \mu)'' = \mathrm{id}\, L^\infty(X, \mathcal{B}_1, \mu)\, \mathrm{id}^*$.

12.4 The spectral theorem and Borel functional calculus

Lemma 12.21. *Let $x_\alpha \in \mathcal{B}(\mathcal{H})$ be an increasing net of positive operators such that $\sup_\alpha \|x_\alpha\| < \infty$, then there exists a bounded operator $x \in \mathcal{B}(\mathcal{H})$ such that $x_\alpha \to x$ in the SOT.*

Proof. We may define a quadratic form on $\mathcal{H}$ by $\xi \mapsto \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$. Since $\sup_\alpha \|x_\alpha\| < \infty$ we have that this quadratic form is bounded and hence there exists a bounded positive operator $x \in \mathcal{B}(\mathcal{H})$ such that $\|\sqrt{x}\xi\|^2 = \lim_\alpha \|\sqrt{x_\alpha}\xi\|^2$, for all $\xi \in \mathcal{H}$. Note that $x_\alpha \leq x$ for all α, and $\sup_\alpha \|(x - x_\alpha)^{1/2}\| < \infty$. Thus for each $\xi \in \mathcal{H}$ we have

$$\begin{aligned}\|(x - x_\alpha)\xi\|^2 &\leq \|(x - x_\alpha)^{1/2}\|^2 \|(x - x_\alpha)^{1/2}\xi\|^2 \\ &= \|(x - x_\alpha)^{1/2}\|^2 (\|\sqrt{x}\xi\|^2 - \|\sqrt{x_\alpha}\xi\|^2) \to 0.\end{aligned}$$

Hence, $x_\alpha \to x$ in the SOT. □

Corollary 12.22. *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra. If $\{p_\iota\}_{\iota \in I} \subset A$ is a collection of pairwise orthogonal projections then $p = \sum_{\iota \in I} p_\iota \in A$ is well defined as a SOT limit of finite sums.*

Let K be a locally compact Hausdorff space and let $\mathcal{H}$ be a Hilbert space. A **spectral measure** E on K relative to $\mathcal{H}$ is a mapping from the Borel subsets of K to the set of projections in $\mathcal{B}(\mathcal{H})$ such that

(*i*) $E(\emptyset) = 0, E(K) = 1$.

(*ii*) $E(B_1 \cup B_2) = E(B_1) + E(B_2)$ for all disjoint Borel sets B_1 and B_2.

(*iii*) For all $\xi, \eta \in \mathcal{H}$ the function
$$B \mapsto E_{\xi,\eta}(B) = \langle E(B)\xi, \eta \rangle$$
is a finite Radon measure on K.

Example 12.23. If K is a locally compact Hausdorff space and μ is a σ-finite Radon measure on K, then the map $E(B) = 1_B \in L^\infty(K, \mu) \subset \mathcal{B}(L^2(K, \mu))$ defines a spectral measure on K relative to $L^2(K, \mu)$.

We denote by $B_\infty(K)$ the space of all bounded Borel functions on K. This is clearly a C^*-algebra with the sup norm.

For each $f \in B_\infty(K)$ it follows that the map

$$(\xi, \eta) \mapsto \int f \, dE_{\xi,\eta}$$

gives a continuous sesqui-linear form on $\mathcal{H}$ and hence it follows that there exists a bounded operator T such that $\langle T\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$. We denote this operator T by $\int f \, dE$ so that we have the formula $\langle (\int f \, dE)\xi, \eta \rangle = \int f \, dE_{\xi,\eta}$, for each $\xi, \eta \in \mathcal{H}$.

Theorem 12.24. *Let K be a locally compact Hausdorff space, let $\mathcal{H}$ be a Hilbert space, and suppose that E is a spectral measure on K relative to $\mathcal{H}$. Then the association*

$$f \mapsto \int f\, dE$$

defines a continuous unital $*$-homomorphism from $B_\infty(K)$ to $\mathcal{B}(\mathcal{H})$. Moreover, the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, and if $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty$, then $\int f_n dE \to \int f dE$ in the SOT.

Proof. It is easy to see that this map defines a linear contraction which preserves the adjoint operation. If $A, B \subset K$ are Borel subsets, and $\xi, \eta \in \mathcal{H}$, then denoting $x = \int 1_A dE$, $y = \int 1_B dE$, and $z = \int 1_{A\cap B} dE$ we have

$$\begin{aligned} \langle xy\xi, \eta\rangle &= \langle E(A)y\xi, \eta\rangle = \langle E(B)\xi, E(A)\eta\rangle \\ &= \langle E(B\cap A)\xi, \eta\rangle = \langle z\xi, \eta\rangle. \end{aligned}$$

Hence $xy = z$, and by linearity we have that $(\int f\, dE)(\int g\, dE) = \int fg\, dE$ for all simple functions $f, g \in B_\infty(K)$. Since every function in $B_\infty(K)$ can be approximated uniformly by simple functions this shows that this is indeed a $*$-homomorphism.

To see that the image of $B_\infty(K)$ is contained in the von Neumann algebra generated by the image of $C(K)$, note that if a commutes with all operators of the form $\int f\, dE$ for $f \in C(K)$ then for all $\xi, \eta \in \mathcal{H}$ we have

$$0 = \langle (a(\int f\, dE) - (\int f\, dE)a)\xi, \eta\rangle = \int f\, dE_{\xi, a^*\eta} - \int f\, dE_{a\xi,\eta}.$$

Thus $E_{\xi,a^*\eta} = E_{a\xi,\eta}$ and hence we have that a also commutes with operators of the form $\int g\, dE$ for any $g \in B_\infty(K)$. Therefore by Theorem 12.16 $\int g\, dE$ is contained in the von Neumann algebra generated by the image of $C(K)$.

Now suppose $f_n \in B_\infty(K)$ is an increasing sequence of non-negative functions such that $f = \sup_n f_n \in B_\infty(K)$. For each $\xi, \eta \in \mathcal{H}$ we have

$$\int f_n\, dE_{\xi,\eta} \to \int f\, dE_{\xi,\eta},$$

hence $\int f_n\, dE$ converges in the WOT to $\int f\, dE$. However, since $\int f_n\, dE$ is an increasing sequence of bounded operators with $\| \int f_n\, dE\| \leq \|f\|_\infty$, Lemma 12.21 shows that $\int f_n\, dE$ converges in the SOT to some operator $x \in \mathcal{B}(\mathcal{H})$ and we must then have $x = \int f\, dE$. □

The previous theorem shows, in particular, that if A is an abelian C^*-algebra, and E is a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$, then we obtain a unital $*$-representation $\pi : A \to \mathcal{B}(\mathcal{H})$ by the formula

$$\pi(x) = \int \Gamma(x) dE.$$

We next show that in fact every unital $*$-representation arises in this way.

Theorem 12.25 (The spectral theorem)**.** *Let A be an abelian C^*-algebra, $\mathcal{H}$ a Hilbert space and $\pi : A \to \mathcal{B}(\mathcal{H})$ a $*$-representation, which is non-degenerate in the sense that $\xi = 0$ if and only if $\pi(x)\xi = 0$ for all $x \in A$. Then there is a unique spectral measure E on $\sigma(A)$ relative to $\mathcal{H}$ such that for all $x \in A$ we have*

$$\pi(x) = \int \Gamma(x) dE.$$

Proof. For each $\xi, \eta \in \mathcal{H}$ we have that $f \mapsto \langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle$ defines a bounded linear functional on $\sigma(A)$ and hence by the Riesz representation therorem there exists a Radon measure $E_{\xi,\eta}$ such that for all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))\xi, \eta\rangle = \int f dE_{\xi,\eta}.$$

Since the Gelfand transform is a $*$-homomorphism we verify easily that $f dE_{\xi,\eta} = dE_{\pi(\Gamma^{-1}(f))\xi,\eta} = dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta}$.

Thus for each Borel set $B \subset \sigma(A)$ we can consider the sesquilinear form $(\xi, \eta) \mapsto \int 1_B dE_{\xi,\eta}$. We have $|\int f dE_{\xi,\eta}| \leq \|f\|_\infty \|\xi\| \|\eta\|$, for all $f \in C(\sigma(A))$ and hence this sesquilinear form is bounded and there exists a bounded operator $E(B)$ such that $\langle E(B)\xi, \eta\rangle = \int 1_B dE_{\xi,\eta}$, for all $\xi, \eta \in \mathcal{H}$. For all $f \in C(\sigma(A))$ we have

$$\langle \pi(\Gamma^{-1}(f))E(B)\xi, \eta\rangle = \int 1_B dE_{\xi,\pi(\Gamma^{-1}(\overline{f}))\eta} = \int 1_B f dE_{\xi,\eta}.$$

Thus it follows that $E(B)^* = E(B)$, and $E(B')E(B) = E(B' \cap B)$, for any Borel set $B' \subset \sigma(A)$. In particular, $E(B)$ is a projection and since π is non-degenerate it follows easily that $E(\sigma(A)) = 1$, thus E gives a spectral measure on $\sigma(A)$ relative to $\mathcal{H}$. The fact that for $x \in A$ we have $\pi(x) = \int \Gamma(x) dE$ follows easily from the way we constructed E. □

If $\mathcal{H}$ is a Hilbert space and $x \in \mathcal{B}(\mathcal{H})$ is a normal operator, then by applying the previous theorem to the C^*-subalgebra A generated by x and 1, and using the identification $\sigma(A) = \sigma(x)$ we obtain a homomorphism from $B_\infty(\sigma(x))$ to $\mathcal{B}(\mathcal{H})$ and hence for $f \in B_\infty(\sigma(x))$ we may define

$$f(x) = \int f dE.$$

Note that it is straight forward to check that considering the function $f(z) = z$ we have

$$x = \int z dE(z).$$

We now summarize some of the properties of this functional calculus which follow easily from the previous results.

Theorem 12.26 (Borel functional calculus)**.** *Let $A \subset \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and suppose $x \in A$ is a normal operator, then the Borel functional calculus defined by $f \mapsto f(x)$ satisfies the following properties:*

(i) $f \mapsto f(x)$ is a continuous unital $*$-homomorphism from $B_\infty(\sigma(x))$ into A.

(ii) If $f \in B_\infty(\sigma(x))$ then $\sigma(f(x)) \subset f(\sigma(x))$.

(iii) If $f \in C(\sigma(x))$ then $f(x)$ agrees with the definition given by continuous functional calculus.

Exercise 12.27. Suppose that K is a compact Hausdorff space and E is a spectral measure for K relative to a Hilbert space $\mathcal{H}$, show that if $\{f_n\}_n \subset B_\infty(K)$ is a uniformly bounded sequence, and $f \in B_\infty(K)$ such that $f_n(k) \to f(k)$ for every $k \in K$, then $\int f_n \, \mathrm{d}E \to \int f \, \mathrm{d}E$ in the strong operator topology.

12.5 Locally compact groups

If G is a locally compact group there exists a **Haar measure** λ, which is a non-zero Radon measure on G satisfying $\lambda(gE) = \lambda(E)$ for all $g \in G$, and Borel subsets $E \subset G$. When a Haar measure λ on G is fixed, we'll often use the notation $\int f(x) \, \mathrm{d}x$ for the integral $\int f(x) \, \mathrm{d}\lambda(x)$.

Any two Haar measures differ by a constant. If $h \in G$, and λ is a Haar measure then $E \mapsto \lambda(Eh)$ is again a Haar measure and hence there is a scalar $\Delta(h) \in (0, \infty)$ such that $\lambda(Eh) = \Delta(h)\lambda(E)$ for all Borel subsets $E \subset G$. The map $\Delta : G \to (0, \infty)$ is the **modular function**, and is easily seen to be a continuous homomorphism. The group G is **unimodular** if $\Delta(g) = 1$ for all $g \in G$, or equivalently if a Haar measure for G is right invariant. For example, all abelian groups and all compact groups are unimodular.

Example 12.28. (i) Lebesgue measure is a Haar measure for $\mathbb{R}^n$, and similarly for $\mathbb{T}^n$.

(ii) Counting measure is a Haar measure for any discrete group.

(iii) Consider the group $GL_n(\mathbb{R})$ of invertible matrices. As the determinant of a matrix is given by a polynomial function of the coefficients of the matrix it follows that $GL_n(\mathbb{R})$ is an open dense subset of $\mathbb{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. If $x \in GL_n(\mathbb{R})$ then the transformation induced on $\mathbb{M}_n(\mathbb{R})$ by left multiplication has Jacobian given by $\det(x)^n$, and hence if λ denotes Lebesgue measure on $\mathbb{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, then we obtain a Haar measure for $GL_n(\mathbb{R})$ by

$$\mu(B) = \int_B \frac{1}{|\det(X)|^n} \, \mathrm{d}X.$$

Since the linear transformation induced by right multiplication has the same Jacobian we see that this also gives a right Haar measure, and hence $GL_n(\mathbb{R})$ is unimodular.

(iv) If $G \cong G_1 \times G_2$, and λ_i are Haar measures for G_i, then $\lambda_1 \times \lambda_2$ is a Haar measure for G. For example, consider the group $SL_n(\mathbb{R})$ of matrices with determinant 1. Then $SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$, and we also have an embedding of $\mathbb{R}^*$ in $GL_n(\mathbb{R})$ as constant diagonal matrices. The subgroups $SL_n(\mathbb{R})$, and $\mathbb{R}^*$ are both closed and normal, and their intersection is $\{e\}$, hence we have an isomorphism $GL_n(\mathbb{R}) \cong \mathbb{R}^* \times SL_n(\mathbb{R})$. Thus, for suitably chosen Haar measures μ on $GL_n(\mathbb{R})$, and λ on $SL_n(\mathbb{R})$ we have $\mathrm{d}\mu(tx) = \frac{1}{|t|} \mathrm{d}t \, d\lambda(x)$. Note that $SL_n(\mathbb{R})$ is again unimodular.

(v) Consider the group N of upper triangular $n \times n$ matrices with real coefficients and diagonal entries equal to 1. We may identify N with $\mathbb{R}^{n(n-1)/2}$ by means of the homeomorphism

$N \ni n \mapsto (n_{ij})_{1\leq i<j\leq n} \in \mathbb{R}^{n(n-1)/2}$. Under this identification, Lebesgue measure on $\mathbb{R}^{n(n-1)/2}$ is a Haar measure on N. Indeed, if $n, x \in N$, then for $i < j$ we have

$$(nx)_{ij} = n_{ij} + x_{ij} + \sum_{i<k<j} n_{ik}x_{kj}.$$

If we endow the set of pairs (i, j), $1 \leq i < j \leq n$ with the lexicographical order, then it is clear that the Jacobi matrix corresponding to the transformation $x \mapsto nx$ is upper triangular with diagonal entries equal to 1, and hence the Jacobian of this transformation is 1.

The same argument shows that the transformation $n \mapsto nx$ also has Jacobian equal to 1 and hence Lebesgue measure is also right invariant, i.e., N is unimodular.

(vi) Suppose K, and H are locally compact groups with Haar measures $\mathrm{d}k$, and $\mathrm{d}h$ respectively. Suppose also that $\alpha : K \to \mathrm{Aut}(H)$ is a homomorphism which is continuous in the sense that the map $K \times H \ni (k, h) \mapsto \alpha_k(h) \in H$ is jointly continuous. Then for each $k \in K$, the push-forward of $\mathrm{d}h$ under the transformation α_k is again a Haar measure and hence must be of the form $\delta(k)^{-1}\mathrm{d}h$, where $\delta : K \to \mathbb{R}_{>0}$ is a continuous homomorphism.

The **semi-direct product** of K with H is denoted by $K \ltimes H$. Topologically, it is equal to the direct product $K \times H$, however the group law is given by

$$(k_1, h_1)(k_2, h_2) = (k_1k_2, \alpha_{k_2}^{-1}(h_1)h_2); \qquad k_1, k_2 \in K, \quad h_1, h_2 \in H.$$

A Haar measure for $K \ltimes H$ is given by the product measure $\mathrm{d}k\mathrm{d}h$. Indeed, if $f \in C_c(G \ltimes H)$, and $k' \in K$, $h' \in H$ then we have

$$\begin{aligned}\int f((k', h')(k, h))\,\mathrm{d}k\mathrm{d}h &= \int f(k'k, \alpha_k^{-1}(h')h)\,\mathrm{d}h\mathrm{d}k\\ &= \int f(k'k, h)\,\mathrm{d}k\mathrm{d}h = \int f(k, h)\,\mathrm{d}k\mathrm{d}h.\end{aligned}$$

The modular function for $K \ltimes H$ is given by $\Delta_{K\ltimes H}(k, h) = \delta(k)\Delta_K(k)\Delta_H(h)$. Indeed, if $f \in C_c(K \ltimes H)$ then

$$\begin{aligned}\int f((k, h)^{-1})\mathrm{d}k\mathrm{d}h &= \int f(k^{-1}, \alpha_k(h^{-1}))\mathrm{d}h\mathrm{d}k\\ &= \int \delta(k)f(k^{-1}, h^{-1})\mathrm{d}h\mathrm{d}k\\ &= \int \delta(k)\Delta_K(k)\Delta_H(h)f(k, h)\mathrm{d}k\mathrm{d}h.\end{aligned}$$

A specific case to consider is when $G = A$ is the group of diagonal matrices in $\mathbb{M}_n(\mathbb{R})$ with positive diagonal coefficients, and $H = N$ is the group of upper triangular matrices in $\mathbb{M}_n(\mathbb{R})$ with diagonal entries equal to 1. In this case A acts on N by conjugation and the resulting semi-direct product can be realized as the group B of upper triangular matrices with positive diagonal coefficients.

Both N and A are unimodular, and for each $a = \text{diag}(a_1, a_2, \ldots, a_n) \in A$ we see from the previous example that conjugating N by a multiplies the Haar measure on N by a factor of $\delta(a) = \prod_{1 \leq i < j \leq n} \frac{a_i}{a_j}$. Hence, the modular operator for B is given by $\Delta_B(g) = \prod_{1 \leq i < j \leq n} \frac{g_{ii}}{g_{jj}}$.

In the case when $n = 2$ we have the group $G = \{ \left(\begin{smallmatrix} x & y \\ 0 & x^{-1} \end{smallmatrix} \right) \mid x \in \mathbb{R}_+^*, y \in \mathbb{R} \}$, so that $G = \mathbb{R}_+^* \ltimes \mathbb{R}$, $\delta : \mathbb{R}_+^* \to \mathbb{R}_+^*$ is given by $\delta(x) = x^2$, and a left Haar measure for G is given by $\frac{1}{x} \mathrm{d}x \, \mathrm{d}y$, while a right Haar measure is given by $\frac{1}{x^3} \mathrm{d}x \, \mathrm{d}y$.

Theorem 12.29. *Let G be a locally compact group and $H < G$ a closed subgroup. Then there exists a unique measure class on G/H which is quasi-invariant. Moreover, G/H has a σ-finite invariant measure if and only if the restriction of Δ_G to H agrees with Δ_H.*

Proof. See Theorem 2.49 in [Fol95]. □

12.6 Positive definite functions

Let G be a locally compact group with a Haar measure λ. A function $\varphi \in L^\infty G$ is of **positive definite** if for all $f \in L^1 G$ we have $\int \varphi(x)(f^* * f)(x) \, \mathrm{d}\lambda(x) \geq 0$. Note that this condition is weak$*$-closed in $L^\infty G$ so that the space of positive definite functions is weak$*$-closed.

Proposition 12.30. *Suppose $\pi : G \to \mathcal{U}(\mathcal{H})$ is a continuous representation of G, and $\xi_0 \in \mathcal{H}$, then the function defined by $\varphi(x) = \langle \pi(x)\xi_0, \xi_0 \rangle$ is of positive definite.*

Proof. If we consider the associated representation of $L^1 G$, then for $f \in L^1 G$ we have

$$\begin{aligned} \int \langle \pi(x)\xi_0, \xi_0 \rangle (f^* * f)(x) \, \mathrm{d}\lambda(x) &= \langle \pi(f^* * f)\xi_0, \xi_0 \rangle \\ &= \|\pi(f)\xi_0\|^2 \geq 0. \end{aligned}$$

Thus, φ is positive definite. □

We also have the following converse to Proposition 12.30.

Theorem 12.31 (The GNS-construction)**.** *If $\varphi \in L^\infty G$ is positive definite there exists a continuous representation $\pi : G \to \mathcal{U}(\mathcal{H})$, and a vector $\xi_0 \in \mathcal{H}$, such that $\varphi(g) = \langle \pi(g)\xi_0, \xi_0 \rangle$, for locally almost every $g \in G$. In particular, every function positive definite agrees with a continuous function locally almost everywhere.*

Proof. See Theorem 3.20 in [Fol95]. □

Corollary 12.32. *Every positive definite function on G agrees locally almost everywhere with a continuous positive definite function.*

Corollary 12.33. *If $\varphi \in C_b G$ is positive definite then φ is uniformly continuous, and the following statements hold:*

- $\|\varphi\|_\infty = f(e)$;
- $\varphi(x^{-1}) = \overline{\varphi(x)}$, *for all $x \in G$;*

- $|\varphi(y^{-1}x) - \varphi(x)|^2 \leq 2\varphi(e)\mathrm{Re}(\varphi(e) - \varphi(y))$, *for all* $x, y \in G$.

Proof. Suppose $\varphi(x) = \langle \pi(x)\xi_0, \xi_0 \rangle$ as in the GNS-construction. For the first statement we have $\varphi(e) \leq \|\varphi\|_\infty = \sup_{x \in G} |\langle \pi(x)\xi_0, \xi_0\rangle| \leq \|\xi_0\|^2 = \varphi(e)$.

For the second statement we have $\varphi(x^{-1}) = \langle \pi(x^{-1})\xi_0, \xi_0\rangle = \langle \xi_0, \pi(x)\xi_0\rangle = \overline{\varphi(x)}$.

For the third statement we have

$$\begin{aligned} |\varphi(y^{-1}x) - \varphi(x)|^2 &= |\langle \pi(x)\xi_0, \pi(y)\xi_0 - \xi_0\rangle|^2 \leq \|\xi_0\|^2 \|\pi(y)\xi_0 - \xi_0\|^2 \\ &= 2\|\xi_0\|^2(\|\xi_0\|^2 - \mathrm{Re}(\langle \pi(y)\xi_0, \xi_0\rangle)) \\ &= 2\varphi(e)\mathrm{Re}(\varphi(e) - \varphi(y)). \end{aligned}$$

Since φ is continuous at e, this also shows that φ is left uniformly continuous, and right uniform continuity then follows since $\varphi(x^{-1}) = \overline{\varphi(x)}$, for all $x \in G$. □

We let $\mathcal{P}(G) \subset L^\infty G$ denote the convex cone of positive definite functions, $\mathcal{P}_1(G) = \{\varphi \in \mathcal{P}(G) \mid \varphi(e) = 1\}$, and $\mathcal{P}_{\leq 1}(G) = \{\varphi \in \mathcal{P}(G) \mid \varphi(e) \leq 1\}$. We let $\mathcal{E}(\mathcal{P}_1(G))$, and $\mathcal{E}(\mathcal{P}_{\leq 1}(G))$ denote, respectively, the extreme points of these last two convex sets.

Theorem 12.34 (Raikov)**.** *The weak*-topology on $\mathcal{P}_1(G)$ agrees with the topology of uniform convergence on compact sets.*

Proof. See Theorem 3.31 in [Fol95]. □

12.7 Duality and Bochner's theorem

Let G be a locally compact abelian group. A (unitary) **character** on G is a continuous homomorphism $\chi : G \to \mathbb{T}$. We denote the set of all characters by $\hat{G}$, which by Corollary **??** agrees with $\mathcal{E}(\mathcal{P}_1(G))$, which is canonically identified with $\sigma(C^*G)$, and we endow $\hat{G}$ with the weak*-topology, which by Theorem 12.34 is the same as the topology of uniform convergence on compact sets. The set of all characters is also a group under pointwise operations, and these operations are clearly continuous and hence $\hat{G}$ is an abelian locally compact group, which is the **Pontryagin dual group** of G. If $\chi \in \hat{G}$, and $x \in G$, then we'll also use the notation $\langle x, \chi\rangle = \chi(x)$.

We leave it to the reader to verify the following.

Example 12.35.

- $\hat{\mathbb{R}} \cong \mathbb{R}$ with the pairing $\langle x, \xi\rangle = e^{2\pi i x\xi}$.
- $\hat{\mathbb{Z}} \cong \mathbb{T}$ with the pairing $\langle n, \lambda\rangle = \lambda^n$.
- $\hat{\mathbb{T}} \cong \mathbb{Z}$ with pairing $\langle \lambda, n\rangle = \lambda^n$.
- $\widehat{\mathbb{Z}/n\mathbb{Z}} \cong \mathbb{Z}/n\mathbb{Z}$ with the pairing $\langle j, k\rangle = e^{2\pi jki/n}$.
- If G_1, G_2, are locally compact abelian groups, then we have $\widehat{G_1 \times G_2} \cong \hat{G}_1 \times \hat{G}_2$ under the pairing $\langle (x, y), (\chi, \omega)\rangle = \langle x, \chi\rangle\langle y, \omega\rangle$.
- If G is a finite abelian group then $\hat{G} \cong G$.

The Gelfand transform gives an isomorphism $C^*G \cong C_0(\hat{G})$, which on the dense subspace $L^1G \subset C^*G$ is given by

$$\Gamma(f)(\chi) = \int f(x)\chi(x)\,\mathrm{d}\lambda(x).$$

We introduce the closely related **Fourier transform** $\mathcal{F} : L^1G \to C_0(\hat{G})$ given by $\mathcal{F}(f)(\chi) = \hat{f}(\chi) = \int f(x)\overline{\chi(x)}\,\mathrm{d}\lambda(x)$. Note that since $\chi \mapsto \overline{\chi}$ is a homeomorphism on $\hat{G}$ it follows that $\mathcal{F}$ also extends continuously to an isomorphism from C^*G onto $C_0(\hat{G})$.

Theorem 12.36 (Bochner's Theorem). *Let G be a locally compact abelian group. If $\varphi \in \mathcal{P}(G)$ then there exists a unique positive measure $\mu \in M(\hat{G})$ such that $\varphi(x) = \int \chi(x)\mathrm{d}\mu(\chi)$, for all $x \in G$.*

Proof. See Theorem 4.18 in [Fol95]. ☐

Theorem 12.37 (The Plancherel Theorem). *The Fourier transform on $L^1G \cap L^2G$ extends uniquely to a unitary isomorphism from L^2G to $L^2\hat{G}$.*

Proof. See Theorem 4.25 in [Fol95]. ☐

Theorem 12.38 (Pontrjagin Duality). *The map $\Phi : G \mapsto \hat{\hat{G}}$ given by $\Phi(x)(\chi) = \chi(x)$ gives an isomorphism of topological groups.*

Proof. See Theorem 4.31 in [Fol95]. ☐

12.8 Lattices

Let G be a locally compact group. A **lattice** in G is a discrete subgroup $\Gamma < G$ such that G/Γ has a G-invariant probability measure. Note that by Theorem 12.29 only unimodular groups may have lattices.

Fix $n \in \mathbb{N}$, and set $G = SL_n(\mathbb{R})$. We also set $K = SO(n) < G$, A the abelian subgroup of G consisting of diagonal matrices in G with positive diagonal coefficients, and N the subgroup of upper triangular matrices in G with diagonal entries equal to one. We also set $\Gamma = SL_n(\mathbb{Z})$. We denote by $\{e_i\}_{1 \leq i \leq n}$, the standard basis vectors for $\mathbb{R}^n$. For $1 \leq i, j \leq n$, $i \neq j$, we denote by $E_{ij} \in \Gamma$ the elementary matrix which has diagonal entries and the ijth entry equal to 1, and all other entries equal to 0.

Proposition 12.39. *For every $v \in \mathbb{Z}^n \setminus \{0\}$, there exists $\gamma \in SL_n(\mathbb{Z})$ such that $\gamma v \in \mathbb{N}e_1$.*

Proof. The proposition is trivially true for $n = 1$. For $n = 2$, suppose $v_0 \in \mathbb{Z}^2 \setminus \{0\}$, multiplying by $-I$ if necessary we may obtain a vector v_1 with at least 1 positive entry, multiplying by an appropriate power of E_{12}, or E_{21} we obtain a vector $v_2 = (\alpha_1, \alpha_2)$ with both entries non-negative. We now try to minimize these entries as follows: If $0 < \alpha_1 \leq \alpha_2$ we multiply by E_{21}^{-1}, and if $0 < \alpha_2 < \alpha_1$ we multiply by E_{12}^{-1}. Repeating this procedure we eventually obtain a vector v_3 with one positive entry and the other entry equal to 0. Thus, either $v_3 \in \mathbb{N}e_1$, or $v_3 \in \mathbb{N}e_2$. In the latter case, multiplying by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ gives a vector $v_4 \in \mathbb{N}e_1$.

If $n > 2$ then for each $1 \leq i < j \leq n$, we may realize $SL_2(\mathbb{Z})$ as the subgroup of $SL_n(\mathbb{Z})$ which fixes e_k, for $k \neq i, j$. By considering the embeddings corresponding to $(1, j)$ as j decreases from n to 2, then for $v \in \mathbb{Z}^n \setminus \{0\}$ we may inductibely find elements $\gamma_j \in SL_n(\mathbb{Z})$ such that $\gamma_2\gamma_3 \cdots \gamma_n v \in \mathbb{N}e_1$. ☐

Theorem 12.40 (Iwasawa decomposition for $SL_n(\mathbb{R})$)**.** *The product map $K \times A \times N \to G$ is a homeomorphism.*

Proof. Fix $g \in G$, and suppose that g has column vectors $x_1, \dots, x_n$. Using the Gram-Schmidt process we inductively construct unit orthogonal vectors $\tilde{y}_1, \dots, \tilde{y}_n$ by setting $y_i = x_i - \sum_{1 \le j < i} \langle \tilde{y}_j, x_j \rangle \tilde{y}_j$, and then $\tilde{y}_i = y_i / \|y_i\|$, for $1 \le i \le n$.

We may then consider the orthogonal transformation $k \in O(n)$, such that $ke_i = \tilde{y}_i$. It is then easy to check that $k^{-1}g$ is an upper triangular matrix with diagonal entries equal to $\|y_i\| > 0$. Note that we have $k \in SO(n)$, since $\det(k^{-1}) = \det(k^{-1}g) = \prod_{i=1}^n \|y_i\| \neq -1$.

If $a = \mathrm{diag}(\|y_1\|, \|y_2\|, \dots, \|y_n\|)$ then we have $g = kan$ where $n \in N$. This association is clearly continuous and it is easy to see that it is an inverse to the product map $K \times A \times N \to G$. Hence, this must be a homeomorphism. □

Given the decomposition above, it is then natural to see how Haar measures for K, A, and N relate to Haar measures for G.

Theorem 12.41. *Suppose $\mathrm{d}k$, $\mathrm{d}a$, and $\mathrm{d}n$, are Haar measure for K, A, and N respectively. Then, with respect to the Iwasawa decomposition, a Haar measure for G is given by $\mathrm{d}g = \delta(a)\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$, where $\delta(a) = \prod_{1 \le i < j \le n} \frac{a_i}{a_j}$, for $a = \mathrm{diag}(a_1, a_2, \dots, a_n) \in A$.*

Proof. From Example 12.28, we see that a right Haar measure for $B = AN$ is given by $\delta(a)\,\mathrm{d}a\mathrm{dn}$. Hence a right Haar measure for $K \times B$ is given by $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{dn}$.

We denote by $\mathrm{d}x$ a measure on $K \times B$ which is obtained from a Haar measure on G through the homeomorphism $K \times B \ni (k, b) \mapsto kb \in G$. Since G is unimodular it follows that $\mathrm{d}x$ is right invariant under the actions of B, and left invariant under the action of K. Hence $\psi_* \mathrm{d}x$ is a right invariant measure on $K \times B$, where $\psi : K \times B \to K \times B$ is given by $\psi(k, b) = (k^{-1}, b)$, and hence must be a scalar multiple of $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{dn}$. But K is unimodular, and hence it follows that $\mathrm{d}x = \psi_* \mathrm{d}x$ is a scalar multiple of $\delta(a)\,\mathrm{d}k\mathrm{d}a\mathrm{dn}$. □

For $t > 0$ we let A_t be the subset of A given by diagonal matrices a such that $a_{ii}/a_{(i+1)(i+1)} \le t$, for all $1 \le i \le n-1$. For $u > 0$ we let N_u be the subset of N consisting of those matrixes (n_{ij}) such that $|n_{ij}| \le u$, for all $1 \le i < j \le n$. Note that N_u is compact. A **Segal set** in $SL_n\mathbb{R}$ is a set of the form $\Sigma_{t,u} = KA_tN_u$.

Lemma 12.42. *We have $N = N_{1/2}(N \cap \Gamma)$.*

Proof. We will prove this by induction on n. Note that for $n \in \{1, 2\}$ this is easy. If $n > 2$, and $u \in N$, then we have $u = \left(\begin{smallmatrix} 1 & * \\ 0 & u_0 \end{smallmatrix}\right)$, where $u_0 \in SL_{n-1}(\mathbb{R})$ is upper triangular with diagonal entries equal to one. Thus, by induction there exists $\gamma_0 \in SL_{n-1}(\mathbb{Z})$ upper triangular with diagonal entries equal to one, such that the non-diagonal entries in $u_0\gamma_0$ have magnitude at most $1/2$. We may then write $u \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & x \\ 0 & u_0\gamma_0 \end{smallmatrix}\right)$.

If we take $y \in \mathbb{Z}^{n-1}$, such that $y + x$ has entires with magnitude at most $1/2$, then we have $u \left(\begin{smallmatrix} 1 & 0 \\ 0 & \gamma_0 \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & y \\ 0 & I \end{smallmatrix}\right) \in N_{1/2}$. □

Lemma 12.43. *If $g \in AN$ such that $\|ge_1\| \le \|gv\|$, for all $v \in \mathbb{Z}^n \setminus \{0\}$, then $g_{11}/g_{22} \le 2/\sqrt{3}$.*

Proof. Suppose $g \in AN$ is as above. Note that since N stabilizes e_1, if $\gamma \in \Gamma \cap N$, then $g\gamma$ again satisfies $\|g\gamma e_1\| \leq \|g\gamma v\|$, for all $v \in \mathbb{Z}^n \setminus \{0\}$. Also, g and $g\gamma$ have the same diagonal entries if $\gamma \in \Gamma \cap N$. Thus, from the previous lemma it is enough to consider the case $g = an$, with $a \in A$, and $n \in N_{1/2}$.

In this case we have $ge_1 = a_{11}e_1$, and $ge_2 = a_{11}n_{12}e_1 + a_{22}e_2$, with $|n_{12}| \leq 1/2$. Hence,

$$a_{11}^2 = \|ge_1\|^2 \leq \|ge_2\|^2 = a_{11}^2|n_{12}|^2 + a_{22}^2 \leq a_{11}^2/4 + a_{22}^2.$$

Hence, $\frac{3}{4}g_{11}^2 = \frac{3}{4}a_{11}^2 \leq a_{22}^2 = g_{22}^2$. □

Theorem 12.44. *For $t \geq 2/\sqrt{3}$, and $u \geq 1/2$ we have $G = \Sigma_{t,u}\Gamma$.*

Proof. By Lemma 12.42 it is enough to show $G = KA_{2/\sqrt{3}}N\Gamma$, which we will do by induction, with the case $n = 1$ being trivial.

Assume therefore that $n > 1$, and this holds for $n - 1$. We fix $g \in G$. Since $g(\mathbb{Z}^n \setminus \{0\})$ is discrete, there exists $v_0 \in \mathbb{Z}^n \setminus \{0\}$ which achieves the minimum of $\{\|gv\| \mid v \in \mathbb{Z}^n \setminus \{0\}\}$. Note that we cannot have $v_0 = \alpha v$, for some $v \in \mathbb{Z}^n \setminus \{0\}$, and $\alpha \in \mathbb{Z}$, unless $\alpha = \pm 1$. Hence, by Proposition 12.39 there exists $\gamma \in \Gamma$ such that $\gamma e_1 = v_0$.

We consider the Iwasawa decomposition $g\gamma = kan$, and write $an = \begin{pmatrix} \lambda & * \\ 0 & \lambda^{-1}h_0 \end{pmatrix}$, where $h_0 \in SL_{n-1}(\mathbb{R})$. By the induction hypothesis there then exists $k_0 \in SO(n-1)$, and $\gamma_0 \in SL_{n-1}(\mathbb{Z})$ such that $k_0^{-1}h_0\gamma_0$ is upper triangular and whose positive diagonal entries $\{a_{i,i}\}_{i=1,n-1}$ satisfy $a_{i,i}/a_{i+1,i+1} \leq 2/\sqrt{3}$, for $1 \leq i \leq n-2$.

Thus, if we consider $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & k_0^{-1} \end{pmatrix} k^{-1}g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix}$, then we see that $\tilde{g} \in AN$, and the diagonal entries of $\tilde{g}$ satisfy $\tilde{g}_{i,i}/\tilde{g}_{i+1,i+1} \leq 2/\sqrt{3}$ for all $2 \leq i \leq n-1$. Thus, to finish the proof it suffices to show that we also have $\tilde{g}_{1,1}/\tilde{g}_{2,2} \leq 2/\sqrt{3}$. However, this follows from Lemma 12.43 since for all $v \in \mathbb{Z}^n \setminus \{0\}$ we have

$$\begin{aligned} \|\tilde{g}e_1\| &= \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} e_1\| = \|g\gamma e_1\| = \|gv_0\| \\ &\leq \|g\gamma \begin{pmatrix} 1 & 0 \\ 0 & \gamma_0 \end{pmatrix} v\| = \|\tilde{g}v\|. \end{aligned}$$

□

Theorem 12.45. *$SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$.*

Proof. By the previous theorem it suffices to show that $\Sigma_{t,u}$ has finite Haar measure for $t = 2/\sqrt{3}$, and $u = 1/2$. By Theorem 12.41 a Haar measure for G is given by $\mathrm{d}g = \delta(a)\mathrm{d}k\,\mathrm{d}a\,\mathrm{d}n$, where $\delta(a) = \prod_{1 \leq i < j \leq n} \frac{a_i}{a_j}$ for $a = \mathrm{diag}(a_1, \ldots, a_n)$. Hence, $\int_{\Sigma_{t,u}} \mathrm{d}g = (\int_K \mathrm{d}k)(\int_{A_t} \delta(a)\,\mathrm{d}a)(\int_{N_u} \mathrm{d}u)$. Note that since K and N_u are compact, this integral is finite if and only if the integral $\int_{A_t} \delta(a)\,\mathrm{d}a$ is finite.

Consider the isomorphism $\mathbb{R}^{n-1} \to A$ given by

$$(x_1, x_2, \ldots, x_{n-1}) \mapsto \lambda \mathrm{diag}(e^{x_1+x_2+\cdots+x_{n-1}}, e^{x_2+\cdots+x_{n-1}}, \ldots, e^{x_{n-1}}, 1),$$

where $\lambda^n = \prod_{i=1}^{n-1} e^{-ix_i}$. We then have an explicit Haar measure for A given by the push forward of Lebesgue measure on $\mathbb{R}^{n-1}$. Moreover the preimage of A_t under this map is given by $E = (-\infty, \log(2/\sqrt{3})]^{n-1}$. Thus, we may compute directly

$$\int_{A_t} \delta(a)\,\mathrm{d}a = \int_E \prod_{1 \leq i \leq j \leq n-1} e^{x_i + \cdots + x_j}\,\mathrm{d}x_1 \cdots \mathrm{d}x_{n-1} = \prod_{i=1}^{n-1} \int_{-\infty}^{\log(2/\sqrt{3})} e^{b_i x}\,\mathrm{d}x,$$

where b_i are positive integers. Hence, $\int_{A_t} \delta(a)\, \mathrm{d}a < \infty$. □

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