Topology of positive scalar curvature 2

Answer to a question of last time:

K3-surface(s) $K3 = \{[x_1 : \dots : x_4] ; \sum_{i=1}^{4} x_i^4 = 0\} \subset \mathbb{C}P^3$

Fact: from point of view of diff. topol. this is unique. $\hat{A}(K3) \neq 0$!

Survey on top. of psc

For a spin mfd M, variants of index of Dirac exist, among them

Rosenberg index $\alpha(M) \in K_{\dim}(C^*\Gamma)$, $\Gamma = \pi_1 M$

Important: use as much structure as possible to obtain optimal info, here $\Gamma = \pi_1 M$

still M has psc $\Rightarrow \alpha(M) = 0 \in K_{\dim M}(C^*\Gamma)$

Gromov-Lawson-Rosenberg question

For which M is "$\Leftarrow$" true?

i.e.: when does $\alpha(M) = 0$ imply: M has pscm?

History: originally it was a conjecture that this is always so (if $\dim M \neq 4$)

Remark: 4-dim mfd are very special

Prop: Start with $T^5 = (S^1)^5$ with $\pi_1 T^5 = \mathbb{Z}^5$,

do "surgery" to obtain a new mfd

M^5 with $\pi_1 M = \mathbb{Z}^3 \times \mathbb{Z}/5 \times \mathbb{Z}/5$

and obtain

a) $\alpha(M) \in K_5(C^*(\mathbb{Z}^3 \times \mathbb{Z}/5 \times \mathbb{Z}/5))$

an abelian group which only contains 2-torsion elements

but on the other hand: the structure of M forces $\alpha(M)$ to be 5-torsion.

$$\Rightarrow \alpha(M) = 0$$

b) Use different method from index, namely "Schoen-Yau minimal hypersurface method"

to see that M does not admit psc.

<u>Theorem</u> (Stolz): If $\Gamma = \{1\}$ then $\alpha(M) = 0 \Rightarrow M$ has pscm.

<u>Tool</u>:
- surgery
- transfer of problem to algebraic topology

Survey on results about topology of $R^+(M)$ and $R^+(M)/\mathrm{Diffeo}(M)$

For this, we need ways to "construct" pscm and to detect structure.

<u>Hitchin</u>: construction via Diffeos:

$$\begin{tikzcd} \mathrm{Diffeo}(D^n, \partial D^n) \arrow[r, "\text{extend}"', "\text{by id}"] \arrow[drr, "\text{constant map}"'] & \mathrm{Diffeo}(M) \arrow[r, "\text{pullback } g_0"] & R^+(M) \arrow[d] \\ & & R^+(M)/\mathrm{Diffeo}(M) \end{tikzcd}$$

$$\varphi \longmapsto \varphi^* g_0$$

well studied & understood

M

Theorem (Hitchin, Crowley - S. - Steimle)

the map

$$\begin{array}{c} \pi_{n-d-1}(\mathrm{Diffeo}(D^d, \partial)) \longrightarrow \pi_{n-d-1}(R^+(M); g_0) \\ \downarrow \mathrm{ind}\ \mathrm{rel} \\ KO_n \end{array}$$

is non-zero in the following cases

$n > d = \dim M$; $n \equiv 1,2\ (8)$ $\Rightarrow$ $KO_n = \mathbb{Z}/2$

[Hitchin : $n-d-1 = 0, 1$]

Theorem (Piazza - S.) : If $\Gamma = \pi_1(M)$ contains non-trivial torsion

M^{4k-1} spin

then $\pi_0(R^+(M)) \longrightarrow \pi_0(R^+(M)/\mathrm{Diffeo}(M))$

$$\downarrow \varrho_{(2)}$$

$$\mathbb{R}$$

has infinite image, construction uses "surgery"

Theorem (Botvinnik - Ebert - (Randall-Williams))
(corr, Hanke - S. - Steimle)

based on "surgery construction"

$\mathrm{ind}\ \mathrm{rel} : \pi_k(R^+(M^n); g_0) \longrightarrow KO_{n+k+1}$

is always with non-trivial image

if $\dim(M) \geq 6$

and $n+k+1 \equiv \begin{cases} 0,4 & (8) \\ 1,2 & (8) \\ \text{else} & \end{cases}$ then $KO_{n+k+1} \cong \mathbb{Z}$, " $\cong \mathbb{Z}/2\mathbb{Z}$, " $= \{0\}$.

we would enter the theory of cobordism categories and deep theorem of Galatius-RW on the homology of Diffeo-groups.

Philosophy of index method.

Let's look at basic cases.

we saw C^*-algebras A

and their K-theory

Reminder: A is a complex C^*-algebra if it is a norm-closed and $*$-closed subalgebra of $\mathcal{B}(H = \text{Hilbert space})$

Ex: • $\mathcal{B}(H) \supset K(H) =$ closure of operators with finite dim image

• $\mathcal{C}(\underset{\text{compact}}{M}) \subset$ multiplication ops $\mathcal{B}(L^2(M))$

• $\mathcal{C}\ell_n$ later

Reminder/Def: $A \ni 1$

$K_0(A) =$ Grothendieck group of semigroup of projections in $A \otimes K(\ell^2\mathbb{N}) \leftarrow$ a C^*-alg.

$$K_1(A) = \pi_0\left(\bigcup_{n=1}^{\infty} Gl\left(A \otimes M_n(\mathbb{C})\right)\right)$$

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

Let M compact spin mfd,

Then we can construct AS Dirac operator

$$D : \Gamma(\$) \longrightarrow \Gamma(\$)$$
$$\overset{\downarrow}{\cap}$$

which has nice analytic properties:

- <u>elliptic</u>

which implies that it has an unique extension to an <u>self-adjoint</u> unbounded operator $\not{D}$

$$\begin{array}{ccc} & L^2(\$) & \\ & \cup & \\ \not{D} : & \mathrm{dom}(\not{D}) & \longrightarrow L^2(\$) \\ & \cup & \\ & \Gamma(\$) & \end{array}$$

Important consequence: functional calculus is available for $\not{D}$, i.e. we have "substitute $\not{D}$" homomorphism

$$\mathcal{C}_{\text{bounded}}(\mathbb{R}) \longrightarrow B(L^2(\$))$$
$$f \longmapsto f(\not{D})$$

e.g.: have $\dfrac{\not{D}}{\sqrt{1+\not{D}^2}} \in B(L^2(\$))$

Some detail = $\boxed{\text{extra structure}}$

If $\dim M \equiv 0 \ (2) \implies \$ = \$^+ \oplus S^-$

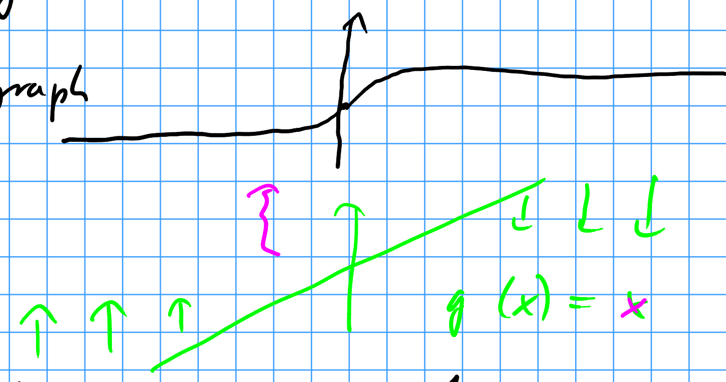
and $\not{D} = \begin{pmatrix} 0 & \not{D}^- \\ \not{D}^+ & 0 \end{pmatrix}$

Analytic consequence of ellipticity (+ compactness of M):

- $f \in \mathcal{C}_0(\mathbb{R}) \implies f(\not{D}) \in K(L^2(\$))$

We now define the index of $\not{D}$ in a somewhat indirect way:

$f(x) = \dfrac{x}{\sqrt{1+x^2}}$ has graph



$f(\not{D})$

$\not{D}$

$f^2(x) = \dfrac{x^2}{1+x^2}$ this differs from 1

by $h(x) \in \mathcal{C}_0(\mathbb{R})$

$$\Rightarrow \quad f(\not{D})^2 = f^2(\not{D}) = 1 - \underbrace{h(\not{D})}_{\in K(H)}.$$

$H = L^2(\$)$.

So: $[f(\not{D})]$ is invertible $\in \mathcal{B}(H)/K(H)$

$\uparrow$ a C^*-algebra.

More precisely: $\underset{\text{odd}}{f(D)} = \begin{pmatrix} 0 & f(\not{D})_- \\ f(\not{D})_+ & 0 \end{pmatrix}$

Choose a unitary $U : L^2(\$_-) \xrightarrow{\cong} L^2(\$_+)$

(choice not relevant because inside a contractible set).

and consider $[U \circ f(\not{D}_+)] \in B(L^2(\$_+))/K(L^2(S_+))$

an invertible element.

so defines $[\not{D}] \in K_1(B(L^2(\$_+))/K(L^2(S_+)))$

a "fundamental class" of M / $\not{D}$.

Reminder / Fact on K-theory of C^*-algebras:

Given an exact sequence

$$0 \to I \hookrightarrow A \twoheadrightarrow Q = A/I \to 0$$

there is a LES of K-theory groups

$$\begin{tikzcd} K_1(A) \arrow[r] & K_1(Q) \arrow[r, "\delta"] & K_0(I) \arrow[d] \\ K_1(I) \arrow[u] & K_0(Q) \arrow[l, "\delta"'] & K_0(A) \arrow[l] \end{tikzcd}$$

For us:

$$\begin{tikzcd} 0 \arrow[d, phantom, "\shortparallel"] & \mathbb{Z} \arrow[rr, "\cong"] \arrow[d, phantom, "\cong"] & & \mathbb{Z} \arrow[d, phantom, "\cong"] \\ K_1(B(L^2)) \arrow[r] & K_1\left(B(L^2)/K(L^2)\right) \arrow[rr, "\delta"] & & K_0(K(L^2)) \\ \rho(g) \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & {[\not{D}]} \arrow[u, phantom, "\cup"] \arrow[rr, mapsto] & & \mathrm{ind}(\not{D}) \arrow[u, phantom, "\cup"] \\ & & & 0 \arrow[u, phantom, "\shortparallel"] \end{tikzcd}$$

defines $\mathrm{ind}(\not{D})$.

We know already: psc $g \Rightarrow \mathrm{ind}(\not{D}) = 0$.

We reprove this now,
along the way implement the idea
"Don't just prove a vanishing result,
construct a new invariant out of the
reason for vanishing".

Further facts on functional calculus

$f \in \mathcal{C}_{\text{bounded}}(\mathbb{R})$, then $f(\not{D})$ only depends on $f|_{\mathrm{spec}(\not{D})}$

even better, we can form $f(\not{D})$ $\forall f \in \mathcal{C}_{\text{bounded}}(\mathrm{spec}(\not{D}) \subset \mathbb{R})$

<u>Fact</u>: (Schrödinger-Lichnerowicz implies

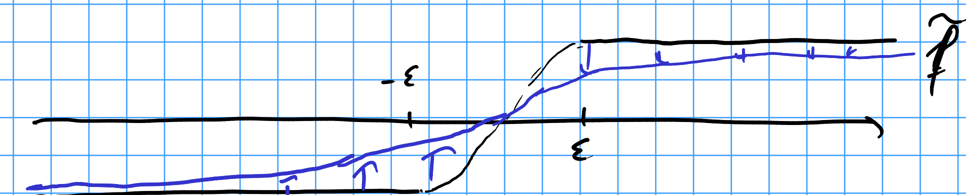
$\mathrm{spec}(\not{D}) \cap [-\varepsilon, \varepsilon] = \emptyset$ for some $\varepsilon > 0$.

if $\mathrm{scal}_g > 4\varepsilon > 0$.

<u>This means</u>: if we use instead of

$f(x) = \dfrac{x}{\sqrt{1+x^2}}$ a $\overset{\text{odd}}{\text{function}}$ $\tilde{f}$

s.t. $\tilde{f}(x) = \begin{cases} 1, & x \geq \varepsilon \\ -1, & x \leq -\varepsilon \end{cases}$



$\tilde{f}(\not{D})^2 = \mathbb{1}$ (because $\tilde{f}^2 \equiv 1$ on $\mathrm{spec}(\not{D})$)

Consequence: the construction of $[\not{D}]$

if f is replaced by $\tilde{f}$

produces $\rho(g) \in K_1(B(L^2(S_+)))$

by homotopy invariance

$$[\rho(D)] = [D] \in K_n(B/K).$$

To make better use of this idea:

<u>add more structure</u> (we already used the $S = S_+ \oplus S_-$)

e.g. $\pi_1 M = \Gamma$

more about $\dim(M)$ $\qquad KO_{\dim M}$

or ...

produce more interesting C^*-algebras

than $K(L^2(S_+)) \subset B(L^2(S_+))$

with non-zero K-theory, ...