

The topology of psc, Lecture 3: Clifford + spin geometry

Schedule: Today + Monday lecture per video, end of first half.

→ Continuation (in classroom) from Sep 13...

The story of "extra structure".

Look out for it and use it!

Ex: Imagine you only would know about \mathbb{C} -v.s.
[if today you see an \mathbb{R} -vector space, you complexity] \leftrightarrow any matrix in $M_n(\mathbb{R})$ you would $\in M_n(\mathbb{C})$

Problem: you wouldn't know about orientation
Why not? $A \in GL_n(\mathbb{R})$ is orientation preserving if $\det(A) > 0$.

Properties: invariant under deformation, conjugate
Of course, in \mathbb{C} " > 0 " doesn't make sense.

Ex: find deformations / conjugation in \mathbb{C}
from $\det > 0$ to $\det < 0$.

This has a shadow for \mathbb{C}^* -algebras

Def: A \mathbb{C} -algebra A (e.g. \mathbb{C}^* -algebra) is Real ("has a Real structure") if it has an conjugate involution $\bar{\cdot}: A \rightarrow A$
linear + multiplicative

$$\overline{\overline{a}} = a \quad ; \quad \overline{a \cdot b} = \overline{a} \cdot \overline{b}$$

Basic example: if $A_{\mathbb{R}}$ is an \mathbb{R} -algebra

$$A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}, \quad \bar{\cdot}: A \rightarrow A; \quad \overline{a \otimes h} := a \otimes \bar{h}$$

other example: If X manifold
with involution $\tau: X \rightarrow X$

Then $\mathcal{C}_0(X; \mathbb{C})$ is a Real algebra
with $\overline{f}(x) := \overline{f(\tau(x))}$ $\in \mathbb{C}$

Note: if $\tau = \text{id}$ then this is the Real algebra $\mathcal{C}_0(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$,
but in general not. Ex: Find example

Promise: our indices will be with Real C^* -algebra. We need to make use of this real structure (K-theory)

2nd bit of extra structure $\hat{=}$ Clifford geom.
will a) fundamental to construct Dirac op. (spin geometry)
b) take care of the dimension of the mfd (with out case distinctions)

Def: Clifford algebra $\mathcal{C}\ell_{p,q}$ is a unital Real \mathbb{C} -algebra
with a $\mathbb{Z}/2$ -graded.

Defined by generators and relations

generators $e_1, \dots, e_p, \varepsilon_1, \dots, \varepsilon_q$ (also 1)

with relations: • any two different generators

anticommute: $xy = -yx \Leftrightarrow xy + yx = 0$.

• $e_i^2 = -1$; $\varepsilon_j^2 = 1$

Recall: an algebra A is graded

$(\Leftrightarrow) A = A_0 \oplus A_1$ with $A_i \cdot A_j \subset A_{i+j}$

For us: all generators are in A_1 $\stackrel{\uparrow}{\hat{=}}$ are odd mod 2

Real structure is determined by:

$$\bar{x} = x \quad \text{for all generators}$$

[Really this means: $\mathfrak{ll}_{p,q} = \mathfrak{ll}_{p,q}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$]

$\mathfrak{ll}_{p,q}$ becomes a C^* -algebra

setting $e_i^* = -e_i$; $\epsilon_j^* = \epsilon_j$.

Ex: $\mathfrak{ll}_{p,q}$ is a finite dim algebra over \mathbb{C} of dimension $2^{p+q} = \dim(\wedge^*(\mathbb{C}^{p+q}))$

Tensor products for graded \mathbb{C} -algebras:

Def: If $A = A_0 \oplus A_1$; $B = B_0 \oplus B_1$ are graded algebras, the graded tensor prod. is

$A \hat{\otimes}_{\mathbb{C}} B := A \otimes_{\mathbb{C}} B$ with multiplicative

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b| \cdot |a'|} \cdot (aa' \otimes bb')$$

where $|b|$ is the degree of b i.e.

$ b =0$	if	$b \in B_0$
$ b =1$	"	$b \in B_1$

Consequence / Ex: In $A \hat{\otimes} B$

elements $a \otimes 1$ and $1 \otimes b$

do not always commute on the nose, they "graded commute",

i.e. $a \otimes 1 \cdot 1 \otimes b = - (1 \otimes b) \cdot (a \otimes 1)$

if $a \in A_1$, $b \in B_1$.

Lemma / Ex: $\mathfrak{ll}_{p,q} \hat{\otimes} \mathfrak{ll}_{p',q'} = \mathfrak{ll}_{p+p', q+q'}$

with e -generators $\underline{e_1 \otimes 1}, \dots, e_p \otimes 1, 1 \otimes \underline{e'_1}, \dots, 1 \otimes e'_{p'}$

Ex :

$$\begin{aligned} \mathcal{C}\ell_{1,0} &= \mathbb{C} \otimes_{\mathbb{R}} (\underbrace{\mathbb{R}[1, i]}_{= \mathbb{C}}) \\ \mathcal{C}\ell_{0,1} &= \mathbb{C} \otimes_{\mathbb{R}} (\underbrace{\mathbb{R}[1, e]}_{= \mathbb{R}[z/2z]}) \\ \mathcal{C}\ell_{2,0} &= \mathbb{C} \otimes_{\mathbb{R}} (\underbrace{\mathbb{R}[1, i, j, ij=k]}_{\|1\|}) \\ &\vdots \end{aligned}$$

Ex :

$$\mathcal{C}\ell_{8,0} = \mathbb{C} \otimes_{\mathbb{R}} (M_{16}(\mathbb{R}) = M_2(M_8(\mathbb{R}))$$

with standard grading

on $M_2(A)$ with have standard grading

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \text{ even; } \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix} \text{ odd.}$$

Consequence : in this sense : $\mathcal{C}\ell_{p,0}$ is "8-periodic" :

there is $\mathcal{C}\ell_{0,0}, \dots, \mathcal{C}\ell_{7,0}$

$$\underbrace{\mathbb{C} \otimes_{\mathbb{R}} M_2(M_8(\mathbb{R}))}_{st} \cong \mathcal{C}\ell_{7,0}, \quad st \otimes \mathcal{C}\ell_{2,0}, \dots$$

Spin geometry from Clifford geometry

make use of the group $\text{Spin}(n)$, related to $SO(n), \dots$

Def : $\text{Spin}(n) :=$ Subgroup of invertible elements in $\mathcal{C}\ell_{n,0}$ generated by :

[the vector space $\mathbb{R}^p \otimes \mathbb{R}^q$ generated by $e_1, \dots, e_p \leftarrow \text{ONB}$ is contained in $\mathcal{C}\ell_{p,q}$ and are invertible if $||\text{norm} \neq 0$.]

be vector in the unit sphere of $\mathbb{R}^p \subset \mathbb{R}^{p,0}$

but then only the even part of this.

$$\text{i.e. } \text{Spin}(p) = \left\{ v_1 v_2 \cdots v_{2k-1} v_{2k} \mid v_j \in S_1(\mathbb{R}^p) \right\}$$

this is a Lie subgroup of $\mathbb{R}^{p,0}$
 this is fin. dim. euclidean vector space $\mathbb{R}^{p,0}$.

There is an adjoint action of $\text{Spin}(n)$ on $\mathbb{R}^n \subset \mathbb{R}^{n,0}$

by conjugation, using the multiplication in $\mathbb{R}^{n,0}$: $g \in \text{Spin}(n) \subset \mathbb{R}^{n,0}$

$$\text{ad}(g)(v) : v \in \mathbb{R}^n \subset \mathbb{R}^{n,0} \rightarrow g v g^{-1} \in \mathbb{R}^n$$

$$\text{ad}(g) \in \text{SO}(\mathbb{R}^n) \quad \text{Exercise}$$

Fact: This defines short exact seq. of

$$\text{groups } 1 \rightarrow \mathbb{Z}/2 \rightarrow \text{Spin}(n) \xrightarrow{\text{ad}} \text{SO}(n) \rightarrow 1$$

$$\{1, -1 = e^2\}$$

General facts imply: the map ad is a covering projection, with 2 leaves.

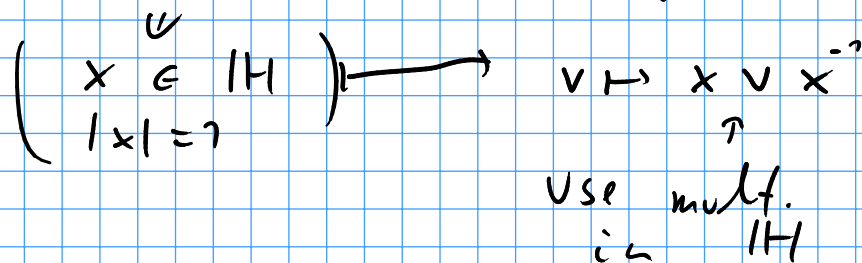
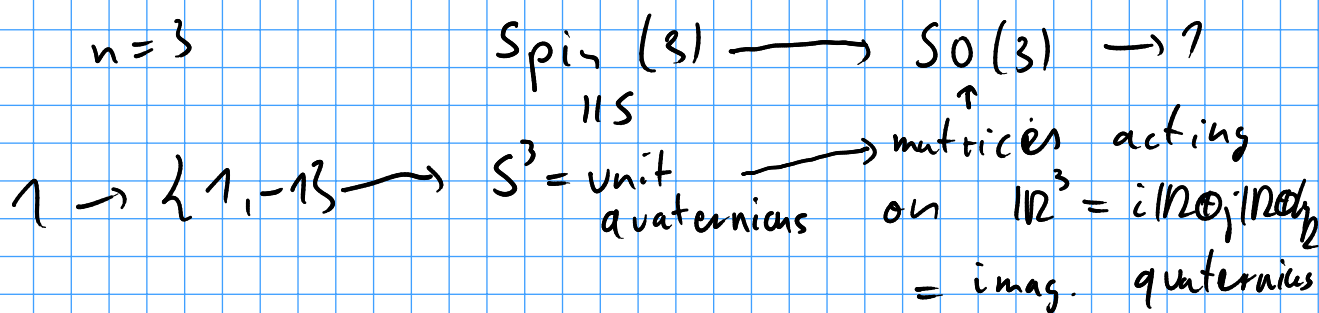
Ex: $n=2$

$$\text{Spin}(2) \rightarrow \text{SO}(2)$$

$$1 \rightarrow \{1, -1\} \xrightarrow{\cong} S^1 \xrightarrow{\cong} S^1 \rightarrow 1$$

$$\cong \mathbb{Z} \xrightarrow{\cong} \mathbb{Z}^2$$

Ex: $n=3$



Generally: Spin(n) is connected, and indeed if $n \geq 2$ is the universal covering of SO(n)

$$\left[\begin{array}{l} \pi_1(\text{Spin}(n)) = \mathbb{Z}/2 \\ \pi_1(\text{SO}(n)) \cong \mathbb{Z}/2 \end{array} \right]$$

Recall: For a smooth manifold M , orientation + Riemannian metric

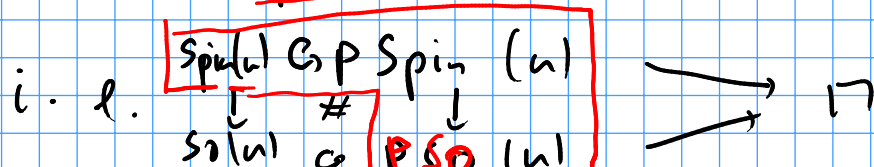
\implies

SO(n) - principal frame bundle

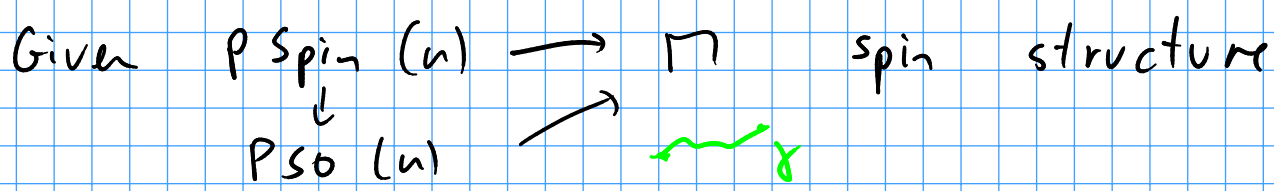
$$\begin{array}{ccc} \text{PSO}(n) & \longrightarrow & M^n \\ & & \cup \\ & & x \\ \text{PSO}(n) \times & \longrightarrow & x \\ \text{"} & & \end{array}$$

{ oriented orthonormal basis of $T_x M$ }
= { isometr. isom. $\mathbb{R}^n \xrightarrow{\sim} T_x M$ }

A spin structure on M is a "Spin(n) - reduction" of this $\text{PSO}(n)$



Fundamental construction of $\mathbb{C}\ell_{n,0}$ - linear Real graded Dirac operator:



observe: the Levi-Civita connection \cong a parallel transport in $PSO(n)$ for smooth paths in M lifts to $PSpin(n)$

we can form associated vector bundles:

$$PSO(n) \times_{SO(n)} \mathbb{R}^n = TM$$

$$PSpin(n) \times_{ad} \mathbb{R}^n = TM$$

$$PSpin(n) \times_{\substack{Spin(n) \subset \mathbb{C}\ell_{n,0} \\ \text{left mult.}}} \mathbb{C}\ell_{n,0} =: \$$$

\downarrow
 M

with extra structure: $\$$ is a bundle

- with fibers the structure of free $\mathbb{C}\ell_{n,0}$ right modules of rank 1, with Real struct and grading.
- and with a connection.
- with a Clifford multiplication $TM \times \$ \rightarrow \$$ ($\mathbb{C}\ell_n$ -linear)

$$(PSpin(n) \times_{ad} \mathbb{R}^n) \times (\underbrace{PSpin(n) \times_{\substack{Spin(n) \\ \subset \mathbb{C}\ell_n}} \mathbb{C}\ell_n}_{\mathbb{C}\ell_n})$$

$$(g_{ij}^{-1}, g^{ij}) \mapsto (g_{ij}^{-1}, g^{ij})$$

Ex: well defined. in \mathcal{L}_n

Def \mathcal{L}_n -lic. Dirac op.

$$D : \Gamma(S) \rightarrow \Gamma(S)$$

$$s \mapsto (Ds)(x) :=$$

$$\sum_{e_i \text{ ONB of } T_x M} e_i \cdot (\nabla_{e_i} s)(x)$$

this is 1st order diff. op.

- \mathcal{L}_n linear \neq
- [Real $\hat{=} (Ds = \overline{Ds})$]
- odd for grading \in

slides on my web-page.

