The topology of psc. Lecture 3: Clifford + spin geometry...

Schedule: Today + Monday lecture per video,
end of first half.
→ Continuation (in classroom) from Sep 13 ...

The story of "extra structure".
Look out for it and use it!

Ex: Imagine you only would know about $\mathbb{C}$-v.s.
[if today you see an $\mathbb{R}$-vectorspace, you complexify] $\longleftrightarrow$ any matrix in $M_n(\mathbb{R})$ you would $\in M_n(\mathbb{C})$

Problem: you wouldn't know about orientation
Why not? $A \in Gl_n(\mathbb{R})$ is orientation preserving if $\det(A) > 0$.
Properties: invariant under deformation, conjugate
Of course, in $\mathbb{C}$ ">0" doesn't make sense.
Ex: find deformations / conjugation in $\mathbb{C}$ from $\det > 0$ to $\det < 0$.

This has a shadow for $\mathbb{C}$-C^*-algebras

Def: A $\mathbb{C}$-algebra A (e.g. C^*-algebra) is **Real** ("has a Real structure") if it has an conjugate linear + multiplicative involution $\bar{\ }: A \to A$

$$\overline{\lambda a} = \bar{\lambda} \cdot \bar{a} \quad ; \quad \overline{ab} = \bar{a} \cdot \bar{b}.$$

Basic example: if $A_{\mathbb{R}}$ is an $\mathbb{R}$-algebra

$$A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \quad , \quad \bar{\ }: A \to A \; ; \; \overline{a \otimes \lambda} := a \otimes \bar{\lambda}.$$

other example: If X manifold
with involution $\tau : X \to X$
Then $C_0(X; \mathbb{C})$ is a Real algebra
with $\overline{f}(x) := \overline{f(\tau(x))}$ $\in \mathbb{C}$

Note: if $\tau = id$ then this is the Real algebra $C_0(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$,
but in general not. Ex: Find example

Promise: our indices will be with Real C^*-algebra. We need to make use of this Real structure (K-theory)

2nd bit of extra structure $\hat{=}$ Clifford geom.
will a) fundamental to construct Dirac op. (spin geometry)
b) take care of the dimension of the mfd (without case distinctions)

Def: Clifford algebra $\mathcal{Cl}_{p,q}$; $p, q \in \mathbb{N}_{\geq 0}$
is a unital Real $\mathbb{C}$-algebra
with a $\mathbb{Z}/2$-graded.
Defined by generators and relations

generators $e_1, \dots, e_p, \varepsilon_1, \dots, \varepsilon_q$ (also 1)
with relations: • any two different generators anticommute: $xy = -yx \iff xy + yx = 0$.
• $e_i^2 = -1$, $\varepsilon_j^2 = 1$

Recall: an algebra A is graded
$\iff A = A_0 \oplus A_1$ with $A_i \cdot A_j \subset A_{i+j}$ mod 2

For us: all generators are in A_1
$\hat{=}$ are odd

Real structure is determined by:

$\bar{x} = x$ for all generators

[Really this means: $Cl_{p,q} = Cl_{p,q}^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$]

$Cl_{p,q}$ becomes a C^*-algebra

setting $e_i^* = -e_i$, $\varepsilon_j^* = \varepsilon_j$.

Ex: $Cl_{p,q}$ is a finite dim algebra over $\mathbb{C}$ of dimension $2^{p+q} = \dim(\Lambda^*(\mathbb{C}^{p+q}))$

Tensor products for graded $\mathbb{C}$-algebras:

Def: If $A = A_0 \oplus A_1$, $B = B_0 \oplus B_1$ are graded algebras, the graded tensor prod. is $A \hat{\otimes}_{\mathbb{C}} B := A \otimes_{\mathbb{C}} B$ with multiplic

$$(a \otimes b) \cdot (a' \otimes b') := (-1)^{|b| \cdot |a'|} \cdot (aa' \otimes bb')$$

where $|b|$ is the degree of b i.e.

$|b| = 0$ if $b \in B_0$

$|b| = 1$ " $b \in B_1$.

Consequence / Ex: In $A \hat{\otimes} B$ elements $a \otimes 1$ and $1 \otimes b$ do not always commute on the nose, they "graded commute",

i.e. $a \otimes 1 \cdot 1 \otimes b = -(1 \otimes b) \cdot (a \otimes 1)$

if $a \in A_1$, $b \in B_1$.

Lemma / Ex: $Cl_{p,q} \hat{\otimes} Cl_{p',q'} = Cl_{p+p', q+q'}$

with e-generators $e_1 \otimes 1, \dots, e_p \otimes 1, 1 \otimes e_1', \dots, 1 \otimes e_{p'}'$

$\underline{\text{Ex}}$: $\quad Cl_{1,0} = \mathbb{C} \otimes_{\mathbb{R}} (\underbrace{\mathbb{R}[1, i]}_{= \mathbb{C}})$

$\quad Cl_{0,1} = \mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R}[1, e] = \mathbb{R}[\mathbb{Z}/2\mathbb{Z}])$

$\quad Cl_{2,0} = \mathbb{C} \otimes_{\mathbb{R}} (\underbrace{\mathbb{R}[1, i, j, ij = k]}_{\mathbb{H}})$

$\quad \vdots$

$\underline{\text{Ex}}$: $\quad Cl_{8,0} = \mathbb{C} \otimes_{\mathbb{R}} (M_{16}(\mathbb{R}) = M_2 (M_8(\mathbb{R}))$

with standard grading

on $M_2(A)$ with have $\underline{\text{standard}}$ grading

$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ even ; $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ odd .

$\underline{\text{Consequence}}$: in this sense : $Cl_{p,0}$ is "8-periodic" :

there is $Cl_{0,0}, \dots, Cl_{7,0}$,

$\underbrace{\mathbb{C} \otimes_{\mathbb{C}} M_2(M_8(\mathbb{R}))}_{st}$, $st \,\hat{\otimes}\, Cl_{1,0}$, $st \,\hat{\otimes}\, Cl_{2,0}$, ...

Spin geometry from Clifford geometry

make use of the group Spin (n),

related to $SO(n)$, ...

$\underline{\text{Def}}$: $\quad$ Spin (n) := Subgroup of invertible elements in $Cl_{n,0}$ generated by :

[the vector space $\mathbb{R}^p \oplus \mathbb{R}^q$ generated by $e_1, \dots, e_p \leftarrow$ ONB

is contained in $Cl_{p,q}$

and are invertible if of norm $\neq 0$.]

be vector in the unit sphere of $\mathbb{R}^p \subset Cl_{p,0}$

but then only the even part of this.

i.e. $\mathrm{Spin}(p) = \{ v_1 v_2 \cdots v_{2k-1} v_{2k} \mid v_j \in S_1(\mathbb{R}^p) \}$

$\uparrow$

this is a Lie subgroup of this fin. dim. euclidean vector space $Cl_{p,0}$.

There is an adjoint action of $\mathrm{Spin}(n)$ on $\mathbb{R}^n \subset Cl_{n,0}$

by conjugation, using the multiplication in $Cl_{n,0}$) : $g \in \mathrm{Spin}(n) \subset Cl_{n,0}^{\times}$

$v \in \mathbb{R}^n \subset Cl_{n,0}$

$\mathrm{ad}(g)(v) : g v g^{-1} \underset{\text{Exercise}}{\in} \mathbb{R}^n$

$\mathrm{ad}(g) \in SO(\mathbb{R}^n)$.

<u>Fact</u>: This defines short exact seq. of groups

$$1 \to \underset{\substack{\shortparallel \\ \{1, -1 = e_i^2\}}}{\mathbb{Z}/2} \to \mathrm{Spin}(n) \xrightarrow[\mathrm{ad}]{} SO(n) \to 1$$

General facts imply: the map ad is a covering projection, with 2 leaves.

<u>Ex</u>: $n=2$ $\qquad \mathrm{Spin}(2) \to SO(2)$

$$1 \to \{1,-1\} \xrightarrow{\subset} \overset{\cong}{S^1} \longrightarrow \overset{\shortparallel}{S^1} \to 1$$

$$z \longmapsto z^2$$

Ex: $n=3$

$$\begin{array}{ccccccc} & & \mathrm{Spin}(3) & \longrightarrow & SO(3) & \to 1 \\ & & \text{\rotatebox{90}{$\cong$}} & & \uparrow & \\ 1 \to \{1,-1\} & \longrightarrow & S^3 = \text{unit quaternions} & \longrightarrow & \text{matrices acting on } \mathbb{R}^3 = i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R} & \\ & & & & = \text{imag. quaternions} & \end{array}$$

$$\left(\begin{array}{c} x \in \mathbb{H} \\ |x| = 1 \end{array} \right) \longmapsto \left(v \mapsto x v x^{-1} \right)$$

use mult. in $\mathbb{H}$

Generally: $\mathrm{Spin}(n)$ is connected, and indeed if $n > 2$ is the universal covering of $SO(n)$

$$\left[\begin{array}{l} \pi_1(\mathrm{Spin}(n)) = \{1\} \\ \pi_1(SO(n)) \cong \mathbb{Z}/2 \end{array} \right]$$

Recall: For a smooth manifold M,

orientation + Riemannian metric

$\leadsto$

$SO(n)$-principal frame bundle

$$P\,SO(n) \longrightarrow M^n$$

$$P\,SO(n)_x \longrightarrow x \in M$$

$= \{$ oriented orthonormal basis of $T_xM \}$

$= \{$ isomet. isom. $\mathbb{R}^n \xrightarrow{\cong} T_xM \}$

A spin structure on M is a "Spin(n)-reduction" of this $P\,SO(n)$
$\downarrow$
M

i.e.

$$\begin{array}{ccccc} \mathrm{Spin}(n) & \hookrightarrow & P\,\mathrm{Spin}(n) & \longrightarrow & M \\ \downarrow & \# & \downarrow & \nearrow & \\ SO(n) & \hookrightarrow & P\,SO(n) & & \end{array}$$

Fundamental construction of $\mathcal{C}\ell_{n,0}$ - linear Real graded Dirac operator:

Given $P\mathrm{Spin}(n) \longrightarrow M$ spin structure

$$\begin{array}{l} P\mathrm{Spin}(n) \longrightarrow M \\ \downarrow \qquad \nearrow \\ PSO(n) \end{array}$$

<u>observe</u>: the Levi-Civita connection

$\hat{=}$ a parallel transport in $PSO(n)$ for smooth paths in M

lifts to $P\mathrm{Spin}(n)$

we can form associated vector bundles:

$$PSO(n) \underset{SO(n)}{\times} \mathbb{R}^n = TM$$

$$\begin{array}{c} \Vert \\ P\mathrm{Spin}(n) \underset{ad}{\times} \mathbb{R}^n = TM \end{array}$$

$$P\mathrm{Spin}(n) \underset{\substack{\mathrm{Spin}(n) \subset \mathcal{C}\ell_{n,0} \\ \text{left mult.}}}{\times} \mathcal{C}\ell_{n,0} =: \begin{array}{c} \$ \\ \downarrow \\ M \end{array}$$

with extra structure: $\$$ is a bundle

- with fibers the structure of free $\mathcal{C}\ell_{n,0}$ right modules of rank 1, with Real struct and grading.
- and with a connection.
- with a Clifford multiplication

$$TM \times \$ \longrightarrow \$ \qquad (\mathcal{C}\ell_n\text{-linear})$$

$$\Big(P\mathrm{Spin}(n) \underset{ad}{\times} \underset{\subset \mathcal{C}\ell_n}{\mathbb{R}^n}\Big) \times \Big(P\mathrm{Spin}(n) \underset{\mathrm{Spin}(n)}{\times} \mathcal{C}\ell_n\Big)$$

$$((q g_1^{-1}, g v g_1^{-1}), (q g_1^{-1}, g x)) \longmapsto (q g_1^{-1}, g v g_1^{-1} g x)$$

<u>Ex</u>: well defined. in Cl_n

Def Cl_n-lin. Dirac op.

$$\not{D} : \Gamma(\$) \longrightarrow \Gamma(\$)$$

$$s \longmapsto (\not{D} s)(x) := \sum_{e_i \text{ OND of } T_x M} e_i \cdot (\nabla_{e_i} s)(x)$$

this is 1st order diff. op.

- Cl_n linear
- Real $\Leftrightarrow$ $(\not{D} \bar{s} = \overline{\not{D} s})$
- odd for grading

Slides on my web-page.