TOPOLOGY OF PSC Rom 21

End result of constr. last time:

Cl_n-spinor bundle $S := PSpin(n) \times_{Spin(n)} Cl_{n,0}$

- $\mathbb{Z}/2$-grading
- Real structure
- each fiber is $Cl_{n,0}$-right module ($\mathbb{Z}/2$-graded, Real)

- $D\!\!\!/$ on sections of S

$$D\!\!\!/ : \Gamma(S) \to \Gamma(S)$$

Cl_n-linear, odd, Real $\leftarrow$

How does this produce ind/fund. class / secondary index $\in K_*$(C^*-algebra) ?

Two facets: (a) which C^*-algebras ?
(b) how to define + compute K-theory ?

to b): we saw

$$K_0(A) = \text{"projections"}$$

$$K_1(A) = \text{"unitaries"}$$

Advantage: relativ. straight forward

Disadvantage:

- Doesn't work so easily with
  - grading
  - Real struct.
- constr. of ind(D), ... is not so natural

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to a) observe that $Cl_n := Cl_{n,0}$ is a C^*-algebra.

to use this:

- we treat Cl_n as our new scalars, replacing $\mathbb{C}$ ($\mathbb{R}$).

Note: $\mathbb{C}$ is the range of the

inner product on $L^2(\$)$ for classical Dirac, and $L^2(\$)$ is based on inner product on fibers of $\$$.

<u>So</u>: Define Cl_n-valued inner product on fibers of $\$$, $\rightsquigarrow$ on sections $\Gamma(\$)$.

<u>Def</u>: If A is C^*-algebra a Hilbert A-module H_A is:

- right A-module
- with map $H_A \times H_A \to A$, $\langle \cdot, \cdot \rangle$ satisfying
  - A-linear in 2nd entry
    $$\langle v, w \cdot a \rangle = \langle v, w \rangle \cdot a$$
  - $\langle v, w \rangle = \langle w, v \rangle^* \quad \forall v, w \in H_A$
  - $\langle v, v \rangle \geq 0$ in A
    $\langle v, v \rangle = 0 \iff v = 0$

- M_A is complete for norm $\|x\| = |\langle x, x\rangle|_A^{1/2}$

Rem: M_A is a Banach space but in general not Hilbert.

Ex 1 $A^n = M_A$

with $\left\langle \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} a_1' \\ \vdots \\ a_n' \end{pmatrix} \right\rangle := \sum a_i^* a_i'$

$l^2(A) :=$ completion of $\bigoplus_{k=1}^{\infty} A$.

For us: the fibers of $\$ = P\,\mathrm{spin}(n) \times_{\mathrm{spin}(n)} Cl_n$

have canonical Cl_n-Hilbert module structure from Example 1.

Ex: is well defined

$\rightsquigarrow L^2(\$) =$ Hilbert Cl_n-module

Observe: $\begin{array}{c} P\mathrm{Spin}(n) \\ \downarrow \\ M \end{array}$

is trivial over a subset $U \subset M$ of full measure: $\mu(M \setminus U) = 0$.

$$\Rightarrow \quad L^2(\mathcal{S}) = L^2(\mathcal{S}|_U)$$

$$= L^2(U \times \cancel{\mathrm{Spin}(n)} \times_{\cancel{\mathrm{Spin}(n)}} \mathcal{Cl}_n)$$

$$= L^2(U) \otimes \mathcal{Cl}_n$$

<u>However!</u> $\not{D}$ sees the non-triviality of $\mathcal{S}$.

<u>Conclusion</u>: $\not{D}$ acts as unbounded op. on Hilbert $\mathcal{Cl}_n$-module $L^2(\mathcal{S})$.

Def: If H_A is Hilbert A-module

$$B_A(H_A) := \{ T : H_A \to H_A \mid T \ A \text{ linear}, \ T \text{ has adjoint } T^* \}$$

$\Rightarrow T$ bounded

where $\langle v, Tw \rangle = \langle T^* v, w \rangle$

$B_A(H_A)$ are a C^*-algebra with $* : T^*$

$K_A(H_A)$ is closed subalgebra generated by $\forall v, w \in H_A$

$$T_{v,w} : H_A \to H_A$$
$$x \longmapsto w \cdot \langle v, x \rangle$$

Exercise $T_{v,w}^* = T_{w,v}$.

Fact: $K_A(H_A)$ are ideal in $B_A(H_A)$

If A is finite dim

A compact $\Leftrightarrow$ Banach space compact.

Now in our situation:

The relevant algebra is

$$\boxed{\begin{array}{c} \mathcal{B}_{\mathrm{Cl}_n}(L^2(\$)) \\ \cup \\ \mathcal{K}_{\mathrm{Cl}_n}(L^2(\$)) \end{array}}$$

<u>Extra structure</u>:

Assume that a group Γ acts isometrically on M, preserving also $\mathrm{Pspin}(n)$

(≙ preserving spin structure)

$\Rightarrow$ Γ acts on $L^2(\$)$

& $\not{D}$ commute with Γ-action

Then the relevant algebras will be:

$$\boxed{\begin{matrix} B_{\ell\ell_n}(L^2(\mathcal{G}))^\Gamma \\ \cup \\ K_{\ell\ell_n}(L^2(\mathcal{G}))^\Gamma \end{matrix}} \overset{\text{subalgebra}}{\subset} B_{\ell\ell_n}(L^2(\mathcal{G}))$$

Exercise:

$$B_{\ell\ell_n}(L^2(\mathcal{G}) = L^2(M) \otimes \ell\ell_n)$$

$$\cong$$

$$B(L^2(M)) \otimes \underbrace{B_{\ell\ell_n}(\ell\ell_n^{\,!})}_{\cong\, \boxed{\ell\ell_n}}$$

$$\cup$$

$$K_{\ell\ell_n}(L^2(\mathcal{G})) \cong K(L^2(M)) \otimes \boxed{\ell\ell_n}$$

Appropriate picture of K-theory

A graded (Real) C^*-algebra

Consider the

$$\boxed{\begin{array}{l} C^*\text{-algebra} \quad S := C_0(\mathbb{R}; \mathbb{C}) \\ \text{with} \cdot \text{even/odd fct grading} \\ \quad\quad \cdot \text{Real struct by complex conj. in } \mathbb{C} \end{array}}$$

<u>Def</u>: $\underline{KO}(A) :=$

$\{ f : S \longrightarrow A \hat{\otimes} \mathbb{K}(\underbrace{\ell^2 \oplus \ell^2}_{=: \mathbb{H}})$

C^*-algebra homom., preserving grading, Real structure $\}$

$\mathbb{Z}/2$-graded Hilbert space

topol. space with point-norm convergence: $f_n \longrightarrow f \iff \forall x \in S$

$f_n(x) \longrightarrow f(x)$ in norm in $A \hat{\otimes} \ldots$

$$\widehat{KO}_0(A) := \pi_0(\underline{KO}(A))$$

$$\widehat{KO}_n(A) := \pi_n(\underline{KO}(A))$$

[base point is 0-homom.]

Ex: If A is ungraded, unital then there is an isom. (homotopy classes)

$$KO_0(A) \longrightarrow [S, A \hat{\otimes} K(\ell^2 \oplus \ell^2)]$$

$$\left([p_0] - [p_1] \quad p_{0,1} \in M_n(A) \subset A \otimes K(\ell^2)\right) \longmapsto f \mapsto f(0) \cdot \begin{pmatrix} p_0 & 0 \\ 0 & p_1 \end{pmatrix}$$

$\underline{\text{Ex}}$: Dirac operator.

Recall the FA fact:

$f \in C_0(\mathbb{R}; \mathbb{C})$

(vanishing at $\pm \infty$)

$\Rightarrow f(\not{D})$ is <u>compact</u>

Therefore:

$$\mathcal{S}' \longrightarrow \mathbb{K}_{\mathcal{C}\ell_n}(L^2(\not{S}))$$

$$\boxed{f \longmapsto f(\not{D})}$$

is a homomorphism of C^*-alg$_s$.

grading + Real structure preserving

$\rightsquigarrow$ Defines $\boxed{\operatorname{ind}(\not{D}) \in K_0(\mathbb{K}_{\mathcal{C}\ell_n}(L^2(\not{S})))}$

Further structure is taken care of automatically.

Prop: If $\not{D}$ is invertible,

i.e. $0 \notin \operatorname{spec}(\not{D})$

e.g. if $\operatorname{scal}_g \geq \varepsilon > 0$

then: For $t \in [0,1]$, $f \in S'$

set $f_t(x) := f\left(\frac{x}{t}\right)$

Then consider the path of homomorphisms

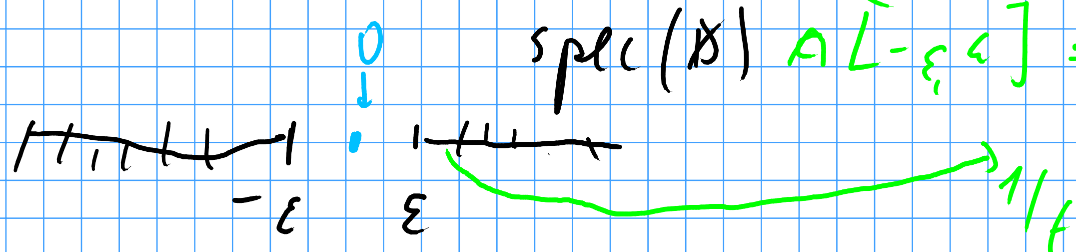
$$S' \longrightarrow \mathbb{K}_{ell_n}(L^2(\$))$$

$$(f, t) \longmapsto f_t(\not{D})$$

Clearly: for $t = 1$ this represents $\operatorname{ind}(\not{D})$.

For $t \searrow 0$ observe

$\operatorname{spec}(\not{D}) \cap [-\varepsilon, \varepsilon] = \emptyset$



$\Rightarrow \quad f_t(D) \xrightarrow{t \to 0} 0$

$\Rightarrow$ the path of homom. converges to 0 as $t \to 0$.

$\Rightarrow \quad \mathrm{ind}(D) = 0 \in K_0(\mathbb{K}_{\ell^2_n}(|\beta|))$.

Observe: this argument is "universal"

Exercise: If D is not invertible but has discrete spectrum

the path of homomorphisms will not necc. converge to 0, but to

$$f \mapsto f_0(0) \begin{pmatrix} pr_{\ker(D_+)} & 0 \\ 0 & pr_{\ker(D_-)} \end{pmatrix}$$

Remember our philosophy:

we should improve a vanishing theorem by constructing a (secondary) invariant giving the reason for vanishing.

$\boxed{\text{Here: we have a path from } \operatorname{ind}(\not{D}) \text{ to } 0}$

Recall: $\widehat{KO}_n(A) = \pi_n(\underline{KO}(A))$

So: $n=1$ $\hat{=}$ paths

(problem: need closed paths)

Def [Construction of second path from "ind(D)" to 0]

which works however not in $\mathbb{K}_{\ell\ell}(L^2(\mathcal{S}))$ but in $\mathbb{B}_{\ell\ell}(L^2(\mathcal{S}))$ $\longleftarrow$

Doing the following:

$$(f, s) \longmapsto f(sD)$$

$s \in [0,1]$

for $s = 1$: ind (D)

$s = 0$: $f \mapsto \underbrace{f(0) \cdot id_{L^2(\mathcal{S})}}_{\notin \mathbb{K}_{\ell\ell}}$

$$= f(0) \cdot \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}$$

Ex. ~ 0 in $\mathbb{B}_{\ell\ell}(L^2(\mathcal{S}))$

putting these 2 paths together defines a secondary invariant

$$\begin{tikzcd}
\rho(g) \arrow[d, mapsto, "\text{psc metric)}"'] & \in & \widehat{KO}_1(\mathbb{B}_{\ell_n}(\emptyset)) \arrow[d] \\
{[D]} \arrow[d, mapsto] & \in & \widehat{KO}_1(\mathbb{B}_{\ell_n} / \mathbb{K}_{\ell_n}) \arrow[d] \\
\mathrm{ind}(D) & \in & \widehat{KO}_0(\mathbb{K}_{\ell_n})
\end{tikzcd}$$

Further aspect of

$\widehat{KO} = \pi_n(\underline{KO})$:

develop this exact sequence!

Easy property:

$$\underline{KU}(A \otimes \ell_0(\mathbb{R}^n))$$

$$\cong$$

$$\ell_0(\mathbb{R}^n, \underline{KU}(A))$$

$$\overset{!!}{=}$$

$$\Omega^n(\underline{KU}(A))$$

<u>Fact</u>: $\pi_k(\Omega^n X) = \pi_{k+n}(X)$

$$\Rightarrow \widehat{KO}_k(A \otimes \ell_0(\mathbb{R}^n))$$

$$\cong$$

$$\widehat{KO}_{n+k}(A)$$

Now the Bott periodicity can be proven using only the $\ell\ell_n(\mathbb{R}^n)$-factor

$$\boxed{\widehat{KO}_k(A \otimes \ell_0(\mathbb{R}^n)) \cong \widehat{KU}_k(A \hat{\otimes} \ell\ell_{n,0})}$$