Positive scalar — lectures at Universita La Sapienza, Roma. March 2021

Thomas Schick, Mathematisches Institut, Universität Göttingen

Abstract

In these notes, we address the question if a given smooth manifold M admits a Riemannian metric with positive scalar curvature. If so, we also introduce the tools to classify all such metrics.

One principal tool to invistigate these problems is the Schrödinger-Lichnerowicz formula, which connects these problems to index theory — more specifically to the index of the Dirac operator on a spin manifold. A decisive role in the calculation of these invariants is played by the Atiya-Singer index theorem and its higher variants, and also by its relative version. The latter involves also ideas around the Atiyah-Patodi-Singer index theorem. It is crucial to use "higher" variants of these invariants to get optimal information. In particular, we will use the K-theory of C^*-algebras and index obstructions in these K-theory groups.

For the classification results one relies on the Atiyah-Patodi-Singer index theorem and the calculation of eta-invariants; aspects not covered in the course.

The course has been divided in two parts. The first part studied concentrated on the K-theory and index theory underlying the most powerful obstructions to the existence of metrics of postivie scalar curvature.

The second part, which is not relying on specific results and constructions from the first part, changes gears. Given a compact smooth manifold M without boundary which admits Riemannian metrics of positive scalar curvature, we will look in more detail at the space of all such metrics.

We will discuss the basic tool to construct metrics of positive scalar curvature wich is "surgery". This allows to investigate the geometric problem of understanding the space of metrics of positive scalar curvature with tools from algebraic topology, in particular based on bordism theory.

We will therefore also spend some time discussing the underlying algebraic topology.

In particular, we discuss relevant aspects of the classification of simply connected manifolds which admit metrics of positive scalar curvature (due to Gromov-Lawson and Stolz).

We end short of giving details of some of the various results which show that the space of metrics of positive scalar curvature has a rich topology, with infinitely many components (and infinite higher homotopy groups).

1 Organisation of the talks

There are 8 lectures of 90 minutes each.

(1) First lecture: Survey, including the main results to be discussed in the course.

- Riemannian geometry of positive scalar curvature
- spin structures and Dirac operator
- Schrödinger-Lichnerowicz formula (repetition)
- Atiyah-Singer index theorem and first obstructions to positive scalar curvature: $K3$-surfaces and the Bott-manifold do not admit metrics with positive scalar curvature
- Hitchin and Rosenberg higher index

(2) Second lecture: K-theory obstructions to positive scalar curvature

- flash on constructions of positive scalar curvature metrics, and on non-triviality results for $\pi_k(R^+(M))$.
- The K-theory philosophy:
  - Invertibility implies vanishing in K-theory
  - special geometry implies invertibility
  - better: special geometry gives a reason for invertibility
  - this reason gives a secondary invariant which classifies special geometry.

(3) Third lecture: Clifford and Spin geometry

- The real and complex Clifford algebras
- gradings
- algebraic Bott periodicity in Clifford algebras
- spin geometry from Clifford eometry
- the Clifford linear real graded Dirac operator

(4) Fourth lecture: the universal primary and secondary index

- K-theory of general graded and Real C^*-algebras
- Hilbert A-modules and operators on them
- Hilbert A-module bundles
- the higher index of the Clifford linear graded Dirac operator twisted by a Hilbert A-module bundle
- the secondary index coming measuring positive scalar curvature

(5) Lectures 5-7: Classification of metrics of positive scalar curvature and bordism theory

- basic definition of surgery
- computing and adjusting homotopy groups in surgery constructions
- the role of the spin condition: making normal bundles trivial
- connection to Morse theory
- the Gromov-Lawson, Schoen-Yau, Hajac-Stolz psc surgery result
- its family versions: Surgery constructions of Chernysh, Walsh (Ebert-Frenck, Bär-Hanke)

(6) Lecture 8: The Stolz positive scalar curvature exact sequence and non-triviality of the homotopy type of the space of metrics of positive scalar curvature

(7) Topics *not covered*in the course:

- The improved surgery constructions of Ebert-Wiemeler: homotopy type of $R^+(M)$ for closed simply connected spin manifold M depends at most on the dimension.
- Non-triviality results for homotopy groups of $R^+(M)$ and $R^+(M)/D(M)$:
  - Hitchin's construction via the action of the diffeomorphism group: π_0, π_1, π_2 via exotic spheres and the Gromoll filtration.
  - Crowley-S. and Crowley-S.-Steimle: improvement to higher homotopy groups (even infinitely many at once)
  - Hanke-Schick-Steimle
  - Botvinnik-Ebert-Randal-Williams and Randal-Williams constructions
- Higher index theory
  - (Family index theory)
  - C^*-algebras and their K-theory
  - Mishchenko-Fomenko index theory
  - Refined obstructions to positive scalar curvature (Rosenberg index)
  - The Gromov-Lawson-Rosenberg conjecture about positive scalar curvature

  - Enlargeability
  - The Gromov-Lawson obstruction to positive scalar curvature: existence of negative sectional curvature prevents positive scalar curvature
  - codimension 2 obstructions to break the symmetry (a reinterpretation of another result of Gromov-Lawson in the context of higher index theory)
  - Kubota and Kubota-S. connection of the codimension 2 obstruction with the Rosenberg index and the index universality conjecture
- Stephan Stolz proof of the stable Gromov-Lawson-Rosenberg conjecture
  - the stable Gromov-Lawson-Rosenberg conjecture
  - Kasparov's KK-theory
  - index in KK-theory
  - K-homology; the Baum-Connes conjecture
  - HP^2-bundles and the corresponding cohomology theory
  - Stephan Stolz' proof of the stable Gromov-Lawson-Rosenberg conjecture
- minimal hypersurface obstructions to positive scalar curvature (after Schoen-Yau)
  - (stable) minimal hypersurfaces
  - geometric measure theory force the existence of minimal surfaces
  - positive scalar curvature and minimal surfaces (Schoen-Yau)
  - Counterexamples to the (unstable) Gromov-Lawson-Rosenberg conjecture
- (universally) invertible Dirac versus truely positive scalar curvature
- positive versus non-negative scalar curvature
- Families of metrics distinguished by rho-invariants (beyond S^0-parametrized ones, which we have "by accident" from the lens spaces

2 Talk 1: Surveying "The topology of positive scalar curvature"

2.1 Basics from Riemannian geometry

(1) Riemannian metric, metric tensor g_{ij} in local coordinates

(2) length of curves, distances

(3) areas of 2-dimensional hypersurfaces

(4) volume element

(5) curvature tensor R^l_{ijk} with many symmetries. Tracing gives Ricci tensor Ric_{ij}, taking trace a second time gives scalar curvature $\mathrm{scal}\colon M \to \mathbb{R}$.

(6) Quick and dirty definition/basic relation:

$$\frac{\mathrm{vol}(B_r(x) \subset M)}{\mathrm{vol}(B_r(0) \subset \mathbb{R}^m)} = 1 - \frac{\mathrm{scal}(x)}{6(m+2)} r^2 + O(r^4) \qquad \text{small } r.$$

(7) For S^n with standard metric: $\mathrm{scal} = n(n-1)$.

(8) Physical relevance: scalar curvature features in Einstein's general relativity.

2.2 Gauß-Bonnet Theorem

2.1 Theorem (Gauß-Bonnet)**.** *If F is a 2-dimensional compact Riemannian manifold without boundary,*

$$\int_F \mathrm{scal}(x)\, d\,\mathrm{vol}(x) = 4\pi\chi(F).$$

2.2 Corollary. $\mathrm{scal} > 0$ *on F implies $\chi(F) > 0$, i.e. $F = S^2, \mathbb{R}P^2$ if F is connected.*

Philosophically: generically, we might have negative curvature, but positive curvature is rare and special.

2.3 Negative scalar curvature can almost always be achieved

2.3 Theorem (Kazdan-Warner)**.** *Let M be a connected closed smooth manifold. Assume* $\dim(M) \geq 3$.

Then if $f(x) < 0$ somewhere, a Riemannian metric with $\mathrm{scal}_g = f$ *exists. Moreover, precisely one of the following cases occurs:*

(1) M admits a metric with $\mathrm{scal} > 0$. *Then for every $f\colon M \to \mathbb{R}$ smooth there is a Riemannian metric g with* $\mathrm{scal}_g = f$

(2) M does not admit a Riemannian metric with $\mathrm{scal} > 0$, *but one with* $\mathrm{scal} = 0$.

If $\mathrm{scal}_g \geq 0$ *then automatically even* $\mathrm{Ric}_g = 0$ *(this is not due to Kazdan-Warner).*

(3) M does not admit a metric with $\mathrm{scal} \geq 0$.

2.4 The players and main questions

2.4 Definition.

- $R^+(M) := \{g \mid \mathrm{scal}_g > 0\}$
- moduli spaces $R^+(M)/D(M)$ where $D(M) \subset \mathrm{Diffeo}(M)$ is a subgroup of the diffeomorphism group (all diffeomorphisms, orientation preserving diffeomorphisms, diffeomorphisms fixing a point and its tangent space,. . .)
- $\pi_0(R^+(M))$, the set of components of the space of psc metrics
- $\pi_0(R^+(M)/D(M))$ the set of components of the moduli space of psc metrics
- weaker equivalence relation than "connected":
  - isotopy
  - bordism (with reference space X)
- $Pos(X)$, the group of bordism classes of manifolds with psc metric and with map to X, group by disjoint union. If $X = M$ contains a coset of metrics realized on M with identity.

2.5 Question. What is the topological type of $R^+(M)$.
Subquestion: $R^+(M) = \emptyset$?
Subquestion: how many components? Higher homotopy groups?

To answer these questions, we need

(1) obstructions and invariants (to distinguish)

(2) consturction methods (to obtain mentrics)

2.5 Spin and Dirac

(1) Dirac introduced the (flat) Dirac operator as a square root of the matrix Laplacian, using Pauli matrices:

$$D = \sum X_i \cdot \partial/\partial x_i$$

where the matrices X_i satisfy the Clifford relations $X_iX_j + X_jX_i = -2\delta_{ij}$.

(2) Schrödinger ("Das Diracsche Elektron im Schwerfeld") developped a version (in local coordinates) for curved space-time. He observed (a local computation)

$$D^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4}$$

This has consequences for the spectrum of the Dirac operator (which is what physicists are interested in: the spectrum describes the energy levels of the electron.

(3) Atiyah-Singer establish the geometric additional structure for this operator to exists globally on a Riemannian manifold (acting on sections of a vector bundle):

one needs a spin structure with associated spinor bundle

(4) Lichnerowicz rediscoverd the Schrödinger formula, now formulated for the global Dirac operator

Main consequence: uniformly positive scalar curvature implies that the Dirac operator is invertible

(5) Atiyah-Singer establihs the index theorem, giving in particular

$$\mathrm{ind}(D) = \dim\ker(D^+) - \dim\ker(D^-) = \int_M \hat{A}(M).$$

The right hand side is independent of the metric, so gives an obstruction to psc.

Example: Kumemr surface.

Non-example: $\mathbb{C}P^{2n}$, T^n.

(6) Hitchin uses a real version of the Dirac operator and of the index to obtain a finer invariant

$$\alpha(M) \in KO_{\dim M}(\mathbb{R}),$$

which still has topological meaning and is an obstruction to psc on M.

Example: half of the exotic spheres in dimension $1, 2 \pmod 8$

(7) Indeed, there is a K-theory philosophy to be discussed here!

(8) Rosenberg twists with the Mishchenko bundle, a canonical flat bundle of modules over the group C^*-algebra $C^*\Gamma$ with $\Gamma = \pi_1(M)$.

One has an index $\alpha(M) \in KO_{\dim M}(C^*\Gamma)$ which still obstructs psc. The Mishchenko-Fomenko index theory applies to this index and shows that it still is topological and an obstruction to psc.

Example: T^n.

(9) Former Gromov-Lawson-Rosenberg conjecture: a spin manifold M has psc if and only if the Rosenberg index vanishes

(10) Counterexample (by S.): Surgery on the 5-torus forces vanishing of the index.

Schoen-Yau minimal hypersurface method shows that the example does not admit a psc metric.

(11) Stable GLR conjecture: a spin manifold stably has psc if and only if the Rosenberg index vanishes.

(12) Stolz: the strong Novikov conjecture implies the stable GLR conjecture

(13) **Index universality conjecture**: whenever we obtain information on psc using index theory (on a spin manifold) of some sorts, this is also detected by the Rosenberg index.

(14) Kubota, Kubota-S.: the Gromov-Lawson codimenison 2 obstruction satisfies the index universality conjecture

2.6 Surgery constructions of psc

2.6 Definition. Surgery; the construction, codimension, connection to Morse theory and to bordism.

2.7 Proposition. *Weak homotopy equivalence of inclusion*

$$R^+_{torp,V}(M) \hookrightarrow R^+(M)$$

if V is a tubular neighborhood of a codim ≥ 3 submanifold with trivial normal bundle. Recent work (Ebert-Wiemeler) generalize this even further.

Then we can cut and repaste and "transport" psc from one manifold to another, and do this even in (compact) families.

Topology/surgery theory:

If M is spin bordant to N with a reference map to $B\Gamma$ which is a π_1-iso for M, then M is obtained from N by surgeries of codimesion ≥ 3.

Applications:

(1) Gromov-Lawson: non-spin simply connected oriented mf carries psc, compare [2, Theorem C].

(2) Rosenberg: understanding which (simply connected) spin manifolds carry psc upto dimension x (x in twenties)

(3) Stolz: simply connected spin manifold carries psc if and only if Rosenberg index vanishes.

2.7 Non-trivial higher homotopy classes of $R^+(M)$ and its moduli space

2.8 Theorem. *Results on connected components*

2.9 Theorem (Crowley-S.)**.** *. If M is a closed spin manifold with $\dim(M) \geq 7$ and $g_0 \in R^+(M)$, $\mathrm{ind}_{rel}\colon \pi_{n-\dim(M)-1}(R^+(M), g_0) \to KO_n = \mathbb{Z}/2$ is surjective, if $n \equiv 2 \pmod 8$, $n \geq 10$.*

These were the first examples of non-trivial higher homotopy groups of R^+.

2.10 Theorem (Crowley-S.-Steimle)**.** *If M is closed spin of dimension $d \geq 6$ and $g_0 \in R^+(M)$ then $\mathrm{ind}_{rel}\colon \pi_{n-d-1}(R^+(M), g_0) \to KO_n = \mathbb{Z}/2$ is surjective if $n \equiv 1, 2 \pmod 8$, $n > d$.*

Here, all the elements in $\pi_*(R^+(M))$ are obtained by the action of $\mathrm{Diffeo}(M)$, more specifically $\mathrm{Diffeo}(D^m, \partial)$.

2.11 Theorem. *If M is closed spin, $dim(M) \geq 6$ then M has a Riemannian metric with non-trivial harmonic spinors. The complementary set of metrics without such a spinor has non-trivial homotopy groups. (Hitchin: $-1, 0, 1 \pmod 8$, Bär $3 \pmod 4$)*

2.12 Theorem (Hanke, S., Steimle)**.** *For each k, if $d = \dim(M) \geq N(k)$ and $n + k + 1 \equiv 0 \pmod 8$, M is closed spin with $g_0 \in R^+(M)$ then*

$$\mathrm{ind}_{rel} \colon \pi_k(R^+(M), g_0) \to KO_{n+k+1} \cong \mathbb{Z} \qquad \text{has infinite image.}$$

For M a homotopy sphere, this factors through the moduli space $R^+(M)/\,\mathrm{Diffeo}_{x_0}(M)$, i.e. the homotopy classes are definitely not obtain by the action of the diffeomorphism group.

2.13 Theorem (Botvinnik, Ebert, Randall-Williams)**.** *If $\dim(M) \geq 6$ is closed spin and $g_0 \in R^+(M)$ then*

$$\mathrm{ind}_{rel} \colon \pi_k(R^+(M), g_0) \to KO_{k+\dim(M)+1}$$

surjects whenever $KO_{k+\dim(M)+1} = \mathbb{Z}/2$, and has infinite image if $KO_{k+\dim(M)+1} \cong \mathbb{Z}$.

3 Talk 2: Spin and Dirac and the K-theory philosophy on index

In the following, for easy of exposition we concentrate on even dimensional manifolds and only use complex C^*-algebras.

3.1 Definition. A C^*-algebra A is a norm-closed $*$-subalgebra of the algebra of bounded operators on a Hilbert space.

We have for a (stable) C^*-algebra A:

- $K_1(A)$ are homotopy classes of invertible elements of A.
- $K_0(A)$ are homotopy classes of projections in A.
- 6-term long exact K-theory sequence for ideal $I \subset A$:

$$\to K_0(A/I) \xrightarrow{\delta} K_1(I) \to K_1(A) \to K_1(A/I) \xrightarrow{\delta} K_0(I) \to \\ \to K_0(A) \to K_0(A/I) \to$$

3.1 Framework of index

- Using functional calculus, the Dirac operator gives the bounded operator $\chi(D)$, contained in the C^**-algebra* A of bounded operators on $L^2(S)$.

  Here $\chi\colon \mathbb{R} \to [-1,1]$ is any odd functions with $\chi(x) \xrightarrow{x\to\infty} 1$ (contractible choice).

- $\dim M$ even: $S = S^+ \oplus S^-$, $\chi(D) = \begin{pmatrix} 0 & \chi(D)^- \\ \chi(D)^+ & 0 \end{pmatrix}$.

- If M is *compact*, ellipticity implies that

  $$\chi^2(D) - 1 \in I, \text{ the ideal of compact operators on } L^2(S),$$

  so also $U\chi(D)^+$ is invertible in A/I (with any unitary $U\colon L^2(S^-) \to L^2(S^+)$: contractible choice).

- It therefore defines a "fundamental class" $[D] \in K_1(A/I)$.

3.2 Elliptic operators and index

- Analytic fact: an elliptic operator D on a *compact* manifold is *Fredholm*: it has a quasi-inverse P such that

  $$DP - 1 = Q_1; \qquad PD - q = Q_2$$

  and Q_1, Q_2 are compact operators.

- Consequence of Fredholm property: null-space of the operator and of the adjoint are finite dimensional, and then

  $$\mathrm{ind}(D) := \dim(\ker(D)) - \dim(\ker(D^*)).$$

- We apply this to the Dirac operator (strictly speaking to its restriction D^+ to positive spinors).

Apply the boundary map $\delta\colon K_1(A/I) \to K_0(I)$ of the long exact K-theory sequence $0 \to I \to A \to A/I \to 0$ to obtain

$$\mathrm{ind}(D) := \delta([D]) \in K_0(I) = \mathbb{Z}.$$

We have the celebrated

3.2 Theorem (Atiyah-Singer index theorem)**.**

$$\mathrm{ind}(D) = \hat{A}(M)$$

Here $\hat{A}(M)$ is a differential -topological invariant, given in terms of the Pontryagin classes of TM, which can be efficiently computed.

It does not depend on the metric (the Dirac operator does).

3.3 Definition. Schrödinger's local calculation relates the Dirac operator to scalar curvature: $D^2 = \nabla^*\nabla + \operatorname{scal}/4 \geq \operatorname{scal}/4$. It implies:
if $\operatorname{scal} > c > 0$ everywhere, $\operatorname{Spec}(D) \cap (-\sqrt{c}/2, \sqrt{c}/2) = \emptyset$.

Choose then $\chi = \pm 1$ on $\operatorname{Spec}(D)$, therefore $\operatorname{ch}^2(D) = 1$ and $U\chi(D)^+$ is invertible in A, representing a structure class

$$\rho(D_g) \in K_1(A),$$

mapping to $[D] \in K_1(A/I)$.

Potentially, $\rho(D_g)$ contains information about the positive scalar curvature metric g.

3.4 Theorem.

- *If M has positive scalar curvature, then* $\operatorname{ind}(D) = 0 \implies \hat{A}(M) = 0$*;*
- $\hat{A}(M) \neq 0$ *is an obstruction to positive scalar curvature!*

Example: Kummer surface. **Non-examples**: $\mathbb{C}P^{2n}$, T^n.

Proof.

$$\begin{array}{ccccc} K_1(A) & \longrightarrow & K_1(A/I) & \xrightarrow{\delta} & K_0(I) \\ \rho(D_g) & \mapsto & [D] & \mapsto & \operatorname{ind}(D) = 0 \end{array}$$

using exactness of the K-theory sequence. □

3.3 Generalized index situation

General goal: find sophisticated algebras $I \subset A$ to arrive at similar index situations. Criteria:

- index construction must be possible (operator in A, invertible modulo an ideal I)
- calculation tools for $K_*(A)$, $K_*(I)$ and the index
- positive scalar curvature must imply vanishing of index (and give structure class $\rho \in K_*(A)$)

Useful/crucial is the context of *C^*-algebras*, where *positivity implies invertibility*.

4 Talk 3 and 4: Clifford and spin geometry and a spectral view on K-theory

We want to elaborate a bit on "extra structure". We should look out for it and use it whereever possible.

4.1 Example. First basic example: imagine you had been brought up only with complex vector spaces, and each time some real vector space normally would occur you would automatically complexify it. All your matrices (even if all entries are real) you would consider as matrices over $\mathbb{C}$.

Consequence: you would not know of orientation. In fact, and orientation preserving real $n \times n$ matrix is one which has positive determinant, and it is important that this property is preserved under deformations and under conjugation.

Considered as matrices over $\mathbb{C}$, this makes no sense: positivity is not well defined for complex non-zero numbers, we can "turn" a matrix with determinant one inside $Gl_n(\mathbb{C})$ to one with determinant -1, and also conjugate to one with opposite determinant.

The definition of "orientation" of a smooth manifold depends on having a well defined orientation of invertible matrices, so also this would be lost.

On the other hand, looking at elements of $M_2(\mathbb{R})$ as real real linear self-maps of $\mathbb{R}^2 = \mathbb{C}$, it is very special if such a matrix is complex linear.

E.g.: if we have a C^1 map : $\mathbb{R}^2 \to \mathbb{R}^2$ whose differential is everywhere complex linear, this is really a holomorphic map, with its great special properties.

4.2 Definition. The Real story.

An $\mathbb{C}$ algebra A (e.g. a C^*-algebra) is Real (equipped with a Real structure) is it has a complexs conjugate algebra involution $\overline{\cdot} : AtoA$. The fixed element under this involution are the real elements.

4.3 Example. Most important example (for us): if $A_{\mathbb{R}}$ is a $\mathbb{R}$-algebra, then $A := A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a Real algebra with complex conjugation induced by complex conjugation in the second tensor factor.

Second example: if X is manifold with an involution τ, then $C_0(X;\mathbb{C})$ becomes a Real algebra with $\overline{f}(x) := \overline{f(\tau(x))}$. In fact, every commutative Real algebra is of this form. If τ is not the identity, this is in general not of the first form (of course it is if $\tau = \mathrm{id}$).

We will be able to make the Dirac operator Real, we have to make sense of the K-theory of Real C^*-algebras, on top of the grading we already saw.

4.1 Change of scalars to non-commutative C^*-algebras

- Throughout, we can replace the complex numbers by any C^*-algebra A; get algebras $C^*(M;A), D^*(M;A)$ (Mishchenko, Fomenko, Higson, Pedersen, Roe,...).
- In particular: $C^*\pi_1(M)$, a C^*-closure of the group ring $\mathbb{C}\pi_1(M)$.
- The whole story then relates to the *Baum-Connes conjecture*.
- Throughout, we can use the Dirac operator twisted with a flat bundle of Hilbert A-modules, e.g. the Mishchenko bundle. All constructions and results carry over (in our setup without too much extra work).

- For compact M, we get Rosenberg index

$$\mathrm{ind}(D) \in K_{\dim M}(C^*\pi_1 M)$$

refining $\mathrm{ind}(D) \in \mathbb{Z}$ we started with.

5 Bordism and surgery constructions

5.1 Definition. Bordism: without orientation, with orientation, with framing, with trivialization, leading to the groups/rings Ω_*^O, Ω_*^{SO}, Ω_*^{fr}, also to Ω_*^{spin}.

In this definition, a framing is an equivalence class of trivializations of the stabilized normal bundle or equivalently the stabilized tangent bundle.

5.2 Theorem. *Pontryagin-Thom construction as a tool for computing bordism groups.*

Example: $\Omega_*^{fr} = \pi_*^s(S^0)$*: framed bordism is stable homotopy.* $\Omega_*^{fr} \to \pi_*^s(S^0 = \lim_{n\to\infty} \pi_{*+n}(S^n)$*: embed the framed manifold in Euclidean space with a tubular neighborhood. Use the trivialization of the normal bundle to map the fibers of the tubular neighborhood to* $\mathbb{R}^n \subset S^n$*, and the rest to the point* $+\infty \in S^n$*. The inverse map represents the stable homotopy class by a smooth map* $f\colon S^{n+*} \to S^n$ *and takes as framed manifold the inverse image of a regular point with trivializaiton of the normal bundle from the implicit function theorem.*

Pontryagin had great hopes to have now a way to compute $\pi_*(S^0)$ *geometrically. It turns out that the other way around is true: there are tools in algebraic topology to compute homotopy groups which allow to compute many bordism groups.*

Better example: $\Omega_*^{SO} = \pi_*(MSO)$*, the homotopy groups of the Thom space for oriented vector bundles.*

One gets $\Omega_*^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^{2n}, n \in \{0,1,\dots\}]$*, a polynomial ring in the generators* $\mathbb{C}P^{2n}$*.*

There is a (somewhat complicated) criterion in terms of characteristic numbers which determines when a list of closed oriented manifolds is generating the ring Ω_*^{SO}*, and one can find these manifolds explicitly. Concretely: one has a set of generators consiting of complex projective spaces, Milnor manifolds and Dold manifolds.*

The Milnor manifolds are bundles with fiber projective spaces over projective spaces (shrinking fibers, the structure group being the isometry group). The Dold manifolds are obtained from spheres, circles and projective spaces by taking quotients by isometries and (twisted) products which also have evidently positive scalar curvature.

5.3 Definition. Surgery: cuts out an embedded $S^k \times D^{n-k}$ out of an n-dimensional manifold M and replaces this by $D^{k+1} \times S^{n-k-1}$ to obtain a manifold M'. Here, $n-k$ is called the *codimention* of the surgery.

Example: connected sum is surgery of codimension n.

Attaching $D^{k+1}\times D^{n-k}$ to $M\times[0,1]$ along the part $S^k\times D^{n-k}$ of its boundary produces a bordism between M and M'. We see that a surgery is producing a bordism.

5.4 Theorem (Gromov-Lawson,Hajduk, Chernysh,Ebert-Frenck)**.** *Starting with a Riemannian metric of positive scalar curvature on M, if M' is obtained from M be surgeries of codimension ≥ 3,*

- *then one can construct a metric of positive scalar curvature on M' (Gromov-Lawson).*
- *One can even construct a Riemannian metric of positive scalar curvature on the canonical bordism between M and M' which is of product form near the boundary*
- *the construction is canonical enough that this works in continuous families (Chernysh)*
- *mistakes in the original arguments and details can all be corrected (several authors, a nice reference is Ebert-Frenck)*
- *Indeed, one can deform within the space of metrics of positive scalar curvature to a metric which has standard torpedo form near the surgery locus: in $S^k \times D^{n-k}$ think of D^{n-k} as a cap over S^{n-k-1} and eqip it with a corresponding cap-torpedo metric (a suitable warped product over S^{n-k-1}. Then the inclusion of psc metrics which are product of (any) metric on S^k with torpedo on D^{n-k} into all psc metrics is a weak homotopy equivalence. One can then cut out and glue back with psc.*

Proof. We don't give the technical proof. The most detailed reference (where also all the mistakes of previous articles should be addressed) is [1].

Mnemo: one needs some independent source of psc where one does the surgery. This is provided by the soul S^{n-k-1} so that we need $n-k-1\ge 2$ to have positive curvature there. The details require care (20 pages in Ebert-Frenck). □

We just saw that surgeries give rise to bordisms. A main point of Morse theory is: all bordisms are obtained by a sequence of surgeries.

5.5 Definition. A Morse function on a smooth manifold W (it is allowed that W is a bordism between a manifold $M_0 = \partial W_0$ and $M_1 = \partial W_1$ is a smooth function $f\colon W \to \mathbb{R}$ which is constant on M_0, constant on M_1 and has only non-degenerate critical points (critical points are points with $df = 0$. They are non-degenerate if the Hessian, say computed in local coordinates, is non-degenerate i.e. a symmetric matrix without kernel). Moreover, all critical points are assumed to lie in the interior of W.

The (Morse) index of a critical point is the number of negative eigenvalues of the Hessian (i.e. 0 at a minimum and $\dim(W)$ at a maximum)

5.6 Example. On the bordism of a surgery we can define a Morse function which grows from 0 on the original boundary to 1 on the result of surgery boundary and which has precisely one critical point, which is non-degenerate and of index $k+1$.

In local coordinates on $D^{k+1} \times D^{n-k}$ the origin is the only critical point and near it the function has the form $-(x_0^2 + \dots x_k^2) + (x_{k+1}^2 + \dots x_n^2)$.

5.7 Theorem (Morse lemma)**.** *One always finds local coordinates around a non-degenerate critical point such that around it a Morse function has the form of our example (if the index is $k+1$).*

In particular, passing from the level set $c-\epsilon$ to the level set $c+\epsilon$ for a critical point of index $k+1$ with value c of the Morse function (and no other critical point in the inverse image of $[c-\epsilon, c+\epsilon]$) always is a bordism of a surgery of codimension $n-k$.

Proof. This is the first cornerstone of Morse theory, covered in all its treatments. Compare e.g. [4, Theorem 2.16] for the first and [4, Theorem 3.1, Theorem 3.2] for the second assertion. Alternatively, compare [6, Section 3].

In the neighborhood of the critical point this is just the picture. Outside, one uses the flow of the vector field grad f which provides the required diffeomorphism to the product with the interval, as the vector field has no zeros. □

5.8 Lemma. *The set of Morse functions on W is dense in the space of all smooth functions (with given constant values on M_0 and M_1). We can also achieve that critical values always have different values. Upto homotopy, we can also assume that the index is non-decreasing in the value. Alternatively, one can make the Morse function self-indexing (value=index).*

As a consequence: every bordism is a sequence of surgeries (exercise: why).

Proof. This is a rather standard result of Morse theory and of course one of its backbones: we can actually always use them and the theory coming with them!

Being standard, the result is explained in all texts on Morse theory, e.g. for the basics in [4, Section 2] and the more refined statements in [4, Theorem 3.27] (with preparation lemmas).

Alternatively, compare [6, Theorem 2.5] for the existence and [6, Section 4] for the rearrangement of the indices. □

5.9 Question. How does a surgery change the manifold, in particular its homotopy type, its homotopy groups?

Give here a very quick reminder of homotopy groups, if necessary.

Answer 1: it is almost impossible to describe the homotopy type completely, and to compute all homotopy groups. Therefore, it is also not possible to describe the full effect of surgeries.

However, if one does surgeries of small dimension/large codimension, one as a very definite effect: one kills or creates homotopy classes.

5.10 Proposition. *Let M be an n-dimensional manifold and let $S^k \subset M$ represent $\alpha \in \pi_k(M)$. The inclusion induced map $\pi_j(M \setminus S^k) \to \pi_j(M)$ is an isomorphism if $j + k + 1 < n$ and a surjection for $j + k < n$.*

In the same way, the inclusion map $M \setminus S^k \times D^{n-k} \to M'$ is an iso on π_j for $j + n - k - 1 < n - 1$, i.e. $j < k$ and an epi for $j = k$. Here, it maps α (and therefore also the subgroup generated by α and even the $\mathbb{Z}\pi_1$-submodule generated by α) to zero.

If the inclusion map α is null-homotopic and $2k + 1 < n$, the map $D^{k+1} \times \{*\} \hookrightarrow M'$ can be extended to a map from S^{k+1} which generates an additional free factor of $\pi_{k+1}(M')$ compared to $\pi_{k+1}(M)$.

Proof. This is a result of homotopy invariance and general position. For the precise computation of the changed group, it is van Kampen for $k = 1$ and the pair sequence in homotopy together with a relative Hurewicz isomorphism for $k > 1$. □

Slogan: surgeries (of dimension lower than the middle dimension) have the potential to adjust homotopy groups.

We want to apply this to a bordism W from M_0 to M_1 and want to make the inclusion map $M_0 \to W$ an isomorphism on the first homotopy groups.

5.11 Question. Surgery always comes with an associated bordism (where we see a Morse function with a critical point which is passed). We just discussed how the homotopy type of the level sets changes. A related question: how do the sublevel sets change.

5.12 Proposition. *Given a Morse function $f\colon W^n \to \mathbb{R}$ such that $f^{-1}(c)$ contains a single critical point of index k, the homotopy type of $f^{-1}(c + \epsilon)$ is the one of $f^{-1}(c - \epsilon)$ with a single cell (here called handle) of dimension k, with attaching map the inclusion map $S^k \to f^{-1}(c - \epsilon)$.*

Proof. Deform the $D^k \times D^{n-k}$ which is attached along $S^k \times D^{n-k}$ to the central D^k, using that D^{n-k} is contractible. □

5.13 Corollary. *A smooth bordism W from M_0 to M_1 is homotopy equivalent to a relative CW-complex relative M_0, with one cell of dimension k attached for each critical point of index k of a Morse function for W from M_0 to M_1.*

5.14 Proposition. *A curious, but important observation: if $f\colon W \to \mathbb{R}$ is a Morse function for W from M_0 to M_1 then $-f$ is one for W as a bordism from M_1 to M_0. The critical points of f are precisely the critical points of $-f$, but the index changes from k to $\dim(W) - k$.*

So: avoiding surgeries of small codimension to go from M_0 to M_1 corresponds the task to avoid surgeries of small dimension when going from M_1 to M_0. This is used a lot in the business.

We can then apply "critical point cancellation":

5.15 Theorem. *If in a bordism W (of dimension $n+1$) from M_0 to M_1 the inclusion map $M_0 \hookrightarrow W$ induces an isomorphism on π_j for $j < k < n-2$ and an epimorphism on π_k (the topologist then say that the inclusion map is a k-equivalence. equivalently, the relative homotopy groups vanish in degree $j \le k$) then we can change the Morse functions for W so that there are no critical points of index $\le k$ by "critical point cancellation and trading" (or "handle cancellation and trading").*

Remark: along the way, one might have to create new $k+1$ and $k+2$-handles.

The strategy is: a given k-handle can be cancelled against a conveniently located $(k+1)$-handle. So: simply add $k+1$-handles which can be used to cancel against the k-handles. Before doing this, add also $k+2$-handles which would cancel with the new $k+1$-handles such that, with the new $k+1$ and $k+2$-handles, we didn't change the diffeomorphism type (alternatively, we changed the Morse function by creating new critical points of index $k+1$ to cancel our critical points of index k, and with new critical points of index $k+2$ which cancel the new ones of index $k+1$.

Then do the k versus (new) $k+1$ cancellation.

Proof. This is a rather powerful theorem, and it should not be a surprise that its proof is quite involved.

We need the basic result on handle cancellation as proved e.g. in [4, Theorem 3.28, Theorem 3.34] or [6, Theorem 5.4]. Note that Milnor states that the proof of [6, Theorem 5.4] is "quite formidable".

However, the handle cancellation theorem requires a pair of handles (critical points) which are in a rather special configuration. That these exist is hard to check, or rather not always true when one would need it.

First, one shows that one indeed can always us the handle cancellation result to get rid of handles of index 0, compare [6, Theorem 8.1] or [3, p. 35]. Next, one shows that one can trade 1-handles for 3-handles using calculations with the (non-abelian) fundamental group, compare [3, p. 35-36] and finally, with considerations of homology groups that one can trade q-handles for $q+2$-handles for $q = 2, \dots, k$. This requires a bit more work, compare [3, p. 36 ff]. Milnor proceeds slightly differentily, but only for simply connected bordisms (and boundaries): using [6, Theorem 7.6] he changes with a lot of effort the Morse function such that the handles represent a preferred basis of the relevant relative homology and then directly carries out the desired cancellation in [6, Theorem 7.8], without introducing new handles of higher degree. This works for $n \ge 5$ and the index of the handle to be cancelled between 2 (cancelled against one of index 3) and $n-2$ (cancelled against one of index $n-1$).

□

5.16 Theorem. *In the following statements,* $\dim(M_0) \ge 5$.

(1) Given a simply connected spin manifold M_0 and a spin bordism to M_1, if $\dim(M_0) \ge 5$ *we can choose the bordism W so that the inclusion $M_0 \to W$ is 2-connected.*

(2) *Given a simply connected oriented manifold M_0 which is not spin and an oriented bordism to M_1, then we can find such a bordism W such that the inclusion $M_0 \hookrightarrow W$ is 2-connected.*

(3) *Given a connected $\dim(M) \geq 5$ dimensional spin manifold M_0 with fundamental group π and a $B\pi$ spin-bordism to M_1, then there is such a bordism W such that the inclusion $M_0 \to W$ is 2-connected.*

(4) *correspondingly for an oriented manifolds which is non-spin.*

Proof. We want to use surgeries to modify the bordism. For this, two important problems have to be addressed:

- we can only do finitely many surgeries, so we have to show that after finitely many steps we are done: we need to check that the group we want to kill is finitely generated (in the appropriate category)
- We can only do surgery on embedded spheres with trivial normal bundle, so we need to make sure that these two conditions can be satisfied.

  This second condition is the reason for the distinction between the spin case and the non-spin case!

Let us start with the classical simply connected case. Recall that a manifold is is orientable if and only if the normal bundle of every embedded circle is trivial. Moreover, recall that an oriented manifold is spin if and only if the normal bundle of every embedded oriented 2-dimensional submanifold is trivial.

Start with a bordism W' from M_0 to M_1. As W' has finitely many components, we can perform finitely many connected sum operations in the interior to make it connected. The resulting connected compact bordism has (as every compact manifold) finitely generated fundamental group. Because the dimension of the bordism is big enough, we can realize generators by disjoint embedded circles, because of orientability, they have trivial normal bundle. So we can perform the surgeries to kill the fundamental group of W

Next, by the Hurewicz theorem for the simply connected compact W, $\pi_2(W) = H_2(W)$ is a finitely generated abelian group. Because of the dimension and the spin condition, we can realize a finite set of generators by embedded 2-spheres with trivial normal bundle. Surgeries change W such that also $\pi_2(W) = 0$ and therefore $M_0 \hookrightarrow W$ is 2-connected.

The relevant condition of finite generation are discussed in a pedestrian way (and in the greater generality of non-simply connected spaces) in [7, Section 3]

Let us next look at the case where M_0 (and therefore also W) are non-spin with M_0 simply connected. As before, we can arrange W simply connected, as well. Because W is non-spin there then will exist an embedded 2-sphere whose normal bundle is non-trivial and indeed, it is impossible to perform surgeries such that $\pi_2(W) = 0$.

Let $a \in \pi_2(M_0) = H_2(M_0)$ be represented by a sphere with non-trivial normal bundle. Consider the group $\pi_2(W)/\langle a \rangle$. As quotient of a finitely generated abelian group this is finitely generated. Choose finitely many generators and represent them by embedded spheres in the interior of W. If w_2 of the normal bundle is non-zero (the normal bundle is not trivial), perform a connected sum with a sphere representing a. Details of this can be found in [2, Proof of Theorem C].

More details (and greater generality) is also contained in the treatment of [1, Proposition 6.3] or in [8, Lemma 5.6]. □

As a corollary, we obtain the classification of simply connected oriented non-spin manifolds which admit positive scalar curvature of Gromov and Lawson [2, Theorem C].

5.17 Corollary. *Let M a a compact simply-connected manifold without boundary. If M is oriented and non-spin and of dimension $n \geq 5$, then M admits a metric of positive scalar curvature.*

6 Stolz positive scalar curvature exact sequence

General feature: index is a cobordism invariant. Topologist's approach: organize the existence and classification problem in appropriate LES of groups and then start computations.

6.1 Definition. (1) (spin) bordism groups $\Omega_n^{spin}(X)$ for a reference space X:

Cycles are closed n-dimensional spin manifolds M^n of dimension n with a reference map $f\colon M \to X$. The equivalence relation is "bordism with an extension of the reference maps to X": a spin compact manifold W with a map $F\colon W \to X$ with $\partial W = M_1 \amalg (-M_2)$ defines a spin bordism between $(M_1, f_1 := F|_{M_1}\colon M_1 \to X)$ and $(M_2, f_2 := F|_{M_2}\colon M_2 \to X)$, and then (M_1, f_1) and (M_2, f_2) define the same element of $\Omega_n^{spin}(X)$. Disjoint union defines a group structure, postcomposition with a map $u\colon X \to Y$ makes this a functor from the category of topological spaces to the category of abelian groups. Cartesian product defines a Ω_*^{spin}-module structure.

Fact: this actually defines a generalized homoloy theory (taken all the Ω_*^{spin} together). This way, there are many tools for computation. For example, we have general suspension-type isomorphisms for the product with a circle:

6.2 Example. $\Omega_*^{spin}(X \times S^1) \cong \Omega_*^{spin}(X) \oplus \Omega_{*-1}^{spin}(X)$. For the first summand, the isomorphism sends uses the functoriality along the inclusion $X = X \times \{1\} \hookrightarrow X \times S^1$. For the second summand, $f\colon M^{n-1} \to X$ is mapped to $f \times \mathrm{id}\colon M \times S^1 \to X \times S^1$ with the spin structure on S^1 which is the boundary of D^2.

6.3 Remark. The corresponding works for oriented bordism or other bordism theories.

(2) Next, we define a group $\mathrm{Pos}_n^{spin}(X)$ which encodes the geometric data of positive scalar curvature. Its cycles are like the cycles for $\Omega_n^{spin}(X)$,

equipped in addition with a fixed metric g of positive scalar curvature on M. Similarly, the bordism relation requires bordisms with a metric of positive scalar curvature extended over W (which is of product form near the boundary and restricts to the given psc metrics on the boundary).

Note that this is again a group by disjoint union and covariantly functorial in X.

(3) There is a canonical "forget structure" natural transformation $\mathrm{Pos}_*^{spin}(X) \to \Omega_*^{spin}(X)$ which simply forgets the metric.

(4) In such a situation, it is possible in rather large generality to define a relative group which interpolates between the other two and finally leads to a long exact sequence. Here, these are the R-groups of Stolz, with cycles for $R_{n+1}^{spin}(X)$ given by a compact spin manifold W with boundary and with reference map $f\colon W \to X$ together with a positive scalar curvature metric on ∂W. The equivalence relation is bordism of such manifolds (which involves to deal with bordism of manifolds with boundary): we start with a psc bordism X between ∂W_1 and ∂W_2. We can then glue W_1 to X along ∂W_1 and further glue W_2 to this along ∂W_2. The result is a closed spin manifold, and we ask for a bordism of this. One can also introduce corners to realize this as bordism of manifolds with boundary (with psc on the boundary strata).

Again, this is a group by disjoint union and functorial in X.

We obtain canonical natural transformation $R_{n+1}(X) \to \mathrm{Pos}_n(X)$ just taking the boundary, and a canonical transformation $\Omega_n^{spin}(X) \to R_n(X)$ by realizing that a closed manifold is also a manifold with boundary, which is empty (and therefore also carries tautoligically a metric of positive scalar curvature).

6.4 Proposition. *The groups and transformations just defined form functorial long exact sequences of abelian groups*

$$\to R_{n+1}(X) \to \mathrm{Pos}_n(X) \to \Omega_n^{spin}(X) \to R_n(X) \to$$

Recall that a sequence of abelian groups is exact if at each place the kernel of the outgoing map is precisely the image of the incoming one.

Proof. Many steps are tautological and left to you. Given $(M, f\colon M \to X, g) \in \mathrm{Pos}_n(X)$, its image in $R_n(X)$ is zero via the bordism from $\emptyset = \partial M$ to $\emptyset$ given by $(-M, g)$, and then the zero bordism $M \times [0,1]$ of the union of M and this $-M$ (glued along $\emptyset$, i.e. disjoint union).

As a second example, if $(W, f, g) \in R_{n+1}(X)$ is mapped to zero in $\mathrm{Pos}_n(X)$, i.e. if there is a psc bordism V of ∂W, we can glue W and $-V$ along ∂W to obtain the closed spin manifold B (with reference map to X. Moreover, the product $X \times [0,1]$ provides a bordism in $R_{n+1}(X)$ between (W, g) and X (considered as manifold with empty boundary, using V with its psc metric as the boundary stratum. □

6.5 Remark. We are certainly very intersted in the groups in the Stolz sequence: they encode a lot of information about existence (and classification) of psc metrics. In particular, $\mathrm{Pos}_n(X)$ is relevant.

Starting with a closed spin manifold M, there are two particularly natural choices for X: $X = M$ with most canonical cycles of the form $(\mathrm{id}\colon M \to M, g)$: this concentrates on the psc metric g (provided one exists!)

The second natural choice is $X = B\pi_1(M)$ with $f\colon M \to B\pi_1(M)$ inducing an isomorphism of fundamental groups (this map is unique upto homotopy, therefore also upto bordism). This concentrates on all manifolds with a given fundamental group and is universal for those.

Good news. The group $R_n(X)$ depends only on the fundamental group of X (compare the following theorem).

Bad news. Nonetheless there is no single interesting example where we can compute it. An alternative, algebraic way to do this would be desirable (and indeed, this is what happens in parallel situations, e.g. in surgery theory when trying to classify manifolds upto diffeomorphism).

The following theorem is essentially due to Stolz and implicit in [8, Section 5].

6.6 Theorem. *Let $f\colon X \to Y$ be a 2-connected map (iso on π_0, π_1 and surjective on π_2). Then the induced map $R_n(X) \to R_n(Y)$ is an isomorphism.*

Proof. It suffices to do this for connected spaces and with $Y = B\pi$, $\pi = \pi_1(X)$ (the general case follows by considering one component at a time, and by comparing both X and Y to $B\pi$).

Let us show surjectivity (which follows also directly from [8, Proposition 5.8]). We hope to give a proof which is more transparent and direct.

Given a cycle $(W, f\colon W \to B\pi, g)$ we need to change it to an equivalent cycle such that the map factors over X. Here, we can use the following principles:

(1) restricted to the 2-skeleton, the map $X \to B\pi$ has a split $B\pi^{(2)} \to X$, therefore it suffices to factor the map over $B\pi^{(2)}$.

(2) A map from a space homotopy equialent to a 2-dimensional CW-complex to $B\pi$ factors (upto homotopy) over the inclusion $B\pi^{(2)} \hookrightarrow B\pi$ of the 2-skeleton (and homotopies are rather trivial bordisms).

Choose now a Morse function for W considered as bordism from the empty set to ∂W. and order the critical points according to their index.

Decompose W as bordism W_1 consisting only of handles of dimension $0, 1, 2$ from the empty set to M_1 and W_2 from M_2 to ∂W. By this decomposition, there are no critical points of low index from M_2 to ∂W or, dually, there are no surgeries of codimesion < 3 in W_2 as bordism from ∂W to M_1. Consequently, we can use the Gromov-Lawson-Hajac surgery theorem to extend the psc metric from ∂W to W_2, ending in one such metric g_1 on M_1.

Now, this structure provides a bordism between (W, g, f) and $(W_1, g_1, f|_{W_1})$ (the bordism is again given by $W \times [0, 1]$).

Finally, observe that W_1 consists only of 0, 1, and 2-handles, i.e. is homotopy equivalent to a 2-dimensional CW-complex, so that $f|_{W_2}$ automatically factors over $B\pi^{(2)}$.

Similarly, to prove injectivity, start with cycle $(W, g, f\colon W \to X)$ with a zero bordism B over $B\pi$. We can think of the resulting object (after pulling a tubular neighborhood of ∂W in the bordism out) as follows: B is a manifold with corners. It has three boundary parts: W at the bottom, then $\partial W \times [0,1]$ vertically, then W_2 on top. There is a map to $B\pi$, which factors through X (or equivalently $B\pi^{(2)}$) on the bottom and on the vertical part $M \times [0,1]$ of the boundary.

Because the vertical boundary is constant, we can obtain B by attaching handles in the interior starting from W. As before, we can decompose into B_1 which only contains handles of dimension 0, 1, and 2 and which is a bordism from W to W_1 (with constant vertical boundary $M \times [0, 1/2]$) and a second bordism B_2 from W_1 to W_2. On the latter, we can extend the psc metric on W_2 over all of B_2 (staying constant on the constant vertical boundary $M \times [1/2, 1]$, as the construction is local). This way, we transport the psc metric to W_1.

On the other hand, the map to $B\pi$ restricts to on B_1 to a map which factors over $B\pi^{(2)}$ and therefore over X. This way, B_1 provides the required 0-bordism of (W, f, g) already for $R_n(X)$. ☐

6.7 Remark. Note that this construction works in all dimensions and does not require any handle cancellations. This is not true for the next result of Stephan Stolz [8, Theorem 6.8].

6.8 Theorem. *The group $R_{n+1}(B\pi)$ acts freely and transitively on the set of concordance classes of metrics of positive scalar curvature on M if M is a closed connected spin manifold of dimension n with fundamental group π and if $n \geq 5$ (and if M admits a psc metric).*

Proof. For more details on the proof compare [8, Theorem 6.8].

We start with the disjoint union of $M \times [0,1]$ and a cycle W of $R_{n+1}(M)$. This is a bordism from $M \amalg \partial W$ to M, and we have a given psc metric on $M \amalg \partial W$. Now we do surgery on the interior to make the inclusion map of M into this bordism a 2-connected map (involving connected sum,...). This is possible because of the assumptions, but requires extra care (finite generation of the relevant groups to kill, spin structure to have trivial normal bundles, high enough dimension to represent by embeddings and be able to do the required handle trading). Then the Gromov-Lawson surgery theorem allows to transport the metric. Further work with bordisms shows that this is well defined when mapping to concordance classes of psc metrics.

Transitivy follows by acting with $W = M \times [0,1]$ with metric $g \amalg g_2$ on $\partial W = M \amalg -M$. ☐

Although unfortunately, until now we have not been able to completely compute $R_n(\pi)$ in any relevant case, we are able to get quite a bit of information by explicit constructions and by detection, obtaining lower bounds on the complexity of objects like $R_n(\pi)$, $\mathrm{Pos}_n^{spin}(M)$, $\mathrm{Pos}_n^{\mathrm{spin}}(B\pi)$ or also the set of components, concordance classes or bordism classes of psc metrics on a given manifold M (admitting at least one such metric). A host of results in this direction can be found for example in [7] and in [9].

7 Not covered in course: Non-negative versus positive scalar curvature

The Ricci flow is an extremely powerful tool to study positive curvature.

This is the following non-linear evolution equation for the metric:

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}_{g(t)}\,.$$

This is one of the hot tools in geometric analysis. It plays a pivotal role in Perelman's proof of the Poincaré conjecture (proposed by Hamiton).

Fundamental properties:

(1) a priori estimates are possible

(2) in particular, one has short-time existence for solutions of the initial value problem with arbitrary initial values on a compact manifold

(3) Ricci flow preserves positivity in several ways, in particular as follows:

If M is a closed manifold and g_0 is a metric on M with non-negative scalar curvature, consider the Ricci flow $g(t)$ with $g(0) = g_0$. Suppose that the flow exists for all $t \in [0, T]$. Then $g(t)$ has non-negative scalar curvature for all $t \in [0, T]$. Moreover, $g(t)$ has positive scalar curvature for all $t \in (0, T]$ unless g_0 is Ricci flat, in which case $g(t) = g_0$ for all $t \in [0, T]$.

(4) blowup in finite time does occur: the round sphere shrinks to a point. Therefore, rescaling is applied. Even after rescaling, in general blowup does occur, and necessarily the (scalar) curvature blows up. Understanding the blow-up regions is key to the topological use of the Ricci flow.

References

[1] Johannes Ebert and Georg Frenck, *The Gromov-Lawson-Chernysh surgery theorem*, Bol. Soc. Mat. Mex. (3) **27** (2021), no. 2, Paper No. 37, 43, DOI 10.1007/s40590-021-00310-w. MR4236952 ↑5, 5

[2] Mikhael Gromov and H. Blaine Lawson Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), no. 3, 423–434, DOI 10.2307/1971103. MR577131 ↑(1), 5

[3] Michel A. Kervaire, *Le théorème de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1965), 31–42, DOI 10.1007/BF02564363 (French). MR189048 ↑5

[4] Yukio Matsumoto, *An introduction to Morse theory*, Translations of Mathematical Monographs, vol. 208, American Mathematical Society, Providence, RI, 2002. Translated from the 1997 Japanese original by Kiki Hudson and Masahico Saito; Iwanami Series in Modern Mathematics. MR1873233 ↑5, 5, 5

[5] J. Milnor, *Morse theory*, Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963. Based on lecture notes by M. Spivak and R. Wells. MR0163331 ↑

[6] John Milnor, *Lectures on the h-cobordism theorem*, Princeton University Press, Princeton, N.J., 1965. Notes by L. Siebenmann and J. Sondow. MR0190942 ↑5, 5, 5

[7] Thomas Schick and Vito Felice Zenobi, *Positive scalar curvature due to the cokernel of the classifying map*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), Paper No. 129, 12, DOI 10.3842/SIGMA.2020.129. MR4184617 ↑5, 6

[8] Stephan Stolz, *Concordance classes of positive scalar curvature metrics*, 1998. http://www3.nd.edu/ stolz/concordance.ps. ↑5, 6, 6, 6.7, 6

[9] Zhizhang Xie, Guoliang Yu, and Rudolf Zeidler, *On the range of the relative higher index and the higher rho-invariant for positive scalar curvature*, Adv. Math. **390** (2021), Paper No. 107897, 24, DOI 10.1016/j.aim.2021.107897. MR4292958 ↑6