Transverse Index Theory and Hopf Algebras

Henri Moscovici

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## Classical geometries

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# Geometries modulo primitive Cartan-Lie pseudogroups and the corresponding Hopf algebras

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5. Volume preserving up to constant
6. Symplectic form preserving up to constant
7. Complex analytic version of the above
Q [H. Hopf] Is there a completely integrable plane field on $S^3$?

A [G. Reeb, 1948] $S^3 = D^2 \times S^1 \bigsqcup S^1 \times D^2 / \sim$

Q [A. Haefliger] Is any $E \subset TV$ homotopic to an integrable one?

A [R. Bott, 1970] Cohomological obstructions:

$$\text{Pont}^{>2n}(TV/E) = 0, \quad n = \text{rank}(TV/E).$$
The vanishing theorem is obtained by applying the Chern-Weil theory to $E$-flat connections.

This vanishing gives rise by transgression to new, $\mathbb{R}$-valued and continuously varying, characteristic classes.

The Lie algebra $\mathfrak{a}_n$ of formal vector fields on $\mathbb{R}^n$ plays the role of the classifying object, and its Gelfand-Fuks cohomology gives universal transverse characteristic classes for foliations.

In the dual, $K$-homological context, the universal object turned out to be a Hopf algebra $\mathcal{H}_n$, canonically associated to the group $\text{Diff} \mathbb{R}^n$, and its Hopf cyclic cohomology delivers the universal transverse characteristic classes.
Holonomy gives rise to a pseudogroup on a complete transversal, or equivalently to a transverse étale groupoid.

Thus, while ordinary, locally compact, topological spaces $X$ can be equivalently described in terms of their function algebras $C_0(X)$, one needs to allow noncommuting coordinates to adequately describe ‘spaces of leaves’, and replace the notion of isomorphism by that of Morita equivalence.

For instance, $C(S^1 \rtimes \Gamma) \equiv C(S^1) \rtimes \Gamma$, captures the transverse space to a codimension 1 foliation ($\Gamma = \mathbb{Z}$ for the Kronecker foliation).

If $\Gamma$ acts properly, e.g. $\Gamma = \mathbb{Z}/N\mathbb{Z}$, then $C(S^1 \rtimes \Gamma)$ and $C(S^1/\Gamma)$ are Morita equivalent; but e.g. for $\Gamma = \text{PSL}(2, \mathbb{R})$ the quotient $S^1/\Gamma$ is practically meaningless.
Transverse $K$-homology class [A. Connes & H.M., 1995]

- Hypoelliptic signature operator: $M^n = \text{oriented smooth manifold}$, 
  \[ \pi : PM = F^+ M / SO(n) \to M, \quad \mathcal{V} = \text{Ker } \pi_* , \quad \mathcal{N} = T(PM) / \mathcal{V} \]
  \[ Q := (d_\mathcal{V}^* d_\mathcal{V} - d_\mathcal{V} d_\mathcal{V}^*) \oplus \gamma_\mathcal{V} (d_H + d_H^*), \quad H = \text{connection}. \]

- Defines a spectral triple $(\mathcal{A}_\Gamma, \mathcal{H}, D)$, for any $\Gamma \subset \text{Diff}^+(M)$, where 
  \[ \mathcal{A}_\Gamma = C_c^\infty(PM) \rtimes \Gamma, \quad \mathcal{H} = L^2(\wedge \mathcal{V}^* \otimes \wedge \mathcal{N}^*, \text{vol}), \]
  and with $D$ determined by funct. eq. $Q = D|D|$; specifically, 
  \[ D = D^*, \quad [D, a] := D a - a D \quad \text{bounded} \quad \forall a \in \mathcal{A}_\Gamma, \]
  \[ f(D + i)^{-1} \in \mathcal{L}^p(\mathcal{H}), \quad p = v + 2n, \quad \forall f \in C_c^\infty(PM). \]

- Zeta functions: $\forall P \in \Psi DO'$, the function $z \mapsto \text{Tr}(P|D|^{-z})$ has meromorphic continuation with simple poles and 
  \[ \int P := \text{Res}_{z=0} \text{Tr}(P|D|^{-z}) = \frac{1}{(2\pi)^{v+n}} \int_{S' T^* M} \sigma'_{-v-2n}(P) \]

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Theorem (AC & HM, 1995)

Assume \((\mathcal{A}, \mathfrak{H}, D) = (\text{odd})\) spectral cycle, such that \(\exists\) residues

\[
\int T := \text{Res}_{s=0} \text{Tr}(T|D|^{-2s}), \quad T \in \Psi\{\mathcal{A}, [D, \mathcal{A}], |D|^{-z}; z \in \mathbb{C}\}.
\]

1. \([(\varphi_n)_{n=1, 3, \ldots}] \text{ is a cocycle in the } (b, B)-\text{bicomplex of } \mathcal{A},

\[
\varphi_n(a^0, \ldots, a^n) = \sum_k c_{n,k} \int a^0 [D, a^1]^{(k_1)} \cdots [D, a^n]^{(k_n)} |D|^{-n-2|k|}
\]

\[
\nabla(T) = [D^2, a], \quad T^{(k)} = \nabla^k(T), \quad |k| = k_1 + \ldots + k_n,
\]

\[
c_{n,k} = \frac{(-1)^{|k|} \Gamma \left( |k| + \frac{n}{2} \right)}{k_1! \cdots k_n!(k_1 + 1) \ldots (k_1 + \cdots + k_n + n)}.
\]

2. \([(\varphi_n)_{n=1, 3, \ldots}] = ch^*(\mathfrak{H}, F) \in HC^*(\mathcal{A}).\]
The zeta functions associated to the Dirac spectral triple 
\( (C^\infty(M^m), L^2(S), \mathcal{D}) \) are meromorphic with simple poles;

\[ \int P \simeq \int_{S^*_M} \sigma_{-n}(P). \quad \forall P \in \Psi DO^\infty(M^n); \]

(Guillemin-Wodzicki residue)

\[ \int f^0[\mathcal{D}, f^1]^{(k_1)} \ldots [\mathcal{D}, f^n]^{(k_n)} |\mathcal{D}|^{-(n+2|k|)} = 0, \quad \text{if} \quad |k| > 0; \]

\[ \int f^0[\mathcal{D}, f^1] \ldots [\mathcal{D}, f^n] |\mathcal{D}|^{-n} = c_n \int_M \hat{A}(R) \wedge f^0 \, df^1 \wedge \ldots \wedge df^n; \]

under the Connes isomorphism \( HP^*(C^\infty(M^m)) \cong H^*_dR(M, \mathbb{C}), \)

\[ ch^*(\mathcal{H}, \mathcal{D}) \equiv [(\varphi_n)] \cong [\hat{A}(R)] \equiv \text{Chern}_*(\mathcal{D}). \]
Transverse Index Theorem \((\text{mod } \text{Diff})\)

There are canonical constructions for the following entities:

1. a Hopf algebra \(\mathcal{H}_n\) associated to \(\text{Diff}(\mathbb{R}^n)\), with modular character \(\delta\), and with \((\delta, 1)\) modular pair in involution;

2. a co-cyclic structure for any Hopf algebra \(\mathcal{H}\) endowed with a modular pair in involution \((\delta, \sigma)\);

3. an isomorphism \(\kappa_n^*\) between the Gelfand-Fuks cohomology \(H_{\text{cont}}^*(\mathfrak{a}_n)\), resp. \(H_{\text{cont}}^*(\mathfrak{a}_n, \text{SO}_n)\), and \(H_{\text{P}}^*(\mathcal{H}_n; \mathbb{C}_\delta)\), resp. \(H_{\text{P}}^*(\mathcal{H}_n, \text{SO}_n; \mathbb{C}_\delta)\);

4. an action of \(\mathcal{H}_n\) on \(A_\Gamma(\mathbb{P}\mathbb{R}^n)\), inducing a characteristic map \(\chi_\Gamma^*: H_{\text{P}}^*(\mathcal{H}_n, \text{SO}_n; \mathbb{C}_\delta) \to H_{\text{P}}^*(1)(A_\Gamma(\mathbb{P}\mathbb{R}^n)) \cong H_*(\mathbb{P}\mathbb{R}^n \times_\Gamma \mathcal{E}_\Gamma)\);

5. a cohomology class \(L_n \in H_{\text{cont}}^*(\mathfrak{a}_n, \text{SO}_n)\), such that \(\chi_{\text{cont}}^*(A_\Gamma(P\mathbb{R}^n), J(P\mathbb{R}^n), D)(1) = (\chi_\Gamma^* \circ \kappa_n^*)(L_n)\).
Example: the Hopf algebra $\mathcal{H}_1$

- **As algebra:** generated by $\{X, Y, \delta_1, \delta_2, \ldots\}$ subject to

  \[
  [Y, X] = X, \quad [Y, \delta_k] = k\delta_k, \quad [X, \delta_k] = \delta_{k+1}, \quad [\delta_k, \delta_\ell] = 0.
  \]

- **As coalgebra:**
  \[
  \Delta(Y) = Y \otimes 1 + 1 \otimes Y,
  \]
  \[
  \Delta(X) = X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y,
  \]
  \[
  \Delta(\delta_1) = \delta_1 \otimes 1 + 1 \otimes \delta_1,
  \]
  \[
  \Delta(\delta_2) = \delta_2 \otimes 1 + \delta_1 \otimes \delta_1 + 1 \otimes \delta_2,
  \]
  \[
  \Delta(\delta_3) = \delta_3 \otimes 1 + \delta_2 \otimes \delta_1 + 3\delta_1 \otimes \delta_2 + \delta_1^2 \otimes \delta_1 + 1 \otimes \delta_3, \ldots
  \]

- **Action** on $C^\infty(F^+ S^1) \rtimes \Gamma$:
  \[
  \varphi(x, y) = (\varphi(x), \varphi'(x) y),
  \]
  \[
  Y(fU^*_\varphi) = y \frac{\partial f}{\partial y} U^*_\varphi, \quad X(fU^*_\varphi) = y \frac{\partial f}{\partial x} U^*_\varphi,
  \]
  \[
  \delta_n(fU^*_\varphi) = y^n \frac{d^n}{dx^n} \left( \log \frac{d\varphi}{dx} \right) fU^*_\varphi.
  \]
**Modular pair** $= \delta \in \mathcal{H}^*$ character, $\sigma \in \mathcal{H}$ group-like element, $\varepsilon(\sigma) = 1$, $\delta(\sigma) = 1$, such that $\tilde{S}^2 = \text{Ad}(\sigma)$, where $\tilde{S}(h) = \sum_{(h)} \delta(h(1)) S(h(2))$, $\forall h \in \mathcal{H}$.

**Cyclic object** $\mathcal{H}^\mathbb{H}_{\text{Hopf}} = \{ C^q_{\text{Hopf}}(\mathcal{H}) \}_{q \geq 0}$

\[
\begin{align*}
\partial_0(h^1 \otimes \ldots \otimes h^n) &= 1 \otimes h^1 \otimes \ldots \otimes h^n, \\
\partial_j(h^1 \otimes \ldots \otimes h^n) &= h^1 \otimes \ldots \otimes \Delta h^j \otimes \ldots \otimes h^n, \\
\partial_n(h^1 \otimes \ldots \otimes h^n) &= h^1 \otimes \ldots \otimes h^n \otimes \sigma, \\
\sigma_i(h^1 \otimes \ldots \otimes h^n) &= h^1 \otimes \ldots \otimes \varepsilon(h^{i+1}) \otimes \ldots \otimes h^n, \\
\tau_n(h^1 \otimes \ldots \otimes h^n) &= \tilde{S}(h^1) \cdot h^2 \otimes \ldots \otimes h^n \otimes \sigma.
\end{align*}
\]

**Example** $(H\mathcal{P}^*(\mathcal{H}_1; \mathbb{C}_\delta))$

$TF = X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y$ (universal fundamental),

$GV = \delta_1$ (universal Godbillon-Vey).
\( \mathcal{H} = \) Hopf algebra, \( M = \mathcal{H}\)-module/comodule is an SAYD if:

\[
m_{\langle 0 \rangle} m_{\langle -1 \rangle} = m, \\
(mh)_{\langle -1 \rangle} \otimes (mh)_{\langle 0 \rangle} = S(h_{(3)}) m_{\langle -1 \rangle} h_{(1)} \otimes m_{\langle 0 \rangle} h_{(2)}.\]

\( \forall \mathcal{H}\)-module coalgebra \( C \), \( \exists \) canonical co-cyclic structure:

\[
C^n := C^n_{\mathcal{H}}(C, M) = M \otimes_{\mathcal{H}} C^{\otimes n+1}, \quad n \geq 0
\]

\[
\partial_i : C^n \rightarrow C^{n+1}, \quad \sigma_j : C^n \rightarrow C^{n-1}, \quad \tau_n : C^n \rightarrow C^n
\]

\[
\partial_i(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes \Delta(c_i) \otimes \ldots \otimes c^n
\]

\[
\partial_{n+1}(m \otimes_{\mathcal{H}} \tilde{c}) = m_{\langle 0 \rangle} \otimes_{\mathcal{H}} c^0_{(2)} \otimes c^1 \otimes \ldots \otimes c^n \otimes m_{\langle -1 \rangle} c^0_{(1)}
\]

\[
\sigma_j(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \ldots \otimes \epsilon(c^{i+1}) \otimes \ldots \otimes c^n
\]

\[
\tau_n(m \otimes_{\mathcal{H}} \tilde{c}) = m_{\langle 0 \rangle} \otimes_{\mathcal{H}} c^1 \otimes \ldots \otimes c^n \otimes m_{\langle -1 \rangle} c^0.
\]
Hopf cyclic cohomology of a Hopf algebra:
\[ C := \mathcal{H}, \mathcal{H} \text{ acts on itself by multiplication} \Rightarrow C^n_\mathcal{H}(\mathcal{H}, M) \cong M \otimes \mathcal{H}^\otimes n. \]

Hopf cyclic cohomology relative to a sub-Hopf algebra \( \mathcal{K} \subset \mathcal{H} \):
\[ C := \mathcal{H} \otimes_\mathcal{K} \mathbb{C} \Rightarrow C^n_\mathcal{H}(\mathcal{H}, M) = M \otimes_\mathcal{K} \mathcal{H}^\otimes n; \]
alternatively,
\[
\begin{align*}
\partial_0(m \otimes h^1 \otimes \ldots \otimes h^n) &= m \otimes 1 \otimes h^1 \otimes \ldots \otimes h^n \\
\partial_i(m \otimes h^1 \otimes \ldots \otimes h^n) &= m \otimes h^1 \otimes \ldots \otimes \Delta(h^i) \otimes \ldots \otimes h^n \\
\partial_{n+1}(m \otimes h^1 \otimes \ldots \otimes h^n) &= m_{<\bar{0}>} \otimes h^1 \otimes \ldots \otimes h^n \otimes m_{<-1>} \\
\sigma_j(m \otimes h^1 \otimes \ldots \otimes h^n) &= m \otimes h^1 \otimes \ldots \otimes 1 \otimes \ldots \otimes h^n \\
\tau_n(m \otimes h^1 \otimes \ldots \otimes h^n) &= m_{<\bar{0}>} h^1_{(1)} \otimes S(h^1_{(2)})(h^2 \otimes \ldots \otimes h^n \otimes m_{<-1>})
\end{align*}
\]
Hopf algebras associated to classical groups of diffeomorphisms  [H.M. & B. Rangipour]

Splitting à la G.I. Kac:  \( \Pi = \text{primitive Lie pseudogroup of infinite type}, \)
\( \text{Diff}_\Pi = \text{globally defined diffeomorphisms of type } \Pi, \text{ then set-theoretically} \)

- \( \text{Diff}_\Pi = G_\Pi \cdot N_\Pi, \quad G_\Pi \cap N_\Pi = \{e\}, \) where

  - flat case : \( G_\Pi := \mathbb{R}^n \ltimes G_0^\Pi, \) with \( G_0^\Pi := \text{linear transformations in } \text{Diff}_\Pi, \)
  \( N_\Pi := \{\phi \in \text{Diff}_\Pi; \quad \phi(0) = 0, \quad \phi'(0) = \text{Id}\}; \)

- non-flat case : identifying \( \mathbb{R}^{2n+1} \) with the Heisenberg group \( H_n, \)
  \( G_\Pi := H_n \ltimes G_0^\Pi, \) with \( G_0^\Pi := \text{linear contact transformations,} \)
  \( N_\Pi := \text{all contact diffeomorphisms preserving the origin to order 1 in the sense of Heisenberg tangency, i.e. } \phi'(0) = \text{Id}^H = \text{the identity map of the Heisenberg tangent bundle}, \)
  \( T^H\mathbb{R}^{2n+1} := (T\mathbb{R}^{2n+1}/D) \oplus D, \) where \( D = \text{contact distribution}. \)
Mutual actions:

- The factorization $\text{Diff}_\Pi = G_\Pi \cdot N_\Pi$ allows to represent uniquely any $\phi \in \text{Diff}_\Pi$ as a product $\phi = \varphi \cdot \psi$, with $\varphi \in G_\Pi$ and $\psi \in N_\Pi$.

- Factorizing the product of any two elements $\varphi \in G_\Pi$ and $\psi \in N_\Pi$ in the reverse order, $\psi \cdot \varphi = (\psi \triangleright \varphi) \cdot (\psi \triangleleft \varphi)$, one obtains a left action $\psi \mapsto \tilde{\psi}(\varphi) := \psi \triangleright \varphi$ of $N_\Pi$ on $G_\Pi$, and a right action $\triangleleft$ of $G_\Pi$ on $N_\Pi$.

- These actions are restrictions of the natural actions of $\text{Diff}_\Pi$ on the coset spaces $\text{Diff}_\Pi / N_\Pi \cong G_\Pi$ and $G_\Pi \setminus \text{Diff}_\Pi \cong N_\Pi$. 
Fix basis \( \{ X_i \}_{1 \leq i \leq m} \) for the Lie algebra \( g_\Pi \) of \( G_\Pi \). Each \( X \in g_\Pi \) gives rise to a left-invariant vector field on \( G_\Pi \), which is then extended to a linear operator on \( A_\Pi := C^\infty(G_\Pi) \ltimes \text{Diff}_\Pi \),

\[
X(f \ U_{\phi^{-1}}) = X(f) \ U_{\phi^{-1}}, \quad f \in C^\infty(G_\Pi), \quad \phi \in \text{Diff}_\Pi.
\]

One has

\[
U_{\phi^{-1}} \ X_i \ U_\phi = \sum_{j=1}^{m} \Gamma^j_i(\phi) \ X_j, \quad \text{with} \quad \Gamma^j_i(\phi) \in C^\infty(G_{cn});
\]

define corresponding multiplication operators on \( A_\Pi \) by

\[
\Delta^j_i(f \ U_{\phi^{-1}}) = (\Gamma(\phi)^{-1})^j_i f \ U^{*}_\phi, \quad \text{where} \quad \Gamma(\phi) = (\Gamma^j_i(\phi))_{1 \leq i,j \leq m}.
\]
As an algebra, $\mathcal{H}_\Pi$ is generated by the operators $X_k$'s and $\Delta^j_i$'s. In particular, $\mathcal{H}_\Pi$ contains all iterated commutators

$$\Delta^j_{i,k_1...k_r} := [X_{k_r}, \ldots, [X_{k_1}, \Delta^j_i] \ldots],$$

which are multiplication operators by the functions

$$\Gamma^j_{i,k_1...k_r}(\phi) := X_{k_r} \ldots X_{k_1}(\Gamma^j_i(\phi)), \quad \phi \in \text{Diff}_\Pi.$$

For any $a, b \in A(\Pi_{cn})$, one has

$$X_k(ab) = X_k(a) b + \sum_j \Delta^j_k(a) X_j(b), \quad \Delta^j_i(ab) = \sum_k \Delta^k_i(a) \Delta^j_k(b),$$

Every $h \in \mathcal{H}(\Pi_{cn})$ satisfies a Leibniz rule of the form

$$h(ab) = \sum h_{(1)}(a) h_{(2)} b), \quad \forall a, b \in A_\Pi.$$

The operators $\Delta^\bullet ... \bullet$ satisfy the following Bianchi-type identities:

$$\Delta^k_{i,j} - \Delta^k_{j,i} = \sum_{r,s} c^{k}_{rs} \Delta^r_i \Delta^s_j - \sum_{\ell} c_{ij}^\ell \Delta^k_\ell,$$

where $c^{i}_{jk}$ are the structure constants of the Lie algebra $g_\Pi$. 

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**Transverse Index Theory and Hopf Algebras**

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Theorem (Realization as Hopf algebra)

Let $\mathfrak{H}_\Pi$ be the abstract Lie algebra generated by the operators
\[ \{ X_k, \Delta^j_{i,k_1\ldots k_r} \} \] and their commutation relations.

1. The algebra $\mathfrak{H}_\Pi$ is isomorphic to the quotient of the universal enveloping algebra $\mathcal{U}(\mathfrak{S}_\Pi)$ by the ideal $B_\Pi$ generated by the Bianchi identities.

2. The Leibniz rule determines uniquely a coproduct, with respect to which $\mathfrak{H}_\Pi$ a Hopf algebra and $A_\Pi$ an $\mathfrak{H}_\Pi$-module algebra.

Example (Projective pseudogroup)

$\mathcal{G} := \text{SL}_2(\mathbb{R}) = G \cdot N$, with $G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}$, and $N = \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right\}$, $c \in \mathbb{R}$. Then $\mathcal{H}(\mathcal{G}) \cong \mathcal{H}_1/(\delta'_2)$.
Bicrossed product definition of $\mathcal{H}_\Pi$

- $\mathcal{F}_\Pi :=$ the algebra of functions on $N$ generated by the jet ‘coordinates’

$$\eta^j_{i,k_1...k_r}(\psi) := \Gamma^j_{i,k_1...k_r}(\psi)(e), \quad \psi \in N;$$

- $\mathcal{F}_\Pi$ is a Hopf algebra with coproduct

$$\Delta f(\psi_1, \psi_2) := f(\psi_1 \circ \psi_2), \quad \forall \psi_1, \psi_2 \in N,$$

- with antipode

$$Sf(\psi) := f(\psi^{-1}), \quad \psi \in N,$$

- and with counit evaluation at $e,$

$$\psi \in N \mapsto \psi(e).$$

Clearly, the definition is independent of the choice of basis for $g_\Pi.$
\( \mathcal{F}_\Pi \) coacts on \( \mathcal{U}_\Pi = \mathcal{U}(\mathfrak{g}_\Pi) \) and makes it a comodule coalgebra,
\[
\nabla : \mathcal{U}_\Pi \to \mathcal{U}_\Pi \otimes \mathcal{F}_\Pi,
\]
as follows:

with \( \{ X_I = X^{i_1}_1 \cdots X^{i_m}_m ; i_1, \ldots, i_m \in \mathbb{Z}^+ \} \) PBW basis of \( \mathcal{U}_\Pi \),
\[
U_{\psi^{-1}} X_I U_{\psi} = \sum_J \beta^J_I(\psi) X_J,
\]
with \( \beta^J_I(\psi) \) in the algebra of functions on \( G_\Pi \) generated by \( \Gamma^j_{i,K}(\psi) \).

Define \( \nabla : \mathcal{U}_\Pi \to \mathcal{U}_\Pi \otimes \mathcal{F}_\Pi \) by
\[
\nabla(X_I) = \sum_J X_J \otimes \beta^J_I(\cdot)(e).
\]

The right action \( \triangleleft \) of \( G_\Pi \) on \( N_\Pi \) induces an action of \( G_\Pi \) on \( \mathcal{F}_\Pi \), hence a left action of \( \mathcal{U}_\Pi \) on \( \mathcal{F}_\Pi \), that makes \( \mathcal{F}_\Pi \) a left \( \mathcal{U}_\Pi \)-module algebra.
Theorem (Realization as bicrossed product)

With the above operations, $\mathcal{U}_\Pi$ and $\mathcal{F}_\Pi$ form a matched pair of Hopf algebras, and their bicrossed product $\mathcal{F}_\Pi \rhd \mathcal{U}_\Pi$ is canonically isomorphic to the Hopf algebra $\mathcal{H}_\Pi^{\text{cop}}$.

$$\varepsilon(u \triangleright f) = \varepsilon(u)\varepsilon(f),$$
$$\Delta(u \triangleright f) = u_{(1)}<_0> \triangleright f_{(1)} \otimes u_{(1)}<_1>(u_{(2)} \triangleright f_{(2)}),$$

$$\nabla(1) = 1 \otimes 1,$$
$$\nabla(uv) = u_{(1)}<_0> v<_0> \otimes u_{(1)}<_1>(u_{(2)} \triangleright v<_1>),$$
$$u_{(2)}<_0> \otimes (u_{(1)} \triangleright f)u_{(2)}<_1> = u_{(1)}<_0> \otimes u_{(1)}<_1>(u_{(2)} \triangleright f).$$
The bicrossed product structure allows to replace the original Hopf co-cyclic module by a simpler bicomplex, which combines the Chevalley-Eilenberg cohomology complex of the Lie algebra $g = g_\Pi$ with coefficients in $C_\delta \otimes F^{\otimes \bullet}$ and the coalgebra cohomology complex of $F = F_\Pi$ with coefficients in $\wedge^\bullet g$: 

\begin{align*}
\cdots & \xrightarrow{\partial_g} C_\delta \otimes \wedge^2 g & \xrightarrow{\beta_F} & C_\delta \otimes F \otimes \wedge^2 g & \xrightarrow{\beta_F} & C_\delta \otimes F^{\otimes 2} \otimes \wedge^2 g & \xrightarrow{\beta_F} & \cdots \\
\downarrow \partial_g & & \downarrow \partial_g & & \downarrow \partial_g & & \downarrow \partial_g \\
\cdots & \xrightarrow{\beta_F} C_\delta \otimes g & \xrightarrow{\beta_F} & C_\delta \otimes F \otimes g & \xrightarrow{\beta_F} & C_\delta \otimes F^{\otimes 2} \otimes g & \xrightarrow{\beta_F} & \cdots \\
\downarrow \partial_g & & \downarrow \partial_g & & \downarrow \partial_g & & \downarrow \partial_g \\
\cdots & \xrightarrow{\beta_F} C_\delta \otimes C & \xrightarrow{\beta_F} & C_\delta \otimes F \otimes C & \xrightarrow{\beta_F} & C_\delta \otimes F^{\otimes 2} \otimes C & \xrightarrow{\beta_F} & \cdots \\
\end{align*}
Theorem (Hopf cyclic bicomplex)

1. The above bicomplex computes the periodic Hopf cyclic cohomology \( HP^* (\mathcal{H}_\Pi; \mathbb{C}_\delta) \).

2. There is a relative version of the above bicomplex that computes the relative periodic Hopf cyclic cohomology \( HP^* (\mathcal{H}_\Pi, U(\mathfrak{h}); \mathbb{C}_\delta) \), for any reductive subalgebra \( \mathfrak{h} \) of the linear isotropy Lie algebra \( \mathfrak{g}_\Pi^0 \).

Corollary

Explicit cocycle representatives for basis of \( HP^* (\mathcal{H}_n, \mathfrak{g}_n; \mathbb{C}_\delta) \), i.e. universal Hopf cyclic Chern classes.

For each partition \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_k) \) of \( \{1, \ldots, p\} \), where \( 1 \leq p \leq n \), let \( \lambda \in S_p \) also denotes a permutation whose cycles have lengths \( \lambda_1 \geq \ldots \geq \lambda_k \). Define

\[
C_{p,\lambda} := \sum (-1)^\mu 1 \otimes \eta_{\mu(1),j(1)}^{j_1} \wedge \cdots \wedge \eta_{\mu(p),j(p)}^{j_p} \otimes X_{\mu(p+1)} \wedge \cdots \wedge X_{\mu(n)},
\]

where the summation is over all \( \mu \in S_n \) and all \( 1 \leq j_1, j_2, \ldots, j_p \leq n \).
Theorem (Hopf cyclic Chern classes)

The cochains \( \{ C_p, \lambda ; 1 \leq p \leq n, [\lambda] \in [S_p] \} \) are cocycles and their classes form a basis of the group \( HP^\epsilon(\mathcal{H}_n, \mathcal{U}(\mathfrak{gl}_n); \mathbb{C}_\delta) \), where \( \epsilon \equiv n \mod 2 \);
\( HP^{1-\epsilon}(\mathcal{H}_n, \mathcal{U}(\mathfrak{gl}_n); \mathbb{C}_\delta) = 0 \).

Correspondence with the universal Chern classes:

\[
P_n[c_1, \ldots c_n] = \mathbb{C}[c_1, \ldots c_n]/\mathcal{I}_n
\]

where \( \deg(c_j) = 2j \), and \( \mathcal{I}_n \) is the ideal generated by the monomials of degree \( > 2n \); to each partition \( \lambda \) as above, one associates the degree \( 2p \) monomial

\[
c_{p, \lambda} := c_{\lambda_1} \cdots c_{\lambda_k}, \quad \lambda_1 + \cdots + \lambda_k = p;
\]

the corresponding classes \( \{ c_{p, \lambda} ; 1 \leq p \leq n, \lambda \in [S_p] \} \) form a basis of the vector space \( P_n[c_1, \ldots c_n] \).