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## ON MANIFOLDS HOMEOMORPHIC TO THE 7-SPHERE

BY JOHN MILNOR<sup>1</sup>

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The object of this note will be to show that the 7-sphere possesses several distinct differentiable structures.

In §1 an invariant  $\lambda$  is constructed for oriented, differentiable 7-manifolds  $M^7$  satisfying the hypothesis (\*)  $H^3(M^7) = H^4(M^7) = 0$ . (Integer coefficients are to be understood.) In §2 a general criterion is given for proving that an  $n$ -manifold is homeomorphic to the sphere  $S^n$ . Some examples of 7-manifolds are studied in §3 (namely 3-sphere bundles over the 4-sphere). The results of the preceding two sections are used to show that certain of these manifolds are topological 7-spheres, but not differentiable 7-spheres. Several related problems are studied in §4.

All manifolds considered, with or without boundary, are to be differentiable, orientable and compact. The word *differentiable* will mean differentiable of class  $C^\infty$ . A closed manifold  $M^n$  is *oriented* if one generator  $\mu \in H_n(M^n)$  is distinguished.

### §1. The invariant $\lambda(M^7)$

For every closed, oriented 7-manifold satisfying (\*) we will define a residue class  $\lambda(M^7)$  modulo 7. According to Thom [5] every closed 7-manifold  $M^7$  is the boundary of an 8-manifold  $B^8$ . The invariant  $\lambda(M^7)$  will be defined as a function of the index  $\tau$  and the Pontrjagin class  $p_1$  of  $B^8$ .

An orientation  $\nu \in H_8(B^8, M^7)$  is determined by the relation  $\partial\nu = \mu$ . Define a quadratic form over the group  $H^4(B^8, M^7)/(\text{torsion})$  by the formula  $\alpha \rightarrow \langle \nu, \alpha^2 \rangle$ . Let  $\tau(B^8)$  be the index of this form (the number of positive terms minus the number of negative terms, when the form is diagonalized over the real numbers).

Let  $p_1 \in H^4(B^8)$  be the first Pontrjagin class of the tangent bundle of  $B^8$ . (For the definition of Pontrjagin classes see [2] or [6].) The hypothesis (\*) implies that the inclusion homomorphism

$$i: H^4(B^8, M^7) \rightarrow H^4(B^8)$$

is an isomorphism. Therefore we can define a "Pontrjagin number"

$$q(B^8) = \langle \nu, (i^{-1}p_1)^2 \rangle.$$

**THEOREM 1.** *The residue class of  $2q(B^8) - \tau(B^8)$  modulo 7 does not depend on the choice of the manifold  $B^8$ .*

Define  $\lambda(M^7)$  as this residue class.<sup>2</sup> As an immediate consequence we have:

**COROLLARY 1.** *If  $\lambda(M^7) \neq 0$  then  $M^7$  is not the boundary of any 8-manifold having fourth Betti number zero.*

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Let  $B_1^8, B_2^8$  be two manifolds with boundary  $M^7$ . (We may assume they are disjoint.) Then  $C^8 = B_1^8 \cup B_2^8$  is a closed 8-manifold which possesses a differentiable structure compatible with that of  $B_1^8$  and  $B_2^8$ . Choose that orientation  $\nu$  for  $C^8$  which is consistent with the orientation  $\nu_1$  of  $B_1^8$  (and therefore consistent with  $-\nu_2$ ). Let  $q(C^8)$  denote the Pontrjagin number  $\langle \nu, p_1^2(C^8) \rangle$ .

According to Thom [5] or Hirzebruch [2] we have

$$\tau(C^8) = \langle \nu, \frac{1}{45} (7p_2(C^8) - p_1^2(C^8)) \rangle;$$

and therefore

$$45\tau(C^8) + q(C^8) = 7\langle \nu, p_2(C^8) \rangle \equiv 0 \pmod{7}.$$

This implies

$$(1) \quad 2q(C^8) - \tau(C^8) \equiv 0 \pmod{7}.$$

LEMMA 1. *Under the above conditions we have*

$$(2) \quad \tau(C^8) = \tau(B_1^8) - \tau(B_2^8)$$

and

$$(3) \quad q(C^8) = q(B_1^8) - q(B_2^8).$$

Formulas 1, 2, 3 clearly imply that

$$2q(B_1^8) - \tau(B_1^8) \equiv 2q(B_2^8) - \tau(B_2^8) \pmod{7};$$

which is just the assertion of Theorem 1.

PROOF OF LEMMA 1. Consider the diagram

$$\begin{array}{ccc} H^n(B_1, M) \oplus H^n(B_2, M) & \xleftarrow[\approx]{h} & H^n(C, M) \\ \downarrow i_1 \oplus i_2 & & \downarrow j \\ H^n(B_1) \oplus H^n(B_2) & \xleftarrow{k} & H^n(C) \end{array}$$

Note that for  $n = 4$ , these homomorphisms are all isomorphisms. If  $\alpha = jh^{-1}(\alpha_1 \oplus \alpha_2) \in H^4(C)$ , then

$$(4) \quad \langle \nu, \alpha^2 \rangle = \langle \nu, jh^{-1}(\alpha_1^2 \oplus \alpha_2^2) \rangle = \langle \nu_1 \oplus (-\nu_2), \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle.$$

Thus the quadratic form of  $C^8$  is the "direct sum" of the quadratic form of  $B_1^8$  and the negative of the quadratic form of  $B_2^8$ . This clearly implies formula (2).

Define  $\alpha_1 = i_1^{-1}p_1(B_1)$  and  $\alpha_2 = i_2^{-1}p_1(B_2)$ . Then the relation

$$k(p_1(C)) = p_1(B_1) \oplus p_1(B_2)$$

implies that

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<sup>2</sup> Similarly for  $n = 4k - 1$  a residue class  $\lambda(M^n)$  modulo  $s_{k\mu}(L_k)$  could be defined. (See [2] page 14.) For  $k = 1, 2, 3, 4$  we have  $s_{k\mu}(L_k) = 1, 7, 62, 381$  respectively.

$$jh^{-1}(\alpha_1 \oplus \alpha_2) = p_1(C).$$

The computation (4) now shows that

$$\langle \nu, p_1^2(C) \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle,$$

which is just formula (3). This completes the proof of Theorem 1.

The following property of the invariant  $\lambda$  is clear.

LEMMA 2. *If the orientation of  $M^7$  is reversed then  $\lambda(M^7)$  is multiplied by  $-1$ .*

As a consequence we have

COROLLARY 2. *If  $\lambda(M^7) \neq 0$  then  $M^7$  possesses no orientation reversing diffeomorphism<sup>3</sup> onto itself.*

### §2. A partial characterization of the $n$ -sphere

Consider the following hypothesis concerning a closed manifold  $M^n$  (where  $R$  denotes the real numbers).

(H) *There exists a differentiable function  $f: M^n \rightarrow R$  having only two critical points  $x_0, x_1$ . Furthermore these critical points are non-degenerate.*

(That is if  $u_1, \dots, u_n$  are local coordinates in a neighborhood of  $x_0$  (or  $x_1$ ) then the matrix  $(\partial^2 f / \partial u_i \partial u_j)$  is non-singular at  $x_0$  (or  $x_1$ .)

THEOREM 2. *If  $M^n$  satisfies the hypothesis (H) then there exists a homeomorphism of  $M^n$  onto  $S^n$  which is a diffeomorphism except possibly at a single point.*

Added in proof. This result is essentially due to Reeb [7].

The proof will be based on the orthogonal trajectories of the manifolds  $f = \text{constant}$ .

Normalize the function  $f$  so that  $f(x_0) = 0, f(x_1) = 1$ . According to Morse ([3] Lemma 4) there exist local coordinates  $v_1, \dots, v_n$  in a neighborhood  $V$  of  $x_0$  so that  $f(x) = v_1^2 + \dots + v_n^2$  for  $x \in V$ . (Morse assumes that  $f$  is of class  $C^3$ , and constructs coordinates of class  $C^1$ ; but the same proof works in the  $C^\infty$  case.) The expression  $ds^2 = dv_1^2 + \dots + dv_n^2$  defines a Riemannian metric in the neighborhood  $V$ . Choose a differentiable Riemannian metric for  $M^n$  which coincides with this one in some neighborhood<sup>4</sup>  $V'$  of  $x_0$ . Now the gradient of  $f$  can be considered as a contravariant vector field.

Following Morse we consider the differential equation

$$\frac{dx}{dt} = \text{grad } f / \|\text{grad } f\|^2.$$

In the neighborhood  $V'$  this equation has solutions

$$(v_1(t), \dots, v_n(t)) = (a_1(t)^{\frac{1}{2}}, \dots, a_n(t)^{\frac{1}{2}})$$

for  $0 \leq t < \epsilon$ , where  $a = (a_1, \dots, a_n)$  is any  $n$ -tuple with  $\sum a_i^2 = 1$ . These can be extended uniquely to solutions  $x_a(t)$  for  $0 \leq t \leq 1$ . Note that these solutions satisfy the identity

<sup>3</sup> A diffeomorphism  $f$  is a homeomorphism onto, such that both  $f$  and  $f^{-1}$  are differentiable.

<sup>4</sup> This is possible by [4] 6.7 and 12.2.

$$f(x_a(t)) = t.$$

Map the interior of the unit sphere of  $R^n$  into  $M^n$  by the map

$$(a_1(t)^{\frac{1}{2}}, \dots, a_n(t)^{\frac{1}{2}}) \rightarrow x_a(t).$$

It is easily verified that this defines a diffeomorphism of the open  $n$ -cell onto  $M^n - (x_1)$ . The assertion of Theorem 2 now follows.

Given any diffeomorphism  $g: S^{n-1} \rightarrow S^{n-1}$ , an  $n$ -manifold can be obtained as follows.

CONSTRUCTION (C). Let  $M^n(g)$  be the manifold obtained from two copies of  $R^n$  by matching the subsets  $R^n - (0)$  under the diffeomorphism

$$u \rightarrow v = \frac{1}{\|u\|} g\left(\frac{u}{\|u\|}\right).$$

(Such a manifold is clearly homeomorphic to  $S^n$ . If  $g$  is the identity map then  $M^n(g)$  is diffeomorphic to  $S^n$ .)

COROLLARY 3. A manifold  $M^n$  can be obtained by the construction (C) if and only if it satisfies the hypothesis (H).

PROOF. If  $M^n(g)$  is obtained by the construction (C) then the function

$$f(x) = \|u\|^2 / (1 + \|u\|^2) = 1 / (1 + \|v\|^2)$$

will satisfy the hypothesis (H). The converse can be established by a slight modification of the proof of Theorem 2.

### §3. Examples of 7-manifolds

Consider 3-sphere bundles over the 4-sphere with the rotation group  $SO(4)$  as structural group. The equivalence classes of such bundles are in one-one correspondence<sup>5</sup> with elements of the group  $\pi_3(SO(4)) \approx Z + Z$ . A specific isomorphism between these groups is obtained as follows. For each  $(h, j) \in Z + Z$  let  $f_{h,j}: S^3 \rightarrow SO(4)$  be defined by  $f_{h,j}(u) \cdot v = u^h v u^j$ , for  $v \in R^4$ . Quaternion multiplication is understood on the right.

Let  $\iota$  be the standard generator for  $H^4(S^4)$ . Let  $\xi_{h,j}$  denote the sphere bundle corresponding to  $(f_{h,j}) \in \pi_3(SO(4))$ .

LEMMA 3. The Pontrjagin class  $p_1(\xi_{h,j})$  equals  $\pm 2(h - j)\iota$ .

(The proof will be given later. One can show that the characteristic class  $\bar{c}(\xi_{h,j})$  (see [4]) is equal to  $(h + j)\iota$ .)

For each odd integer  $k$  let  $M_k^7$  be the total space of the bundle  $\xi_{h,j}$  where  $h$  and  $j$  are determined by the equations  $h + j = 1$ ,  $h - j = k$ . This manifold  $M_k^7$  has a natural differentiable structure and orientation, which will be described later.

LEMMA 4. The invariant  $\lambda(M_k^7)$  is the residue class modulo 7 of  $k^2 - 1$ .

LEMMA 5. The manifold  $M_k^7$  satisfies the hypothesis (H).

Combining these we have:

<sup>5</sup> See [4] §18.

**THEOREM 3.** For  $k^2 \not\equiv 1 \pmod 7$  the manifold  $M_k^7$  is homeomorphic to  $S^7$  but not diffeomorphic to  $S^7$ .

(For  $k = \pm 1$  the manifold  $M_k^7$  is diffeomorphic to  $S^7$ ; but it is not known whether this is true for any other  $k$ .)

Clearly any differentiable structure on  $S^7$  can be extended through  $R^8 - (0)$ . However:

**COROLLARY 4.** There exists a differentiable structure on  $S^7$  which cannot be extended throughout  $R^8$ .

This follows immediately from the preceding assertions, together with Corollary 1.

**PROOF OF LEMMA 3.** It is clear that the Pontrjagin class  $p_1(\xi_{hj})$  is a linear function of  $h$  and  $j$ . Furthermore it is known that it is independent of the orientation of the fibre. But if the orientation of  $S^3$  is reversed, then  $\xi_{hj}$  is replaced by  $\xi_{-j-h}$ . This shows that  $p_1(\xi_{hj})$  is given by an expression of the form  $c(h - j)\iota$ . Here  $c$  is a constant which will be evaluated later.

**PROOF OF LEMMA 4.** Associated with each 3-sphere bundle  $M_k^7 \rightarrow S^4$  there is a 4-cell bundle  $\rho_k: B_k^8 \rightarrow S^4$ . The total space  $B_k^8$  of this bundle is a differentiable manifold with boundary  $M_k^7$ . The cohomology group  $H^4(B_k^8)$  is generated by the element  $\alpha = \rho_k^*(\iota)$ . Choose orientations  $\mu, \nu$  for  $M_k^7$  and  $B_k^8$  so that

$$\langle \nu, (i^{-1}\alpha)^2 \rangle = +1.$$

Then the index  $\tau(B_k^8)$  will be  $+1$ .

The tangent bundle of  $B_k^8$  is the "Whitney sum" of (1) the bundle of vectors tangent to the fibre, and (2) the bundle of vectors normal to the fibre. The first bundle (1) is induced (under  $\rho_k$ ) from the bundle  $\xi_{hj}$ , and therefore has Pontrjagin class  $p_1 = \rho_k^*(c(h - j)\iota) = ck\alpha$ . The second is induced from the tangent bundle of  $S^4$ , and therefore has first Pontrjagin class zero. Now by the Whitney product theorem ([2] or [6])

$$p_1(B_k^8) = ck\alpha + 0.$$

For the special case  $k = 1$  it is easily verified that  $B_1^8$  is the quaternion projective plane  $P_2(K)$  with an 8-cell removed. But the Pontrjagin class  $p_1(P_2(K))$  is known to be twice a generator of  $H^4(P_2(K))$ . (See Hirzebruch [1].) Therefore the constant  $c$  must be  $\pm 2$ , which completes the proof of Lemma 3.

Now  $q(B_k^8) = \langle \nu, (i^{-1}(\pm 2k\alpha))^2 \rangle = 4k^2$ ; and  $2q - \tau = 8k^2 - 1 \equiv k^2 - 1 \pmod 7$ . This completes the proof of Lemma 4.

**PROOF OF LEMMA 5.** As coordinate neighborhoods in the base space  $S^4$  take the complement of the north pole, and the complement of the south pole. These can be identified with euclidean space  $R^4$  under stereographic projection. Then a point which corresponds to  $u \in R^4$  under one projection will correspond to  $u' = u/\|u\|^2$  under the other.

The total space  $M_k^7$  can now be obtained as follows.<sup>5</sup> Take two copies of  $R^4 \times S^3$  and identify the subsets  $(R^4 - (0)) \times S^3$  under the diffeomorphism

$$(u, v) \rightarrow (u', v') = (u/\|u\|^2, u^h v u^j / \|u\|)$$

(using quaternion multiplication). This makes the differentiable structure of  $M_k^7$  precise.

Replace the coordinates  $(u', v')$  by  $(u'', v')$  where  $u'' = u'(v')^{-1}$ . Consider the function  $f: M_k^7 \rightarrow R$  defined by

$$f(x) = \Re(v)/(1 + \|u\|^2)^{\frac{1}{2}} = \Re(u'')/(1 + \|u''\|^2)^{\frac{1}{2}};$$

where  $\Re(v)$  denotes the real part of the quaternion  $v$ . It is easily verified that  $f$  has only two critical points (namely  $(u, v) = (0, \pm 1)$ ) and that these are non-degenerate. This completes the proof.

#### §4. Miscellaneous results

**THEOREM 4.** *Either (a) there exists a closed topological 8-manifold which does not possess any differentiable structure; or (b) the Pontrjagin class  $p_1$  of an open 8-manifold is not a topological invariant.*

(The author has no idea which alternative holds.)

**PROOF.** Let  $X_k^8$  be the topological 8-manifold obtained from  $B_k^8$  by collapsing its boundary (a topological 7-sphere) to a point  $x_0$ . Let  $\bar{\alpha} \in H^4(X_k^8)$  correspond to the generator  $\alpha \in H^4(B_k^8)$ . Suppose that  $X_k^8$ , possesses a differentiable structure, and that  $p_1(X_k^8 - (x_0))$  is a topological invariant. Then  $p_1(X_k^8)$  must equal  $\pm 2k\bar{\alpha}$ , hence

$$2q(X_k^8) - \tau(X_k^8) = 8k^2 - 1 \equiv k^2 - 1 \pmod{7}.$$

But for  $k^2 \not\equiv 1 \pmod{7}$  this is impossible.

Two diffeomorphisms  $f, g: M_1^n \rightarrow M_2^n$  will be called *diffeomorphically isotopic* if there exists a diffeomorphism  $M_1^n \times R \rightarrow M_2^n \times R$  of the form  $(x, t) \rightarrow (h(x, t), t)$  such that

$$h(x, t) = \begin{cases} f(x) & (t \leq 0) \\ g(x) & (t \geq 1). \end{cases}$$

**LEMMA 6.** *If the diffeomorphisms  $f, g: S^{n-1} \rightarrow S^{n-1}$  are diffeomorphically isotopic, then the manifolds  $M^n(f), M^n(g)$  obtained by the construction (C) are diffeomorphic.*

The proof is straightforward.

**THEOREM 5.** *There exists a diffeomorphism  $f: S^6 \rightarrow S^6$  of degree +1 which is not diffeomorphically isotopic to the identity.*

**Proof.** By Lemma 5 and Corollary 3 the manifold  $M_3^7$  is diffeomorphic to  $M^7(f)$  for some  $f$ . If  $f$  were diffeomorphically isotopic to the identity then Lemma 6 would imply that  $M_3^7$  was diffeomorphic to  $S^7$ . But this is false by Lemma 4.

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#### REFERENCES

1. F. HIRZEBRUCH, *Ueber die quaternionalen projektiven Räume*, S.-Ber. math.-naturw. Kl. Bayer. Akad. Wiss. München (1953), pp. 301-312.
2. ———, *Neue topologische Methoden in der algebraischen Geometrie*, Berlin, 1956.

3. M. MORSE, *Relations between the numbers of critical points of a real function of  $n$  independent variables*, Trans. Amer. Math. Soc., 27 (1925), pp. 345-396.
4. N. STEENROD, *The topology of fibre bundles*, Princeton, 1951.
5. R. THOM, *Quelques propriétés globale des variétés différentiables*, Comment. Math. Helv., 28 (1954), pp. 17-86.
6. WU WEN-TSUN, *Sur les classes caractéristiques des structures fibrées sphériques*, Actual. sci. industr. 1183, Paris, 1952, pp. 5-89.
7. G. REEB, *Sur certain propriétés topologiques des variétés feuilletées*, Actual. sci. industr. 1183, Paris, 1952, pp. 91-154.