COARSE GEOMETRY AND K-HOMOLOGY

By
Behnam Norouzizadeh
Supervisor
Prof. N. P. Landsman

Contents

Acknowledgements

I would like to acknowledge and thank Prof. Landsman for his help and for guiding me in such way that I got very interested in the subject. Also I would like to thank MRI for their financial support during this year and mathematical institute of Utrecht University for very nice time that I spent there.

INTRODUCTION

In this thesis we try to explain some aspects of the relationship between the coarse geometry and K-theory of C^*-algebra associated to a coarse space. Chapter 1 develops a general theory of coarse spaces. This provides a framwork in which one can discuss various kinds of control on a uniform footing: in particular, a metric on a space X defines one kind of coarse structure on it, a compactipication defines a different kind, and there are the expected relations between them. In the first section of chapter 2 we introduce the C^*-algebra associted to a coarse space, and discuss their functorial properties. In section 2 we compute K-theory for special examples of bounded coarse spaces and in the last section we relate K-theory and K-homology for continuously controlled coarse structures.

Chapter 1

Introduction to Coarse Geometry

Let X be a metric space, with metric d. For a topologist, the signficance of the metric lies in the collection of open sets it generates. This passage from the metric to its associated topology loses a good deal of information; in fact only the 'very small scale structure' of the metric is reflected in the topology. For example the metric

$$d'(x, y) = min\{d(x, y), 1\}$$

defines the same topology as d.

There is a dual procedure, in which one studies 'very large scale structure'. Coarse geometry is the study of the 'very large scale' properties of spaces. Indeed, coarse geometry arises from the study of metric spaces by looking at when points are far apart. Small scale structure does not matter in coarse geometry; every space of a finite size is equivalent to a single point as far as coarse geometry is concerned. All that matters is the large scale geometry of infinitely large spaces.

As topological property can be defined entirely in terms of open sets. Analogously, a large scale property can be defined entirely in terms of controlled sets. To do this, as one can define the notion of an abstract topological space by axiomatizing the properties of open sets in a metric space, one can define an abstract coarse space by axiomatizing the properties of controlled sets in a metric space.

In this chapter we define what a coarse space is, and we study a number of ways of constructing a coarse on a set so as to make it into a coarse space. We also consider some of the elementary concepts associated with coarse spaces.

1.1 Coarse structures

Definition 1.1.1. Let X, Y be pseudometric spaces and $f : X \longrightarrow Y$ a not necessarily continuous map.

(a) f is called *coarsely proper(metrically proper)*, if the inverse image of bounded sets is bounded.

(b) f is called *coarsely uniform(uniformely bornologous)*, if for every $r > 0$ there is $s(r) > 0$ such that for all $x_1, x_2 \in X$

$$d(x_1, x_2) \leq r \Rightarrow d(f(x_1), f(x_2)) \leq s(r).$$

(c) f is called a *coarse map*, if it is coarsely proper and coarsely uniform.

(d) Let S be a set. Two maps $f, g : S \longrightarrow X$ are called close if there is $C > 0$ such that for all $s \in S$

$$d(f(s), g(s)) < C$$

(e) Let $E \subseteq X \times X$. E is called *controlled(entourage)*, if the coordinate projection maps $\pi_1, \pi_2 : E \longrightarrow X$ are close.

For example, if $X = Y = \mathbb{N}$, the natural numbers, then the map $n \mapsto 14n + 78$ is coarse, but the map $n \mapsto 1$ is not coarse (it fails to be coarsely proper), and the map $n \mapsto n^2$ is not coarse either (it fails to be coarsely uniform).

Also suppose that $\mathbb{R}^2$ and $\mathbb{R}$ are given their natural metric coarse structures. Then the absolute value map $v \mapsto |v|$, from $\mathbb{R}^2$ to $\mathbb{R}$, is a coarse map, but the coordinate projections from $\mathbb{R}^2$ to $\mathbb{R}$ are not.

It is easy to see that the controlled sets associated to a metric space X have the following properties

(i) Any subset of a controlled set is controlled;

(ii) The transpose $E^t = \{(x, y) : (y, x) \in E\}$ of a controlled set E is controlled;

(iii) The composition $E_1 \circ E_2$ of controlled sets E_1 and E_2 is controlled:

$$E_1 \circ E_2 := \{(x, z) \in X \times X : \exists y \in X, (x, y) \in E_1 and (y, z) \in E_2\}$$

(iv) A finite union of controlled sets is controlled;

(v) The diagonal $\triangle_X := \{(x, x) : x \in X\}$ is controlled:

If our metrics are not allowed to take the value $+\infty$, then the controlled sets will have the following additional property:

(vi) The union of all controlled sets is $X \times X$. Now we try to axiomatize the situation in a more metric independent way:

Definition 1.1.2 (coarse structure). Let X be a set. A collection $\mathcal{E}$ of subsets of $X \times X$ is called a coarse structure- the elements of $\mathcal{E}$ will be called *controlled*-, if the axioms (i)-(iv) holds for E.

The pair $(X, \mathcal{E})$ is called a coarse space. The coarse space is called *unital*, if (v) holds and a coarse structure having the property (vi) will be called *connected.*

Definition 1.1.3 (bounded coarse structure). Let (X, d) be a metrics space, as we saw the metric d induces a coarse structure on X, which is called bounded coarse structure

More clear, we can define bounded coarse structure induced by the metric d as follow: Set $D_r := \{(x, y) \in X \times X : d(x, y) < r\}$. Then $E \subseteq X \times X$ is controlled, if $E \subseteq D_r$ for some $r > 0$.

Definition 1.1.4 (coarse structure generated by $\mathcal{M}$). Let X be a set and $\mathcal{M}$ a collection of subsets of $X \times X$. Since any intersection of coarse structures on X is itself a coarse structure, we can make the following definition. By $cs(\mathcal{M})$ we denote the smallest coarse structure containing $\mathcal{M}$, i.e. the intersection of all coarse structures containing $\mathcal{M}$. We call $cs(\mathcal{M})$ the *coarse structure generated by* $\mathcal{M}$.

Definition 1.1.5 (close maps). Let $(X, \mathcal{E})$ be a coarse space and S a set. The maps $f, g : S \longrightarrow X$ are called *close* if $\{(f(s), g(s)) : s \in S\} \subseteq X \times X$ is controlled.

Proposition 1.1.1. *Let $(X, \mathcal{E})$ be a coarse space, and let B be a subset of X. The following are equivalent:*

(a) $B \times B$ is controlled;

(b) $B \times \{p\}$ is controlled for some $p \in X$;

(c) The inclusion map $B \longrightarrow X$ is close to a constant map.

Proof. $(a) \Rightarrow (b)$ is obvious. For $(b) \Rightarrow (a)$, since $B \times \{p\}$ is controlled, so $\{p\} \times B$ is also controlled, But

$$B \times B \subseteq (B \times \{p\}) \circ (\{p\} \times B)$$

i.e. $B \times B$ is controlled.

$(b) \Leftrightarrow (c)$ is also obvious. $\square$

Definition 1.1.6. Let $(X, \mathcal{E})$ be a coarse space.

(i) $B \subseteq X$ is called *bounded* if B satisfying one of the above conditions;

(ii) a collection $\mathcal{U}$ of subsets of X is called *uniformly bounded* if $\bigcup_{U\in\mathcal{U}} U \times U$ is controlled;

(iii) the coarse space X is called *seperable* if it has a countable uniformly bounded cover.

For example, in a bounded coarse structure, the bounded sets are just metrically bounded ones: $B \subseteq X$ is bounded if and only if $B \times \{p\}$ is controlled for some $p \in X$ and it holds if and only if $B \times \{p\} \subseteq D_r$ for some $r > 0$ which means $B \subseteq B(p; r)$.

Lemma 1.1.2. *In a connected coarse structure, the union of two bounded sets is bounded.*

Proof. Suppose B_1 and B_2 are two bounded sets in a connected coarse structure X. Since B_1 and B_2 are bounded, $B_1 \times \{p\}$ and $B_2 \times \{q\}$ are controlled for some $p, q \in X$. But X is connected, so there exists an controlled set E such that $(q, p) \in E$. Hence $(B_2 \times \{q\}) \circ E$ is controlled and obviously $B_2 \times \{p\} \subseteq (B_2 \times \{q\}) \circ E$, therefore $B_2 \times \{p\}$ is controlled. On the other hand

$$(B_1 \cup B_2) \times \{p\} = (B_1 \times \{p\}) \cup (B_2 \times \{p\})$$

So $(B_1 \cup B_2) \times \{p\}$ is controlled, i.e. $B_1 \cup B_2$ is bounded. □

Definition 1.1.7 (coarse equivalence). Let X, Y be coarse spaces and $f : X \longrightarrow Y$ a map

(a) We call f *coarsely proper*, if the inverse image of bounded sets is bounded.

(b) We call f *coarsely uniform*, if the image of controlled sets under the map

$$f \times f : X \times X \longrightarrow Y \times Y$$

is a controlled set.

(c) We call f a *coarse map*, if it is coarsely proper and coarsely uniform.

(d) The map $f : X \longrightarrow Y$ is called a *coarse equivalence*, if it is a coarse map and if there exists another coarse map $g : Y \longrightarrow X$ such that $g \circ f$ is close to id_X and $f \circ g$ is close to id_Y.

(e) X and Y are called *coarsely equivalent*, if there exists a coarse equivalence from X to Y.

Example 1.1.1. Let $\mathbb{Z}$ equiped with the coarse structure inherited as a subset of the metric space $\mathbb{R}$. It is clear that the spaces $\mathbb{R}$ and $\mathbb{Z}$ are coarsely equivalent. Precisely, the inclusion $\mathbb{Z} \to \mathbb{R}$ is a coarse equivalence and an inverse up to closeness is provided by the map $\mathbb{R} \to \mathbb{Z}$ which is the integer part function.

Sometimes, the coarse space X will in addition be equipped with a locally compact Hausdorff topology. It will be natural to require a certain compatibility between the coarse structure and the topology:

Definition 1.1.8. Let X be a coarse space. Then X is called a *coarse topological space*, if it is equipped with a locally compact Hausdorff topology such that

(i) X has a uniformly bounded open cover;

(ii) every bounded subset of X has compact closure.

Remark 1.1.1. Suppose that X is a locally compact metrizable space (indeed proper metric space) with a bounded coarse structure. If X is seperable in the usual topological sense, then it is seperable in the coarse sense. The coarse structure induced by a proper metric is a coarse topological structure so there is a uniformly bounded open cover; and every open cover of a seperable metrizable space has a countable subcover.

Definition 1.1.9. If X is a coarse space, equipped with some topology, we say that the topology and coarse structure are *compatible* when X is a coarse topological space.

Example 1.1.2. Let R be the subset of $\mathbb{R}^2$ shown in the following figure equipped with the bounded coarse structure defined by the metric inherited as a subset of the metric space $\mathbb{R}^2$. Suppose B is a bounded set in coarse space R, so since R is closed in $\mathbb{R}^2$, $\bar{B}^R$ is closed in $\mathbb{R}^2$. On the other hand, B is metrically bounded, hence $\bar{B}^R$ is compact, i.e. the closure of every bounded set in coarse space R is compact. Therefore the bounded coarse structure in R is compatible with the subspace topology.

Proposition 1.1.3. *Any coarse topological space is locally compact, and the bounded sets are precisely those which are precompact.*

Proof. It is enough to show that if X is a coarse topological space and $B \subseteq X$ is precompact, then B is bounded.Since X is a coarse topological space so it has a uniformly bounded open cover $\bigcup_\alpha U_\alpha$. By compactness of $\bar{B}$

$$\bar{B} \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$$

On the other hand, since $U_{\alpha_i} \times U_{\alpha_i} \subseteq \bigcup_\alpha U_\alpha \times U_\alpha$ so U_{α_i} is bounded for each i. Hence by Lemma 1.1.2, $W = U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$ is bounded, i.e. $W \times W$ is controlled. But $B \times B \subseteq \bar{B} \times \bar{B} \subseteq W \times W$ so $B \times B$ is controlled, i.e. B is bounded. □

1.2 Some standard constructions of coarse structure

In this section we study a number ways of constructing a coarse on a set, and we try to know more about their structures for example their bounded sets, compatibility, etc.

We start with this fact that bounded coarse structures are more general that they might at first appear because of the following result:

Proposition 1.2.1. *A coarse structure on X is metrizable if and only if it is countably generated in the sense that there is a sequence of controlled sets (E_n) such that every controlled set E is contained in a finite composition of the form $E_1 \circ E_2 \circ \ldots \circ E_n$.*

Proof. If $\mathcal{E}$ is given by the metric d, then it is generated by the sets $E_n = \{(x,x') : d(x,x') \leq n\}$ for $n = 1, 2, \ldots$. Conversely suppose that $\mathcal{E}$ is generated by a countable family of sets E_n, Inductively define $F_0 = \triangle_X$ and

$$F_n = (F_{n-1} \circ F_{n-1}) \cup E_n \cup E_n^t$$

So $\triangle_X \subseteq F_1$, since $\triangle_X = F_0 \circ F_0$, hence $\triangle_X \subseteq F_1 \circ F_1$, i.e. $\triangle_X \subseteq F_2$ so $\triangle_X \subseteq F_n$ for each n. Suppose $(x,y) \in F_{n-1}$, since $\triangle_X \subseteq F_{n-1}$ so $(y,y) \in F_{n-1}$, i.e. $(x,y) \in F_{n-1} \circ F_{n-1}$, therefore $F_{n-1} \subseteq F_{n-1} \circ F_{n-1}$, hence we obtain that

$$\triangle_X \subseteq F_{n-1} \subseteq F_{n-1} \circ F_{n-1} \subseteq F_n$$

Further, the F_n are symmetric ($F_n = F_n^t$), so the collection of all subsets of $X \times X$ that are contained in some F_n is a coarse structure $\mathcal{F}$. Since each generator E_n belongs to $\mathcal{F}$, we have $\mathcal{E} \subseteq \mathcal{F}$; on the other hand, each F_n belongs to $\mathcal{E}$ and so $\mathcal{F}$. Hence $\mathcal{E} = \mathcal{F}$ and the F_n generate $\mathcal{E}$.

Now set

$$d(x,x') = inf\{n : (x,x') \in F_n\}.$$

We claim that d is a metric. We check the triangle inequality: suppose $x \neq x' \neq x''$ and $\epsilon > 0$, there exist $n_1, n_2 \in \mathbb{N}$ such that $n_1 < d(x,x') + \frac{\epsilon}{2}$, $(x,x') \in F_{n_1}$ and

$n_2 < d(x', x'') + \frac{\epsilon}{2}$, $(x', x'') \in F_{n_2}$, where without loss of generality $n_1 \geq n_2$, then $(x, x'') \in F_{n_1} \circ F_{n_1} \subseteq F_{n_1+1}$, therefore

$$d(x, x'') \leq n_1 + 1 \leq n_1 + n_2 \leq d(x, x') + d(x', x'') + \epsilon$$

Notice that d may take the value $+\infty$; it does so if and only if the coarse structure $\mathcal{E}$ is disconnected. $\square$

Definition 1.2.1 (continuously controlled coarse structure). Let X be a locally compact Hausdorff space equipped with a metrizable compactification $\dot{X}$. The collection

$$\mathcal{E} := \{E \subseteq X \times X : \bar{E}^{\dot{X} \times \dot{X}} \subseteq (X \times X) \cup \triangle_{\dot{X}}\}$$

of all subsets $E \subseteq X \times X$, the closure of which meet the boundary $(\dot{X} \times \dot{X}) \setminus (X \times X)$ only in the diagonal, is a coarse structure on X.

For later uses, we present equivalent conditions that define a controlled set in a continuously controlled coarse space:

Lemma 1.2.2. *Let X be a continuously controlled coarse space. each of the following conditions assert that E is controlled:*
(i) For each sequence (x_n, x'_n) in E if one of the sequences (x_n),(x'_n) converges to the $y \in \partial X$, then the other one also converges to $y \in \partial X$.
(ii) For each open set $U \subseteq \dot{X}$ containing y, there is another open subset $V \subseteq \dot{X}$ containing y such that if $(x, x') \in E$ and one of x or x' belong to V, the other must belong to U.

Proof. Obviously, E is controlled if and only if (i) holds. To show that (ii) is equivalent to controlledness of E we act as follows:

$(ii) \Rightarrow (i)$: Suppose (x_n) is a sequence in X converging to y and let U be a open set in $\dot{X}$ containing y. By (ii), there is another open subset $V \subseteq \dot{X}$ containing y satisfies in (ii). On the other hand since $x_n \to y$, there exists $N > 0$ such that $x_n \in V$ for every $n \geq N$ now, by (ii), $x'_n \in U$ for every $n \geq N$, i.e. $x'_n \to y$.
Let E be controlled we show (ii) holds. suppose $U \subseteq \dot{X}$ is an open set containing y. Thus there exists a compact $K \subseteq X$ such that $U = \dot{X} - K$, let x belong to K. Then $(x, y) \notin \bar{E}^{\dot{X}}$, and so there exist disjoint open sets U_x and V_x containing x and y respectively, such that $(U_x \times V_x \cup V_x \times U_x) \cap \bar{E}^{\dot{X}} = \emptyset$ (we also use the fact that E^t is also controlled.) Using compactness, take a finite cover of K by the set U_x: $K \subseteq \bigcup_{i=1}^{n} U_{x_i}$ now define $V = \bigcap_{i=1}^{n} V_{x_i}$. Suppose $(z, z') \in E$ and z belongs to V so $z \in V_{x_i}$ for $i = 1, \cdots, n$. Since $(z, z') \in E \subset \bar{E}^{\dot{X}}$ so $(z, z') \notin (U_{x_i} \times V_{x_i} \cup V_{x_i} \times U_{x_i})$, therefore $z' \notin U_{x_i}$ for $i = 1, \cdots, n$, hence $z' \notin K$, i.e. $z' \in U$. □

Proposition 1.2.3. *Any continuously controlled coarse space is a coarse topological space.*

Proof. Fix a countable dense set $\{z_n\}$ in X, and let $d_n > 0$ be the distance (in a metric defining the topology of $\dot{X}$) from z_n to ∂X. Suppose $(x_i, y_i) \in \bigcup_n (B(z_n; d_n/2) \times B(z_n; d_n/2))$ so for each i there exist n_i such that $x_n, y_n \in B(z_{n_i}; d_{n_i})$. By condition (i) of Lemma 1.2.2, it is clear that $\bigcup_n B(z_n; d_n/2) \times B(z_n; d_n/2)$ is controlled, therefore $\mathcal{U} = \{B(z_n; d_n/2)\}$ is a countable open cover of X which is uniformly bounded for the coarse structure.
Now suppose that $B \subseteq X$ has compact closure, we show that B is bounded. Let $p \in \bar{B}$, we show that $B \times \{p\}$ is controlled. Suppose $U \subseteq \dot{X}$ is an open set containing y, so there exists a compact set $K \subseteq X$ such that $U = \dot{X} \setminus K$. Let $V = \dot{X} \setminus (K \cup \bar{B})$. For every $b \in B$ we have $b \notin V$, also $p \notin V$, so it is clear that V satisfies the condition

(ii) of Lemma 1.2.2. hence $B \times \{p\}$ is controlled.

Conversely, suppose B is bounded so there exists $p \in X$ such that $B \times \{p\}$ is controlled. Choose open set $U = \dot{X} \setminus \{p\}$ in $\dot{X}$ containing y, by (ii) of Lemma 1.2.2, there is open subset $V \subseteq \dot{X}$ containing y which satisfies in condition (ii). On the other hand, there exists a compact set $K' \subseteq X$ such that $V = \dot{X} \setminus K'$. Suppose $b \in V$. by (ii), $p \in U$ which is contradiction so $b \notin V$ for every $b \in B$, i.e. $b \in K'$. Therefore $B \subseteq K'$, i.e. B has compact closure. □

Now we present an example of two coarse structure which are not coarsely equivalent. It is also an example of non-metrizable coarse spaces:

Example 1.2.1. Let $\mathbb{R}^+$ be the space $[0, \infty)$ equipped with the bounded coarse structure defined by the metric and let R be the space $[0, \infty)$ equipped with continuously controlled coarse structure arising from a metrizable compactification $[0, \infty)$. We will show that the coarse structure on the space R is not metrizable. Suppose that the coarse structure on the space R is generated by a metric. By proposition 1.2.1, there is a sequence, (M_n), of controlled sets such that every controlled $E \subseteq R \times R$ belongs to some member of the sequence M_n.

Choose points $z_n = (x_n, y_n) \in R \times R$ such that $(x_n, y_n) \notin M_n, x_i \neq x_j$ for $i \neq j$, and $y_i \neq y_j$ for $i \neq j$. Let

$$E = \bigcup_{n \in \mathbb{N}} B(z_n; 1)$$

where $B(z_n; 1)$ is the open ball of radius 1 in the metric space $[0, \infty) \times [0, \infty)$. Then according to the definition of the coarse structure on the space R, the open set E is a controlled set. But there is no set in the sequence M_n that contains E.

Now we asserts a point-set topology calculation which we need later:

Lemma 1.2.4. *Let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of ∂X and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of open sets in $\dot{X}$ such that*
(a) the closure of U_n is disjoint from K_n,
(b) for every point $y \in \partial X$ there is a subsequnce $\{n_i\}$ of the natural numbers for which

$$\{y\} = K_{n_1} \cap K_{n_2} \cap ...$$

and

$$X \backslash \{y\} = U_{n_1} \cap U_{n_2} \cap ...$$

If $S \subseteq X \times X$ and if , for every n, there is an open neighborhood V_n (in X) of K_n such that

$$S \cap (U_n \times V_n) = \emptyset, S \cap (V_n \times U_n) = \emptyset$$

then S is controlled.

Definition 1.2.2 (word metric). Let G be a group with a finite generating set F. The distance $d_{(G,F)}(g_1, g_2)$ of $g_1, g_2 \in G$ in the word metric associated to the pair (G,F) is the lenght of the shortest word in F representing $g_1^{-1}g_2$ which is denoted by $|g_1^{-1}g_2|_F$.

The word metric of course depends on the choice of generating set but:

Proposition 1.2.5. *Let F and F' be two finite generating sets for the same group G, and let d and d' be the associated word metrics. Then the identity map $(G, d') \longrightarrow (G, d)$ is a coarse equivalence.*

Proof. For each $h \in F'$ choose a word in F of minimal length representing h. Define λ to be the maximum of the length of these words. Given any word in F' representing

$g \in G$, we replace each letter by the chosen word in F and count letters, therefore

$$d(x, x') \leq \lambda d'(x, x').$$

This shows that the identity map from (Γ, d') to (Γ, d) is a coarse map. Equally, the identity map from (Γ, d') to (Γ, d) is a coarse map, and the result follows. $\square$

So the word metrics arising from different finite generating systems induce the same bounded coarse structure on G.

Definition 1.2.3 (length space). A metric space X is a *length space* if $d(x, y)$ is equal to the infimum of the lengths of closed curves joining x and y.

Definition 1.2.4 (proper metric space). A metric space X is a *proper metric space* if closed bounded subsets of X are compact.

Definition 1.2.5. Let Γ be a group acting on a topological space X. (i) The action is called *cocompact* if there is a compact set $K \subseteq X$ such that

$$\cup_\gamma \gamma K = X$$

(ii) It is called *proper* if each $x \in X$ has a neighborhood U such that $\gamma U \cap U = \emptyset$ for all but finitely many γ.

Remark 1.2.1. We recall some facts about length spaces:
(i) Any length space that admits a proper, cocompact group action is itself proper;
(ii) any group that acts properly and cocompactly on a length space is finitely generated;
(iii) let X be a length space, Y any metric space. Then the map f is coarsely uniform if and only if there exist $R, S > 0$ such that $d(x_1, x_2) < R \Rightarrow d(f(x_1), f(x_2)) < S$.

Let M be a compact manifold, with universal cover $\tilde{M}$ and fundamental group Γ which acting properly and cocompactly on $\tilde{M}$ by deck transformations. As above remark Γ is finitely generated so the word metric induces a coarse structure on Γ. On the other hand, as we know $\tilde{M}$ is a proper length space, so we have a bounded coarse structure on $\tilde{M}$. The interesting fact is that coarse spaces Γ and $\tilde{M}$ are coarsely equivalent. We prove this fact in more general case:

Theorem 1.2.6 ($\check{S}varc - Milnor$)**.** *Suppose that X is a length space and Γ a group acting properly and cocompactly by isometries on X. Then Γ is coarsely equivalent to X.*

Proof. Since Γ acts cocompactly on X so there is a compact set $K \subseteq X$ such that $X = \cup_{\gamma\in\Gamma}\gamma K$. Consider open cover $K \subseteq \cup_{x\in K} N_1(x)$ so there exist $x_1, \cdots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} N_i(x_i)$. So there exist $x_0 \in X$ and $r > 0$ such that $K \subseteq N_r(x_0)$, hence $X = \cup_{\gamma\in\Gamma}\gamma N_r(x_0)$. Define $f : \Gamma \to X$ by $f(\gamma) = \gamma x_0$ and $g : X \to \Gamma$ by sending $x \in X$ to any $\gamma \in \Gamma$ such that $x \in \gamma N_r(x_0)$.

We must show that f and g are coarse maps and that their composites are close to the identity.

(i) g is a coarse map: Since X is proper, $\bar{B}(x_0; 4r)$ is compact, and therefore there are only finitely many group elements γ for which $\gamma B(x_0; 4r) \cap B(x_0; 4r) \neq$. Let k be the maximum of their word lengths. Suppose now that $x, x' \in X$ with $d(x, x') \leq r$ and $g(x) = \gamma$, $g(x') = \gamma'$, hence $x \in \gamma B(x_0; r)$ and $x' \in \gamma' B(x_0; r)$, i.e. there exist $y, y' \in B(x_0; r)$ such that $x = \gamma y$ and $x' = \gamma' y'$. Then, since the action of Γ is by isometries we can write

$$d(\gamma x_0, \gamma' x_0) \leq d(\gamma x_0, x) + d(x, x') + d(x', \gamma' x_0)$$
$$\leq d(x_0, \gamma^{-1} x) + d(x, x') + d(\gamma'^{-1} x', x_0) \leq 3r$$

Then $d(x_0, \gamma^{-1}\gamma' x_0) \leq 3r$ and so $\gamma^{-1}\gamma' B(x_0; 4r) \cap B(x_0; 4r) \neq$. we conclude that $|\gamma^{-1}\gamma'| \leq K$, so $d(\gamma.\gamma') \leq k$. We have proved that

$$d(x, x') \leq r \Rightarrow d(g(x), g(x')) \leq k.$$

Because X is a length space, by above remark, this suffices to show that g is coarsely uniform. To show that g is coarsely proper, suppose L is a bounded set in Γ, hence there exist open ball $B(\gamma_0; s)$ such that $L \subseteq B(\gamma_0; s)$. There are only finitely many $\gamma \in \Gamma$ with $|\gamma| \leq s$, so we may define $c = max\{d(x_0, \gamma x_0) : |\gamma| \leq s\}$. We claim that $g^{-1}(L) \subseteq B(\gamma_0 x_0; c + r)$. To prove our claim, assume $x \in g^{-1}(L)$. If $g(x) = \gamma'$, then $x \in \gamma' B(x_0; r)$, i.e. there exists $y \in B(x_0; r)$ such that $x = \gamma' y$. On the other hand since $x \in g^{-1}(L)$, $g(x) \in L \subseteq B(\gamma_0; s)$, so $\gamma' \in B(\gamma_0; s)$, i.e. $|\gamma_0^{-1}\gamma'| \leq s$, hence $d(x_0, \gamma_0^{-1}\gamma' x_0) < c$. Therefore

$$d(\gamma_0 x_0, x) \leq d(\gamma_0 x_0, \gamma' x_0) + d(\gamma' x_0, x) < c + r$$

i.e. $x \in B(\gamma_0 x_0; c + r)$.

(ii) f is a coarse map: For every $r > 0$ there are only finitely many $\gamma \in \Gamma$ with $|\gamma| \leq r$, so we may define $s = max\{d(x_0, \gamma x_0) : |\gamma| \leq r\}$. Therefore

$$d(\gamma, \gamma') \leq r \Rightarrow |\gamma^{-1}\gamma'| \leq r \Rightarrow d(x_0, \gamma^{-1}\gamma' x_0) \leq s \Rightarrow d(\gamma x_0, \gamma' x_0) \leq s$$

So f is coarsely uniform.

(iii) $g \circ f$ is close to id_Γ: If $g(f(\gamma)) = \gamma'$, then $f(\gamma) \in \gamma' B(x_0; r)$, i.e. there exists

$y \in B(x_0; r0$ such that $F(\gamma) = \gamma x_0 = \gamma' y$. Thus

$$d(\gamma' x_0, \gamma x_0) = d(x_0, \gamma'^{-1} \gamma x_0) = d(x_0, y) < r$$

By the same argument as we saw in (i) it follows that $|\gamma^{-1}\gamma'| \leq k$, hence $g \circ f$ is close to id_Γ.

(iv) $f \circ g$ is close to id_X: Suppose $g(x) = \gamma$ so $x \in \gamma B(x_0; r)$, i.e. there exists $y \in B(x_0; r)$ such that $x = \gamma y$. Now assume $f(g(x)) = x'$ so $f(\gamma) = x'$, i.e. $\gamma x_0 = x'$, hence we can write

$$d(x', x) = d(\gamma x_0, x) = d(x_0, \gamma^{-1} x) = d(x_0, y) < r$$

Therefore $f \circ g$ is close to id_X. $\square$

Chapter 2

Coarse Geometry and K-Theory

In this chapter, we introduce the C^*-algebra associted to a coarse space. This C^*-algebra was introduced by Roe to study index theory on open manifolds. Indeed, coarse structures are studied analytically via this C^*-algebra. To define this C^*-algebra, we need some elementary definitions and results which we mention them in the first section without proofs. In the two last section, we compute the K-theory groups of the C^*-algebras associated to several bounded coarse structures and continuously controlled coarse structures.

2.1 C^*-algebra associated to a coarse structure

We assert a few facts from K-theory which we shall need in the remaining sections. Throught this section A is a C^*-algebra and all our modules will be right A-module.

Lemma 2.1.1. *let A be a C^*-algebra and J an ideal in A. Suppose that the sequence*

$$0 \to J \to A \to A/J \to 0$$

*is split by a *-homomorphism $\gamma : A/J \longrightarrow A$. Then the associated K-theory sequences*

$$0 \to K_p(J) \to K_p(A) \to K_p(A/J) \to 0$$

are split exact.

Proof. We know that for every short exact sequence $0 \longrightarrow J \xrightarrow{\iota} A \xrightarrow{\pi} A/J \to 0$ we get a natural semi-infinite exact sequence of abelian groups and homomorphisms

$$K_1(J) \xrightarrow{\iota_*} K_1(A) \xrightarrow{\pi_*} K_1(A/J) \xrightarrow{\delta} K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J) \longrightarrow 0$$

where the last arrow comes from the fact that $\pi_* \circ \gamma_* = id_{K_0(A/J)}$ implies surjectivity of π_*. Now, as also $\pi_* : K_1(A) \longrightarrow K_1(A/J)$ is surjective in this split exact case, we see by exactness that the kernel of γ is everything. Thus γ is the zero map, and the above implies split exactness of

$$0 \longrightarrow K_0(J) \xrightarrow{\iota_*} K_0(A) \xrightarrow{\pi_*} K_0(A/J) \longrightarrow 0$$

as wanted. Invoking the natural transformation $\theta : K_1(B) \longrightarrow K_0(SB)$ we also get split exactness of the K_1-sequence. $\square$

Lemma 2.1.2. *Suppose that A is any C^*-algebra (not necessarily unital) and that u is a unitary in a unital C^*-algebra that contains A as an ideal. Then Ad_u induces the identity on $K_p(A)$ for all p.*

Proof. If A is an ideal in the unital C^*-algebra E. then form the C^*-algebra

$$D = \{e_1 \oplus e_2 \in E \oplus E : e_1 - e_2 \in A\},$$

which is sometimes called the double of E along A. The C^*-algebra A is included in D as the ideal $A \oplus 0$, and the quotient is clearly isomorphic to E. But the formula $e \mapsto e \oplus e$ gives a splitting $E \to D$ of the quotient map, and so by Lemma 2.1.1 $K_0(A)$ injects into $K_0(D)$. There is a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "Ad_v"] & D \arrow[d, "Ad_w"] \\ A \arrow[r] & D \end{tikzcd}$$

where w is the unitary $u \oplus u$ in D. Since the horizontal maps induce injectons on K_0 and the map Ad_w induces the identity on $K_0(D)$, the map Ad_u must induce the identity on $K_0(A)$. The results for K_p follows from that for K_0. $\square$

Lemma 2.1.3. *If v is an isometry in A (or in a unital C^*-algebra containing A as an ideal) then the endomorphism $Ad_v(a) = vav^*$ induces the identity map on K-theory.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "Ad_v"] & M_2(A) \arrow[d, "Ad_w"] \\ A \arrow[r] & M_2(A) \end{tikzcd}$$

where the horizontal arrows denote the stabilization maps and

$$w = \begin{pmatrix} v & 1 - vv^* \\ v^*v - 1 & v^* \end{pmatrix}$$

Since the matrix w is a unitary in $M_2(A)$ (or in a C^*-algebra containing $M_2(A)$ as an ideal), Lemma 2.1.2 shows that it induces the identity map on K-theory. $\square$

Lemma 2.1.4. *If $\alpha_1, \alpha_2 : A \to B$ are *-homomorphisms with $\alpha_1[A]\alpha_2[A] = 0$, so that $\alpha_1 + \alpha_2$ is also a *-homomorphism, then $(\alpha_1)_* + (\alpha_2)_* = (\alpha_1 + \alpha_2)_*$ on K-theory.*

Proof. It follows from the fact that if $p, q \in M_n(A)$ are projections such that $pq = qp = 0$, then $p + q$ is also a projection and $[p + q] = [p] + [q]$. $\square$

Lemma 2.1.5. *If J_0 and J_1 are ideals in a C^*-algebra A, with $J_0 + J_1 = A$, then there is a six-term exact sequence*

$$\begin{tikzcd} K_1(J_0 \cap J_1) \arrow[r] & K_1(J_0) \oplus K_1(J_1) \arrow[r] & K_1(A) \arrow[d] \\ K_0(A) \arrow[u] & K_0(J_0) \oplus K_0(J_1) \arrow[l] \arrow[r, leftarrow] & K_0(J_0 \cap J_1) \arrow[l] \end{tikzcd}$$

Proof. It is enough to consider the six-term exact sequence in K-theory arising from the algebra

$$B = \{f \in C[0,1] \otimes A : f(0) \in J_0, f(1) \in J_1\}$$

and the ideal

$$I = SA = \{f \in C[0,1] \otimes A : f(0) = f(1) = 0\}$$

in B. $\square$

Definition 2.1.1. A *pre-Hilbert A-module* is a right A-module E (which is at the same time a complex vector space) equipped with an A-valued inner product $(.,.) : E \times E \longrightarrow A$ that is sesquilinear, positive definite, and respects the module action. In other words:

(i) $(x, y_1 + y_2) = (x, y_1) + (x, y_2)$ for $x, y_1, y_2 \in E$;

(ii) $(x, ya) = (x, y)a$ for $x, y \in E$, $a \in A$;

(iii) $(x, \lambda y) = \lambda(x, y)$ for $x, y \in E$, $\lambda \in \mathbb{C}$;

(iv) $(x, y) = (y, x)^*$ for $x, x \in E$;

(v) $(x, x) \geq 0$ for $x \in E$, and $(x, x) = 0 \Leftrightarrow x = 0$.

Definition 2.1.2. The norm of an element $x \in E$ is defined as

$$||x|| := (||(x, x)||)^{1/2}$$

If a pre-Hilbert A-module is complete with respect to its norm, it is said to be a *Hilbert A-module*. A *Hilbert submodule* of a Hilbert module E is a closed submodule of E.

If E and F are both Hilbert A-modules, by a *Hilbert module map* (or just a module map) from E to F we mean a linear map $\phi : E \longrightarrow F$ that respects the module action: $\phi(xa) = \phi(x)a$. We call ϕ *unitary* when it is an isomorphism which preserves inner products: $(\phi(x), \phi(y)) = (x, y)$.

Example 2.1.1 (The Hilbert A-module A). The C^*-algebra A itself can be re-organinized to become a Hilbert A-module if we define the inner product

$$(a, b) := a^*b.$$

The corresponding norm is just the norm on A because of the uniquness of norm of C^*-algebra.

Definition 2.1.3. Let E be Hilbert A-module. A map $T : E \longrightarrow E$ (a priori neither linear nor bounded) is said to be *adjointable* if there exists a map $T^* : E \longrightarrow E$ satisfying

$$(x, Ty) = (T^*x, y)$$

for all x,y in E. Such a map T^* is then called the *adjoint* of T.

By $\mathcal{L}(E)$ we denote the set of all adjointable module maps in E and its elements are called *Hilbert module operator*, whereas $\mathbb{B}_b(E)$ is the set of all bounded module maps in E.

Lemma 2.1.6. *If T is adjointable, then its adjoint is unique and adjointable with $T^{**} = T$. If both T and S are adjointable, then so is ST with $(ST)^* = T^*S^*$.*

Proposition 2.1.7. *When equipped with the operator norm*

$$||T|| = sup\{||Tx|| : ||x|| \leq 1\}$$

$\mathbb{B}_b(E)$ is a Banach algebra and $\mathcal{L}(E)$ a C^*-algebra.

Definition 2.1.4. If $x, y \in E$, define $\theta_{x,y} : E \longrightarrow E$ by

$$\theta_{x,y}(z) := x(y, z)$$

and let $\Theta := \{\theta_{x,y} : x, y \in E\}$.

The set of *compact Hilbert module operators* on E, denoted by $\mathbb{K}(E)$, is the closed subspace of $\mathcal{L}(E)$ generated by the maps $\theta_{x,y}$, i.e.

$$\mathbb{K}(E) := \overline{Span\Theta}$$

We then immediately get:

Lemma 2.1.8. *Let $x, y \in E$ and $T \in \mathcal{L}(E)$. Then $\theta_{x,y}$ is adjintable with*

$$\theta^*_{x,y} = \theta_{y,x}$$

Moreover,

$$T\theta_{x,y} = \theta_{Tx,y}, \theta_{x,y}T = \theta_{x,T^*y}$$

and

$$||\theta_{x,y}|| \leq ||x||.||y||$$

Proposition 2.1.9. $\mathbb{K}(E)$ *is a C^*-algebra which is an ideal in $\mathcal{L}(E)$.*

Definition 2.1.5. The quotient C^*-algebra $\mathcal{L}(E)/\mathbb{K}(E)$ is called the *Calkin* algebra, and is denoted by $\mathcal{D}(E)$. If T is a Hilbert module operator, then we shall denote by $\pi(T)$ its image in $\mathcal{D}(E)$.

Definition 2.1.6. Let X be a locally compact Hausdorff topological space. Then an *(X,A)-module* is a Hilbert A-module E equipped with a morphism $\rho : C_0(X) \longrightarrow \mathcal{L}(E)$ of C^*-algebras. We say that $C_0(X)$ is represented *non-degenerately* on Hilbert module E if $\overline{\rho[C_0(X)]E} = E$.

We usually omit explicit mention of the morphism ρ when talking about (X,A)-modules. For instance, we write $fTg = 0$ instead of writting $\rho(f)T\rho(g)$.

Definition 2.1.7. Let E be an (X,A)-module.

(i) Let $e \in E$. The *support* of e, denoted by *supp(e)*, is the complement, in X, of the union of all open subsets $U \subseteq X$ such that $fe = 0$ for all $f \in C_0(U)$;

(ii) let $T \in \mathcal{L}(E)$. The *support* of T, denoted by *supp(T)*, is the complement, in $X \times X$, of the union of all open subsets $U \times V \subseteq X \times X$ such that $fTg = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$. More generally, if E and F are (X, A) and (Y, A)-modules, respectively, then the *support* of a Hilbert module operator $T : E \to F$ is the complement, in $Y \times X$, of the union of all open sets $U \times V \subseteq Y \times X$ such that $\rho_Y(f)T\rho_X(g) = 0$ for all $f \in C_0(U)$ and $g \in C_0(V)$.

Definition 2.1.8. Let X be a separable coarse topological space , and let E be an (X,A)-module. Consider an operator $T \in \mathcal{L}(E)$.

(i) The operator T is said to be *locally compact* if the operators Tf and fT are both compact(in the sense of Hilbert module operators) for all functions $f \in C_0(X)$;

(ii) the operator T is said to be *pseudolocal* if the commutator $fT - Tf$ is compact for all functions $f \in C_0(X)$;

(iii) the operator T is said to be *controlled* if its support, supp(T), is controlled.

Now we try to construct a C^*-algebra of Hilbert module operators on E which reflects the coarse structure of X. To do this we need some notions and results. First of all, we restrict our attention to (X, A)-modules for which $C_0(X)$ is represented nondegenerately on Hilbert A-module E and the quotient to the Calkin algebra $C_0(X) \to \mathcal{D}(E)$ is injective, i.e. we can define:

Definition 2.1.9. An (X, A)-module E is called *adequte* if $\overline{\rho[C_0(X)]E} = E$, and no non-zero element in $C_0(X)$ acts on E as a compact operator.

For subsets $A \subseteq Y \times X$ and $B \subseteq X$, denote by $A \circ B$ the subset

$$\{y \in Y : \exists x \in B, (y, x) \in A\}$$

Then we have:

Lemma 2.1.10. *Let $T : E \longrightarrow F$ be a Hilbert module operator. For every compactly supported $e \in E$ we have*

$$supp(Te) \subseteq supp(T) \circ supp(e)$$

Moreover, $supp(T)$ is the smallest closed subset of $Y \times X$ that has this property.

Proposition 2.1.11. *Let X be a separable coarse topological space, and suppose E is a adequte (X, A)-module. The collection of controlled operators is a unital *-subalgebra of $\mathcal{L}(E)$, and the locally compact and controlled operators form a *-ideal in it.*

Proof. Let $S, T \in \mathcal{L}(E)$ and $e \in E$. By the definition of the support we have:

$$supp(T + \lambda S) \subseteq supp(T) \cup supp(S)$$

$$supp(T^*) = (supp(T))^t$$

$$supp(Te) \times supp(Te) \subseteq supp(T) \circ (supp(e) \times supp(e)) \circ (supp(T))^t$$

Let T be a controlled operator and suppose e is compactly supported. Since for a coarse topological structure, the bounded sets are axactly those having compact closure so $supp(e)$ is bounded, i.e. $supp(e) \times supp(e)$ is controlled, hence, by above, $supp(Te) \times supp(Te)$ is controlled, i.e. $supp(Te)$ is bounded. Therefore $supp(Te)$ is compact by the closedness of $supp(Te)$. Now let S, T be controlled operators. By Lemma 2.1.10

$$\begin{aligned} supp(ST)e = suppS(Te) &\subseteq supp(S) \circ supp(Te) \\ &\subseteq supp(S) \circ (supp(T) \circ supp(e)) \\ &= (supp(S) \circ supp(T)) \circ supp(e) \end{aligned}$$

Then by the second statement of Lemma 2.1.10

$$supp(ST) \subseteq supp(S) \circ supp(T)$$

i.e. $ST \in \mathcal{L}(E)$ is a controlled operators. Now, by Proposition 2.1.9, the locally compact,controlled operators form a *-ideal in the *-algebra of controlled operators. $\square$

Now we define the C^*-algebra of a separable coarse topological space X as follows:

Definition 2.1.10. Let X be a separable coarse topological space, and let E be an adequate (X, A)-module. Then we define the C^*-algebra $C^*_\rho(X)$, *C^*-algebra associated to X*, to be the norm closure of the algebra of locally compact, controlled operators on E.

In general the definition of $C^*_\rho(X)$ depends on the choice of "(X,A)-module" but we shall prove that K-theory of the C^*-algebra $C^*_\rho(X)$ does not depend on the choice of adequate (X,A)-module E.

Definition 2.1.11. Let X and Y be separable coarse topological spaces and suppose that E and F are adequte (X, A) and (Y, A)-modules, respectively. Let $f : X \to Y$

be a coarse map. A Hilbert module operator $T : E \to F$ *covers* f if there is an controlled set D for Y such that

$$supp(T) \subseteq \{(y, x) : (y, f(x)) \in D\}.$$

In other words, $T : E \to F$ covers f if the map π_1 and $f \circ \pi_2$, from $supp(T) \subseteq Y \times X$ to Y, are close.

Lemma 2.1.12. *Let X and Y be separable coarse topological spaces and suppose that E and F are adequte (X, A) and (Y, A)-modules, respectively. Let $f : X \to Y$ be a coarse map. If $V : E \to F$ is an isometry which covers f, then the *-homomorphism $Ad_V(T) = VTV^*$ maps $C^*(X)$ to $C^*(Y)$.*

Proof. Let $T \in C^*(X)$. To check that VTV^* is controlled, let $S \subseteq Y \times X \times X \times Y$ be the set of 4-tuples (y, x, x', y') such that $(y, x) \in supp(V)$, $(x, x') \in supp(T)$, $(x', y') \in supp(V^*)$. Let $\pi_i, i = 1, \ldots 4$, denote the coordinate projections of S. Then, since V covers q, the map π_1 and $q \circ \pi_2$, from S to Y, are close, and similarly so are the maps $q \circ \pi_3$ and π_4. Since T is controlled the map π_2 and π_3 are close, and since q is a coarse this implies that the maps $q \circ \pi_2$ and $q \circ \pi_3$ are close. Since closeness is transitive, π_1 and π_4 are close on S. But $supp(VTV^*) \subseteq (\pi_1, \pi_4)(S)$, so VTV^* is controlled. $\square$

Lemma 2.1.13. *Let Y be a separable coarse topological space. Then Y can be written as the disjoint union of a countable, uniformly bounded collection of Borel subsets each having non-empty interior.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be countable uniformly bounded open cover of Y. We know that the cover $\{\bar{U}_n\}_{n=1}^{\infty}$ is also uniformly bounded (by the same argument as Proposition

1.1.3). Put $V_1 = U_1$ and

$$V_n = U_n \backslash (U_1 \cup \cdots \cup U_{n-1});$$

Then the V_n form a countable uniformly bounded family of disjoint Borel sets covering Y. Some of the sets V_n may however have empty interior. Discard these and let $V_{n_1}, V_{n_2}, \ldots$ be the sets that remain. The closure $\bar{V}_{n_i}$ of the sets V_{n_i} cover Y, since any point y belonging to one of the discard sets V_n must be the limit of a sequence belonging to some V_m for $m < n$. Finally put $W_1 = \bar{V}_{n_1}$ and

$$W_i = \bar{V}_{n_i} \backslash (\bar{V}_{n_1} \cup \cdots \cup \bar{V}_{n_{i-1}}).$$

then each W_i has non-empty interior and the disjoint family $W = \{W_i\}_{i=1}^{\infty}$ covers Y. Moreover W is uniformly bounded, since $W_i \subseteq \bar{U}_{n_i}$. $\square$

Proposition 2.1.14. *Let X and Y be separable coarse topological spaces and suppose that E and F are adequte (X, A) and (Y, A)-modules, respectively. Then any coarse map $f : X \to Y$ can be covered by an isometry.*

Proof. Choose an controlled set S for Y and partition Y into a countable union of Borel components W_i having non-empty interior and such that $\cup W_i \times W_i \subseteq S$ which we know exist by Lemma 2.1.13 Then write

$$E = \oplus_i \chi_{f^{-1}(W_i)} E, F = \oplus_i \chi_{W_i} F.$$

By the Stabilization Theorem there exist isometries $V_i : \chi_{f^{-1}(W_i)} E \longrightarrow \chi_{W_i}$; define

$$V = \oplus_i V_i$$

By construction, $(x_1, x_2) \in supp(V)$ only if $(f(x_1), x_2) \in \cup_i W_i \times W_i$, so V covers f. $\square$

Now we can assert and prove the main theorem of this section which makes $K_p(C^*(X))$ functorial in X:

Proposition 2.1.15. *Let X and Y be separable coarse topological spaces and let E and F be adequte (X, A) and (Y, A)-modules, respectively. Let $f : X \to Y$ be a coarse map which is covered by isometries V_1 and V_2 from E to F. Then V_1 and V_2 induce the same map on K-theory:*

$$(Ad_{V_1})_* = (Ad_{V_2})_* : K_p(C^*(X)) \to K_p(C^*(Y))$$

Proof. The maps

$$T \mapsto \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$T \mapsto \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$$

taking $C^*(X)$ into $M_2(C^*(X))$, induce the same isomorphism on K-theory. So it suffices to show that the maps

$$T \mapsto \begin{pmatrix} V_1TV_1^* & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$T \mapsto \begin{pmatrix} 0 & 0 \\ 0 & V_2TV_2^* \end{pmatrix}$$

which take $C^*(X)$ to $M_2(C^*(Y))$, induce the same map on K-theory. But the second is obtained from the first by conjugating with the unitary

$$\begin{pmatrix} I - V_1V_1^* & V_1V_2^* \\ V_2V_1^* & I - V_2V_2^* \end{pmatrix}$$

which is an element of $M_2(C^*(Y))$. So uniqueness follows from Lemma 2.1.2. $\square$

In particular taking f to be the identity map we have:

Corollary 2.1.16. *Let X be a separable coarse topological space, and let E and F be two adequte (X, A)-module. Then the K-theory groups $K_p(C^*_\rho(X))$ and $K_p(C^*_{\rho'}(X))$ are canonically isomorphic.*

It is in this sense that the K-theory of $C^*_\rho(X)$ does not depend on the choice of adequte (X, A)-module so we shall usually omit mention of the representation ρ and for each separable coarse topological space X, we allow to fix an adequte (X, A)-module and use it in forming the C^*-algebra $C^*(X)$.

Definition 2.1.12. If $f : X \to Y$ is a coarse map, then we define

$$f_* : K_p(C^*(X)) \to K_p(C^*(Y))$$

to be the map $(Ad_{V_f})_*$, where $V_f : E \to F$ is any isometry that cover f.

Proposition 2.1.17. *If $f, g : X \to Y$ are close, then $f_* = g_* : K_p(C^*(X)) \to K_p(C^*(Y))$. In particular, a coarse equivalence induces an isomorphism on K-theory.*

Proof. It is clear that if V cover f, then it also covers any coarse map close to f, i.e. in this case V covers g. So the statement follows from the prevoius definition and Proposition 2.1.15. $\square$

Since $V_f V_g$ is an isometry that covers fg, we have:

Lemma 2.1.18. *The correspondence $f \mapsto f_*$ is a covariant functor from the category of separable coarse topological spaces and coarse maps to the category of ablian groups and homomorphisms.*

2.2 K-theory for bounded coarse structures

In this section we are going to compute $K_p(C^*(X))$ for certain bounded coarse structures on space X. From now on, for simpilisity, we consider adequte representation of $C_0(X)$ on a Hilbert spaces instead of Hilbert modules. The first lemma however, applies in general:

Lemma 2.2.1. *If the proper separable metric space X (with bounded coarse structure arising from this metric) is also compact then*

$$K_p(C^*(X)) \cong \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ 0 & \text{if } p = 1 \end{cases}$$

Proof. Since every nondegenerate representation is unital we have $C^*(X) \subseteq K(H)$. On the other hand , since X is compact so every set $E \subseteq X \times X$ is controlled, i.e. $supp(T)$ is controlled for every $T \in \mathcal{L}(E)$. Hence $K(H) \subseteq C^*(X)$ and the result follows. □

Since X is metric space we can define the *propagation* of T as follows

$$prop(T) = sup\{d(x,y) : (x,y) \in supp(T)\}$$

It is clear that controlled operators on H_X are then those of finite propagation.
Now we shall compute the coarse K-theory for ray $\mathbb{R}^+$ which we introduced in Example 1.2.1.

Lemma 2.2.2. *For all p we have*

$$K_p(C^*(\mathbb{R}^+)) = 0.$$

Proof. Let ρ be the representation of $C_0(\mathbb{R}^+)$ by multiplication operators on $H = L^2(\mathbb{R}^+)$, if $g \in L^2(\mathbb{R}^+)$ and $f \in C_0(\mathbb{R}^+)$, then there exist a compact set $K \subseteq C_0(\mathbb{R}^+)$ such that $|f(x)| < \epsilon$ for each $x \in \mathbb{R}^+ \backslash K$, therefore

$$\begin{aligned}\int |fg|^2 dx &= \int_K |fg|^2 dx + \int_{\mathbb{R}^+\backslash K} |fg|^2 dx \\ &\leq M \int_K |g|^2 dx + \int_{\mathbb{R}^+\backslash K} |g|^2 dx \\ &\leq max(M,1) \int_{\mathbb{R}^+} |g|^2 dx < \infty\end{aligned}$$

Now let $H' = H \oplus H \oplus \ldots$ be the direct sum of infinitely many copies of H, with corresponding representation ρ'. Let V be the inclusion of H as the first summand in H'. Then both ρ and ρ' are adequte representations, and V is an isometry covering the identity map. It follows from Proposition 2.1.15 that the *-homomorphism $\alpha_1 = Ad_V$ induces an isomorphism

$$(\alpha_1)_* = K_p(C^*_\rho(\mathbb{R}^+)) \longrightarrow K_p(C^*_{\rho'}(\mathbb{R}^+)).$$

We shall show that this induced isomorphism is the zero map. It will follow that $K_p(C^*(R^+)) = 0$.

Let $U : H \longrightarrow H$ be the right-translation isometry given by

$$Uf(t) = \begin{cases} f(t-1) & \text{if } t \geq 1 \\ 0 & \text{if } 0 \leq t < 1 \end{cases}$$

Define a *-homomorphism $\alpha_2 : B(H) \longrightarrow B(H')$ by the formula

$$\alpha_2(T) = 0 \oplus Ad_U(T) \oplus Ad_U^2(T) \oplus \ldots$$

We make two claims about this *homomorphism:

(a) if T is controlled, then $\alpha_2(T)$ is controlled, and

(b) if T is locally compact, then $\alpha_2(T)$ is locally compact.

Thus α_2 restricts to a *-homomorphism $C^*_\rho(\mathbb{R}^+)) \longrightarrow C^*_{\rho'}(\mathbb{R}^+)$.

To prove (a), notice that the support of $Ad_U(T)$ is just a translate of the support of T, and that therefore T and $AD_U(T)$ have the same propagation. The propagation of the direct sum

$$\alpha_2(T) = 0 \oplus Ad_U(T) \oplus Ad_U^2(T) \oplus \dots$$

is the supremum of the propagation of the individual summands, and hence is also equal to the propagation of T.

To prove (b), suppose that T is locally compact. Let f be a compactly supported function on $\mathbb{R}^+$. By definition of isometry U it is clear that there is an integer N such that $\rho(f)U^n = 0$ for $n > N$. It follows that, in the direct sum

$$\rho'(f)\alpha_2(T) = 0 \oplus \rho(f)Ad_U(T) \oplus \rho(f)Ad_U^2(T) \oplus \dots$$

all the summands $\rho(f)Ad_U^n(T)$ for $n > N$ are zero. Moreover, all the summands without exception are compact, since T is locally compact. It follows that $\rho'(f)\alpha_2(T)$ is compact for every compactly supported function f on $\mathbb{R}^+$. Hence $\alpha_2(T)$ is locally compact.

We have

$$\alpha_2 = Ad_W \circ (\alpha_1 + \alpha_2)$$

where $W : H' \longrightarrow H'$ is the isometry

$$W(f_1, f_2, \dots) = (0, Uf_1, Uf_2, \dots)$$

The isometry W covers the identity map and so induces the identity on K-theory. Now by Lemma 2.1.4:

$$(\alpha_2)_* = (\alpha_1 + \alpha_2)_* = (\alpha_1)_* + (\alpha_2)_*$$

on K-theory, hence $(\alpha_1)_* = 0$. $\square$

The argument is applicable more widely. In fact, suppose now that Y is a proper seperable metric space and that $X = \mathbb{R}^+ \times Y$ is equipped with the product metric d_X defined by

$$d_X((t,y),(t',y'))^2 = |t-t'|^2 + d_Y(y,y')^2.$$

We can form $H = L^2(\mathbb{R}^+) \otimes H_Y$, and $H' = H \otimes H \otimes \ldots$; these Hilbert spaces carry adequte representations of $C_0(X)$. The definitions of the isometries U, V and W, and of the *-homomorphisms α_1 and α_2 now extend directly to the more general case, and the proof of Lemma 2.2.2 goes through word for word. We obtain:

Proposition 2.2.3. *Let Y be a proper separable metric space and let $X = \mathbb{R}^+ \times Y$, equipped with the product metric and its associated coarse structure. Then $K_p(C^*(X)) = 0$ for all p.*

Let X be a proper seperable metric space and $Y \subseteq X$ a closed subspace. For each $n \in \mathbb{N}$ let Y_n denote the closure of $\{x \in X : d(x,Y) < n\}$. Note that the inclusion map $Y \subseteq Y_n$ is a coarse equivalence, and that Y_n is the closure of its interior.

Definition 2.2.1. A controlled subset $S \subseteq X \times X$ is *near* Y if it is contained in $Y_n \times Y_n$ for some n. A controlled operator T is *near* Y if its support is near Y.

The operators near Y form an ideal in the algebra of all controlled operators, and similarly the locally compact operators near Y form an ideal in the algebra of all locally compact controlled operators.

Definition 2.2.2. Let X be a proper seperable metric space, and let $Y \subseteq X$ be a closed subspace. The ideal I_Y of $C^*(X)$ *supported near* Y is by definition the norm closure of the set of all locally compact, controlled operators near Y.

Here we take into account a property of Borel function which we need:

Definition 2.2.3. If X is a locally compact separable metrizable space then denote by $B(X)$ the C^*-algebra of bounded Borel functions on X (with the supremum).

Proposition 2.2.4. *Let T be a normal Hilbert space operator. The functional calculus homomorphism $f \mapsto f(T)$, from $C(Spectrum(T))$ to $B(H)$, extends to a C^*-homomorphism from $B(Spectrum(T))$ to $B(H)$.*

Proof. Assume without loss of generality that T is a multiplication operator by x on $L^2(X, \mu)$, where $X = Spectrum(T)$. Then for any bounded Borel function f we may define $f(T)$ to be the operator of multiplication by $f(x)$. It is clear that this produces an operator with the required properties. $\square$

Remark 2.2.1. The same argument shows that any representation of commutative C^*-algebra $C_0(X)$ (X being locally compact Hausdorf and second countable) extends to a representation of $B(X)$.

The above *Borel functional calculus* is the unique extension of the continuous functinal calculus which has the property that if the sequence $\{f_n\}_{n=1}^{\infty}$ of functions converges pointwise to f, and if $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded, then $f_n(T) \longrightarrow f(T)$ in the strong operator topology. This continuity property is a simple consequence of the Dominated Convergence Theorem.

Proposition 2.2.5. *Let X be a proper separable metric space, let $Y \subseteq X$ be a closed subspace, and let I_Y be the ideal of $C^*(X)$ supported near Y. There is an isomorphism*

$$K_p(I_Y) \cong K_p(C^*(Y))$$

between the K-theory of the ideal I_Y and the K-theory of the C^*-algebra associted to Y as a coarse space in its own right.

Proof. Define Y_n as above, and let H_{Y_n} which is the range of the projection operator corresponding to the characteristic function of Y_n under the Borel functional calculus. Since each Y_n is the closure of its interior, the natural representation of $C_0(Y_n)$ on H_{Y_n} is adequte. Thus we may identify $C^*(Y_n)$ with the subalgebra $C^*(X)$ consisting of operators T such that both T and T^* vanish on the orthogonal complement of H_{Y_n}. Moreover, the inclusion $H_{Y_n} \subseteq H_{Y_{n+1}}$ is an isometry which covers the map $Y_n \subseteq Y_{n+1}$ of coarse spaces.

Having made these identifications, we find that $C^*(X)$ contains an increasing sequence of C^*-algebras

$$C^*(Y_1) \subseteq C^*(Y_2) \subseteq \ldots$$

and that the closure of their union is I_Y. By

$$K_p(I_Y) \cong \underrightarrow{lim} K_p(C^*(Y_n)).$$

But each of the inclusion maps $Y \longrightarrow Y_n$ is a coarse equivalence, so by Proposition 2.1.17 the maps in the direct sequence are all isomorphisms, and the result follows. $\square$

Remark 2.2.2. Let $V : H_Y \longrightarrow H_X$ be any isometry covering the inclusion map $Y \subseteq X$. Then Ad_V maps $C^*(Y)$ to $C^*(X)$. In fact, $Ad_V(C^*(Y) \subseteq I_Y$ and the proof above shows that the K-theory isomorphism $K_p(C^*(Y)) \longrightarrow K_p(I_Y)$ is induced by Ad_V.

Suppose now that X is a proper metric space which is written as a union $X = Y \cup Z$ of two closed subspaces. Then $I_Y + I_Z = C^*(X)$.

To compute $K_p(C^*(\mathbb{R}^n))$, we need the followig lemma:

Lemma 2.2.6. *Let $X = \mathbb{R}^n$, let $Y = \mathbb{R}^- \times \mathbb{R}^{n-1}$, and let $Z = \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Then the associated ideals in $C^*(X)$ satisfy the relation $I_{Y \cap Z} = I_Y \cap I_Z$.*

Proof. We use the notation of the proof of Proposition 2.2.5. The algebra $C^*(X)$ contains an increasing sequence of C^*-algebras

$$C^*(Y_1) \cap C^*(Z_1) \subseteq C^*(Y_2) \cap C^*(Z_2) \subseteq \dots$$

Moreover, the closure of their union is $I_Y \cap I_Z$. But in the case at hand

$$Y_n \cap Z_n = (Y \cap Z)_n$$

and so $C^*(Y_n) \cap C^*(Z_n) = C^*((Y \cap Z)_n)$. Since the closure of the union of the subalgebras $C^*((Y \cap Z)_n)$ is the ideal $I_{Y \cap Z}$, the result follows. □

We can now compute the coarse K-theory for Euclidean space.

Theorem 2.2.7. *The groups $K_p(C^*(\mathbb{R}^n))$ are given by*

$$K_p(C^*(\mathbb{R}^n)) \cong \begin{cases} \mathbb{Z} & \text{if } p \equiv n (mod 2) \\ 0 & \text{if } p \equiv n+1 (mod 2) \end{cases}$$

Proof. If $n = 0$ the result follows from Lemma 2.2.1. It suffices then for an inductive proof to establish a 'suspension isomorphism'

$$K_p(C^*(\mathbb{R}^n)) \cong K_{p-1}(C^*(\mathbb{R}^{n-1})).$$

Let Y and Z be the subspaces described in Lemma 2.2.6 above, and let I_Y, I_Z and $I_{Y \cap Z}$ be the ideals supported near Y, Z and $Y \cap Z$. Using the six-term exact sequence of Lemma 2.1.5, there is an exact sequence

$$\begin{tikzcd}
K_1(I_{Y \cap Z}) \arrow[r] & K_1(I_Y) \oplus K_1(I_Z) \arrow[r] & K_1(C^*(\mathbb{R}^n)) \arrow[d] \\
K_0(C^*(\mathbb{R}^n)) \arrow[u] & K_0(I_Y) \oplus K_0(I_Z) \arrow[l] \arrow[r] & K_0(I_{Y \cap Z}) \arrow[l]
\end{tikzcd}$$

The ideals I_Y, I_Z and $I_{Y\cap Z}$ have the same K-theory as the corresponding C^*-algebras $C^*(Y), C^*(Z)$, and $C^*(Y \cap Z)$ by Proposition 2.2.5. But $Y \cap Z = \mathbb{R}^{n-1}$, while $C^*(Y)$ and $C^*(Z)$ have zero K-theory by Proposition 2.2.3. The suspension isomorphism now follows from the above diagram. □

2.3 K-theory for continuously controlled coarse structures

In this section, first we define the dual and relative dual algebra of A associated to a representation of A on a seperable Hilbert space and we show the functoriality of K-theory on these algebras. We also introduce the reduced analytic K-homology groups of a separable, unital C^*-algebra. Then we are going to relate the algebra $C^*(X)$, for continuously controlled coarse structures on X, to the relative dual algebra and at the end of this section as a result we shall be able to express the groups $K_p(C^*(X))$ in terms of K-homology.

Definition 2.3.1. Let A be a C^*-algebra and let $K(H)$ be the C^*-algebra of compact operator on a separable and infinite-dimensional Hilbert space H. an *extension* of A by $K(H)$ is a short exact sequence of C^*-algebras and *-homomorphisms, of the form

$$0 \longrightarrow K(H) \longrightarrow E \longrightarrow A \longrightarrow 0.$$

To say it as plainly as possible, an extension is a one-to-one *-homomorphism of $K(H)$ onto a closed, two-sided ideal of a C*-algebra E, and a *-homomorphism of E onto A whose kernel is this ideal. It is important to point out that the *-homomorphisms, and not just the C^*-algebras, are part of the data comprising an extension. In particular, describing E, up to isomorphism, does not fully describe the extension.

Definition 2.3.2. Two extensions of A by the compact operators, written as

$$0 \longrightarrow K(H) \longrightarrow E \longrightarrow A \longrightarrow 0.$$

and

$$0 \longrightarrow K(H') \longrightarrow E' \longrightarrow A \longrightarrow 0.$$

are *isomorphic* if there are *-homomorphisms $\alpha : K(H) \longrightarrow K(H')$ and $\beta : E \longrightarrow E'$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & K(H) \arrow[r] \arrow[d, "\alpha"'] & E \arrow[r] \arrow[d, "\beta"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(H') \arrow[r] & E' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Remark 2.3.1. Every *-isomorphism $\alpha : K(H) \longrightarrow K(H')$ is conjugation by some unitary isomorphism of Hilbert spaces. Furthermore, if an isomorphism $\alpha = Ad_U : K(H) \longrightarrow K(H')$ extends to an isomorphism $\beta : E \longrightarrow E'$ in such a way that the diagram above commutes, then this extension is unique. Thus an isomorphism of extensions is completely determine by unitary $U : H \longrightarrow H'$, and one can think of isomorphism of extensions as a kind of 'unitary equivalence'. In particular, two essentially normal operators with essential spectrum X are essentially unitarily equivalent if and only if the extensions of $C(X)$ that they determine are isomorphic.

Lemma 2.3.1. *Suppose that φ is a *-homomorphism from A to the Calkin algebra $\mathcal{D}(H)$. There is, up to isomorphism, a unique extension of A by $K(H)$ which fits into a commutative diagram of following sort:*

$$\begin{tikzcd}
0 \arrow[r] & K(H) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & K(H) \arrow[r] & B(H) \arrow[r, "\pi"] & \mathcal{D}(H) \arrow[r] & 0.
\end{tikzcd}$$

Proof. To construct the extension, define E to be the pull-back

$$E = \{T \oplus a \in B(H) \oplus A : \pi(T) = \varphi(a)\}.$$

There are obvious maps from $K(H)$ into E, and from E to A and $B(H)$, and we obtain a commutative diagram of extensions of the required sort. To prove uniqueness, suppose that $0 \longrightarrow K(H) \longrightarrow E' \xrightarrow{\pi'} A \longrightarrow 0$ also fits into the above sort of diagram. If we denote by ρ' the map $E' \longrightarrow B(H)$ we get a diagram

$$\begin{tikzcd}
0 \arrow[r] & K(H) \arrow[r] \arrow[d, equal] & E' \arrow[r, "\pi'"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(H) \arrow[r] & E \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

by mapping $e' \in E'$ to element $\rho'(e') \oplus \pi'(e') \in E$. □

Definition 2.3.3. Two *-homomorphism $\varphi : A \longrightarrow \mathcal{D}(H)$ and $\varphi' : A \longrightarrow \mathcal{D}(H')$, are unitary equivalent, if there is a unitary isomorphism $U : H \longrightarrow H'$ such that

$$\varphi'(a) = Ad_U \varphi(a)$$

for all $a \in A$, where $Ad_U : \mathcal{D}(H) \longrightarrow \mathcal{D}(H')$ is the isomorphism induced by conjugating with the unitary $U : H \longrightarrow H'$.

Let us now show that every extension of A by $K(H)$ arises from some *-homomorphim $\varphi : A \longrightarrow \mathcal{D}(H)$. Suppose we are given an extension

$$0 \longrightarrow K(H) \longrightarrow E \xrightarrow{\pi} A \longrightarrow 0.$$

Let J be the ideal in E which is identified isomorphically with $K(H)$. The identification gives a representation ρ of J on the hilbert space H. It extends to a representation of E on H (which we shall also call by ρ) by the formula

$$\rho(e)(\rho(j)v) = \rho(ej)v(e \in E, j \in J, v \in H).$$

We may now define the required *-homomorphism $\varphi : A \longrightarrow \mathcal{D}(H)$ by the formula

$$\varphi(\pi(e)) = \pi(\rho(e))(e \in E).$$

Using Remark 2.3.1 we see that isomorphic extension give arise to unitarily equivalent *-homomorphism from A to $\mathcal{D}(H)$, and we thus arrive at the following result:

Proposition 2.3.2. *There is a one-to-one correspondence between isomorphism classes of extension of A by $K(H)$ and unitary equivalence classes of *-homomorphisms from A into Calkin algebra, in which an extension and a *-homomorphism correspond if there is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & K(H) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\varphi"'] & 0 \\
0 \arrow[r] & K(H) \arrow[r] & B(H) \arrow[r, "\pi"] & \mathcal{D}(H) \arrow[r] & 0.
\end{tikzcd}$$

From now on we shall use the term "extension of A by $K(H)$" to refer either to a short exact sequence $0 \longrightarrow K(H) \longrightarrow E \longrightarrow A \longrightarrow 0$ or to a *-homomorphism $\varphi : A \longrightarrow \mathcal{D}(H)$. We shall use the term "unitary equivalence to refer either to isomorphism of short exact sequence or to unitary equivalence of *-homomorphism into Calkin algebra.

Definition 2.3.4. (a) A unital extension $\varphi : A \longrightarrow \mathcal{D}(H)$ is *split* if there is a unital *-homomorphism $\tilde{\varphi} : A \longrightarrow B(H)$ such that $\varphi = \pi \circ \tilde{\varphi}$, where π is the quotient map from $B(H)$ to $\mathcal{D}(H)$. We shall $\tilde{\varphi}$ a *multiplicative lifting* of φ.
(b) A unital, injective extentision $\varphi : A \longrightarrow \mathcal{D}(H)$ is *semisplit* if there is another unital and injective extension $\tilde{\varphi} : A \longrightarrow \mathcal{D}(H')$ such that $\varphi \oplus \tilde{\varphi}$ is a split extension.

Definition 2.3.5. Let A be a C^*-algebra with unit and suppose that $\rho : A \longrightarrow B(H)$ is nondegenerate repesentation of A on a separable Hilbert space H. Let P be a projection which commutes with the action A modulo compact operators. In other words, assume that

$$\rho(a)P - P\rho(a) \in K(H), \tag{2.3.1}$$

for all $a \in A$. The *abstract Teoplitz operator* $T_a \in B(PH)$ with symbol $a \in A$, associated to the pair (ρ, P) , is the operator

$$PH \xrightarrow{\text{inclusion}} H \xrightarrow{\rho(a)} H \xrightarrow{\text{projection}} PH \,.$$

The *abstract Teoplitz extension* associted to the pair (ρ, P) is the homomorphism

$$\varphi_P : A \longrightarrow \mathcal{D}(PH).$$

defined by the formula $\varphi_P(a) = \pi(T_a)$.

Proposition 2.3.3. *A unital, injective extension $\varphi : A \longrightarrow \mathcal{D}(H)$ is semisplit if and only if it is unitary equivalent to an abstract Toeplitz extension.*

Proof. Suppose, on one hand, that φ is semisplit and that $\varphi \oplus \varphi'$ lifts to a *-homomorphism $\rho : A : B(H \oplus H')$. If P denotes the orthogonal projection from $H \oplus H'$ onto $H \oplus 0$ then $P(H \oplus H') \cong H$, and the abstract Teoplitz extension $\varphi_P : A \longrightarrow \mathcal{D}(P(H \oplus H'))$ is unitary equivalent to φ. Suppose, on the other hand, φ_P is a Teoplitz extension which is injective, and denote by $\rho : A \longrightarrow B(H)$ the nondegenerate representation from which it is constructed. we may form the abstract Teoplitz extension $\varphi_{P^\perp}$ using the projection $P^\perp$ complementary to P, which also satisfies the condition 2.3.1. The direct sum $\varphi_P \oplus \varphi_{p^\perp}$ is split. Indeed, the representation

ρ is a lifting; this because $P_{\rho(a)}P + P^{\perp}\rho(a)P^{\perp} = \rho(a)$ modulo $K(H)$, thanks to 2.3.1. If $\varphi_{p^{\perp}}$ is injective then we are done; otherwise just form the direct sum of ρ with a suitable nondegenerate representation and repeat the argument to obtain an injective $\varphi_{p^{\perp}}$. □

Definition 2.3.6. A bounded linear map $\sigma : A \longrightarrow B$ between two unital C^*-algebras is *completely positive* if $\sigma(1) = 1$ and

$$\Sigma_{i,j} b_i^* \sigma(a_i^* a_j) b_j \geq 0$$

for all n, all $a_1, \ldots, a_n \in A$, and all $b_1, \ldots, b_n \in B$.

It is easy to check that the other conditions on a completely positive map automatically imply that σ is bounded, so we could have left boundedness out of definition. We could also have given a 'non-unital' definition, simply by disregarding the condition $\sigma(1) = 1$.

Theorem 2.3.4 (Stinespring's Theorem). *Let A be a unital C^*-algebra. A unital linear map $\sigma : A \longrightarrow B(H)$ is completely positive if and only if there are*
(a) an isometry $V : H \longrightarrow H_1$, and
(b) a nondegenerate representation $\rho : A \longrightarrow B(H_1)$,
such that $\sigma(a) = V^*\rho(a)V$ for all $a \in A$.

Proof. The map $\sigma(a) = V^*\rho(a)V$ is completely positive since

$$\Sigma_{i,j} b_i^* \sigma(a_i^* a_j) b_j = (\Sigma_i \rho(a_i) V b_i)^* (\Sigma_i \rho(a_i) V b_i) \geq 0.$$

The proof that every completely positive map is of this form uses a device similar to the GNS construction. Define a sesquilinear form on the algebric tensor product

$A \otimes H$ (taken over $\mathbb{C}$) by

$$\langle \Sigma a_i \otimes \xi_i, \Sigma a'_j \otimes \xi'_j \rangle = \Sigma_{i,j} \langle \xi_i, \sigma(a_i^* a'_j) \xi'_j \rangle.$$

This is positive-semidefinite because σ is a completely positive map. As in the GNS construction, take the quotient of $A \otimes H$ by the subspace N comprising vectors with length zero, to obtain a genuine inner product space, and then complete A on $A \otimes H$ by left multiplication descends to a representation $\rho : A \longrightarrow B(H_1)$, and the map $V : \xi \mapsto 1 \otimes \xi + N$ has the stated properties. $\square$

Lemma 2.3.5. *Let A be a C^*-algebra, and suppose that $\rho : A \longrightarrow B(H \oplus H')$ is a representation of A on a Hilbert space which is decomposed as an orthogonal direct sum. Let*

$$\rho(a) = \begin{pmatrix} \rho_{11}(a) & \rho_{12}(a) \\ \rho_{21}(a) & \rho_{22}(a) \end{pmatrix}$$

*be the matrix representation of ρ relative to this direct sum decomposition. Suppose further that ρ_{11} is a *-homomorphism modulo the compact operators: that is the composite map*

$$A \xrightarrow{\rho_{11}} B(H) \xrightarrow{\pi} \mathcal{D}(H)$$

*is a *-homomorphism. Then $\rho_{12}(a)$ and $\rho_{21}(a)$ are compact for all $a \in A$, and ρ_{22} is also a *-homomorphism modulo the compact operators.*

Proof. Since ρ is a *-homomorphism, matrix multiplication gives

$$\rho_{11}(aa^*) = \rho_{11}(a)\rho_{11}(a)^* + \rho_{12}(a)\rho_{12}(a)^*.$$

Since ρ_{11} is a *-homomorphism modulo the compact operators, it follows that $\rho_{12}(a)\rho_{12}(a)^*$ is compact for each a, and as a result $\rho_{12}(a)$ and $\rho_{21}(a) = \rho_{12}(a^*)^*$ are compact. From this it follows that ρ_{22} is a *-homomorphism modulo the compact operators. $\square$

Theorem 2.3.6. *An extension $\varphi : A \longrightarrow \mathcal{D}(H)$ is semisplit if and only if there is a completely positive map $\sigma : A \longrightarrow B(H)$ such that $\varphi(a) = \pi(\sigma(a))$ for all $a \in A$.*

Proof. If φ is semisplit then there is some φ' such that $\varphi \oplus \varphi'$ lifts to a unital *-homomorphism $\rho : A \longrightarrow B(H \oplus H')$. If $V : H \longrightarrow H \oplus H'$ is the obvious inclusion then the map $\sigma(a) = V\rho(a)V^*$ is a lifting of φ which is completely positive and unital. Conversely, suppose that $\sigma : A \longrightarrow B(H)$ is a completely positive lifting of φ. Stinespring's Theorem provides a nondegenerate representation $\rho : A \longrightarrow B(H_1)$ and an isometry $V : H \longrightarrow H_1$ such that $\sigma(a) = V^*\rho(a)V$ for all $a \in A$. We use this isometry to identify H with a subspace Of H_1. Applying Lemma 2.3.5 we deduce that $\rho(a)$ commutes, modulo compact operators, with the orthogonal projection $P = VV^* : H_1 \longrightarrow H$. It follows that φ is unitarily equivalent to the abstract Toeplitz extension $\varphi_P(a) = \pi(P_\rho(a))P)$. By Proposition 2.3.3,$\varphi$ is semisplit. $\square$

Definition 2.3.7. If T and T' are bounded operators on the same Hilbert space H, we shall write

$$T \sim T'$$

if T and T' differ by a compact operator on H.

Definition 2.3.8. Let $\rho : A \longrightarrow B(H)$ and $\rho' : A \longrightarrow B(H')$ be two representations. We shall write

(a) $\rho' \approx \rho$ if there is a unitary $U : H' \longrightarrow H$ such that $\rho'(a) \sim U^*\rho(a)U$, for all $a \in A$, and

(b) $\rho' \lesssim \rho$ if there is a isometry $V : H' \longrightarrow H$ such that $\rho'(a) \sim V^*\rho(a)V$, for all $a \in A$.

The relation $\approx$ is an equivalence relation. The relation $\lesssim$ is transitive, but it is not a partial order: $\rho' \lesssim \rho$ and $\rho \lesssim \rho'$ do not together imply $\rho' \approx \rho$.
Here we asserts a formulation of Voiculescu 's Theorem which we need:

Theorem 2.3.7. *Let H be separable Hilbert space, let E be a unital separable C^*-algebra, and let $\rho : E \longrightarrow B(H)$ be a nondegenerate representation. Let L be a separable Hilbert space and let $\sigma : E \longrightarrow B(L)$ be a completely positive map. If σ has the property that*

$$\rho(e) \in K(H) \Rightarrow \sigma(e) = 0$$

for every $e \in E$, then $\sigma \lesssim \rho$.

In our present language, Lemma 2.3.5 and Theorem 2.3.6 may be reformulated as follows:

Lemma 2.3.8. *If $\rho : A \longrightarrow B(H)$ and $\rho' : A \longrightarrow B(H')$ are representation of A and if $\rho \lesssim \rho$, then there is a completely positive $\rho'' : A \longrightarrow B(H'')$ such that $\rho \approx \rho' \oplus \rho''$.*

Theorem 2.3.9. *Let $\rho : A \longrightarrow B(H)$ and $\rho' : A \longrightarrow B(H')$ be nondegenerate representations of a separable, unital C^*-algebra on separable Hilbert spaces. Suppose that $\rho[A] \cap K(H') = 0$, then $\rho' \oplus \rho \approx \rho$. Thus if, in addition, $\rho'[A] \cap K(H') = 0$, then $\rho \approx \rho'$.*

Proof. Let L be the direct sum of countably many copies of the Hilbert space H' and let $\sigma : A \longrightarrow B(L)$ be the direct sum of countably many copies of the representation ρ'. It follows from Theorem 2.3.7 that $\sigma \lesssim \rho$. It therefore follows from Lemma 2.3.8 that $\rho \approx \sigma \oplus \rho''$ for some completely positive map ρ''. But it is clear that $\rho' \oplus \sigma \approx \sigma$ and hence

$$\rho \approx \sigma \oplus \rho'' \approx \rho' \oplus \sigma \oplus \rho'' \approx \rho' \oplus \rho$$

which is what we wanted to prove. □

Definition 2.3.9. Let A be a separable, unital C^*-algebra and let $\rho : A \longrightarrow B(H)$ be a representation of A on a separable Hilbert space. The *dual* algebra of A associated to the representation ρ is the following C^*-algebra of $B(H)$:

$$\mathcal{D}(A) = \{T \in B(H) : [T, \rho(a)] \sim 0 \forall a \in A\}.$$

A projection $P \in \mathcal{D}(A)$ determines a Toeplitz extension $\varphi_P : A \longrightarrow \mathcal{D}(PH)$ by the formula

$$\varphi_P(a) = \pi(Pa)$$

If ρ_1 and ρ_2 are two adequte representations of a separable unital C^*-algebra A on two seperable Hilbert spaces H_1 and H_2, then according to Theorem 2.3.9 there is a unitary isomorphism $U : H_1 \longrightarrow H_2$ such that $U\rho_1(a) \sim \rho_2(a)U$ for every $a \in A$. It follows that the *-homomorphism Ad_U maps $\mathcal{D}_{\rho_1}(A)$ isomorphically onto $\mathcal{D}_{\rho_2}(A)$. Therefore , up to *-isomorphism the dual algebra $\mathcal{D}(A)$ is independent of the choice of adequte representation used to define it. Furthermore any two such such *-homomorphism Ad_U induce the same isomorphism of K-theory groups, because inner automorphisms acts trivially on K-theory by Lemma 2.1.2. These observation allow us to make the following definition:

Definition 2.3.10. Let A be a seperable and unital C^*-algebra. We define the reduced analytic K-homology groups of A to be

$$\widetilde{K}^1(A) = K_0(\mathcal{D}(A)), \widetilde{K}^0(A) = K_1(\mathcal{D}(A)),$$

where $\mathcal{D}(A)$ denotes the dual algebra of some arbitrarily chosen adequte representation of A.

Definition 2.3.11. Let A and B be separable, unital C^*-algebras and let $\alpha : A \longrightarrow B$ be a unital *-homomorphism. Let ρ_A and ρ_B be adequte representations of A and B on separable Hilbert spaces H_A and H_B. An isometry $V : H_B \longrightarrow H_A$ *covers* $\alpha : A \longrightarrow B$ if

$$V^* \rho_A(a) V \sim \rho_B(\alpha(a))$$

for every $a \in A$.

Suppose that V covers α. Then Lemma 2.3.5 shows that the projection VV^* belong to $\mathcal{D}(A)$. From this we easily obtain:

Lemma 2.3.10. *If $\alpha : A \longrightarrow B$ is a unital *-homomorphism of separable unital C^*-algebras, and if $V : H_B \longrightarrow H_A$ covers α, then the *-homomorphism $Ad_V(T) = VTV^*$ maps $\mathcal{D}(B)$ into $\mathcal{D}(A)$.*

Lemma 2.3.11. *Every unital *-homomorphism $\alpha : A \longrightarrow B$ is covered by some isometry $V : H_B \longrightarrow H_A$, and any two such isometries, both covering α, induce the same map on K-theory:*

$$(Ad_{V_1})_* = (Ad_{V_2})_* : K_{1-p}(\mathcal{D}(B)) \longrightarrow K_{1-p}(\mathcal{D}(A)).$$

Proof. According to Voiculescu's Theorem 2.3.7, there is an isometry $V : H_B \longrightarrow H_A$ such that $\rho_B(\alpha(a)) \sim V^* \rho_A(a) V$, for all $a \in A$. This takes care of the existence of covering isometries. The uniqueness part of the lemma is exactly the same as Proposition 2.1.15. □

Definition 2.3.12. If $\alpha : A \longrightarrow B$ is a unital *-homomorphism then we define

$$\alpha^* : \widetilde{K}^p(B) = \widetilde{K}^p(A)$$

to be the map $(Ad_{V_\alpha})_* : K_{1-p}(\mathcal{D}(B)) \longrightarrow K_{1-p}(\mathcal{D}(A))$, where V_α is any isometry which covers α.

Lemma 2.3.12. *The correspondence $\alpha \mapsto \alpha^*$ is a contravariant functor.*

Proof. If $V_\alpha : H_B \longrightarrow H_A$ covers $\alpha : A \longrightarrow B$, and if $V_\beta : H_C \longrightarrow H_B$ covers $\beta : B \longrightarrow C$, then $V_\alpha V_\beta$ covers $\beta\alpha$. □

Definition 2.3.13. Let A be a separable C^*-algebra, possibly without unit, and let $\dot{A}$ be the C^*-algebra with a unit adjoined. define the (unreduced) *K-homology groups* of A to be

$$K^p(A) = K_{1-p}(\mathcal{D}(\dot{A}))$$

where $\mathcal{D}(\dot{A})$ is the dual of an adequte representation of $\dot{A}$.

Thus the unreduced K-homology groups of A are the reduced K-homology groups of $\dot{A}$. Since any *-homomorphism $\alpha : A \longrightarrow B$ gives rise to a unital *-homomorphism $\dot{\alpha} : \dot{A} \longrightarrow \dot{B}$, our previous discussion of reduced K-homology makes the unreduced groups contravariantly functorial for arbitrary *-homomorphisms.

Definition 2.3.14. Let $J \subseteq A$ be an ideal in a separable C^*-algebra, and let ρ be a representation of A on a Hilbert space H. We define the *relative dual algebra* $\mathcal{D}_\rho(A//J)$ to be the following ideal in $\mathcal{D}_\rho(A)$:

$$\mathcal{D}_\rho(A//J) = \{T \in \mathcal{D}_\rho(A) : T\rho(a) \sim 0 \sim \rho(a)T \forall a \in J\}$$

We shall sometimes say that an operator $T \in \mathcal{D}(A)$ satisfying the condition appearing in this definition is *locally compact* for J.

Definition 2.3.15. Let say that the short exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow B \longrightarrow 0$$

is *semisplit* if the quotient map $\dot{A} \longrightarrow \dot{B}$ admits a completely positive section.

Proposition 2.3.13. *If the short exact sequence*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

of separable C^*-algebra is semisplit, and if $V : H_{\dot{A}/J} \longrightarrow H_{\dot{A}}$ is any isometry which covers the quotient map $\pi : A \longrightarrow A/J$, then the *-homomorphism

$$Ad_V : \mathcal{D}(A/J) \longrightarrow \mathcal{D}(A//J)$$

induces an isomorphism on K-theory.

Proof. Denote by ρ_A and $\rho_{A/J}$ the given representation of A and A/J on $H_{\dot{A}}$ and $H_{\dot{A}/J}$, respectively. Let σ be a completely positive section of the quotient map π. By Stinespring's Theorem 2.3.4, there is a separable Hilbert space H and a dilation of the completely positive map $\rho_A\sigma : A/J \longrightarrow B(H_{\dot{A}})$ to a representation

$$\rho'_{A/J} = \begin{pmatrix} \rho_A & * \\ * & * \end{pmatrix} : A/J \longrightarrow B(H_{\dot{A}} \oplus H).$$

Now let $H'_{\dot{A}/J} = H_{\dot{A}} \oplus H$ and let $W : H_{\dot{A}} \longrightarrow H'_{\dot{A}/J}$ be the obvious inclusion. Then Ad_W maps the ideal $\mathcal{D}_{\rho_A}(A//J)$ into $\mathcal{D}_{\rho'_{A/J}}(A/J)$. The composition

$$H_{\dot{A}/J} \xrightarrow{V} H_{\dot{A}} \xrightarrow{W} H'_{\dot{A}/J}$$

is an isometry which covers the identity map $\dot{A}/J \longrightarrow \dot{A}/J$. It follows from Lemma 2.3.11 that the composition

$$\mathcal{D}_{\rho_{A/J}}(A/J) \xrightarrow{Ad_V} \mathcal{D}_{\rho_A}(A//J) \xrightarrow{Ad_W} \mathcal{D}_{\rho'_{A/J}}(A/J)$$

induces an isomorphism on K-theory. This shows that Ad_V is injective at the level of K-theory. To prove surjectivity, define a representation ρ'_A of A on the Hilbert space

$$H'_{\dot{A}} = H_{\dot{A}} \oplus H'_{\dot{A}/J} = H_{\dot{A}} \oplus H_{\dot{A}} \oplus H$$

by forming the sum of ρ_A, acting on the first summand, with $\rho'_{A/J}\pi$, acting on $H'_{\dot{A}/J}$. The obvious inclusion $X : H'_{\dot{A}/J} \longrightarrow H'_{\dot{A}}$ covers $\pi : A \longrightarrow A/J$. We are going to show that the composition

$$(*)\mathcal{D}_{\rho_A}(A//J) \xrightarrow{Ad_W} \mathcal{D}_{\rho'_{A/J}}(A/J) \xrightarrow{Ad_X} \mathcal{D}_{\rho'_A}(A//J)$$

induces an isomorphism on K-theory; this will finish the proof. The composition of isometries

$$H_{\dot{A}} \xrightarrow{W} H_{\dot{A}} \oplus H \xrightarrow{X} H'_{\dot{A}} = H_{\dot{A}} \oplus H_{\dot{A}} \oplus H$$

includes $H_{\dot{A}}$ as the second summand. This isometry does not cover the identity map $1_A : A \longrightarrow A$. However, it is homotopic by a rotation to the isometry

$$H_{\dot{A}} \xrightarrow{Y} H_{\dot{A}} \oplus H_{\dot{A}} \oplus H$$

which maps $H_{\dot{A}}$ into the first factor of the triple sum, and this isometry does cover 1_A. The corresponding homotopy of *-homomorphisms

$$T \mapsto \begin{pmatrix} sin^2(\frac{\pi}{2}t)T & sin(\frac{\pi}{2}t)cos(\frac{\pi}{2}t)T & 0 \\ sin(\frac{\pi}{2}t)cos(\frac{\pi}{2}t)T & cos^2(\frac{\pi}{2}t)T & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

connects the composition (*) to the *-homomorphism

$$Ad_Y : \mathcal{D}_{\rho_A}(A//J) \longrightarrow \mathcal{D}_{\rho'_A}(A//J),$$

which induces an isomorphism on K-theory by the argument of Lemma 2.3.11. The proof of the proposition is therefore completed by an appeal to the homotopy invariance of K-theory. □

Definition 2.3.16. Let X be a compact Hausdorff space and suppose that $C(X)$ acts via a nondegenerate representation on a Hilbert space H. An operator $T \in B(H)$ is *pseudolocal* if fTg is a compact operator for every pair of continuous function f and g on X with disjoint supports.

If $T \in \mathcal{D}(C(X))$ then is pseudolocal. Indeed, if f and g have disjoint supports then

$$fTg = f[T, g] \sim 0.$$

The converse is a useful observation of Kasparov:

Lemma 2.3.14 (Kasparov's Lemma). *Let X be a compact Hausdorff space and suppose that $C(X)$ acts via a nondegenerate representation on a Hilbert space H. If $T \in B(H)$ is pseudolocal then $[T, f] \sim 0$ for every $f \in C(X)$. Thus $\mathcal{D}(C(X))$ is composed precisely of pseudolocal operators on H.*

Proof. Let us recall that every representation of $C(X)$ extends to a representation of the bounded Borel functions, and observe that if T is pseudolocal then $fTg \sim 0$ for all bounded Borel functions f, g whose supports are disjoint. This is because we can find continuous functions f', g', with disjoint supports, such that $f = ff'$ and $g = gg'$. Hence $fTg = ff'Tgg' \sim 0$.

It is suffices to show that if f is a real valued continuous function on X then $[f, T]$ may be approximated in norm by compact operators. Let $\epsilon > 0$ and partition the range of f into Borel sets, $U_1, U_2, \ldots, U_n$, each of diameter less than ϵ, in such a way that $\overline{U_i}$ intersects $\overline{U_j}$ if and only if $|i - j| \leq 1$. Let $f_1, \ldots, f_n$ be the characteristic funtions of the Borel sets $f^{-1}(U_1), \ldots, f^{-1}(U_n)$ in X. Observe that

(a) if $|i-j|>1$ then $f_i T f_j \sim 0$, and

(b) if $\tilde{f} = f(x_1)f_1 + \cdots + f(x_n)f_n$, where $x_1, \ldots, x_n$ are points chosen from $f^{-1}(U_1)$ $\ldots, f^{-1}(U_n)$, then $||f-\tilde{f}|| < \epsilon$.

The operator $[f,T]$ is within $2\epsilon||T||$of $[f',T]$, and since $f_1+\cdots+f_n=1$ we have that

$$\begin{aligned}\tilde{f}T - T\tilde{f} &= \sum_{i,j} f(x_i)f_iTf_j - f_iTf(x_j)f_j \\ &\sim \sum_{|i-j|=1} (f(x_i)-f(x_j))f_iTf_j.\end{aligned}$$

Break the last sum into two parts, one where $i=j+1$ and where $i=j-1$. The first part is a direct sum of operators $(f(x_{j+1})-f(x_i))f_{j+1}Tf_j$, from f_jH to $f_{j+1}H$. It follows that the norm of this part is the maximum of the norms of its summanands. Therefore since $|f(x_{j+1})-f(x_j)| < 2\epsilon$ the norm of the first part is no more than $2\epsilon||T||$. Treating, the second part in the same way, we see that last line of the display is of norm less than $4\epsilon||T||$. It follows that $[f,T]$ is a norm limit of compact operators, as required. $\square$

Now we are ready to assert and prove the main target of this section. Let X be a locally compact space, and suppose that X is provided with a merizable compacfication $\dot{X}$. Let $A=C(\dot{X})$ and $J=C_0(X)$, so that $A/J = C(\partial X)$ where $\partial X = \dot{X}\setminus X$. We assume that J is adequtely represented on a Hilbert space H. The representation of J extends to a representation of A. This extended representation is also adequte, since X is dense in $\dot{X}$.

There are two ways of forming a C^*-algebra from the above data:

(a) as in definition 2.3.16, we may form the relative dual algebra $\mathcal{D}(A//J)$, which consists of operators on H which are locally compact for the action J and commute modulo the compact operators with the action of A, or

(b) as in definition 2.1.10, we may form the C^*-algebra $C^*(X)$, which is the norm closure of the collection of operators on H that are locally compact for the action $C_0(X)$ and controlled for the compactfication $\dot{X}$

The important fact is this that the two constructions yield exactly the same result:

Theorem 2.3.15. *Let X be a locally compact Hausdorff space, equipped with a metrizable compactfication $\dot{X}$. Then the algebra $C^*(X)$ associated to the continuously controlled coarse structure on X is equal to the relative dual algebra $\mathcal{D}(A//J)$, where $A = C(\dot{X})$ and $J = C_0(X)$.*

The proof of the theorem is given by the next two lemmas. The first shows that $C^*(X) \subseteq \mathcal{D}(A//J)$, and the second shows that $\mathcal{D}(A//J) \subseteq C^*(X)$.

Lemma 2.3.16. *If $T \in B(H)$ is a locally compact and controlled operator, then $T \in \mathcal{D}(A//J)$.*

Proof. First we show that T is pseudolocal, i.e. if f and g are continuous functions on $\dot{X}$ with disjoint supports then ftg is compact, then by Kasparov 's Lemma 2.3.14, the result will follow.

Note that

$$Support(fTg) \subseteq Support(T) \cap (Support(f) \times Support(g)).$$

Let (x_n, x'_n) be a sequence in Support(fTg). If one of x_n or x'_n were to converge to a point of ∂X, then by Lemma 1.2.2(i) both sequences would have to converge to the same point and this point would have to lie in *Supp*$(f) \cap$ *Supp*(g), contradicting disjointness. We conclude that *Supp*(fTg) contains no sequence either of whose components converges to a point of ∂X, and hence that Support(fTg) is compact subset of $X \times X$. But a compactly supported, locally compact operator is compact. $\square$

Lemma 2.3.17. *If T is an operator belonging to $\mathcal{D}(A//J)$, then for every $\epsilon > 0$ there exists a locally compact and controlled operator T' with $||T - T'|| < \epsilon$.*

Proof. Choose sequences $\{K_n\}$ and $\{U_n\}$ as in the Lemma 1.2.4; our assumption that $\dot{X}$ is merizable implies that such sequence can be found.

Let $T \in \mathcal{D}(A//J)$ and let $\epsilon > 0$. We shall define $T' \in \mathcal{D}(A//J)$, within ϵ of T, whose support satisfies the hypotheses of Lemma 1.2.4. To obtain T', we shall build a sequence $T_0, T_1, \cdots \in \mathcal{D}(A//J)$, starting with $T_0 = T$, and a sequence of open sets $V_1, V_2, \ldots$ in X such that

(a) $K_n \subseteq V_n$,

(b) $||T_n - T_{n-1}|| < 2^{-n}\epsilon$,

(c) $Supp(T_n) \subseteq Supp(T_{n-1})$, and

(d) $Supp(T_n) \cap (U_n \times V_n) = \emptyset$ and $Supp(T_n) \cap (V_n \times U_n) = \emptyset$.

Then we shall define $T' = \lim_{n\to\infty} T_{n-1}$, which has the required propoties.

Supposing that $T_0, \ldots, T_{n-1}$ and $V_1, \ldots V_{n-1}$ have been constructed, we we obtain T_n and V_n as follows. First recall from Remark 2.2.1 that the representation of $A = C(\dot{X})$ on H extends to a representation of the Bounded Borel functions. So each Borel subset $B \subseteq \dot{X}$ determines a projection operator P_B on H. Let $W_1, W_2, \ldots,$ be a decreasing sequence of neighborhoods of K_n in $\dot{X}$, With intersection K_n. We can assume that the closure of W_1 is disjoint from the closure of U_n, and therefore

$$P_{U_n} T_{n-1} P_{W_1} \sim P_{W_1} T_{n-1} P_{U_n} \sim 0,$$

since T_{n-1} is pseudolocal. Since $K_n = \bigcap W_k$ does not meet X, the projections P_{W_k}converge to zero in the strong operator topology. Since $P_{U_n}T_{n-1}P_{W_1}$ is compact, it follows that

$$\lim_{n\to\infty} ||P_{U_n}T_{n-1}P_{W_k}|| = \lim_{n\to\infty} ||(P_{U_n}T_{n-1}P_{W_1})P_{W_k}|| = 0,$$

and similarly $\lim_{n\to\infty} ||P_{W_k}T_{n-1}P_{U_n}|| = 0$. So we may define $V_n = W_k$, where k is so large that $||P_{U_n}T_{n-1}P_{W_k}|| < 2^{-n-1}\epsilon$ and $||P_{W_k}T_{n-1}P_{U_n}|| < 2^{-n-1}\epsilon$, and

$$T_n = T_{n-1} - P_{U_n}T_{n-1}P_{W_k} - P_{W_k}T_{n-1}P_{U_n}.$$

Then items (a) to (d) above hold, as required. □

Corollary 2.3.18. *Let X be a locally compact space equipped with a metrizable compactfication $\dot{X}$, and let $\partial X = \dot{X} \setminus X$. Then there is an isomorphism*

$$K_p(C^*(X)) \cong \widetilde{K}^{1-p}(C(\partial X)).$$

Proof. By Theorem 2.3.15 we have

$$K_p(C^*(X)) = K_p(\mathcal{D}(A//J))$$

But Proposition 2.3.13 implies

$$K_p(\mathcal{D}(A//J)) \cong K_p(\mathcal{D}(A/J)$$

So by Definition 2.3.10 we have

$$K_p(C^*(X)) \cong K_p(\mathcal{D}(A/J) \cong \widetilde{K}^{1-p}(A/J) = \widetilde{K}^{1-p}(C(\partial X)).$$

□

Bibliography

[1] N. Higson and J. Roe, *Analytic K-homology*, Oxford University Press, Oxford, 2000.

[2] N. Higson, E.K.Pedersen, and J. Roe, *C*-algebras and controlled topology*, K-theory, 11(1997), 209-239.

[3] N. Higson and J. Roe, *A homotopy invariance theorem in coarse cohomology and K-theory*, Trans. Amer. Math. Soc. 345 (1994), no. 1, 347–365

[4] N. Higson, J. Roe, and G. Yu, *A coarse Mayer-Vietoris principle*, Math. Proc. Cambridge Philos. Soc. 114 (1993), 85-97.

[5] P. D. Mitchener, *Addendum to "Coarse homology theories"*, Algebraic and Geometric Topology, 1 (2001), 271-297.

[6] P. D. Mitchener, *Addendum to "Coarse homology theories"*, Algebraic and Geometric Topology, 3 (2003), 1089-1101.

[7] J. Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. 104 (1993), no. 497.

[8] J. Roe, *Index theory, coarse geometry, and the topology of manifolds*, Regional Conference Series on Mathematics, vol. 90, CBMS Conference Proceedings, American Mathematical Society, 1996.

[9] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford University Press, Oxford, 1993.

[10] N. Wright, *C_0 coarse geometry and scalar curvature*, Journal of Functional Analysis, 197 (2003) 469-488.