Lecture course „Riemannian Geometry I“ WS 23/24

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1 Course description

Riemannian geometry deals with the geometric aspects of smooth manifolds.

The starting point of this theory are surfaces embedded in Euclidean space $\mathbb{R}^3$ and their intrisic geometric features. We expect intuitively that the sphere is curved while a flat plane is not, and a saddle is curved in a different way, and indeed, that these notions of curvature are intrinsic.

More systematically, a smooth manifold is often naturally equipped with a Riemannian metric: a rule to determine length and angle of tangent vectors. This gives rise to concepts like geodesics (length-minimizing curves) and the Riemann curvature tensor. The features of these and their relation (among themselves and with the global topology) will be studied in the course. Concretely this means (list a bit tentative towards the end):

- We will introduce Smooth manifolds and Riemannian metrics on them.
- We will define the concept ofgeodesic and of energy of a curve.
- We will prove existence (and uniqueness) of geodesics and introduce basic properties (length minimizer, energy minimizer, solving the geodesic OdE)
- We will construct the Riemannian exponential map from the tangent space to the manifold and study its basic properties (diffeomorphism near zero)
- We will introduce the Riemannian Curvature tensor and its smaller cousins: Ricci curvature and scalar curvature
- We will prove the Hopf-Rinov theorem: if we are in negative curvature and all loops are contractible: there are globally unique geodesics.
- We will introduce Jakobi fields governing the variation of geodesics and the cut locus where geodesics cut themselves.
- We will learn about Myer's theorem limiting the diameter if the Ricci curvature is positive.
- We will study Riemannian submersions and compute their curvature tensor.

2 Prerequisites

We assume a solid understanding of the basis courses AGLA I+II and Analysis I+II, or any full first year courses in linear algebra and in analysis.

The coure “Analysis III” is relevant for the lecture and prepares some of the aspects of the courses. In particular, the concept of smooth manifold and tangent space is needed and used from the start. This knowledge could be acquired via individual studies within one or two weeks before the semester and I’m happy to give advise on appropriate literature for that. So: having taken the course “Analysis III” is not strictly a requirement.

3 Literature

There are many nice books introducing into Riemannian geometry. So far, it is not planned that the course will strictly follow a particular one. As additional reading, I suggest to look into several of them and check which style suits best. My favorite is probably Gallot, Hullin, Lafontaine. Note that it all goes back to Gauss, compare the essay of Dombrowski (which include sthe original).

The books on curves and surfaces have a different focus (and also different level), but certainly serve as a nice introduction.

[1] Christian Bär, *Elementary differential geometry*, Cambridge University Press, Cambridge, 2010. Translated from the 2001 German original by P. Meerkamp. MR2664879 ↑

[2] Manfredo Perdigão do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty. MR1138207 ↑

[3] Peter Dombrowski, *150 years after Gauss’ “Disquisitiones generales circa superficies curvas”*, Astérisque, vol. 62, Société Mathématique de France, Paris, 1979. With the original text of Gauss. MR535996 ↑

[4] Sylvestre Gallot, Dominique Hulin, and Jacques Lafontaine, *Riemannian geometry*, 3rd ed., Universitext, Springer-Verlag, Berlin, 2004. MR2088027 ↑

[5] Jürgen Jost, *Riemannian geometry and geometric analysis*, 7th ed., Universitext, Springer, Cham, 2017. MR3726907 ↑

[6] John M. Lee, *Introduction to Riemannian manifolds*, Graduate Texts in Mathematics, vol. 176, Springer, Cham, 2018. Second edition of [MR1468735]. MR3887684 ↑

[7] Peter Petersen, *Riemannian geometry*, 3rd ed., Graduate Texts in Mathematics, vol. 171, Springer, Cham, 2016. MR3469435 ↑