Handout for the workshop on the Baum-Connes conjecture
Useful facts to keep in mind

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1 C^*-algebras

- A C^*-algebra A is
  - a $\mathbb{C}$-algebra
  - together with an involution $*: A \to A$ such that $(\lambda ab)^* = \overline{\lambda} b^* a^*$ for all $\lambda \in \mathbb{C}$, $a, b \in A$.
  - It is equipped with an algebra norm, i.e. $|ab| \leq |a|\,|b|$.
  - A is complete with this norm.
  - The *C^*-identity* $|a^*a| = |a|^2$ is satisfied $\forall a \in A$

  A C^*-algebra homomorphism is a continuous map which respects all the algebraic structure.

- Examples of C^*-algebras:
  - $\mathbb{C}$
  - every closed $*$-subalgebra of the bounded operators $B(H)$ on a Hilbert space H. By the GNS-construction, every C^*-algebra is isomorphic to such an algebra.
  - If X is a (locally compact Hausdorff) space, the algebra $C_0(X)$ of complex valued continuous functions on X, which vanish at infinity, is a commutative C^*-algebra. Every commutative C^*-algebra is isomorphic to such an algebra.
  - If X is a measure space, $L^\infty(X)$ with its norm is another example of a commutative C^*-algebra. (This is usually very different from $C(X)$!).
  - Given an arbitrary C^*-algebra A and X as before, $C_0(X; A)$ is the algebra of continuous functions on X with values in A, which vanish at infinity. The algebraic operations are defined pointwise, the norm is the sup-norm.
  - Given a Hilbert space H, $K(H)$ is the algebra of compact operators $k: H \to H$, i.e. the closure of the algebra of all operators of finite rank (i.e. with finite dimensional image). This is a two-sided ideal inside $B(H)$.

  - Given an arbitrary C^*-algebra A, the algebra of n-by-n-matrices $M_n(A)$ also is a C^*-algebra.

- Short exact sequences of C^*-algebras:

$$0 \to I \to A \to B \to 0,$$

  consist of C^*-algebras together with C^*-algebra homomorphism, such that kernel of outgoing and image of incoming map coincide at each algebra in question. In such a diagram, I is a two-sided closed ideal of A, and $B = A/I$ is the quotient.

  Example: $0 \to K(H) \to B(H) \to C(H) \to 0$, where by definition $C(H) = B(H)/K(H)$ is the *Calkin algebra.*

- $a \in A$ is called
  - self adjoint, if $a = a^*$
  - unitary, if $a^* = a^{-1}$
  - a projection, if $a = a^2 = a^*$.

- For self-adjoint and unitary element a of a C^*-algebra A, functional calculus is defined, i.e. a C^*-homomorphism

$$C_0(\mathbb{C}) \to A \colon f \mapsto f(a)$$

  with $\mathrm{id}(a) = a$. If we replace for the domain of f $\mathbb{C}$ by the spectrum of a (a compact subset of $\mathbb{C}$), this map becomes a C^*-algebra embedding.

- Given a discrete group Γ, the reduced group C^*-algebra $C_r^*\Gamma$ is defined as the norm closure of the (algebraic) group ring $\mathbb{C}\Gamma$ inside the bounded operators on $l^2(\Gamma)$, where $l^2(\Gamma)$ is the Hilbert spaces of L^2-functions on Γ, which we equip with the discrete measure (every element has volume one). Observe that this is a Haar measure on Γ.

- In a similar way, we can form the C^*-algebra of a topological group G (e.g. a Lie group) as completion of the convolution algebra $C_{comp}(G)$ inside $B(L^2(G))$. Attention: $C_{comp}(G)$ consists of functions on G with compact support, but the product is *not* pointwise, it is given by convolution (again using a fixed Haar measure).

- One can form (completed) tensor products of C^*-algebras. Of particular importance is
  - $A \otimes M_n(\mathbb{C}) = M_n(A)$
  - $A \otimes K(l^2(\mathbb{Z}))$, a limit of $M_n(A)$ for $n \to \infty$.
  - $C_0(X) \otimes A = C_0(X; A)$.

2 K-theory

- For a C^*-algebra A, we define the abelian groups
  - $K_0(A)$, consisting of (formal differences of) (equivalence classes of) projections in A, $M_n(A)$ (and $A \otimes K$). One of the (several different) ways to define equivalence is as: homotopic through the spaces of projections.
  - $K_1(A)$ consists of unitary elements of A, $M_n(A)$ (and $A \otimes K$), again module equivalence given by homotopies through the space of unitaries.
- Both K_0 and K_1 are functors from the category of C^*-algebras to the category of abelian groups.
- Examples:

A	$\mathbb{C}$	K	B	C	C_r^*G for G compact	$C_0(X)$	C_r^*G
$K_0(A)$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$Rep(G)$	$K^0(X)$	??
$K_1(A)$	0	0	0	$\mathbb{Z}$	0	$K^1(X)$	??

  $K^0(X)$ is the topological K-theory of X, defined as (equivalence classes of formal differences) of vector bundles over X.
- Morita-Equivalence:
$$K_*(A) = K_*(M_n(A)) = K_*(A \otimes K).$$
- Bott periodicity and six term exact sequence (some people call this excision): Given a short exact sequence $0 \to I \to A \to B \to 0$, we get an induced exact sequence
$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I). \end{array}$$
  The map $\delta\colon K_1(B) \to K_0(I)$ is the *index map*, the map $K_0(B) \to K_1(I)$ involves in addition *Bott periodicity*.
- Example: for $0 \to K \to B \to C \to 0$, we get the K-theory exact sequence
$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$
- Homotopy invariance: two C^*-homomophism $f_0, f_1 : A \to B$ are called homotopic, if they can be joint by a (pointwise continuous) path $f_t : A \to B$ of C^*-homomorphism. In this situation, f_0 and f_1 induce identical maps on K_*.
- Every trace on a C^*-algebra A gives rise to a homomorphism $K_0(A) \to \mathbb{C}$. The usual operator trace is defined on all of K, it gives rise to the isomorphism
$$\mathrm{tr}\colon K_0(K) \to \mathbb{Z}.$$
  For a projection $p \in K$, $\mathrm{tr}(p)$ is the dimension of the image of p.

3 Index and differential operators

- Every element of $K_1(C)$ can be represented by a unitary in C. By definition, $C = B/K$, i.e. $u \in C$ comes from an element U of B. Being unitary in C means that $U^*U = 1 - k_0$ and $UU^* = 1 - k_1$ for suitable $k_0, k_1 \in K$. By Atkinsons theorem, such a bounded operator $U\colon H \to H$ is *Fredholm*, i.e. has finite dimension kernel and cokernel (and closed image). Therefore, its (Fredholm) index

  $$ind(U) := \dim(\ker(U)) - \dim(\operatorname{coker}(U)) = \dim(\ker(U)) - \dim(\ker(U^*))$$

  is defined. Using the index map $\delta\colon K_1(C) \to K_0(K)$, the following holds:

  $$\operatorname{tr}\delta([u]) = \operatorname{ind}(U).$$

  This gives an isomorphism $K_1(C) \cong \mathbb{Z}$.

- In the following, we will encounter *unbounded* operators on a Hilbert space H. Such operators are only defined on a dense subspace of H. The adjoint of an unbounded operator is another unbounded operator. If an unbounded operator A is self adjoint, the functional calculus for A is defined, i.e. there is a $*$-homomorphism

  $$L^\infty(\mathbb{R}) \to B(H)\colon f \mapsto f(A),$$

  with the property that if $f(t) = tg(t)$, the $f(A) = Ag(A)$, where on the right hand side we compose the operators A and $g(A)$. If we replace $\mathbb{R}$ by the spectrum of A, this becomes a C^*-algebra embedding.

- A generalized Dirac operator D on a (Riemannian) manifold M is a certain type of first order differential operator on M (defining a map between smooth section of two (Hermitian) vector bundles E and F over M). The following properties are important:

  - D is a self adjoint unbounded operator on $L^2(E)$, the Hilbert space of L^2-section.
  - In particular, functions like
    * $D(1 + D^2)^{-1/2}$
    * e^{itD}; $t \in \mathbb{R}$
    * $^{-tD^2}$; $t \geq 0$

    are defined. The first one is of the type $\chi(D)$ with $\chi :: \mathbb{R} \to [-1, 1]$ even, $\chi(t) \xrightarrow{t\to\pm\infty} \pm 1$. Such a χ is called a chopping function.
  - D satisfies elliptic regularity. Classically: if Df is smooth (on some open subset V of M, then f is smooth (on V). This can be refraised in terms of operators: if $h \in C_0(\mathbb{R})$, and $f, g \in C_0(M)$, then $fh(D)g$ is compact. Here f and g denote the operators on $L^2(E)$ which are given by pointwise multiplication. If M is compact, this means that $h(D)$ itself is compact.
  - D has finite propagation: $gDf = 0$ if the support of g and f are disjoint (this is simply true because D is a differential operator). It translates to corresponding, but weaker statements for functions like $h(D)$.

- Example: on a Riemannian manifold M, let d be the exterior differential on the direct sum $\Omega^*(M)$ of all spaces of differential forms, and $\delta = d^*$ its adjoint. Then $d+\delta$ is a generalized Dirac operator.

- *Grading* Usually (on even dimensional manifolds) a generalized Dirac operator comes with a grading $E = E_+ \oplus E_-$, and

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0\colon L^2(E_+) \oplus L^2(E_-) \to L^2(E_+) \oplus L^2(E_-). \end{pmatrix}$$

. If χ as above is an odd function, then

$$\chi(D) = \begin{pmatrix} 0 & \chi(D_-) \\ \chi(D_+) & 0\colon L^2(E_+) \oplus L^2(E_-) \to L^2(E_+) \oplus L^2(E_-). \end{pmatrix}$$

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- Example of a grading: on $\Omega^*(M)$ we can define a grading $\Omega^{ev}(M) \oplus \Omega^{odd}(M)$ by distinguishing between differential forms of even and odd degree. Then $d+\delta$ decomposes as described.

- Using a unitary isomorphism $U : L^2(E_-) \to L^2(E_+)$ (which usually exists, we assume this now) and a chopping function, we obtain an operator $T := U\chi(D_+) \colon L^2(E_+) \to L^2(E_+)$.

  If M is a compact manifold, then $T^*T - 1$ and $TT^* - 1$ are compact. Therefore T defines a class in $K_1(C)$. We define the index of D to be $\delta([T]) \in K_0(K)$.

  In this case, identifying $K_0(K)$ with $\mathbb{Z}$ gives a number, which is indeed the difference of the dimension of kernel and cokernel of D, i.e. the Fredholm index of D. Using appropriate C^*-algebras instead of K, B, and C, one can define indices of Dirac operators for non-compact manifolds as elements in K-groups of certain C^*-algebras. Moreover, one can refine the analysis, if additional structure is given, e.g. if there is symmetry (a group acting on M such that D is equivariant), to obtain indices in more interesting K-groups like $K_*(C_r^*\Gamma)$.

- No matter how and where the index lives, if it is non-zero, then zero does belong to the spectrum of D, i.e. D can not have a bounded inverse. This is important for applications to geometry, where D often has a bounded inverse if some particular geometric condition is satisfied.

4 K-homology

- For every C^*-algebra A on can also define K-homology groups $K^0(A)$ and $K^1(A)$. This gives a contravariant functor (i.e. arrows are reversed) from the category of C^*-algebras to the category of abelian groups. Properties very similar to K-theory are satisfied.

- Most important is, that there is a pairing

$$K^*(A) \otimes K_*(A) \to \mathbb{Z}.$$

- For us most important is the case $A = C(X)$. In this case, every Dirac type differential operator D produces an element $[D] \in K^*(C(X))$. (If X is a manifold, one will use operators defined over X, in general, every (proper) map $M \to X$ can be used to push K-homology classes given by such operators from $K^*(C(M))$ to $K^*(C(X))$ (observe that we get a C^*-algebra map $C(X) \to C(M)$).

- In the case $A = C(X)$, the pairing

$$K^*(C(X)) \otimes K_*(C(X)) \to \mathbb{C}$$

is given as follows: if a K-homology class is represented by a Dirac type operator D, and a K-theory class by a vector bundle E, then

$$[D] \otimes [E] \mapsto \mathrm{ind}(D_E).$$