Seminar algebraic topology: Bordism theory and the Hirzebruch signature theorem

- Target: students of mathematics from the fifth semester
- The seminar introduces bordism theory as a generalized cohomology theory, the Pontryagin-Thom construction to carry out computations and gives a proof of the Hirzebruch signature theorem.
- Ort: Sitzungszimmer(probably)
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- **Preliminary meeting: to be determined**

The concept of bordism started out as a way to classify manifolds upto a suitably wide equivalence relation to make the (otherwise impossible) task of classification more doable.

It developped into a full-fledged and beautiful theory which proved very useful in algebraic and geometric topology and also in global analysis and index theory.

Throughout bordism theory, the tangent bundle and constructions based on it are fundamental. This requires to also cover quite a bit of the theory of vector bundles in the seminar.

The idea of bordism is used to collect the bordism classes of compact manifolds into a nice algebraic object: the bordism groups (and even the graded bordism ring). Moreover, taking into account that manifolds often live inside some other space (and generalizing this), one gets to the point to define a full fledged (generalized) homology theory based on the idea of bordism (actually: several, depending on extra structure like orientation which is taken into account).

It turns out that there is a very powerful way to identify the geometric idea of bordism with homotopy theoretic concepts. This goes under the name of “Pontryagin-Thom construction”. The keywords to make it work are “embedding” and “transversality”. They show that bordism groups are homotopy groups of certain spaces, called “Thom spaces”. Big surprise: these can often be computed very well.

The concept of bordism allowed Hirzebruch to give a beautiful relation between the signature of a given oriented manifold and cohomological invariants which can be computed locally from the tangent bundle: the famous Hirzebruch signature theorem. To define thes cohomological invariants one needs a surprising relation between new algebraic concepts (multiplicative sequences) and topology.

The goal of the seminar is now clear:

- introduce bordism
- understand its fundamental properties (all the way to "its a generalized homology theory")
- understand the Pontryagin-Thom construction
- computate bordism using the Thom spaces
- use this computation to prove the Hirzebruch signature theorem
- develop/introduce the relevant background (like vector bundle theory, characteristic classes, Serre theory of homotopy groups,...)

We will follow a mix of several references, and will often start with later chapters. This means that it could be intersting and helpful for the preparation of a presenation to have alook at earlier chapters. As a general rule, we will not be able to prove all the results which will be used in the course. Instead, the presentation will present them and then base further arguments on them.

Literatur

[1] James F. Davis and Paul Kirk, *Lecture notes in algebraic topology*, Graduate Studies in Mathematics, vol. 35, American Mathematical Society, Providence, RI, 2001. MR1841974

[2] Friedrich Hirzebruch, *Topological methods in algebraic geometry*, Classics in Mathematics, Springer-Verlag, Berlin, 1995. Translated from the German and Appendix One by R. L. E. Schwarzenberger; With a preface to the third English edition by the author and Schwarzenberger; Appendix Two by A. Borel; Reprint of the 1978 edition. MR1335917

[3] John W. Milnor, *Topology from the differentiable viewpoint*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Based on notes by David W. Weaver; Revised reprint of the 1965 original. MR1487640

[4] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, No. 76, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. MR0440554