Loop groups and string topology
Lectures for the summer school algebraic groups
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1 Introduction

Let G be a compact Lie group. Then the space LG of all maps from the circle S^1 to G becomes a group by pointwise multiplication. Actually, there are different variants of LG, depending on the classes of maps one considers, and the topology to be put on the mapping space. In these lectures, we will always look at the space of smooth (i.e. C^∞) maps, with the topology of uniform convergence of all derivatives.

These groups certainly are not algebraic groups in the usual sense of the word. Nevertheless, they share many properties of algebraic groups (concerning e.g. their representation theory). There are actually analogous objects which are very algebraic (compare e.g. [1]), and it turns out that those have properties remarkably close to those of the smooth loop groups.

The lectures are organized as follows.

(1) Lecture 1: Review of compact Lie groups and their representations, basics of loop groups of compact Lie groups.

(2) Lecture 2: Finer properties of loop groups

(3) Lecture 3: the representations of loop groups (of positive energy)

The lectures and these notes are mainly based on the excellent monograph "Loop groups" by Pressley and Segal [3].

2 Basics about compact Lie groups

2.1 Definition. A Lie group G is a smooth manifold G with a group structure, such that the map $G \times G \to G; (g, h) \mapsto gh^{-1}$ is smooth.

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The group acts on itself by left multiplication: $l_g(h) = gh$. A vector field $X \in \Gamma(TG)$ is called *left invariant*, if $(l_g)_* X = X$ for each $g \in G$. The space of all left invariant vector fields is called the *Lie algebra* $Lie(G)$. If we consider vector fields as derivations, then the commutator of two left invariant vector fields again is a left invariant vector field. This defines the Lie bracket $[\cdot,\cdot]\colon Lie(G) \times Lie(G) \to Lie(G); [X,Y] = XY - YX$.

By left invariance, each left invariant vector field is determined uniquely by its value at $1 \in G$, therefore we get the identification $T_1G \cong Lie(G)$; we will frequently use both variants.

To each left invariant vector field X we associate its flow $\Psi_X\colon G \times \mathbb{R} \to G$ (a priori, it might only be defined on an open subset of $G \times \{0\}$). We define the *exponential map*

$$\exp\colon Lie(G) \to G; X \mapsto \Psi_X(1,1).$$

This is defined on an open subset of $0 \in G$. The differential $d_0 \exp\colon Lie(G) \to T_1(G) = Lie(G)$ is the identity, therefore on a suitably small open neighborhood of 0, exp is a diffeomorphism onto its image.

A maximal torus T of the compact Lie group G is a Lie subgroup $T \subset G$ which is isomorphic to a torus T^n (i.e. a product of circles) and which has maximal rank among all such. It's a theorem that for a connected compact Lie group G and a given maximal torus $T \subset G$, an arbitrary connected abelian Lie subgroup $A \subset G$ is conjugate to a subgroup of T.

2.2 Example. The group $U(n) := \{A \in M(n,\mathbb{C}) \mid AA^* = 1\}$ is a Lie group, a Lie submanifold of the group $Gl(n,\mathbb{C})$ of all invertible matrices.

In this case, $T_1U(n) = \{A \in M(n,\mathbb{C}) \mid A + A^* = 0\}$. The commutator of $T_1U(n) = L(U(n))$ is the usual commutator of matrices: $[A,B] = AB - BA$ for $A, B \in T_1U(n)$.

The exponential map for the Lie group $U(n)$ is the usual exponential map of matrices, given by the power series:

$$\exp\colon T_1U(n) \to U(n); A \mapsto \exp(A) = \sum_{k=0}^{\infty} A^k/k!.$$

The functional equation shows that the image indeed belongs to $U(n)$.

Similarly, $Lie(SU(n)) = \{A \in M(n,\mathbb{C}) \mid A + A^* = 0, \operatorname{tr}(A) = 0\}$.

2.3 Theorem. *If G is a connected compact Lie group, then* $\exp\colon Lie(G) \to G$ *is surjective.*

2.4 Definition. Every compact connected Lie group G can be realized as a Lie-subgroup of $SU(n)$ for big enough n. It follows that its Lie algebra $Lie(G)$ is a sub-Lie algebra of $Lie(U(n)) = \{A \in M(n,\mathbb{C}) \mid A^* = -A\}$. Therefore, the complexification $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ is a sub Lie algebra of $Lie(U(n)) \otimes_{\mathbb{R}} \mathbb{C} = \{A + iB \mid A, B \in M(n,\mathbb{C}), A^* = -A, (iB)^* = -(iB)\} = M(n,\mathbb{C})$. Note that the bracket (given by the commutator) is complex linear on these complex vector spaces.

The corresponding sub Lie group $G_{\mathbb{C}}$ of $Gl(n,\mathbb{C})$ (the simply connected Lie group with Lie algebra $M(n,\mathbb{C})$) with Lie algebra $Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$ is called the complexification of G.

$G_{\mathbb{C}}$ is a *complex Lie group*, i.e. the manifold $G_{\mathbb{C}}$ has a natural structure of a complex manifold (charts with holomorphic transition maps), and the composition $G_{\mathbb{C}} \times G_{\mathbb{C}} \to G_{\mathbb{C}}$ is holomorphic.

2.5 Exercise. Check the assertions made in Definition 2.4, in particular about the complex structure.

2.6 Example. The complexified Lie algebra $Lie(SU(n))\otimes_{\mathbb{R}}\mathbb{C} = \{A \in M(n,\mathbb{C}) \mid \operatorname{tr}(A) = 0\}$, and the complexification of $SU(n)$ is $Sl(n,\mathbb{C})$.

2.7 Definition. Let G be a compact Lie group. It acts on itself by conjugation: $G \times G \to G; (g,h) \mapsto ghg^{-1}$.

For fixed $g \in G$, we can take the differential of the corresponding map $h \mapsto ghg^{-1}$ at $h = 1$. This defines the adjoint representation $ad\colon G \to Gl(Lie(G))$.

We now decompose $Lie(G)$ into irreducible sub-representations for this action, $Lie(G) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. Each of these are Lie subalgebras, and we have $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

G is called semi-simple if non of the summands is one-dimensional, and *simple* if there is only one summand, which additionally is required not to be one-dimensional.

2.8 Remark. The simply connected simple compact Lie groups have been classified, they consist of $SU(n)$, $SO(n)$, the symplectic groups Sp_n and five exceptional groups (called G_2, F_4, E_6, E_7, E_8).

2.9 Definition. Let G be a compact Lie group with a maximal torus T. G acts (induced from conjugation) on $Lie(G)$ via the adjoint representation, which induces a representation on $Lie(G)\otimes_{\mathbb{R}}\mathbb{C} =: \mathfrak{g}_{\mathbb{C}}$. This Lie algebra contains the Lie algebra $\mathfrak{t}_{\mathbb{C}}$ of the maximal torus, on which T acts trivially (since T is abelian).

Since T is a *maximal* torus, it acts non-trivial on every non-zero vector of the complement.

As every finite dimensional representation of a torus, the complement decomposes into a direct sum $\bigoplus \mathfrak{g}_\alpha$, where on each $\mathfrak{g}_\alpha$, $t \in T$ acts via multiplication with $\alpha(t) \in S^1$, where $\alpha\colon T \to S^1$ is a homomorphism, called the *weight* of the summand $\mathfrak{g}_\alpha$.

We can translate the homomorphisms $\alpha\colon T \to S^1$ into their derivative at the identity, thus getting a linear map $\alpha'\colon \mathfrak{t} \to \mathbb{R}$, i.e. an element of the dual space $\mathfrak{t}^*$, they are related by $\alpha(\exp(x)) = e^{i\alpha'(x)}$.

This way, we think of the group of characters $\hat{T} = \operatorname{Hom}(T, S^1)$ as a lattice in t^*, called the lattice of weights. It contains the set of *roots*, i.e. the non-zero weights occurring in the adjoint representation of G.

2.10 Remark. As $\mathfrak{g}_{\mathbb{C}}$ is the complexificatoin of a real representation, if α is a root of G, so is $-\alpha$, with $\mathfrak{g}_{-\alpha} = \overline{\mathfrak{g}_\alpha}$.

It is a theorem that the subspaces $\mathfrak{g}_\alpha$ are always 1-dimensional.

2.11 Example. For $U(n)$, $Lie(U(n))_{\mathbb{C}} = M(n,\mathbb{C})$. A maximal torus is given by the diagonal matrices (with diagonal entries in S^1). We can then index the roots by pairs (i,j) with $1 \leq i \neq j \leq n$. We have $\alpha_{ij}(\operatorname{diag}(z_1,\dots,z_n)) = z_i z_j^{-1} \in S^1$. The corresponding subspace $\mathfrak{g}_{ij}$ consists of matrices which are zero except for the (i,j)-entry. As an element of $\mathfrak{t}^*$ it is given by the linear map $\mathbb{R}^n \to \mathbb{R}; (x_1,\dots,x_n) \mapsto x_i - x_j$.

2.12 Definition. According to Remark 2.10, each of the spaces $\mathfrak{g}_\alpha$ is 1-dimensional. Choose a vector $0 \neq e_\alpha \in \mathfrak{g}_\alpha$, such that $e_{-\alpha} = \overline{e_\alpha}$. Then $h_\alpha := -i[e_\alpha, e_{-\alpha}] \in \mathfrak{t}$ is non-zero. We can normalize the vector e_α in such a way that $[h_\alpha, e_\alpha] = 2ie_\alpha$. Then h_α is canonically determined by α, it is called the *coroot* associated to α.

We call G a *simply knitted* Lie group, if there is a G-invariant inner product on $Lie(G)$ such that $\langle h_\alpha, h_\alpha\rangle = 2$ for all roots α.

This inner product gives rise to an isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$; this isomorphism maps h_α to α.

2.13 Example. For $SU(2)$, $T = \{\mathrm{diag}(z, z^{-1}) \mid z \in S^1\}$, there is only one pair of roots $\alpha, -\alpha$, obtained by restriction of the roots α_{12} and α_{21} of $U(2)$ to the (smaller) maximal torus of $SU(2)$. We get $\alpha(\mathrm{diag}(z, z^{-1})) = z^2$, $e_\alpha = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $h_\alpha = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$.

All the groups $U(n)$ and $SU(n)$ are simply knitted, as well as $SO(2n)$. The canonical inner product of $Lie(U(n))_{\mathbb{C}} = M(n, \mathbb{C})$ is given by $\langle A, B\rangle = -tr(A^*B)/2$.

2.14 Proposition. *Given any semi-simple Lie group, the Lie algebra is spanned by the vectors $e_\alpha, e_{-\alpha}, h_\alpha$ as α varies over the set of (positive) roots.*

For each such triple one can define a Lie algebra homomorphism $Lie(SU(2)) \to Lie(G)$ which maps the standard generators of $Lie(SU(2))$ to the chosen generators of $Lie(G)$.

We get the technical useful result that for a semi-simple compact Lie group G the Lie algebra $Lie(G)$ is generated by the images of finitely many Lie algebra homomorphisms from $Lie(SU(2))$ to $L(G)$.

2.15 Definition. Let G be a compact Lie group with maximal torus T. Let $N(T) := \{g \in G \mid gTg^{-1} \subset T\}$ be the normalizer of T in G. The *Weyl group* $W(T) := N(T)/T$ acts on T by conjugation, and via the adjoint representation also on $\mathfrak{t}$ and then on $\mathfrak{t}^*$. It is a finite group.

This action preserves the lattice of weights $\hat{T} \subset \mathfrak{t}^*$, and also the set of roots.

Given any root α of G, the Weyl group contains an element s_α of order 2. It acts on $\mathfrak{t}$ by reflection on the hyperplane $H_\alpha := \{X \mid \alpha(X) = 0\}$. More precisely we have $s_\alpha(X) = X - \alpha(X)h_\alpha$, where $h_\alpha \in \mathfrak{t}$ is the coroot associated to the root α.

The reflections s_α together generate the Weyl group $W(G)$.

The Lie algebra $\mathfrak{t}$ is decomposed into the union of the root hyperplanes H_α and into their complement, called the set of *regular* elements. The complement decomposes into finitely many connected components, which are called the *Weyl chambers*. One chooses one of these and calls it the *positive Weyl chamber*. The Weyl group acts freely and transitively on the Weyl chambers.

A root of G is called positive or negative, if it assumes positive or negative values on the positive chamber. A positive root α is called *simple*, if its hyperplane H_α is a wall of the positive chamber.

2.16 Example. For the group $U(n)$, the Weyl group is the symmetric group S_n, it acts by permutation of the diagonal entries on the maximal torus. Lifts of the elements of $W(U(n))$ to $N(T) \subset U(n)$ are given by the permutation matrices.

For $SU(n)$, the Weyl group is the alternating group S_n, again acting by permutation of the diagonal entries of the maximal torus.

Since these groups are simply knitted, we can depict their roots, coroots etc. in a simply Euclidean picture of $\mathfrak{t} \cong \mathfrak{t}^*$. We look at the case $G = SU(3)$ where $Lie(T)$ is 2-dimensional.

In the lecture, a picture is drawn.

In general, the positive roots of $U(n)$ or $SU(n)$ can be chosen to be the roots α_{ij} with $i < j$.

2.17 Theorem. *Let G be a compact semi-simple Lie group. There is a one-to-one correspondence between the irreducible representations of G and the set of dominant weights, where a weight α is called dominant if $\alpha(h_\beta) \geq 0$ for each positive root β of G.*

Proof. We don't proof this theorem here; we just point out that the representation associated to a dominant weight is given in as the space of holomorphic sections of a line bundle over a homogeneous space of $G_{\mathbb{C}}$ associated to the weight. □

2.18 Remark. One of the goals of these lectures is to explain how this results extends to loop groups.

3 Basics about loop groups

3.1 Definition. An infinite dimensional smooth manifold (modeled on a locally convex complete topologically vector space X) is a topological space M together with a collection of charts $x_i\colon U_i \to V_i$, with open subset U_i of M and V_i of X and a homeomorphism x_i, such that the change of coordinate maps $x_j \circ x_i^{-1}\colon V_i \to V_j$ are smooth maps between the topological vector space X.

Differentiability of a map $f\colon X \to X$ is defined in terms of convergence of difference quotients, if it exists, the differential is then a map

$$Df\colon X \times X \to X; (v, w) \mapsto \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t}$$

(where the second variable encodes the direction of differentiation), iterating this, we define higher derivatives and the concept of smooth, i.e. C^∞ maps.

3.2 Definition. Let G be a compact Lie group with Lie algebra $Lie(G) = T_1G$ e(a finite dimensional vector space). For us, the model topological vector space X will be $X = C^\infty(S^1, Lie(G))$. Its topology is defined by the collection of semi-norms $q_i(f) := \sup_{x \in S^1}\{\left|\partial^i\phi/\partial t^k(x)\right|\}$ for any norm on $Lie(G)$.

This way, X is a complete separable (i.e. with a countable dense subset) metrizable topological vector space. A sequence of smooth functions $\phi_k\colon S^1 \to Lie(G)$ converges to $\phi\colon S^1 \to Lie(G)$ if and only if the functions and all their derivatives converge uniformly.

3.3 Lemma. *There is a canonical structure of a smooth infinite dimensional manifold on LG. A chart around the constant loop 1 is given by the exponential map $C^\infty(S^1, U) \to LG; \chi \mapsto \exp \circ \chi$ where U is a sufficiently small open neighborhood of $0 \in T_1(G)$. Charts around any other point $f \in LG$ are obtained by translation with f, using the group structure on LG.*

We define the topology on LG to be the finest topology (as many open sets as possible) such that all the above maps are continuous.

It is not hard to see that the transition functions are then actually smooth, and that the group operations (defined pointwise) are smooth maps.

3.4 Exercise. Work out the details of Lemma 3.3, and show that it extends to the case where S^1 is replaced by any compact smooth manifold N.

3.5 Remark. There are many variants of the manifold LG of loops in G. Quite useful are versions which are based on maps of a given Sobolev degree, one of the advantages being that the manifolds are then locally Hilbert spaces.

3.6 Exercise. Show that for $G = SU(2)$ the group LG is connected, but the exponential map is not surjective

3.7 Exercise. There are other interesting infinite dimensional Lie groups. One which is of some interest for loop groups is $Diffeo(S^1)$, the group of all diffeomorphisms of S^1 (and also its identity component of orientation preserving diffeomorphism).

Show that this is indeed a Lie group, with Lie algebra (and local model for the smooth structure) $Vect(S^1)$, the space of all smooth vector fields. $\exp\colon Vect(S^1) \to Diffeo(S^1)$ maps a vector field to the (time 1) flow generated by it.

Show that there are no neighborhoods $U \subset Vect(S^1)$ of 0 and $V \subset Diffeo(S^1)$ of id_{S^1} such that $\exp|\colon U \to V$ is injective or surjective.

3.8 Lemma. *Consider the subgroup of based loops $\Omega(G) = \{f\colon S^1 \to G \in LG \mid f(1) = 1\} \subset LG$, and the subgroup of constant loops $G \subset LG$.*

The multiplication map $G \times \Omega(G) \to LG$ is a diffeomorphism.

Proof. The inverse map is given by $LG \to G \times \Omega(G); f \mapsto (f(1), f(1)^{-1}f)$. Clearly the two maps are continuous and inverse to each other. The differentiable structure of $\Omega(G)$ makes the maps smooth. □

3.9 Corollary. *It follows that the homotopy groups of LG are easy to compute (in terms of those of G); the maps of Lemma 3.8 give an isomorphism $\pi_k(LG) \cong \pi_k(G) \oplus \pi_{k-1}(G)$.*

In particular, LG is connected if and only if G is connected and simply connected, else the connected components of G are parameterized by $\pi_0(G) \times \pi_1(G)$.

3.10 Definition. Embed the compact Lie group G into $U(n)$. Using Fourier decomposition, we can now write the elements of LG in the form $f(z) = \sum A_k z^k$ with $A_k \in M(n, \mathbb{C})$.

We define now a number of subgroups of LG.

(1) $L_{pol}G$ consists of those loops with only finitely many non-zero Fourier coefficients. It is the union (over $N \in \mathbb{N}$) of the subsets of functions of degree $\leq N$. The latter ones are compact subsets of $M(n, \mathbb{C})^N$, and we give $L_{pol}G$ the direct limit topology.

(2) $L_{rat}G$ consists of those loops which are rational functions $f(z)$ (without poles on $\{|z| = 1\}$).

(3) $L_{an}G$ are those loops where the series $\sum A_k z^k$ converges for some annulus $r \leq |z| \leq 1/r$ with $0 < r < 1$. For fixed $0 < r < 1$ this is Banach Lie group (of holomorphic functions on the corresponding annulus), with the topology of uniform convergence. We give $L_{an}G$ the direct limit topology.

3.11 Exercise. In general, $L_{pol}G$ is not dense in G. Show this for the case $G = S^1$.

3.12 Proposition. *If G is semi-simple, then $L_{pol}G$ is dense in LG.*

Proof. Set $H := \overline{L_{pol}G}$, and $V := \{X \in C^\infty(S^1, Lie(G) \mid \exp(tX) \in H \forall t \in \mathbb{R}\}) \subset C^\infty(S^1, Lie(G))$. Then V is a vector space, as

$$\exp(X+Y) = \lim_{k\to\infty} (\exp(X/k)\exp(Y/k))^k$$

converges in the C^∞-topology if X and Y are close enough to 0. Since exp is continuous, V is a closed subspace. It remains to check that V is dense in $L\mathfrak{g}$.

Because of Proposition 2.14 it suffices to check the statement for $SU(2)$. Here, we first note that the elements $X_n : z \mapsto \begin{pmatrix} 0 & z^n \\ -z^{-n} & 0 \end{pmatrix}$ and $Y_n : z \mapsto \begin{pmatrix} 0 & iz^n \\ iz^{-n} & 0 \end{pmatrix}$ belong to V, since $X_n^2 = Y_n^2 = -1$, so that $\exp(tX_n) = \sum_k (tX_n)^k/k!$ actually is a family of polynomial loops. By linearity and the fact that V is closed, every loop of the form $z \mapsto \begin{pmatrix} 0 & f(z)+ig(z) \\ -f(z)+ig(z) & 0 \end{pmatrix}$ belongs to V, for arbitrary smooth real valued functions $f, g\colon S^1 \to \mathbb{R}$ (use their Fourier decomposition). Since V is finally invariant under conjugation by polynomial loops in $LSU(2)$, it is all of $\mathfrak{su}(2)$. □

3.1 Abelian subgroups of LG

We have already used the fact that each compact Lie group G as a maximal torus T, which is unique upto conjugation.

3.13 Proposition. *Let G be a connected compact Lie group with maximal torus T. Let $A \subset LG$ be a (maximal) abelian subgroup. Then $A(z) := \{f(z) \in Gf \in A\} \subset G$ is a (maximal) abelian subgroup of G.*

In particular, we get a the maximal abelian subgroup A_λ for each smooth map $\lambda\colon S^1 \to \{T \subset G \mid T \text{ maximal torus}\} \cong G/N$, where N is the normalizer of T in G, with $A_\lambda := \{f \in LG \mid f(z) \in \lambda(z)\}$.

The conjugacy class of A_λ depends only on the homotopy class of λ. Since $\pi_1(G/N) = W = N/T$, $[S^1, G/N]$ is the set of conjugacy classes of elements of the Weyl group W.

An element $w \in W$ acts on T by conjugation, and the corresponding A_λ is isomorphic to $\{\gamma\colon \mathbb{R} \to T \mid \gamma(t+2\pi) = w^{-1}\gamma(t)w$ for all $t \in \mathbb{R}$.

Proof. It is clear that all evaluation maps of A have abelian image. If all the image sets are maximal abelian, then A is maximal abelian.

Since all maximal tori are conjugate, the action of G on the set of all maximal tori is transitive, with stabilizer (by definition) the normalizer N, so that this space is isomorphic to G/N.

Next, $W = N/T$ acts freely on G/T, with quotient G/N. Since G/T is simply connected (a fact true for every connected Lie group), we conclude from covering theory that $\pi_1(G/T) \cong W$. But then the homotopy classes of non base-point preserving maps are bijective to the conjugacy classes of the fundamental group.

Given a (homotopy class of maps) $\lambda\colon S^1 \to G/T$, represented by $w \in W = N/T$ (with a lift $w' \in N$ and with $x \in Lie(G)$ such that $\exp(2\pi x) = w'$), we can choose λ with $\lambda(z) = \exp(zx)T\exp(-zx)$ (recall that the bijection between G/N and the set of maximal tori is given by conjugation of T).

We then get a bijection from A_λ to the twisted loop group of the assertion by sending $f \in A_\lambda$ to $\tilde{f}\colon \mathbb{R} \to T : t \mapsto \exp(-tx)f(t)\exp(tx)$. □

3.14 Example. The most obvious maximal abelian subgroup of a compact Lie group G with maximal torus T is LT, which itself contains in particular T (as subgroup of constant loops).

3.15 Exercise. Find other maximal abelian subgroups of LG.

3.16 Example. If $G = U(n)$ with maximal torus T, its Weyl group W is isomorphic to the symmetric group S_n. Given a cycle $w \in W = S_n$, the corresponding maximal abelian subgroup $A_w \subset LU(n)$ is isomorphic to LS^1.

More generally, if w is a product of k cycles (possibly of length 1), then A_w is a product of k copies of LS^1.

Proof. We prove the statement if w consists of one cycle (of length n). Then, in the description of Proposition 3.13, A_w consists of functions $\mathbb{R} \to T$ which are periodic of period $2\pi l$. Moreover, the different components are all determined by the first one, and differ only by a translation in the argument of 2π or a multiple of 2π. □

3.17 Proposition. *If G is semi-simple and compact, then LG_0, the component of the identity of LG, is perfect.*

4 Root system and Weyl group of loop groups

4.1 Definition. Let G a compact Lie group with maximal torus T. Consider its complexified Lie algebra $Lie(LG)_\mathbb{C} = L\mathfrak{g}_\mathbb{C}$.

It carries the action of S^1 by reparametrization of loops: $(z_0 \cdot X)(z) := X(z_0 z)$.

We get a corresponding decomposition of $L\mathfrak{g}_\mathbb{C} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_\mathbb{C} z^k$ into Fourier components (the sums has of course to be completed appropriately).

The action of S^1 used in the above decomposition still commutes with the adjoint representation of the subgroup T of constant loops, so the summands can further be decomposed according to the weights of the action of T, to give a decomposition

$$L\mathfrak{g}_\mathbb{C} = \oplus_{(k,\alpha)\in\mathbb{Z}\times\hat{T}} \mathfrak{g}_{(k,\alpha)} z^k.$$

The index set $\mathbb{Z} \times \hat{T}$ is the Pontryagin dual of $S^1 \times T$ (i.e. the set of all homomorphisms $S^1 \times T \to S^1$). Those homomorphisms which occur (now also with possibly $\alpha = 0$) are called the *roots* of LG.

The (infinite) set of roots of LG is permuted by the so called *affine Weyl group*

$$W_{aff} = N(T \times S^1)/(T \times S^1),$$

considered inside the semidirect product $LG \rtimes S^1$, where we use the action of S^1 by reparametrization (rotation) of loops on LG to construct the semidirect product. This follows because we decompose $L\mathfrak{g}_\mathbb{C}$ as a representation of $LG \rtimes S^1$ with respect to the subgroup $T \times S^1$..

4.2 Proposition. *W_{aff} is a semidirect product $\mathrm{Hom}(S^1, T) \rtimes W$, where W is the Weyl group of G, with its usual action on the target T.*

Proof. Clearly $\mathrm{Hom}(S^1, T)$ is a subgroup of LG which conjugates the constant loops with values in T into itself, and the action of the constant loops with values in the normalizer N does the same (and factors through W). It is also evident that all of S^1 belongs to the normalizer of $T \times S^1$.

On the other hand, if $R_{z_0} \in S^1 \in LG \rtimes S^1$ acts by rotation by $z_0 \in S^1$, then for $f \in LG$ we get $f^{-1} R_{z-0} f = f^{-1}(\cdot) f(\cdot z_0) R_{z_0}$.

This belongs to $T \times S^1$ if and only if $z \mapsto f(z)^{-1} f(z z_0)$ is a constant element of T for each z_0, i.e. if and only if $z \mapsto f(1)^{-1} f(z)$ is a group homomorphism $z \to T$. Additionally, f conjugates T to itself if and only if $f(1) \in N$. Therefore, $N(T \times S^1)$ is the product of N, $\mathrm{Hom}(S^1, T)$ and S^1 inside $LG \rtimes S^1$, and the quotient by $T \times S^1$ is as claimed. □

4.3 Definition. We think of the weights and roots of LG not as linear forms on $\mathbb{R} \times \mathfrak{t}$ (derivatives of elements in $\mathrm{Hom}(S^1 \times T, S^1)$), but rather as *affine linear* functions on $\mathfrak{t}$, where we identify $\mathfrak{t}$ with the hyperplane $1 \times \mathfrak{t}$ in $\mathbb{R} \times \mathfrak{t}$; this explains the notation "affine roots".

Moreover, the group W_{aff} acts linearly on $\mathbb{R} \times \mathfrak{t}$ and preserves $1 \times \mathfrak{t}$, where $\lambda \in \mathrm{Hom}(S^1, T)$ acts by translation by $\lambda'(1) \in \mathfrak{t}$.

An affine root (k, α) is (for $\alpha \neq 0$) determined (upto sign) by the affine hyperplane $H_{k,\alpha} := \{x \in \mathfrak{t} \mid \alpha(x) = -k\} \subset \mathfrak{t}$ which is the set where it vanishes.

The collection of these hyperplanes is called the diagram of LG. It contains as the subset consisting of the $H_{0,\alpha}$ the diagram of G

Recall that the connected components of the complement of all $H_{0,\alpha}$ were called the chambers of G, and we choose one which we call the positive chamber. The components of the complement of the diagram of LG are called *alcoves*. Each chamber contains a unique alcove which touches the origin, and this way the positive chamber defines a positive alcove, the set $\{x \in \sqcup \mid 0 < \alpha(x) < 1 \text{ for all positive roots } \alpha\}$. An affine root is called positive or negative, if it has positive or negative values at the positive alcove. The positive affine roots corresponding to the walls of the positive alcove are called the simple affine roots.

4.4 Example. The diagram for $SU(3)$ is the tessellation of the plane by equilateral triangles.

In general, if G is a simple group, then each chamber is a simplicial cone, bounded by the l hyperplanes $H_{0,\alpha_1}, \dots, H_{0,\alpha_l}$ (where $\alpha_1, \dots, \alpha_l$ are the simple roots of G). There is a highest root α_{l+1} of G, which dominates all other roots (on the positive chamber). The positive alcove is then an l-dimensional simplex cut out of the positive chamber by the wall $H_{1,-\alpha_{l+1}}$, and we get $l+1$ simple affine roots of LG, $(0, \alpha_1), \dots, (0, \alpha_l), (1, -\alpha_{l+1})$.

In general, if G is semi-simple with q simple factors, the positive alcove is a product of q simplices, bounded by the walls of the simple roots of G, and walls $H_{1,-\alpha_i}$, $i = l+1, \dots, l+q$ being the highest weights of the irreducible summands of the adjoint representation.

4.5 Proposition. *Let G be a connected and simply connected compact Lie group. Then W_{aff} is generated by reflections in the hyperplanes (the reflections in the walls of the positive alcove suffice), and it acts freely and transitively on the set of alcoves.*

Proof. From the theory of compact Lie groups, we know that W is generated by the reflections at the $H_{0,\alpha}$, and that $\mathrm{Hom}(S^1, \mathfrak{t})$ is generated by the coroots h_α.

Recall that the reflection s_α was given by $s_\alpha(x) = x - \alpha(x)h_\alpha$. Recall that we normalized such that $\alpha(h_\alpha) = 2$, therefore $-kh_\alpha/2 \in H_{k,\alpha}$.

Now the reflection $s_{k,\alpha}$ in the hyperplane $H_{k,\alpha}$ is given by $s_{k,\alpha}(x) = x + kh_\alpha/2 - \alpha(x + kh_\alpha/2)h_\alpha - kh_\alpha/2 = s_\alpha(x) - kh_\alpha$. Since $s_\alpha \in W$ and h_α in $\mathrm{Hom}(S^1, T)$ (identified with the value of its derivative at 1), $s_{k,\alpha} \in W_{aff} = \mathrm{Hom}(S^1, T) \rtimes W$.

To show that these reflections generate W_{aff}, it suffices to show that they generate the translation by h_α. But this is given by $s_{a,-\alpha}s_{0,\alpha}$.

We now show that W_{aff} acts transitively on the set of alcoves. For an arbitrary alcove A, we have to find $\gamma \in W_{aff}$ such that γA is the positive alcove C_0. Now the orbit $W_{aff}a$ of a point $p \in A$ is a subset S of $\mathfrak{t}$ without accumulation points. Choose a point $c \in C_0$ and one of the points $b \in S$ with minimal distance to c. If b would not belong to C_0, then b and c are separated by a wall of C_0, in which we can reflect b to obtain another point of S, necessarily closer to c than b.

Since W acts freely on the set of chambers, W_{aff} acts freely on the set of alcoves (since each element of W preserves the distance to the origin, and each translation moves the positive alcove away from the origin, only elements of W could stabilize the positive alcove). □

5 Central extensions of LG

We want to study the representations of LG for a compact Lie group G. It turns out, however, that most of the relevant representations are no honest representations but only projective representations., i.e. $U_f U_g = c(f,g)U_{fg}$ for every $f, g \in LG$, where U_f is the operator by which f acts, and $c(f,g) \in S^1$ is a scalar valued function (a cocycle).

More precisely, the actions we consider are actions of central extensions of LG (in some sense defined by this cocycle). We don't want to go into the details of the construction and classification of these central extensions, but only state the main results.

Let G be a compact connected Lie group.

(1) LG has many central extensions $1 \to S^1 \to \tilde{L}G \to LG \to 1$.

(2) The corresponding Lie algebras $\tilde{L}\mathfrak{g}$ are classified as follows: for every symmetric invariant bilinear form $\langle \cdot, \cdot \rangle \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ we get a form

$$\omega \colon L\mathfrak{g} \times L\mathfrak{g} \to \mathbb{R}; (X,Y) \mapsto \frac{1}{2\pi} \int_0^{2\pi} \langle X(z), Y'(z) \rangle \, dz.$$

Then $\tilde{L}\mathfrak{g} = \mathbb{R} \oplus L\mathfrak{g}$ with bracket

$$[(a,X),(b,Y)] = (\omega(X,Y),[X,Y]).$$

(3) The extension $0 \to \mathbb{R} \to \tilde{L}\mathfrak{g} \to L\mathfrak{g} \to 0$ with bracket given by the inner product $\langle , \rangle$ on $\mathfrak{g}$ corresponds to an extension of Lie groups $1 \to S^1 \to \tilde{L}G \to LG \to 1$ if and only if $\langle h_\alpha, h_\alpha \rangle \in 2\mathbb{Z}$ for every coroot h_α of G.

(4) If this integrality condition is satisfied, the extension $\tilde{L}G$ is uniquely determined. Moreover, there is a unique action of $\mathrm{Diffeo}^+(S^1)$ on $\tilde{L}G$ which covers the action on LG. In particular, there is a induced canonical lift of the action of S^1 on LG by rotation of the argument to $\tilde{L}G$.

(5) The integrality condition is satisfied if and only if $\omega/2\pi$, considered as an invariant differential form on LG (which is closed by the invariance of $\langle,\rangle$ and therefore of ω), lifts to an integral cohomology class. It then represents the first Chern class of the principle S^1-bundle $S^1 \to \tilde{L}G \to LG$ over LG. It follows that the topological structure completely determines the group extension.

(6) If G is simple and simply connected there is a *universal* central extension $1 \to S^1 \to \tilde{L}G \to LG \to 1$ (universal means that there is a unique map of extensions to any other central extension of LG).

If G is simple, all invariant bilinear forms on $\mathfrak{g}$ are proportional, and the universal extension corresponds to the smallest non-trivial one which satisfies the integrality condition. We call this also the basic inner product and the basic central extension.

(7) For $SU(n)$ and the other simply laced groups, the basic inner product is the canonical inner product such that $\langle h_\alpha, h_\alpha \rangle = 2$ for every coroot α.

(8) There is a precise formula for the adjoint and coadjoint action of suitable elements, compare Lemma 6.10.

6 Representations of loop groups

6.1 Definition. A representation of a loop group LG (or more generally any topological group) is for us given by a locally convex topological vector space V (over $\mathbb{C}$) with an action

$$G \times V \to V; (g, v) \to gv$$

which is continuous and linear in the second variable.

Two representations V_1, V_2 are called *essentially equivalent*, if they contain dense G-invariant subspaces $V_1' \subset V_1$, $V_2' \subset V_2$ with a continuous G-equivariant bijection $V_1' \to V_2'$

The representation V is called *smooth*, if there is a dense subspace of vectors $v \in V$ such that the map $G \to V$; $g \mapsto gv$ is smooth —such vectors are called *smooth vectors* of the representation.

A representation V is called irreducible if it has no closed invariant subspaces.

6.2 Example. The actions of S^1 on $C^\infty(S^1)$,$C^(S^1)$ and $L^2(S^1)$ by rotation of the argument are all equivalent and smooth.

However, rotation of the argument does not define an action in our sense of S^1 on $L^\infty(S^1)$ because the corresponding map $S^1 \times L^\infty(S^1) \to L^\infty(S^1)$ is not continuous.

6.3 Remark. Given a representation V of LG and $z_0 \in S^1$ (or more generally any diffeomorphism of S^1, z_0 gives rise to a diffeomorphism by translation), then we define a new representation $\phi^* V$ by composition with the induced automorphism of LG.

We are most interested in representations which are symmetric, meaning that $\phi^* V \cong V$. Actually, we require a somewhat stronger condition in the following Definition 6.4.

6.4 Definition. When we consider representation V of LG, we really want to consider actions of $LG \rtimes S^1$, i.e. we want an action of S^1 on V which intertwines the action of LG; for $z_0 \in S^1$ we want operators R_{z_0} on V such that $R_{z_0} f R_{z_0}^{-1} v = f(\cdot + z_0) v$ for all $v \in V$, $f \in LG$.

Moreover, we will study *projective* representations, i.e. representations such that $f_1 \cdot (f_2 \cdot v) = c(f_1, f_2)(f_1 f_2) \cdot v$ with $c(f_1, f_2) \in \mathbb{C} \setminus \{0\}$.

More precisely, these are actions of a central extension $1 \to \mathbb{C}^\times \to \tilde{L}G \to LG \to 1$.

Since S^1 acts on V, we get a decomposition $V = \overline{\bigoplus_{k \in \mathbb{Z}} V(k)}$, where $z \in S^1$ acts on $V(k)$ by multiplication with z^{-k}.

We say V is a representation of *positive energy*, if $V(k) = 0$ for $k < 0$.

6.5 Example. The adjoint representation of LG on $L\mathfrak{g}$, or the canonical representation of $LSU(n)$ on the Hilbert space $L^2(S^1, \mathbb{C}^n)$ are *not* of positive energy, nor of negative energy. However, they are not irreducible.

6.6 Remark. One can always modify the action of S^1 on a representation V of LG by multiplication with a character of S^1, so that "positive energy" and "energy bounded below" are more or less the same.

The complex conjugate of a representation of negative energy is a representation of positive energy.

6.7 Proposition. *An irreducible unitary representation V of $\tilde{L}G \rtimes S^1$ which is of positive energy (and this makes only sense with the action of the extra circle) is also irreducible as a representation of $\tilde{L}G$.*

Proof. Let T be the projection onto a proper $\tilde{L}G$-invariant summand of V. This operator commutes with the action of $\tilde{L}G$. Let R_z be the operator through which $z \in S^1$ acts on V. We define the bounded operators $T_q := \int_{S^1} z^q R_z T R_z^{-1}\, dz$. They all commute with $\tilde{L}G$, and T_q maps $V(k)$ to $V(k+q)$.

Let m be the lowest energy of V. Then $T_q(V(m)) = 0$ for all $q < 0$. Since V is irreducible, V is generated as a representation of $LG \rtimes S^1$ by $V(m)$. Since $V(m)$ is S^1-invariant, $T_q(V) = 0$ for $q < 0$. Since $T_{-q} = T_q^*$, we even have $T_q = 0$ for all $q \neq 0$. Now, the T_q are the Fourier coefficients of the loop $z \mapsto R_z T R_z^{-1}$. It follows that this loop is constant, i.e. that T commutes also with the action of S^1. But since V was irreducible, this implies that $T = 0$.

□

The representation of positive energy of a loop group behave very much like the representation of a compact Lie group. This is reflected in the following theorem.

6.8 Theorem. *Let G be a compact Lie group. Let V be a smooth representation of positive energy of $\tilde{L}G \rtimes S^1$. Then, upto essential equivalence:*

(1) If V is non-trivial then it does not factor through an honest representation of LG, i.e. is truly projective.

(2) *V is a discrete direct sum of irreducible representations (of positive energy)*

(3) *V is unitary*

(4) *The representation extends to a representation of $\tilde{L}G \rtimes \mathrm{Diffeo}^+(S^1)$, where $\mathrm{Diffeo}^+(S^1)$ denotes the orientation preserving diffeomorphisms and contains S^1 (acting by translation).*

(5) *V extends to a holomorphic projective representation of $LG_{\mathbb{C}}$.*

Granted this theorem, it is of particular importance to classify the irreducible representations of positive energy

6.9 Definition. Let $T_0 \times T \times S^1 \subset \tilde{L}G \rtimes S^1$ be a "maximal torus", with $T_0 = S^1$ the kernel of the central extension, T a maximal torus of G and S^1 the rotation group. We can then refine the energy decomposition of any representation V to a decomposition

$$V = \overline{\bigoplus_{n,\alpha,h \in \mathbb{Z}\times\hat{T}\times\mathbb{Z}} V_{n,\alpha,h}}$$

according to the characters of $T_0 \times T \times S^1$. The characters which occur are called the *weights* of V. n is called the *energy* and h the *level*.

6.10 Lemma. *The action of $\xi \in \mathrm{Hom}(S^1, T)$ on a weight (n, α, h) is given by*

$$\xi(n, \alpha, h) = (n + \alpha(\xi) + h\,|\xi|^2/2, \alpha + h\xi^*, h)$$

where we identify ξ with $\xi'(1) \in \mathfrak{t}$.

Moreover, the norm is obtained from the inner product on $L\mathfrak{g}$ which corresponds to the central extension $\tilde{L}G$, and $\xi^* \in \hat{T}$ is the image of ξ under the map $\mathfrak{t} \to \mathfrak{t}^*$ defined by this inner product.

6.11 Remark. Note that T_0 commutes with every element of $\tilde{L}G \rtimes S^1$. Consequently, each representation decomposes into subrepresentations with fixed level, and an irreducible representation has only one level h.

The level is a measure for the "projectivity" of the representation; it factors through an honest representation of LG if and only if the level is 0.

The weights of a representation are permuted by the normalizer of $T_0 \times T \times S^1$, hence by the affine Weyl group $W_{aff} = \mathrm{Hom}(S^1, T) \rtimes W$ (where W is the Weyl group of G).

6.12 Definition. Given a root (n, α) of LG, we define the *coroot* $(-n\,|h_\alpha|^2/2, h_\alpha) \in \mathbb{R} \oplus \mathfrak{t} \subset \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$, where we use again the inner product for the central extension $\tilde{L}G$.

Our irreducible representation have a number of additional important properties.

6.13 Theorem. *Let V be a smooth irreducible representation of LG of positive energy (i.e. we really take a representation of $\tilde{L}G \rtimes S^1$).*

(1) *Then V is of finite type, i.e. for each energy n the subspace $V(n)$ is finite dimensional. In particular, each weight space $E_{h,\lambda,n}$ is finite dimensional.*

(2) *V has a unique lowest weight (h,λ,n) with $E_{h,\lambda,n} \neq 0$.* Lowest weight *means by definition, that for each positive root (α,m) the character $(h,\lambda-\alpha,n-m)$ does not occur as a weight in V.*

This lowest weight is antidominant, *i.e. for each positive coroot $(-m\left|h_\alpha\right|^2/2,h_\alpha)$ we have $\langle(h,\lambda,n),(-m\left|h_\alpha\right|^2/2),h_\alpha,-0\rangle = \lambda(h_\alpha) - hm\left|h_\alpha\right|^2/2 \le 0$.*

Since we have in particular to consider the positive roots $(\alpha,0)$ and $(-\alpha,1)$ (for each positive root α of G), this is equivalent to

$$-h\left|h_\alpha\right|^2/2 \le \lambda(h_\alpha) \le 0 \tag{6.14}$$

for each positive root α of G.

(3) *There is a bijection between isomorphism class of irreducible representations of $\tilde{L}G \rtimes S^1$ as above and antidominant weights.*

6.15 Corollary. *If the level $h=0$, only $\lambda=0$ satisfies Inequality* (6.14)*. In other words, among the representations considered here, only the trivial representation is an honest representation of LG, all others are projective.*

For a given level h, there are only finite possible antidominant weights (with $n=0$), because the h_α generate $\mathfrak{t}$.

6.16 Example. If G is simple and we look at an antidominant weight $(h,\lambda,0)$, then $-\lambda$ is a dominant weight in the usual sense of G, i.e. contained in the corresponding simplicial cone in $\mathfrak{t}^*$, but with the extra condition that it is contained in the simplex cut off by $\{\mu \mid \mu(\alpha_0)=h\}$, where α_0 is the highest weight of G.

We get in particular the so called *fundamental weights*

(1) $(1,0,0)$

(2) $(\langle\omega_i,\alpha_0\rangle,-\omega_i,0)$, with ω_i the fundamental weights of G determined by $\omega_i(h_{\alpha_j})=\delta_{ij}$.

The antidominant weights are exactly the linear combinations of the fundamental weights with coefficients in $\mathbb{N}\cup\{0\}$.

Given an irreducible representation V of $\tilde{L}G \rtimes S^1$ of lowest weight $(h,\lambda,0)$, to get a better understanding of the we want to determine which other weights occur in V (or rather, we want to find restrictions for those weights).

First observation: the whole orbit under W_{aff} occurs. This produces, for the $\eta \in \mathrm{Hom}(S^1,T)$, the weights $(h,\lambda+h\eta^*,\lambda(\eta)+h\left|\eta\right|^2/2)$,

6.17 Example. If $G=SU(2)$, we have the isomorphism $(\hat{T}\subset\mathfrak{t}) \cong (\mathbb{Z}\subset\mathbb{R})$, where $\left(\begin{smallmatrix}2\pi it & 0\\ 0 & -2\pi it\end{smallmatrix}\right) \in \mathfrak{t}$ is mapped to $t\in\mathbb{R}$, $\mathfrak{t}^*$ is identified with $\mathfrak{t}$ using the standard inner product and this way the character $\mathrm{diag}(z,z^{-1}) \mapsto z^\mu$ in $\hat{T}$ is mapped to $\mu\in\mathbb{Z}$.

Under this identification, the lowest weight α_0 is identified with $1\in\mathbb{Z}$.

There are exactly two fundamental weights. The W_{aff}-orbit of the weight $(1,0,0)$ is (with this identification of $\hat{T}$ with $\mathbb{Z}$) $\{(1,2k,m) \mid (2k)^2=2m\}$, the set of all weights of the corresponding irreducible representations is $\{(1,2k,m) \mid (2k)^2 \le 2m\}$. Similarly, the orbit of $(1,-1,0)$ is $\{(1,2k+1,m) \mid (2k+1)^2 = 2m+1\}$, the set of all weights is $\{(1,2k+1,m) \mid (2k+1)^2 \le 2m+1\}$.

In general (for arbitrary G), the orbit of the lowest weight (h,μ,m) (with $|(h,\mu,m)|^2 = |\mu|^2 + 2mh$) is contained in the parabola $\{(h,\mu',m') \mid |\mu'|^2 = |(h,\mu,m)|^2 + 2m'h\}$.

All other weights are contained in the interior of this parabola (i.e. those points with "=" replaced by "$\leq$").

The last statement follows because, by translation with an element of W_{aff} we can assume that (h,μ',m') is antidominant. Then

$$|(h,\mu',m')|^2 - |(h,\mu,m)|^2 = \langle (h,\mu',m') + (h,\mu,m), (h,\mu',m') - (h,\mu,m)\rangle \leq 0,$$

because the first entry is antidominant and the second one is positive, (h,μ,m) being a lowest weight.

We use the fact (not proved here) that the extension inner product on $\mathfrak{t}$ extends to an inner product on $\mathbb{R}\oplus\mathfrak{t}\oplus\mathbb{R}$ which implements the pairing between roots and coroots.

7 Proof for Section 5 and Homogeneous spaces of LG

We now want to indicate the proofs of the statements of Section 6.

The basic idea is that we can mimic the we can mimic the Borel-Weil theorem for compact Lie groups. It can be stated as follows:

7.1 Theorem. *The homogeneous space G/T has a complex structure, because it is isomorphic to $G_{\mathbb{C}}/B^+$, where $G_{\mathbb{C}}$ is the complexification of G and B^+ is the Borel subgroup. In case $G = U(n)$, $G_{\mathbb{C}} = Gl(n,\mathbb{C})$ and $B^+ \subset Gl(n,\mathbb{C})$ is the subgroup of upper triangular matrices; the homogeneous space is the flag variety.*

To each weight $\lambda\colon T \to S^1$ there is a uniquely associated holomorphic line bundle L_λ over $G_{\mathbb{C}}/B^+$ with action of $G_{\mathbb{C}}$.

L_λ has non-trivial holomorphic sections if and only if λ is an antidominant weight. In this case, the space of holomorphic section is an irreducible representation with lowest weight λ.

For loop groups, the relevant homogeneous space is

$$Y := LG/T = LG_{\mathbb{C}}/B^+G_{\mathbb{C}}, \qquad B^+G_{\mathbb{C}} = \{\sum_{k\geq 0}\lambda_k z^k \mid \lambda_0 \in B^+\}$$

Recall that for $GL(n,\mathbb{C})$, the *Borel subgroup* B^+ is the subgroup of upper triangular matrices.

Note that the second description defines on Y the structure of a complex manifold.

We have also $Y = \tilde{L}G/\tilde{T} = \tilde{L}G_{\mathbb{C}}/\tilde{B}^+G_{\mathbb{C}}$.

7.2 Lemma. *Each character $\lambda\colon \tilde{T} \to S^1$ has a unique extension $\tilde{B}^+G_{\mathbb{C}} = \tilde{T}_{\mathbb{C}} \cdot \tilde{N}^+G_{\mathbb{C}} \to \mathbb{C}^x$,, with $N^+G_{\mathbb{C}} := \{\sum_{k\geq 0}\lambda_k z^k \mid \lambda_0 \in N^+G_{\mathbb{C}}\}$, where N^+ is the nilpotent subgroup of $G_{\mathbb{C}}$ whose Lie algebra is generated by the positive root vectors, for $Gl(n,\mathbb{C})$ it is the group of upper triangular matrices with 1s on the diagonal. This way defines a holomorphic line bundle*

$$L_\lambda := \tilde{L}G_{\mathbb{C}} \times_{\tilde{B}^+G_{\mathbb{C}}} \mathbb{C} \text{ over } Y.$$

Write Γ_λ *for the space of holomorphic sections of* $L\lambda$. *This is a representation of* $\tilde{L}G \rtimes S^1$.

7.3 Lemma. *The space* $Y = LG/T$ *contains the affine Weyl group* $W_{aff} = (\mathrm{Hom}(S^1, T) \cdot N(T))/T$, *where* $N(T)$ *is the normalizer of* T *in* $G \subset LG$.

Y *is stratified by the orbits of* W_{aff} *under the action of* $N^- LG_{\mathbb{C}} := \{\sum_{k\le 0} \lambda_k z^k \mid \lambda_0 \in N^- G_{\mathbb{C}}\}$. *Here* $N^- G_{\mathbb{C}}$ *is the nilpotent Lie subgroup whose Lie algebra is spanned by the negative root vectors of* $\mathfrak{g}_{\mathbb{C}}$. *For* $Gl(n, \mathbb{C})$ *this is the group of lower triangular matrices with* 1*s on the diagonal.*

7.4 Theorem. *Assume that* $\lambda \in \mathrm{Hom}(\tilde{T}, S^1)$ *is a weight such that the space* Γ_λ *of holomorphic sections of* L_λ *is non-trivial. Then*

(1) Γ_λ *is a representation of positive energy.*

(2) Γ_λ *is of finite type, i.e. each fixed energy subspace* $\Gamma_\lambda(n)$ *is finite dimensional*

(3) λ *is the lowest weight of* Γ_λ *and is antidominant.*

(4) Γ_λ *is irreducible.*

Proof. We use the stratification of Y to reduce to the top stratum. It turns out that the top stratum is $N^- LG_{\mathbb{C}}$ and that L_γ trivializes here, so Γ_λ restricts to the space of holomorphic functions on $N^- LG_{\mathbb{C}}$. Because holomorphic sections are determined by their values on the top stratum, this restriction map is injective. We can further, with the exponential map (which is surjective in this case), pull back to holomorphic functions on $N^- L\mathfrak{g}_{\mathbb{C}}$, and then look at the Taylor coefficients at 0.

This way we finally map invectively into $\prod_{p\ge 0} S^p(N^- L\mathfrak{g}_{\mathbb{C}})^*$, where $S^p(V)^*$ is the space of p-multilinear maps $V \times \cdots \times V \to \mathbb{C}$. This map is indeed $\tilde{T} \times S^1$-equivariant, if we multiply the obvious action on the target with λ.

It now turns out that $N^- L\mathfrak{g}_{\mathbb{C}}$ has (essentially by definition) negative energy, and therefore the duals $S^p(N^- L\mathfrak{g}_{\mathbb{C}})$ all have positive energy; and the weights are exactly the positive roots (before multiplication with λ). Consequently λ is of lowest weight. If for some positive root (α, n), we had $\lambda(\alpha, n) = m > 0$, then reflection in W_{aff} corresponding to α would map λ to $\lambda - m\alpha$, which on the other hand can not be a root of Γ_λ if λ is a root. Consequently, λ is antidominant.

Explicit calculations also show that the image of the “restriction map” is contained in a subspace of finite type.

To prove that Γ_λ is irreducible, we look at the subspace of lowest energy. This is a representation of $G_{\mathbb{C}}$. Pick a lowest weight vector for this representation, it is then invariant under the nilpotent subgroup N^-. Since it is of lowest energy, it is even invariant under $N^- LG_{\mathbb{C}}$. On the other hand, since the top stratum of Y is $N^- LG_{\mathbb{C}}$, the value of an invariant section at one point completely determines it, so that the space of such sections is 1-dimensional. $B^- LG_{\mathbb{C}}$ acts on this space by multiplication with the holomorphic homomorphism $\lambda\colon B^- LG_{\mathbb{C}} \to \mathbb{C}^\times$, so that λ really occurs as lowest weight.

We now show that this vector is actually a cyclic vector, i.e. generates Γ_λ under the action of $\tilde{L}G \rtimes S^1$. Else choose a vector of lowest energy not in this subrepresentation, of lowest weight for the corresponding action of the compact group G, and we get with the argument as above a second $N^{-}LG_{\mathbb{C}}$ invariant section. □

7.5 Lemma. $\Gamma_\lambda \neq 0$ *if and only If* λ *is antidominant.*

Proof. If α is a positive root of G (($\alpha, 0$ therefore a positive root of $LG \rtimes S^1$), we get a corresponding inclusion $i_\alpha\colon Sl_2(\mathbb{C}) \to \tilde{L}G_{\mathbb{C}}$ whose restriction to $\mathbb{C}^\times \subset Sl_2(\mathbb{C})$ is the exponential of h_α. Since α is positive, $B^+Sl_2(\mathbb{C})$ is mapped to $B^+\tilde{L}G_{\mathbb{C}}$. Therefore, we get an induced map $P^1(\mathbb{C}) = Sl_2(\mathbb{C})/B^+ \to Y$. The pullback of L_λ under this map is the line bundle associated to $\lambda \circ h_\alpha$. If Γ_λ is non-trivial, we can therefore pull back to obtain a non-trivial holomorphic section of this bundle over $P^1(\mathbb{C})$. These exist only if $\lambda \circ h_\alpha = \lambda(h_\alpha) \leq 0$.

This prove half the conditions for antidominance. We omit the prove of the other half, where we have to consider the positive root $(-\alpha, 1)$.

If, on the other hand, λ is antidominant, one constructs a holomorphic section along the stratification of Y. □

7.6 Theorem. *An arbitrary smooth representation of* $\tilde{L}G \rtimes S^1$ *of positive energy splits (upto essential equivalence) as a direct sum of representations of the form* Γ_λ.

In particular, the Γ_λ *are exactly the irreducible representations.*

Proof. If E is a representation of positive energy, so is $\overline{E}^*$. Pick in the G-representation $\overline{E}^*(0)$ a vector ϵ of lowest weight λ. For each smooth vector $v \in E$, the map

$$s_v\colon \tilde{L}G_{\mathbb{C}} \to \mathbb{C}; f \mapsto \epsilon(f^{-1}v)$$

turns out to define a holomorphic section of L_λ. This give a non-trivial map $E \to \Gamma_\lambda$. If E is irreducible it therefore is essentially equivalent to Γ_λ

Using $\overline{\Gamma}_\lambda^*$ and similar constructions, we can split off factor Γ_λ successively from an arbitrary representation E. □

The proofs of the statements about the structure of these homogenous spaces uses the study of related Grassmannians. These we define and study for $LU(n)$; results for arbitrary compact Lie groups follow by embedding into $U(n)$ and reduction to the established case.

Let $H = H_+ \oplus H_-$ be a (polarized) Hilbert space. The important example for us is $H = L^2(S^1, \mathbb{C}^n)$, with H_+ generated by functions z^k for $k \geq 0$ and H_- generated by z^k with $k < 0$ (negative or positive Fourier coefficients vanish).

On this Hilbert space, the complex loop group $LGl(n, \mathbb{C})$ acts by pointwise multiplication.

We define the Grassmannian $Gr(H) := \{W \subset H \mid \mathrm{pr}_+\colon W \to H_+$ is Fredholm, $pr_=\colon W \to H_-$ is Hilbert-Schmidt$\}$. This is a Hilbert manifold.

We define the restricted linear group $Gl_{res}(H) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid b, c$ Fredholm$\}$. (This implies that a, d are Hilbert-Schmidt). Set $U_{res}(H) := U(H) \cap Gl_{res}(H)$. These groups act on $Gr(H)$.

7.7 Lemma. *The image of* $LGl(n, \mathbb{C})$ *in* $Gl(H)$ *is contained in* $Gl_{res}(H)$.

Proof. Write $\gamma = \sum_k \gamma_k e^{ik\theta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in LGl(n, \mathbb{C})$. Then for each $n > 0$, $b(e^{-in\theta}) = \sum_{k \geq n} \gamma_k e^{i(k-n)\theta}$, and for $n \geq 0$, $c(e^{in\theta}) = \sum_{k<-n} \gamma_k e^{i(k+n)\theta}$. Therefore $\|b\|_{HS} = \sum_{n \geq 0} \sum_{k \geq n} |\gamma_k|^2 = \sum_{k \geq 0} (1+k) |\gamma_k|^2 < \infty$ since smooth functions have rapidly decreasing Fourier expansion. Similarly for c. □

$Gr(H)$ has a cell decomposition/stratification in analogy to the finite dimensional Grassmannians (with infinitely many cells of all kinds of relative dimensions). Details are omitted here, but this is an important tool in many proofs.

$Gr(H)$ contains the subspace $Gr_\infty(H) := \{W \in Gr(H) \mid \mathrm{im}(pr_-) \cup \mathrm{im}(\mathrm{pr}_+) \subset C^\infty(S^1, \mathbb{C}^n)\}$. Inside this one we consider the subspace $Gr_\infty^{(n)} := \{W \in Gr_\infty(H) \mid zW \subset W\}$.

It turns out that $Gr_\infty(H)^{(n)} = LGl(n, complexs)/L^+Gl(n, \mathbb{C}) = LU(n)/U(n)$.

The main point of the definition of $Gr(H)$ is that we can define a useful (and fine) virtual dimension for its elements; measuring the dimension of $W \cap z^m H_-$ for every m. This is used to stratify these Grassmannians and related homogeneous spaces, and these stratifications were used in the study of the holomorphic sections of line bundles over these spaces.

References

[1] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc. (JEMS)*, 5(1):41–68, 2003. 1

[2] Karl-Hermann Neeb. Borel-Weil theory for loop groups. In *Infinite dimensional Kähler manifolds (Oberwolfach, 1995)*, volume 31 of *DMV Sem.*, pages 179–229. Birkhäuser, Basel, 2001.

[3] Andrew Pressley and Graeme Segal. *Loop groups*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications. 1