Morse Theory with an Introduction to Homology and Cell Complexes: Learn to stop worrying and love the critical points

Some General Comments

We require that the speakers discuss the plan of their talks and possible questions with us a week before giving the talk. This avoids unnecessary stress and makes the seminar more pleasant for everyone. In some of the talks (specially the later ones), it is not reasonable (or possible) to give all the proofs and details. This means that the speaker has to distinguish between essential and inessential (or technical) details and perhaps avoid talking about the latter. Sometimes it may also be beneficial to give only the idea of a proof. The talks, where a judicious choice of the material to be presented is even more crucial are marked with a *. However, this does not mean that time management in the rest of the talks is trivial.

Talk 1: Of Categories and Functors

The goal of this talk is to introduce the notions of categories and functors. The category and functor of great interest to us are the category of chain complexes and the singular complex functor respectively. The latter is a functor from the category of pairs of topological spaces to the category of chain complexes. Here is a detailed list of the topics which should appear in this talk:

- Definition of categories and some examples: category of abelian group, (pairs of) topological spaces, etc..
- Definition of covariant functors and some examples: identity functor, forgetful functors, free abelian groups generated by sets, etc..
- Natural transformations of functors: natural equivalence between the identity functor and the double dual functor on the category of finite dimensional vector spaces.
- Definition of exact sequences and chain complexes.
- The category of chain complexes of abelian groups
- Standard simplices and the singular complex functor.

References: [1, Chapters 1-3], [3, Chapter 1, Sections 7-8].

Talk 2: Homology of Complexes

The aim of this talk is to define the homology functor from the category of chain complexes of abelian groups to the category of graded abelian groups and present some of its properties. At the end of this talk singular homology is defined as the composition of th singular complex functor and the homology functor Here is a detailed list of the topics which should appear in this talk:

- Definition of the homology groups and related notions (chains, cycles, boundaries etc.)
- Proof of the fact that homology groups define a functor.
- The long exact sequence of homology groups: definition of the boundary map and its naturality
- Chain homotopies, homotopy equivalent chain complexes: discussion of homotopy invariance of the homology functor and the notion of contractibility
- Definition of the singular homology groups

Reference: [1, Chapter 2]

Talk 3: Around Singular Homology

The aim of this talk is to discuss the main properties of the singular homology functor. Here is a detailed list of the topics which should appear in this talk:

- The long exact sequences of pairs and triples.
- Some examples: homology of a point, homology of convex subsets of $\mathbb{R}^n$
- Reduced homology
- Behaviour under disjoint union
- Proof of the homotopy invariance of singular homology
- The excision property (without proof)

Reference: [1, Chapter 3]

Talk 4: Some Computations and Applications of Singular Homology

In this talk the properties of the singular homology functors are used to compute some (relative) homology groups of some simple spaces. These are then used to prove certain classical results, e.g. the Brouwer fixed point theorem. Here is a detailed list of the topics which should appear in this talk:

- Homology of cells an spheres ([1, Chapters 4, Proposition 2.2])
- Homology of product of a space with a sphere: computation of homology of a torus
- The Brouwer fixed point theorem
- Definition of the degree of a map between spheres and statement of its properties
- Definition of local degree [1, Chapters 4, Example 5.4]
- Additivity of local degree

Reference: [1, Chapter 4]

Talk 5: Cellular Spaces and their Homology*

The aim of this talk is to introduce the notion of filtration on a space and to discuss how nice filtrations can be used to compute its homology. First, cellular decompositions and the cellular chain complex are discussed. This is followed by the definition of CW-decompositions, which are special cellular decompositions, which make the computations more doable. Here is a detailed list of the topics which should appear in this talk:

- Definition of the category of cellular spaces
- The cellular chain complex functor
- [1, Chapters 5, Proposition 1.3]
- CW-decompositions and some examples: spheres, disc, real and complex projective spaces
- The cellular chain complex for CW-spaces ([1, Chapters 5, Proposition 4.1])
- Homology of complex projective spaces

- Description of the matrix components of the boundary map of the cellular chain complex for CW-spaces ([1, Chapters 5, Proposition 6.9])
- Homology of real projective spaces

Reference: [1, Chapter 5]

Talk 6: On the Local Structure of Morse Function

The aim of this talk is to introduce the fundamental notions of Morse theory and such as critical points and Hessian of a function and to present the proof of the Morse lemma on the local structure of Morse functions near their critical points. Here is a detailed list of the topics which should appear in this talk:

- Definition of critical and regular points of functions between manifolds
- Definition of critical and regular values
- Definition of Hessian and non-degenerate critical points
- Definition of the index of bilinear forms and non-degenerate critical points
- Definition of Morse functions
- Statement and proof of the Morse lemma on the local structure of Morse functions near the critical points

References: For the definitions: [6, Chapter 1], for the proof of the Morse lemma [4, Chapter 2]

Talk 7: Attaching Handles and the Weak Morse Principle

One of the main aims of this is talk is to introduce the notion of handle attachment and how it can be used to obtain new manifolds out of old ones. After a discussion about gradient-like vector fields, the speaker will state and prove the weak Morse principle ([6, Theorem 2.6]). Here is a detailed list of topics which should appear in this talk:

- Handle Attachment
- Smoothing out corners
- Gradient-like vector fields and how to get them
- The weak Morse principle

Reference: [6, Chapter 2]

Talk 8: The Strong Morse Principle

The title is self-explanatory. The aim of this talk is to state and prove the strong Morse principle ([6, Theorem 2.7]). Afterwards, the speaker will discuss the examples in [5]Part I, Section 4. Here is a detailed list of topics which should appear in this talk:

- The strong Morse Principle
- [5, Theorem 4.1]
- Handle decomposition of the complex projective plane
- Application of the above handle decomposition to computation of the homology of the complex projective plane

References: [5]Part I, [6, Chapter 2]

Talk 9: The Morse Inequalities

The aim of this talk is to obtain some homological information form Morse functions using the previuosly obtained results. Here is a detailed list of topics which should appear in this talk:

- Definition of singular homology with coefficients
- The Poincaré polynomial and Euler Characteristic of a manifold
- The abstract and topological Morse inequalities
- Definition of completable and perfect Morse functions
- Statement of some criteria for a Morse function to be completable or perfect and some examples.

Reference: [6]Chapter 2

Talk 10: Morse Functions, Do They Exist?

The aim of this talk is to prove the existence of Morse function on closed manifolds and to show their abundance. Here is a detailed list of topics which should appear in this talk:

- [4, Theorem 2.20]

Reference: [4, Chapter 2]

Talk 11: Classification of Compact Orientable Surfaces*

The aim of this is talk is to present a proof of the theorem on classification of compact orientable surfaces using Morse theory. Here is a detailed list of topics which should appear in this talk:

- Characterisation of Disks using Morse functions ([2, Chapter 9, Theorem 2.1], without proof)
- Preparations for the proof of [2, Chapter 9, Theorem 3.5] ([2, Chapter 9, Section 3])
- Statement and proof of [2, Chapter 9, Theorem 3.5]

Reference: [2, Chapter 9]

Talk 12: Riemannian Geoemtry 101*

The aim of this talk is to introduce some fundamental concepts of Riemannian geometry. The talk starts with the definition of connections on the tangent bundle and moves on to metric connections on Riemannian manifolds. This is followed by a discussion of related notions such as parallel transport, (minimal) geodesics and the exponential map. Here is a detailed list of topics, which should appear in this talk:

- Definition Connections on the tangent bundle
- Definition of metric connections on Riemannian manifolds
- Statement of the fundamental theorem of Riemannian geometry
- Definition of parallel transport and geodesics
- Definition of the exponential map

Reference: [5, Chapter 2]

Talk 13: Path Spaces and the Energy Functional*

The aim of this talk is to discuss how the spaces of paths between two points on a manifold can be treated like a smooth manifold. In particular, we will see what the tangent space at a path is and how one can define the differential and Hessian of a "smooth function" on such path spaces. Furthermore one example of such functions is introduced and its critical points are seen to be geodesics.The possible degeneracies of the critical points of the energy functional are then studied using Jacobi fields. Here is a detailed list of topics, which should appear in this talk:

- Definition of the path spaces
- Variations and the tangent space at a path
- The energy functional and its differential: characterization of the critical points
- The Hessian of the energy functional
- Jacobi fields and characterization of the non-degenerate points of the energy functional

Reference: [5, Chapter 3]

Talk 14: The Topology of (Subsets of) Path Spaces*

The aim of this talk is to study the topology of the subspace of paths having energy less than some given positive number. This is done by approximating this subspace by some finite dimensional smooth manifold and using Morse theory to obtain topological information from the latter space. Here is a detailed list of topics which should appear in this talk:

- Statement of the theorem of Morse on the index of the Hessian of the energy functional
- Definition of the space Ω^c.
- Definition of the finite dimensional approximation to Ω^c
- [5, Chapter 3, Theorems 16.2 & 16.3]

Reference: [5, Chapter 3]

References

[1] A. Dold, *Lectures on algebraic topology*, Classics in Mathematics, Springer Berlin Heidelberg, 1995. ↑(document)

[2] M.W. Hirsch, *Differential topology*, Graduate Texts in Mathematics, Springer New York, 1997. ↑(document)

[3] S. MacLane, *Homology*, Classics in Mathematics, Springer Berlin Heidelberg, 1995. ↑(document)

[4] Y. Matsumoto, K. Hudson, and M. Saito, *An introduction to morse theory*, Europe and Central Asia Poverty Reduction and Economic Manag, American Mathematical Society, 2002. ↑(document)

[5] J.W. Milnor, *Morse theory*, Annals of mathematics studies, Princeton University Press, 1963. ↑(document)

[6] L. Nicolaescu, *An invitation to morse theory*, Universitext, Springer New York, 2011. ↑(document)