Zyklus „Algebraic Topology“

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We give a short introduction of what the courses are about, with some (rather elementary) example questions and results; more sophisticated ones are developped in the developpmend of the course.

Check out the literature suggested in Section 9 on how the course will really start.

1 Objects of study (important examples

(1) topological spaces

(2) manifolds, e.g. surfaces

(3) knots in $\mathbb{R}^3$

(4) relation between geometry and topology (example: Gauss-Bonnet theorem)

2 Famous questions

(1) Poincaré conjecture: the only 3-dimensional manifold, in which every circle can be contracted to a point is S^3 (Fields medail, Clay Millenium Prize)

(2) find „computable“ invariants, which distinguish an arbitrary knot fro the trivial one

(3) make a complete list of all connected compact surfaces

3 Basic concepts

set theoretic topology:

(1) topological space, convergence, continuity

(2) compactness

(3) separation properties

(4) construction methods for topological spaces

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important basic concepts in essentially all mathematical disciplines (funktional analysis, numerics, algebraic geometry, differenzial geometry, differential equations, stochastics,...).

4 True (?) applications

(1) robotic, motion-planning

(2) topologische criteria for complexity questions

(3) tools for (un)solvable combinatorial problems

(4) knot theory when folding polymers and proteins

5 Inner scientific applications

(1) continuous symmetries (in quantum physics)

(2) phase spaces (in classical mechanics)

(3) space-time —in general relativity, in quantum field theory

(4) Gaus̈ theorem in elektro dynamics! elektrodynamics is best understood as a "topological" theory.

6 Basic questions

6.1 Equivalence problem

In topology we study (among others): (differentiable) manifolds, cell complexes (like simplicial complexes) and the relations homeomorphism, diffeomorphism, homotopy equivalence, ...

To distinguish such spaces we introduce *invariants.*

An example for such an invariant (from linear algebra) is the dimension of a vector spaces.

In topology:

6.1 Example. Euler charakteristic of finite polyhedrons.

6.2 Definition. If K is a finite simplicial complex, c_i the number of i-Simplizes

$$\chi(K) := \sum_i (-1)^i c_i$$

is the *Euler charakteristic* of K, $\dim(K) := \max\{i \mid c_i > 0\}$.

6.3 Theorem. *$\chi(X)$ und $\dim(X)$ are homeomorphism invarians of finite polyhedra, $\chi(X)$ even a homotopy invariant.*

6.2 Classifikation problem

6.4 Question. Given a class K of topological spaces, find invariants I such that $I(X) = I(Y)$ if and only if X and Y are equivalent (where equivalent might mean "homeomorphic" or "homotopy equivalent"), and make a list of all possible values of I.

Then, the elements of K are completely understood (upto the equivalence considered).

6.5 Example. Two connected compact orientierble 2-dimensional manifolds are hoemomorphic if and only if their Euler charakteristics are equal. The possible values of the Euler charakteristic are $\{2, 0, -2, \cdots\}$.

The surface of genus g (i.e. with g "holes“) has Euler charakteristic $2 - 2g$, with $g(S^2) = 0$, $g(T^2) =, \ldots$.

6.6 Example. A famous other example is the *(generalized) Poincaré conjecture*: a compact manifold withbout boundary which is homotopy equivalent to S^n is even homeoomorphic to S^n.

This is correct for $n = 1, 2$ and for $n \geq 5$ (1961 Smale) and for $n = 4$ (1982 Friedmann). Most famous is the (original) case $n = 3$, most recently solved by Perelman.

6.3 Existence of additional structure

Often, the existence of extra structure is useful and intersting, like triangulations. Question then: do such additional structures exist, if yes, in how many ways (upto equivalence).

6.7 Example. (1) which topologiscal manigfolds admit a smooth strcture (not all, and in general the smooth structures are not unique).

(2) does every manifold of dimension n have a triangulation (Hauptvermutung: yes).

Answer: correct for $n = 2, 3$, but wrong for $n \geq 5$ and for $n = 4$.

Fro smooth manifolds the assertion is correct.

7 Methods of algebraic topology

(1) fundamental group

(2) homological algebra;

(3) vector spaces, abelian groups, modues over commutative and non-commutative rings (and their morphisms)

(4) group theory

(5) categories and functors

(6) globale (mostly linear) analysis

(7) funktional analysis

(8) mathematical logic

8 Contents of the first semester

We will start with a quick introduction in set theoretic topology: properties of topological spaces like compactness, separation properties (Hausdorff), existence of continous functions,... are introduced and studies. We learn about important constructions like products, quotients, cones....

Then (early on) we study topological spaces with algebraic methods. We introduce homology groups, Betti numbers, the fundamental group,.... Many important and famous applications are given along the way (invariance of domain, Brouwer fixed point theorem, fundamental theorem of algebra, vector fields on spheres)

The first semester gives a detailed introduction in concepts and facts which essentially every mathematician should know. The later parts will continue and deepen this studies.

9 Literature

There is a huge number of suitable literature on the subject. A very subjective choice of three books I like in particular the following is the following (the course will certainly draw from them, as well).

Essentially all other books with titles "Algebraic topology", "Topology", "Homology theory" should be relevant, as well.

[1] Glen E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original. MR1700700

[2] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR1867354

[3] Wolfgang Lück, *Algebraic topology. Homology and manifolds. (Algebraische Topologie. Homologie und Mannigfaltigkeiten.) (German)*, Vieweg Studium: Aufbaukurs Mathematik. Wiesbaden, 2005.

9.1 Remark. Hatcher's books is, completely officially, freely available on his webpage: http://www.math.cornell.edu/ hatcher/, along with other interesting textbook literature (partially in project form).