Quasitriangular C*-quantum groups and braiding

Sutanu Roy
jointly with Ralf Meyer and Stanisław Lech Woronowicz
Mathematisches Institut, Georg-August Universität Göttingen, Germany

Introduction

We define the notion of ***quasitriangularity*** for C*-quantum groups using ***unitary*** R-matrices which are solutions of the ***Yang-Baxter Equation***. Then the corepresentation category of Hilbert spaces along with the unitary R-matrix becomes a ***braided tensor category***. Finally, we describe the ***quantum codouble*** and the ***Drinfeld double*** in the C*-algebraic setting associated to a C*-quantum group, and construct a unitary R-matrix on the quantum codouble. The theory of C*-quantum groups involved in this game is based on the concept of ***manageable multiplicative unitaries***. In particular, we do not assume the existence of Haar weights

C*-quantum groups $\mathbb{G} = (C, \Delta)$

- A locally compact quantum space (C*-algebra) C.
- A non-degenerate $*$-homomorphism (comultiplication) $\Delta\colon C \to \mathcal{M}(C \otimes C)$ which is coassociative: $(\mathrm{id}_C \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_C) \circ \Delta$.

Example: $C_0(G)$ with $\Delta(f)(g_1, g_2) := f(g_1 g_2)$ for all $f \in C_0(G)$ and $g_1, g_2 \in G$, where G is a locally compact group.

Dual quantum group $\widehat{\mathbb{G}} = (\widehat{C}, \hat{\Delta})$

- A locally compact quantum space (C*-algebra) $\widehat{C}$.
- A non-degenerate coassociative $*$-homomorphism (comultiplication) $\widehat{\Delta}\colon \widehat{C} \to \mathcal{M}(\widehat{C} \otimes \widehat{C})$.

Example: $\mathrm{C}^*_{\mathrm{red}}(G)$ with $\widehat{\Delta}(\lambda_g) = \lambda_g \otimes \lambda_g$, where λ_g is the left regular representation on $L^2(G)$ for $g \in G$ – a locally compact group.

Reduced bicharacter and dual pairing

There exists a unitary bicharacter $\mathrm{W} \in \mathcal{M}(\hat{C} \otimes C)$:
$(\mathrm{id}_{\hat{C}} \otimes \Delta)\mathrm{W} = \mathrm{W}_{12}\mathrm{W}_{13} \qquad (\widehat{\Delta} \otimes \mathrm{id}_C)\mathrm{W} = \mathrm{W}_{23}\mathrm{W}_{13}$.

Notation: We write $\mathrm{W} \in \mathcal{UM}(\widehat{\mathbb{G}} \otimes \mathbb{G})$ for the above reduced bicharacter W.

Quasitriangular C*-quantum groups

A C*-quantum group $\mathbb{G} = (C, \Delta)$ is *quasitriangular* if there exists a unitary bicharacter $\mathrm{R} \in \mathcal{M}(\hat{C} \otimes \hat{C})$:

- Bicharacter condition:
$$(\mathrm{id}_{\hat{C}} \otimes \widehat{\Delta})\mathrm{R} = \mathrm{R}_{12}\mathrm{R}_{13} \qquad (\widehat{\Delta} \otimes \mathrm{id}_{\hat{C}})\mathrm{R} = \mathrm{R}_{23}\mathrm{R}_{13}$$
- R-matrix condition:
$$\mathrm{R}(\sigma \circ \widehat{\Delta}(\hat{c}))\mathrm{R}^* = \widehat{\Delta}(\hat{c}) \text{ for all } \hat{c} \in \hat{C},$$
where $\sigma(\hat{c}_1 \otimes \hat{c}_2) = \hat{c}_2 \otimes \hat{c}_1$ for all $\hat{c}_1, \hat{c}_2$.

Such an R satisfies the *Yang-Baxter Equation*:
$$\mathrm{R}_{23}\mathrm{R}_{13}\mathrm{R}_{12} = \mathrm{R}_{12}\mathrm{R}_{13}\mathrm{R}_{23}$$

Corepresentation of C*-quantum groups

A unitary $U \in \mathcal{M}(\mathbb{K}(\mathcal{H}) \otimes C)$ is said to be a corepresentation of $\mathbb{G} = (C, \Delta)$ on a Hilbert space $\mathcal{H}$ if it is a character in the second leg: $(\mathrm{id} \otimes \Delta)U = U_{12}U_{13}$.

Example:

$$\begin{pmatrix} \pi\colon \hat{C} \to \mathbb{B}(\mathcal{H}) \\ \text{representation of} \\ \hat{C} \text{ on } \mathcal{H} \end{pmatrix} \Longrightarrow \begin{pmatrix} U = (\pi \otimes \mathrm{id}_C)\mathrm{W} \\ \text{corepresentation of} \\ \mathbb{G} \text{ on } \mathcal{H} \end{pmatrix}$$

Remark: Not every corepresentation of $\mathbb{G}$ comes from a representation of $\hat{C}$.

Tensor product of corepresentations

Let $U \in \mathcal{M}(\mathbb{K}(\mathcal{H}) \otimes C)$ and $V \in \mathcal{M}(\mathbb{K}(\mathcal{K}) \otimes C)$ be corepresentations of $\mathbb{G}$ on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then $U \mathbin{\textcircled{\top}} V := U_{13}V_{23} \in \mathcal{M}(\mathbb{K}(\mathcal{H} \otimes \mathcal{K}) \otimes C)$ is a corepresentation of $\mathbb{G}$ on $\mathcal{H} \otimes \mathcal{K}$.

Theorem (Meyer, R., Woronowicz, 2012)

There exists a unitary $X \in \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$ which defines a map $\times^{(\mathcal{H},\mathcal{K})} := \Sigma \circ X$ with the $\mathbb{G}$-equivariant property: $\times^{(\mathcal{H},\mathcal{K})}(U \mathbin{\textcircled{\top}} V) = (V \mathbin{\textcircled{\top}} U)\times^{(\mathcal{H},\mathcal{K})}$, where Σ is the flip operator acting on $\mathcal{H} \otimes \mathcal{K}$.
$\times^{(\cdot,\cdot)}$ defines a *braiding* on the *corepresentation category* of Hilbert spaces associated to a quasitriangular quantum group.

Let, $\pi\colon \hat{C} \to \mathbb{B}(\mathcal{H})$ and $\rho\colon \hat{C} \to \mathbb{B}(\mathcal{K})$ are representations. Then the following are representations of $\hat{C}$

- $\pi \mathbin{\textcircled{\top}} \rho := (\pi \otimes \rho) \circ \sigma \circ \hat{\Delta}$ on $\mathcal{H} \otimes \mathcal{K}$
- $\rho \mathbin{\textcircled{\top}} \pi := (\rho \otimes \pi) \circ \sigma \circ \hat{\Delta}$ on $\mathcal{K} \otimes \mathcal{H}$.

Moreover, $\times^{(\mathcal{H},\mathcal{K})} := \Sigma \circ (\pi \otimes \rho)\mathrm{R}$ satisfies $\times^{(\mathcal{H},\mathcal{K})} \circ \pi \mathbin{\textcircled{\top}} \rho = \rho \mathbin{\textcircled{\top}} \pi \circ \times^{(\mathcal{H},\mathcal{K})}$, where Σ is the flip operator.
In particular, $\times^{(\mathcal{H},\mathcal{K})} U \mathbin{\textcircled{\top}} V = V \mathbin{\textcircled{\top}} U \times^{(\mathcal{H},\mathcal{K})}$ for the corepresentations $U = (\pi \otimes \mathrm{id}_C)\mathrm{W}$ and $V = (\rho \otimes \mathrm{id}_C)\mathrm{W}$ of $\mathbb{G}$ on $\mathcal{H}$ and $\mathcal{K}$ respectively.

Theorem (Woronowicz, 2011)

There exist $\alpha\colon \widehat{C} \to \mathbb{B}(\mathcal{H})$ and $\beta\colon C \to \mathbb{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ with the commutation relation:
$$\mathrm{W}_{1\beta}\mathrm{W}_{13}\mathrm{W}_{\alpha 3} = \mathrm{W}_{\alpha 3}\mathrm{W}_{13}\mathrm{W}_{1\beta} \text{ in } \mathcal{UM}(\hat{C} \otimes \mathbb{K}(\mathcal{H}) \otimes C)$$
and $D := [\alpha(\widehat{C})\beta(C)] = [\beta(C)\alpha(\widehat{C})]$ is a C*-algebra where $[\cdot]$ denotes the closed linear span.
Quantum codouble: $\widehat{\mathrm{D}(\mathbb{G})} := (C \otimes \widehat{C}, \Delta_{\widehat{\mathrm{D}(\mathbb{G})}})$,
$\Delta_{\widehat{\mathrm{D}(\mathbb{G})}}(c \otimes \hat{c}) := \mathrm{W}_{23}((\mathrm{id}_C \otimes \sigma \otimes \mathrm{id}_{\hat{C}})(\Delta(c) \otimes \widehat{\Delta}(\hat{c})))\mathrm{W}^*_{23}$
Drinfel'd double: $\mathrm{D}(\mathbb{G}) := (D, \Delta_{\mathrm{D}(\mathbb{G})})$,
$\Delta_{\mathrm{D}(\mathbb{G})}(\alpha(\hat{c})\beta(c)) := ((\alpha \otimes \alpha)\widehat{\Delta}(\hat{c}))((\beta \otimes \beta)\Delta(c))$.
Reduced bicharacter:
$V := \mathrm{W}_{\alpha 2}\widehat{\mathrm{W}}_{\beta 3} \in \mathcal{UM}(\mathrm{D}(\mathbb{G}) \otimes \widehat{\mathrm{D}(\mathbb{G})})$ where $\widehat{\mathrm{W}} := \sigma(\mathrm{W}^*)$ in $\mathcal{UM}(C \otimes \widehat{C})$.
Hence $\widehat{\mathrm{D}(\mathbb{G})}$ and $\mathrm{D}(\mathbb{G})$ are dual to each other, and the dual pairing is established by V.

R-matrix for $\widehat{\mathrm{D}(\mathbb{G})}$

The bicharacter $\mathrm{R} := (\alpha \otimes \beta)\mathrm{W} \in \mathcal{UM}(D \otimes D)$ satisfies the R-matrix condition. Hence $\widehat{\mathrm{D}(\mathbb{G})}$ is quasitriangular C*-quantum group.

http://www.uni-math.gwdg.de/sutanu sutanu@uni-math.gwdg.de