Homomorphisms of quantum groups (joint work with R. Meyer and S.L.Woronowicz)

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Image: From multiplicative unitaries to locally compact quantum groups

- - Bicharacters
 - Universal bicharacter
 - Right or left coactions as homomorphisms
 - Morphism as a functor between coaction categories

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Image: From multiplicative unitaries to locally compact quantum groups

2 Excursion to history

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Wish you happy Seventieth Birthday Prof. Woronowicz



Sutanu Roy (Göttingen) Homomorphisms of quantum groups



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Definition Legs of a multiplicative unitary Locally compact quantum groups

Locally compact quantum group in different setting

- Algebraic setting (Multiplier Hopf*-algebras).
- Topological setting (*C**-algebras).
- Measure theoritic setting (von Neumann algebras).

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Multiplicative unitary

Definition

Legs of a multiplicative unitary Locally compact quantum groups

Definition

An operator $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be multiplicative unitary on the Hilbert space \mathcal{H} if it satisfies the *pentagon equation*

 $\mathbb{W}_{23}\mathbb{W}_{12}=\mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23}.$

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Observations

One can define two non-degenerate, normal, coassociative *-homomorphisms from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$:

$$\Delta(x) = \mathbb{W}(x \otimes I)\mathbb{W}^*$$

 $\widehat{\Delta}(y) = \operatorname{Ad}(\Sigma) \circ (\mathbb{W}^*(I \otimes y)\mathbb{W}).$

for all $x, y \in \mathbb{B}(\mathcal{H})$ and Σ is the flip operator acting on $\mathcal{H} \otimes \mathcal{H}$. Consider the slices/legs of the multiplicative unitaries:

$$C = \overline{\{(\omega \otimes id)\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}}^{\|.\|}$$
$$\widehat{C} = \overline{\{(id \otimes \omega)\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}}^{\|.\|}$$

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Definition Legs of a multiplicative unitary Locally compact quantum groups

Special class of multiplicative unitaries

Manageability and modularity

- Manageable multiplicative unitary. [Woronowicz, 1997]
- Modular multiplicative unitary. [Sołtan-Woronowicz, 2001]

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Definition Legs of a multiplicative unitary Locally compact quantum groups

Nice legs of modular multiplicative unitaries

Theorem (Sołtan, Woronowicz, 2001)

Let, $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. Then,

- *C* and \widehat{C} are *C*^{*}-sub algebras in $\mathbb{B}(\mathcal{H})$ and $W \in \mathcal{UM}(\widehat{C} \otimes C)$.
- there exists a unique $\Delta_C \in Mor(C, C \otimes C)$ such that
 - $(id_{\widehat{C}} \otimes \Delta)W = W_{12}W_{13}$.
 - Δ_C is coassociative: $(\Delta_C \otimes id_C) \circ \Delta_C = (id_C \otimes \Delta_C) \circ \Delta_C$.
 - $\Delta(C)(1 \otimes C)$ and $(C \otimes 1)\Delta(C)$ are linearly dense in $C \otimes C$.
- There exists an involutive normal antiautomorphism R_C of C.

Definition Legs of a multiplicative unitary Locally compact quantum groups

Locally compact quantum groups

Definition [Soltan-Woronowicz, 2001]

The pair $\mathbb{G} = (C, \Delta_C)$ is said to be a locally compact quantum group if the C*-algebra C and $\Delta_C \in Mor(C, C \otimes C)$ comes from a modular multiplicative unitary \mathbb{W} . We say \mathbb{W} giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$.

Observation

The unitary operator $\widehat{\mathbb{W}} = \operatorname{Ad}(\Sigma)(\mathbb{W}^*)$ gives rise to the quantum group $\widehat{\mathbb{G}} = (\widehat{C}, \Delta_{\widehat{C}})$ which is dual to $\mathbb{G} = (C, \Delta_C)$.

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Definition Legs of a multiplicative unitary Locally compact quantum groups

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Definition Legs of a multiplicative unitary Locally compact quantum groups

From groups to quantum groups

Given a locally compact group G:

- $\mathbb{G} = (C_0(G), \Delta)$ is a locally compact quantum group with $\Delta f(x, y) = f(xy)$.
- $\widehat{\mathbb{G}} = (\mathsf{C}^*_\mathsf{r}(G), \hat{\Delta})$ is the dual quantum group of \mathbb{G} with $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in G$.
- ^ˆG^u = (C^{*}(G), Â^u) is a C^{*}-bialgebra which is known as quantum group C^{*}-algebra of G.

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Definition Legs of a multiplicative unitary Locally compact quantum groups

Notations

Let, \mathbb{W} be a modular multiplicative unitary giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$. We write:

- $\mathbb W,$ when we consider it as an unitary operator action on the Hilbert space $\mathcal H\otimes \mathcal H$
- W, when we consider it as in element of of $\mathcal{UM}(\hat{C} \otimes C)$.
- *f*: *A* → *B*, when we consider *f* ∈ Mor(*A*, *B*) or
 f: *A* → *M*(*B*) where *A* and *B* are C*-algebras.

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From multiplicative unitaries to locally compact quantum groups

2 Excursion to history

- Equivalent pictures of homomorphisms of quantum groups
 Bicharacters
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4 Summary

Back to the *History* of quantum group morphisms/homomorphisms

- 1997: Ng introduced equivalent notion of morphisms between basic multiplicative unitaries in terms of
 - Birepresentations.
 - Mutual coactions.
 - Hopf *-homomorphisms.
- 2001: Kustermans defined equivalent notion of morphisms of quantum groups in terms of
 - Hopf *-homomorphisms between universal quantum group.
 - Special coactions of von Neumann algebraic versions of quantum groups.

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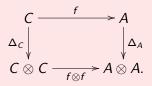
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Hopf *-homomorphisms

Let us consider $\mathbb{G}=(\mathit{C},\Delta_{\mathit{C}})$ and $\mathbb{H}=(\mathit{A},\Delta)$ be two C*-bialgebras.

Definition

A Hopf *-homomorphism between them is a morphism $f: C \rightarrow A$ that intertwines the comultiplications, that is, the following diagram commutes:



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Hopf *-homomorphisms

Let G and H are two locally compact groups.

- Consider a Hopf * homomorphism from $f: C_0(H) \to C_0(G)$.
- f induces a continuous group homomorphism $\phi \colon G \to H$.
- ϕ induces a Hopf *-homomorphism $\hat{f}: C^*(G) \to C^*(H)$

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Hopf *-homomorphisms and it's drawback

Let G and H are two locally compact groups.

- Consider a Hopf * homomorphism from $f: C_0(H) \to C_0(G)$.
- f induces a continuous group homomorphism $\phi \colon \mathcal{G} \to \mathcal{H}$.
- φ induces a Hopf *-homomorphism f̂: C^{*}_r(G) → C^{*}_r(H) if and only if kernel of φ is amenable.

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Hopf *-homomorphisms and it's drawback:example

Let $G = \mathbb{F}_2$ and $H = \{e\}$.

- The Hopf * homomorphism from $f: \mathbb{C} \to C_0(\mathbb{F}_2)$.
- f induces the trival group homomorphism $\phi \colon \mathbb{F}_2 \to \{e\}$.
- ϕ induces trivial Hopf *-morphism $\hat{f}: C^*_r(\mathbb{F}_2) \to \mathbb{C}$ as \mathbb{F}_2 is not amenable.

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Hopf *-homomorphisms and it's drawback

Let G and H are two locally compact groups.

- Consider a Hopf * homomorphism from $f \colon C_0(H) \to C_0(G)$.
- f induces a continuous group homomorphism $\phi \colon G \to H$.
- ϕ induces a Hopf *-homomorphism
 - $\hat{f}: C^*(G) \to C^*(H)$
 - $\hat{f}: C^*_r(G) \to C^*_r(H)$ if and only if kernel of ϕ is amenable.

Conclusion

Hopf *-homomorphisms are not always compatible with the reduced dual but with full dual.

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From multiplicative unitaries to locally compact quantum groups



Equivalent pictures of homomorphisms of quantum groups

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Bicharacters Universal bicharacter Right or left coactions as homomorphisms Morphism as a functor between coaction categories

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Bicharacters

Bicharacters

Universal bicharacter Right or left coactions as homomorphisms Morphism as a functor between coaction categories

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Let, $\mathbb{G} = (C, \Delta_C)$ and $\mathbb{H} = (A, \Delta_A)$ are two quantum groups.

Definition

A unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ a bicharacter from C to A if satisfying

$$\begin{split} &(\Delta_{\hat{\mathcal{C}}}\otimes \operatorname{id}_{\mathcal{A}})V = V_{23}V_{13} \qquad \text{in } \mathcal{UM}(\hat{\mathcal{C}}\otimes \hat{\mathcal{C}}\otimes \mathcal{A}),\\ &(\operatorname{id}_{\hat{\mathcal{C}}}\otimes \Delta_{\mathcal{A}})V = V_{12}V_{13} \qquad \text{in } \mathcal{UM}(\hat{\mathcal{C}}\otimes \mathcal{A}\otimes \mathcal{A}). \end{split}$$

quivalent pictures of homomorphisms of quantum groups Summary

Reduced bicharacter

Bicharacters

Universal bicharacter Right or left coactions as homomorphisms Morphism as a functor between coaction categories

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Let, $\mathbb{G}=\mathbb{H}=(\mathit{C},\Delta_{\mathit{C}})$ be a quantum group.

Definition(Reduced bicharacter)

The unitary $W^C \in \mathcal{UM}(\hat{C} \otimes C)$ a reduced bicharacter of C satisfying

$$\begin{split} &(\Delta_{\hat{C}}\otimes \text{id}_{C}) \mathbb{W}^{C} = \mathbb{W}_{23}^{C} \mathbb{W}_{13}^{C} \qquad \text{in } \mathcal{UM}(\hat{C}\otimes \hat{C}\otimes C), \\ &(\text{id}_{\hat{C}}\otimes \Delta_{C}) \mathbb{W}^{C} = \mathbb{W}_{12}^{C} \mathbb{W}_{13}^{C} \qquad \text{in } \mathcal{UM}(\hat{C}\otimes C\otimes C). \end{split}$$

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Lemma

A unitary $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ comes from a bicharacter $V \in \mathcal{UM}(\hat{C} \otimes A)$ (which is necessarily unique) if and only if

$$\begin{split} \mathbb{V}_{23}\mathbb{W}_{12}^{\mathcal{C}} &= \mathbb{W}_{12}^{\mathcal{C}}\mathbb{V}_{13}\mathbb{V}_{23} & \text{ in } \mathcal{U}(\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}}), \\ \mathbb{W}_{23}^{\mathcal{A}}\mathbb{V}_{12} &= \mathbb{V}_{12}\mathbb{V}_{13}\mathbb{W}_{23}^{\mathcal{A}} & \text{ in } \mathcal{U}(\mathcal{H}_{\mathcal{C}}\otimes\mathcal{H}_{\mathcal{A}}\otimes\mathcal{H}_{\mathcal{A}}). \end{split}$$

From multiplicative unitaries to locally compact quantum groups Excursion to history

Equivalent pictures of homomorphisms of quantum groups
Summary

Modularity revisited

Bicharacters

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Definition [Soltan-Woronowicz, 2001]

A multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be *modular* if there exist positive self-adjoint (may be unbounded) operators \hat{Q} and Q on \mathcal{H} and an operator $\widetilde{\mathbb{W}} \in \mathcal{U}(\bar{\mathcal{H}} \otimes \mathcal{H})$ such that

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$$\hat{Q} = \ker Q = \{0\};$$

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 and $\xi \in \mathcal{D}(Q^{-1})$, $\xi' \in \mathcal{D}(Q)$,

$$(\eta'\otimes\xi'\mid \mathbb{W}\mid\eta\otimes\xi)=(ar\eta\otimes Q\xi'\mid \widetilde{\mathbb{W}}\midar\eta'\otimes Q^{-1}\xi).$$

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An example of bicharacter

Definition [Soltan-Woronowicz, 2001]

A multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be *modular* if there exist positive self-adjoint (may be unbounded) operators \hat{Q} and Q on \mathcal{H} and an operator $\widetilde{\mathbb{W}} \in \mathcal{U}(\bar{\mathcal{H}} \otimes \mathcal{H})$ such that

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If \mathbb{W} gives rise to the quantum group (C, Δ) then $\widetilde{\mathbb{W}}$ is a bicharacter from (C^{op}, Δ) to (C, Δ^{op}) .

Equivalent pictures of homomorphisms of quantum groups
Summary

An important theorem

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Theorem [Woronowicz, 2010]

Let \mathcal{H} be a Hilbert space and let $\mathbb{W} \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. If $a, b \in \mathbb{B}(\mathcal{H})$ satisfy $\mathbb{W}(a \otimes 1) = (1 \otimes b)\mathbb{W}$, then $a = b = \lambda 1$ for some $\lambda \in \mathbb{C}$. More generally, if $a, b \in \mathcal{M}(\mathbb{K}(\mathcal{H}) \otimes D)$ for some C*-algebra D satisfy $\mathbb{W}_{12}a_{13} = b_{23}\mathbb{W}_{12}$, then $a = b \in \mathbb{C} \cdot 1_{\mathcal{H}} \otimes \mathcal{M}(D)$.

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Corollary [Kustermans, Vaes 2000]

Let (C, Δ_C) be a quantum group. If $c \in \mathcal{M}(C)$, then $\Delta_C(c) \in \mathcal{M}(C \otimes 1)$ or $\Delta_C(c) \in \mathcal{M}(1 \otimes C)$ if and only if $c \in \mathbb{C} \cdot 1$. More generally, if D is a C^{*}-algebra and $c \in \mathcal{M}(C \otimes D)$, then $(\Delta_C \otimes id_D)(c) \in \mathcal{M}(C \otimes 1 \otimes D)$ or $(\Delta_C \otimes id_D)(c) \in \mathcal{M}(1 \otimes C \otimes D)$ if and only if $c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$.

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Properties of bicharacters I

Consider $\mathbb{G} = (C, \Delta_C)$, $\mathbb{H} = (A, \Delta_A)$ and $\mathbb{I} = (B, \Delta_B)$ are quantum groups.

- Given a bicharacter $V \in \mathcal{UM}(\hat{C} \otimes A)$ we have:
 - $(R_{\hat{C}}\otimes R_A)V = V.$
 - $\hat{V} = \sigma(V^*) \in \mathcal{UM}(A \otimes \hat{C})$ is the dual bicharacter.

Summary

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Properties of bicharacters II

Given two bicharacters V^{C→A} ∈ UM(Ĉ ⊗ A) and V^{A→B} ∈ UM(Â ⊗ B), there exists unique bicharacter V^{C→B} ∈ UM(Ĉ ⊗ B) satisfying

$$\mathbb{V}_{13}^{C \to B} = (\mathbb{V}_{12}^{C \to A})^* \mathbb{V}_{23}^{A \to B} \mathbb{V}_{12}^{C \to A} (\mathbb{V}_{23}^{A \to B})^*$$

We denote $V^{C \to B} = V^{A \to B} * V^{C \to A}$ as composition of $V^{C \to A}$ and $V^{A \to B}$.

• Identity bicharacter:

$$V^{C \to A} = V^{C \to A} * W^{C}$$
 and $V^{C \to A} = W^{A} * V^{C \to A}$.

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Properties of bicharacters III

• Composition of bicharacters is associative:

$$(\mathsf{V}^{B\to D}*\mathsf{V}^{A\to B})*\mathsf{V}^{C\to A}=\mathsf{V}^{B\to D}*(\mathsf{V}^{A\to B}*\mathsf{V}^{C\to A}).$$

where $V^{B \to D} \in \mathcal{UM}(\hat{B} \otimes D)$ where $\mathbb{J} = (D, \Delta_D)$ is a quantum group.

• Compatibility with duality:

$$\widehat{\mathsf{V}^{C\to B}} = \widehat{\mathsf{V}^{A\to B}} * \widehat{\mathsf{V}^{C\to A}}$$

or equivalently

$$\widehat{\mathbb{V}_{13}^{C \to B}} = \widehat{\mathbb{V}_{12}^{A \to B}}^* \widehat{\mathbb{V}_{23}^{C \to A}} \widehat{\mathbb{V}_{12}^{A \to B}} \widehat{\mathbb{V}_{23}^{C \to A}}^*$$

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Category of locally compact quantum groups

Proposition [Ng, 1997; Meyer, R., Woronowicz, 2011]

The composition of bicharacters is associative, and the multiplicative unitary W^{C} is an identity on C. Thus bicharacters with the above composition and locally compact quantum groups are the arrows and objects of a category. Duality is a contravariant functor acting on this category.

I From multiplicative unitaries to locally compact quantum groups

2 Excursion to history

Equivalent pictures of homomorphisms of quantum groups
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Universal bicharacter

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- Morphism as a functor between coaction categories

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History of quantum group morphisms/homomorphisms revisited

- 1997: Ng introduced equivalent notion of morphisms between basic multiplicative unitaries in terms of
 - Birepresentations.
 - Mutual coactions.
 - Hopf *-homomorphisms.
- 2001: Kustermans defined equivalent notion of morphisms of quantum groups in terms of
 - Hopf *-homomorphisms between universal quantum group.
 - Special coactions of von Neumann algebraic versions of quantum groups.

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Corepresentation and universal bialgebra of a quantum group

Definition

A corepresentation of $(\hat{C}, \Delta_{\hat{C}})$ on a C*-algebra D is a unitary multiplier $V \in \mathcal{UM}(\hat{C} \otimes D)$ that satisfies $(\Delta_{\hat{C}} \otimes id_D)(V) = V_{23}V_{13}.$

Remark

Similarly corepresentation of (C, Δ_C) on a C*-algebra D is a unitary multiplier $V \in \mathcal{UM}(D \otimes C)$ that satisfies $(\mathrm{id}_D \otimes \Delta_C)(V) = V_{12}V_{13}.$

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Equivalent pictures of homomorphisms of quantum groups
Summary

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Universal quantum group C*-algebra

Proposition[Soltan, Woronowicz, 2007]

- There exists a maximal corepresentation *Ṽ* ∈ *UM*(*Ĉ*^u ⊗ *C*) of (*C*, Δ_C) on a C*-algebra *Ĉ*^u such that for any corepresentation *U* ∈ *UM*(*D* ⊗ *C*) there exists a unique φ̂ ∈ Mor(*Ĉ*^u, *D*) such that (φ̂ ⊗ id_C)*Ṽ* = *U*.
- There exists a unique $\Delta_{\hat{C}^{u}} \in Mor(\hat{C}^{u}, \hat{C}^{u} \otimes \hat{C}^{u})$ such that:
 - $(\Delta_{\hat{C}^{u}} \otimes id_{\mathcal{C}})\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{23}\tilde{\mathcal{V}}_{13}$
 - $\Delta_{\hat{\mathcal{L}}^u}(\hat{\mathcal{L}}^u)(1\otimes \hat{\mathcal{L}}^u)$ and $(\hat{\mathcal{L}}^u\otimes 1)\Delta_{\hat{\mathcal{L}}^u}$ are linearly dense in

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Equivalent pictures of homomorphisms of quantum groups
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Universal C^* -bialgebras associated to a quantum group

Universal qunatum groups C*-algebra

 $(\hat{C}^u,\Delta_{\hat{C}^u})$ is known as quantum group C*-algebra or the universal dual of (C,Δ) .

Corollary

There exists a maximal corepresentation $\mathcal{V} \in \mathcal{U}(\hat{C} \otimes C^{u})$ of $(\hat{C}, \Delta_{\hat{C}})$ and C^{*}-bialgebra $(C^{u}, \Delta_{C^{u}})$.

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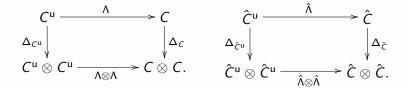
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Reducing morphisms

There exists two Hopf *-homomorphisms $\Lambda \in Mor(C^u, C)$ and $\hat{\Lambda} \in Mor(\hat{C}^u, \hat{C})$ such that



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Preparation results for lifting of bicharacter

Results

- Let (A, Δ_A) be a C*-bialgebra. Bicharacters in UM(Ĉ ⊗ A) correspond bijectively to Hopf *-homomorphisms from (C^u, Δ_{C^u}) to (A, Δ_A).
- There is a unique bicharacter $\mathcal{X} \in \mathcal{UM}(\hat{C}^{u} \otimes C^{u})$ such that

 $\mathcal{V}_{23}\tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12}\mathcal{X}_{13}\mathcal{V}_{23} \qquad \text{in } \mathcal{UM}(\hat{\mathcal{C}}^{\mathsf{u}}\otimes\mathbb{K}(\mathcal{H}_{\mathcal{C}})\otimes\mathcal{C}^{\mathsf{u}}).$

Moreover, \mathcal{X} is universal in the following sense: $(\mathrm{id}_{\hat{C}^{u}} \otimes \Lambda)\mathcal{X} = \tilde{\mathcal{V}}, (\hat{\Lambda} \otimes \mathrm{id}_{C^{u}})\mathcal{X} = \mathcal{V} \text{ and } (\hat{\Lambda} \otimes \Lambda)\mathcal{X} = W.$

• A bicharacter in $\mathcal{UM}(\hat{C} \otimes A)$ lifts uniquely to a bicharacter in $\mathcal{UM}(\hat{C}^u \otimes A^u)$ and hence to bicharacters in $\mathcal{UM}(\hat{C} \otimes A^u)$ and $\mathcal{UM}(\hat{C}^u \otimes A)$.

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$$\mathcal{V}_{23}\tilde{\mathcal{V}}_{12}=\tilde{\mathcal{V}}_{12}\mathcal{X}_{13}\mathcal{V}_{23}\qquad\text{in }\mathcal{UM}(\hat{C}^{\mathsf{u}}\otimes\mathbb{K}(\mathcal{H}_{\mathcal{C}})\otimes C^{\mathsf{u}}).$$

Moreover, \mathcal{X} is universal in the following sense: $(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \Lambda)\mathcal{X} = \tilde{\mathcal{V}}, (\hat{\Lambda} \otimes \mathrm{id}_{C^{\mathrm{u}}})\mathcal{X} = \mathcal{V} \text{ and } (\hat{\Lambda} \otimes \Lambda)\mathcal{X} = W.$

• A bicharacter in $\mathcal{UM}(\hat{C} \otimes A)$ lifts uniquely to a bicharacter in $\mathcal{UM}(\hat{C}^{u} \otimes A^{u})$ and hence to bicharacters in $\mathcal{UM}(\hat{C} \otimes A^{u})$ and $\mathcal{UM}(\hat{C}^{u} \otimes A)$.

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Preparation results for lifting of bicharacter

Results

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Category of universal objects

Theorem [Ng, 1997; Meyer, R., Woronowicz, 2011]

There is an isomorphism between the categories of locally compact quantum groups with bicharacters from *C* to *A* and with Hopf *-homomorphisms $C^{u} \rightarrow A^{u}$ as morphisms $C \rightarrow A$, respectively. The bicharacter associated to a Hopf *-homomorphism $\varphi: C^{u} \rightarrow A^{u}$ is $(\Lambda_{\hat{\mathcal{C}}} \otimes \Lambda_{A}\varphi)(\mathcal{X}^{C}) \in \mathcal{UM}(\hat{\mathcal{C}} \otimes A)$. Furthermore, the duality on the level of bicharacters corresponds to the duality $\varphi \mapsto \hat{\varphi}$ on Hopf *-homomorphisms, where $\hat{\varphi}: \hat{A}^{u} \rightarrow \hat{\mathcal{C}}^{u}$ is the unique Hopf *-homomorphism with $(\hat{\varphi} \otimes id_{A^{u}})(\mathcal{X}^{A}) = (id_{\hat{\mathcal{C}}^{u}} \otimes \varphi)(\mathcal{X}^{C}).$

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4 Summary

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History of quantum group morphisms/homomorphisms revisited

- 1997: Ng introduced equivalent notion of morphisms between basic multiplicative unitaries in terms of
 - Birepresentations.
 - Mutual coactions.
 - Hopf *-homomorphisms.
- 2001: Kustermans defined equivalent notion of morphisms of quantum groups in terms of
 - Hopf *-homomorphisms between universal quantum group.
 - Special coactions of von Neumann algebraic versions of quantum groups.

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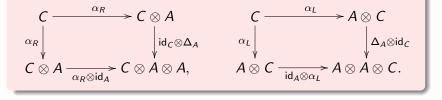
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Right/Left coactions

Definition

A right or left coaction of (A, Δ_A) on a C*-algebra C is a morphism $\alpha_R \colon C \to C \otimes A$ or $\alpha_L \colon C \to A \otimes C$ for which following diagram in the left or the right hand side commutes:



Equivalent pictures of homomorphisms of quantum groups
Summary

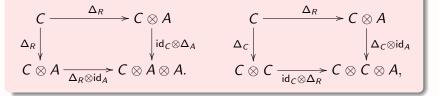
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Right quantum group homomorphisms

Definition

A right quantum group homomorphism from (C, Δ_C) to (A, Δ_A) is a morphism $\Delta_R \colon C \to C \otimes A$ for which following two diagram commute:



Equivalent pictures of homomorphisms of quantum groups
Summary

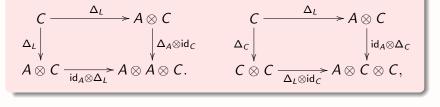
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Left quantum group homomorphisms

Definition

A left quantum group homomorphism from (C, Δ_C) to (A, Δ_A) is a morphism $\Delta_L: C \to A \otimes C$ for which following two diagram commute:



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Right quantum group homomorphisms and bicharacters

Theorem [Meyer, R., Woronowicz, 2011]

For any right quantum group homomorphism $\Delta_R \colon C \to C \otimes A$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

 $(\mathrm{id}_{\hat{C}}\otimes \Delta_R)(\mathsf{W})=\mathsf{W}_{12}V_{13}.$

This unitary is a bicharacter.

Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_R(x) := \mathbb{V}(x \otimes 1)\mathbb{V}^*$$
 for all $x \in C$

defines a right quantum group homomorphism from *C* to *A*. These two maps between bicharacters and quantum group homomorphisms are inverse to each other.

Left quantum group homomorphisms and bicharacters

Theorem [Meyer, R., Woronowicz, 2011]

For any left quantum group homomorphism $\Delta_L \colon C \to A \otimes C$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

 $(\mathrm{id}_{\hat{C}}\otimes\Delta_L)(\mathsf{W})=V_{12}\mathsf{W}_{13}.$

This unitary is a bicharacter.

Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

 $\Delta_L(x) := (R_A \otimes R_C)(\hat{\mathbb{V}}^*(1 \otimes R_C(x))\hat{\mathbb{V}}) \qquad \text{for all } x \in C$

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$$\Delta_L(x) := \sigma \circ (R_C \otimes R_A)(\mathbb{V}(R_C(x) \otimes 1)\mathbb{V}^*)$$
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Left quantum group homomorphisms and bicharacters

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$$\Delta_L(x) := \sigma \circ (R_C \otimes R_A) \Delta_R(R_C(x))$$
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These two maps between bicharacters and quantum group homomorphisms are inverse to each other.

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Commutation relation between left and right homomorphisms

Lemma

Let $\Delta_L \colon C \to A \otimes C$ and $\Delta_R \colon C \to C \otimes B$ be a left and a right quantum group homomorphism. Then the following diagram commutes:

$$C \xrightarrow{\Delta_L} A \otimes C$$

$$\Delta_R \downarrow \qquad \qquad \downarrow^{\operatorname{id}_A \otimes \Delta_R}$$

$$C \otimes B \xrightarrow{\Delta_L \otimes \operatorname{id}_B} A \otimes C \otimes B.$$

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Commutation relation between left and right homomorphisms

Lemma

 Δ_L and Δ_R are associated to the same bicharacter $V \in \mathcal{UM}(\hat{C} \otimes A)$ if and only if the following diagram commutes:

$$C \xrightarrow{\Delta_C} C \otimes C$$

$$\Delta_C \downarrow \qquad \qquad \downarrow^{\mathrm{id}_C \otimes \Delta_L}$$

$$C \otimes C \xrightarrow{\Delta_R \otimes \mathrm{id}_C} C \otimes A \otimes C.$$

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Coaction category

Lemma

Right or left quantum group homomorphisms are injective and satisfies

 $\Delta_R(C)(1 \otimes A)$ is linearly dense in $C \otimes A$

 $\Delta_L(C)(A \otimes 1)$ is linearly dense in $A \otimes C$

Equivalently right and left quantum group homomorphisms are injective and continuous as coactions.

- Let C*alg(A) or C*alg(A, Δ_A) denote the category of C*-algebras with a continuous, injective A-coaction.
- A-equivariant morphisms as arrows in C*alg(A).

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Assumptions

- (A, Δ_A) and (B, Δ_B) be locally compact quantum groups.
- $\alpha : C \to C \otimes A$ be an injective and continuous right coaction of (A, Δ_A) on a C*-algebra C.
- $\Delta_R \colon A \to A \otimes B$ be a right quantum group homomorphism.
- $\mathfrak{For}: \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}$ be the functor that forgets the *A*-coaction.

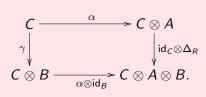
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Theorem [Meyer, R., Woronowicz, 2011]

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This construction is a functor $F : \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}(B)$ with $\mathfrak{Fot} \circ F = \mathfrak{Fot}$ as any A-equivariant morphisms $D \to D'$ are also B-equivariant for $D, D' \in \mathfrak{C}^*\mathfrak{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism Δ_R .

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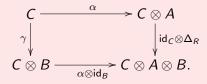
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This construction is a functor $F : \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}(B)$ with $\mathfrak{For} \circ F = \mathfrak{For}$ as any A-equivariant morphisms $D \to D'$ are also B-equivariant for $D, D' \in \mathfrak{C}^*\mathfrak{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism Δ_R .

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From multiplicative unitaries to locally compact quantum groups Excursion to history Equivalent pictures of homomorphisms of quantum groups Summary Morphism as a functor between coaction categories

Assumptions

- (A, Δ_A) and (B, Δ_B) be locally compact quantum groups.
- $\alpha: C \to C \otimes A$ be a right quantum group homomorphism where (C, Δ_C) is a quantum group.
- $\beta: A \to A \otimes B$ be another right quantum group homomorphism.
- $F_{\alpha}: \mathfrak{C}^*\mathfrak{alg}(C) \to \mathfrak{C}^*\mathfrak{alg}(A)$ and $F_{\beta}: \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}(B)$ be the associated functors.

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$$V^{C \to B} = V^{A \to B} * V^{C \to A}$$
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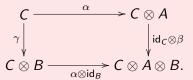
Bicharacters Universal bicharacter Right or left coactions as homomorphisms Morphism as a functor between coaction categories

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Composition of right quantum group homomorphism

Proposition

There exists $\gamma: C \to C \otimes B$ which is the unique right quantum group homomorphism that makes the following diagram commute:



which satisfies $F_{\beta} \circ F_{\alpha} = F_{\gamma}$. Moreover, $V^{C \to B}$ is the bicharacter associated to γ .

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$$\begin{array}{c} C & \xrightarrow{\alpha} & C \otimes A \\ \gamma & & \downarrow^{\operatorname{id}_C \otimes \beta} \\ C \otimes B & \xrightarrow{\alpha \otimes \operatorname{id}_P} & C \otimes A \otimes B. \end{array}$$

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2 Excursion to history

- 3 Equivalent pictures of homomorphisms of quantum groups• Bicharacters
 - Universal bicharacter
 - Right or left coactions as homomorphisms
 - Morphism as a functor between coaction categories





Summary

- Multiplicative unitaries are the fundamental objects.
- Every modular/manageable multiplicative unitary W ∈ UM(Ĉ ⊗ C) admits a unique lift to X ∈ UM(Ĉ^u ⊗ C^u). Hence they are *basic* in sense of Ng and hence the *birepresentations* (bicharacters in our terminology) are indeed the correct notion of homomorphisms between quantum groups.

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Summary

- Kustermans introduced the notion of homomorphisms between quantum groups (von Neumann algebraic setting) as Hopf*-homomorphisms between universal C*-bialgebras which is equivalent to the bicharacters.
- Last but not least, bicharacters induces a functor between coaction categories via left/right quantum group homomorphism which is a new realization of homomorphisms between quantum groups.

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More details.....

http://arxiv.org/abs/1011.4284/v2

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Thank you for your attention!

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