Homomorphisms of quantum groups (joint work with R. Meyer and S.L.Woronowicz)

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Outline

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1. From multiplicative unitaries to locally compact quantum groups
2. Excursion to history
3. Equivalent pictures of homomorphisms of quantum groups
   - Bicharacters
   - Universal bicharacter
   - Right or left coactions as homomorphisms
   - Morphism as a functor between coaction categories
4. Summary

Wish you happy Seventieth Birthday Prof. Woronowicz

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Locally compact quantum group in different setting

- Algebraic setting (Multiplier Hopf*-algebras).
- Topological setting (C^*-algebras).
- Measure theoritic setting (von Neumann algebras).

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Multiplicative unitary

Definition

An operator $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be multiplicative unitary on the Hilbert space $\mathcal{H}$ if it satisfies the *pentagon equation*

$$\mathbb{W}_{23}\mathbb{W}_{12} = \mathbb{W}_{12}\mathbb{W}_{13}\mathbb{W}_{23}.$$

Observations

One can define two non-degenerate, normal, coassociative $*$-homomorphisms from $\mathbb{B}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H} \otimes \mathcal{H})$:

$$\Delta(x) = \mathbb{W}(x \otimes I)\mathbb{W}^*$$
$$\widehat{\Delta}(y) = \mathrm{Ad}(\Sigma) \circ (\mathbb{W}^*(I \otimes y)\mathbb{W}).$$

for all $x, y \in \mathbb{B}(\mathcal{H})$ and Σ is the flip operator acting on $\mathcal{H} \otimes \mathcal{H}$.

Consider the slices/legs of the multiplicative unitaries:

$$C = \overline{\{(\omega \otimes \mathrm{id})\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}}^{\|\cdot\|}$$
$$\widehat{C} = \overline{\{(\mathrm{id} \otimes \omega)\mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}}^{\|\cdot\|}.$$

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Special class of multiplicative unitaries

Manageability and modularity

- Manageable multiplicative unitary. [Woronowicz, 1997]
- Modular multiplicative unitary. [Sołtan-Woronowicz, 2001]

Nice legs of modular multiplicative unitaries

Theorem (Sołtan, Woronowicz, 2001)

Let, $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. Then,

- *C and $\widehat{C}$ are C^*-sub algebras in $\mathbb{B}(\mathcal{H})$ and $\mathrm{W} \in \mathcal{UM}(\widehat{C} \otimes C)$.*
- *there exists a unique $\Delta_C \in \mathrm{Mor}(C, C \otimes C)$ such that*
    - $(\mathrm{id}_{\widehat{C}} \otimes \Delta)\mathrm{W} = \mathrm{W}_{12}\mathrm{W}_{13}$.
    - *Δ_C is coassociative: $(\Delta_C \otimes \mathrm{id}_C) \circ \Delta_C = (\mathrm{id}_C \otimes \Delta_C) \circ \Delta_C$.*
    - *$\Delta(C)(1 \otimes C)$ and $(C \otimes 1)\Delta(C)$ are linearly dense in $C \otimes C$.*
- *There exists an involutive normal antiautomorphism R_C of C.*

Locally compact quantum groups

Definition [Sołtan-Woronowicz, 2001]

The pair $\mathbb{G} = (C, \Delta_C)$ is said to be a locally compact quantum group if the C*-algebra C and $\Delta_C \in \mathrm{Mor}(C, C \otimes C)$ comes from a modular multiplicative unitary $\mathbb{W}$. We say $\mathbb{W}$ giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$.

Observation

The unitary operator $\widehat{\mathbb{W}} = \mathrm{Ad}(\Sigma)(\mathbb{W}^*)$ gives rise to the quantum group $\widehat{\mathbb{G}} = (\widehat{C}, \Delta_{\widehat{C}})$ which is dual to $\mathbb{G} = (C, \Delta_C)$.

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From groups to quantum groups

Given a locally compact group G:

- $\mathbb{G} = (C_0(G), \Delta)$ is a locally compact quantum group with $\Delta f(x, y) = f(xy)$.
- $\widehat{\mathbb{G}} = (\mathrm{C}^*_\mathrm{r}(G), \hat{\Delta})$ is the dual quantum group of $\mathbb{G}$ with $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in G$.
- $\widehat{\mathbb{G}}^\mathrm{u} = (C^*(G), \hat{\Delta}^\mathrm{u})$ is a C^*-bialgebra which is known as *quantum group* C^**-algebra* of $\mathbb{G}$.

Notations

Let, $\mathbb{W}$ be a modular multiplicative unitary giving rise to the quantum group $\mathbb{G} = (C, \Delta_C)$. We write:

- $\mathbb{W}$, when we consider it as an unitary operator action on the Hilbert space $\mathcal{H} \otimes \mathcal{H}$
- W, when we consider it as in element of of $\mathcal{U}\mathcal{M}(\hat{C} \otimes C)$.
- $f\colon A \to B$, when we consider $f \in \mathrm{Mor}(A, B)$ or $f\colon A \to \mathcal{M}(B)$ where A and B are C^*-algebras.

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Hopf *-homomorphisms

Let us consider $\mathbb{G} = (C, \Delta_C)$ and $\mathbb{H} = (A, \Delta)$ be two C*-bialgebras.

Definition

A *Hopf *-homomorphism* between them is a morphism $f\colon C \to A$ that intertwines the comultiplications, that is, the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & A \\ {\scriptstyle \Delta_C}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_A} \\ C \otimes C & \xrightarrow[\quad f \otimes f \quad]{} & A \otimes A. \end{array}$$

Hopf *-homomorphisms

Let G and H are two locally compact groups.

- Consider a Hopf * homomorphism from $f\colon C_0(H) \to C_0(G)$.
- f induces a continuous group homomorphism $\phi\colon G \to H$.
- ϕ induces a Hopf *-homomorphism $\hat{f}\colon C^*(G) \to C^*(H)$

Hopf *-homomorphisms and it's drawback

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- ϕ induces a Hopf *-homomorphism $\hat{f}\colon \mathrm{C}^*_{\mathrm{r}}(G) \to \mathrm{C}^*_{\mathrm{r}}(H)$ if and only if kernel of ϕ is amenable.

Hopf *-homomorphisms and it's drawback:example

Let $G = \mathbb{F}_2$ and $H = \{e\}$.

- The Hopf * homomorphism from $f\colon \mathbb{C} \to C_0(\mathbb{F}_2)$.
- f induces the trival group homomorphism $\phi\colon \mathbb{F}_2 \to \{e\}$.
- ϕ induces trivial Hopf *-morphism $\hat{f}\colon \mathrm{C}^*_\mathrm{r}(\mathbb{F}_2) \to \mathbb{C}$ as $\mathbb{F}_2$ is not amenable.

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- Consider a Hopf * homomorphism from $f\colon C_0(H) \to C_0(G)$.
- f induces a continuous group homomorphism $\phi\colon G \to H$.
- ϕ induces a Hopf *-homomorphism
    - $\hat{f}\colon C^*(G) \to C^*(H)$
    - $\hat{f}\colon \mathrm{C}^*_\mathrm{r}(G) \to \mathrm{C}^*_\mathrm{r}(H)$ if and only if kernel of ϕ is amenable.

Conclusion

Hopf *-homomorphisms are not always compatible with the reduced dual but with full dual.

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Bicharacters

Let, $\mathbb{G} = (C, \Delta_C)$ and $\mathbb{H} = (A, \Delta_A)$ are two quantum groups.

Definition

A unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ a *bicharacter from C to A* if satisfying

$$(\Delta_{\hat{C}} \otimes \mathrm{id}_A)V = V_{23}V_{13} \qquad \text{in } \mathcal{UM}(\hat{C} \otimes \hat{C} \otimes A),$$
$$(\mathrm{id}_{\hat{C}} \otimes \Delta_A)V = V_{12}V_{13} \qquad \text{in } \mathcal{UM}(\hat{C} \otimes A \otimes A).$$

Reduced bicharacter

Let, $\mathbb{G} = \mathbb{H} = (C, \Delta_C)$ be a quantum group.

Definition(Reduced bicharacter)

The unitary $\mathrm{W}^C \in \mathcal{UM}(\hat{C} \otimes C)$ a *reduced bicharacter* of C satisfying

$$(\Delta_{\hat{C}} \otimes \mathrm{id}_C)\mathrm{W}^C = \mathrm{W}^C_{23}\mathrm{W}^C_{13} \quad \text{in } \mathcal{UM}(\hat{C} \otimes \hat{C} \otimes C),$$
$$(\mathrm{id}_{\hat{C}} \otimes \Delta_C)\mathrm{W}^C = \mathrm{W}^C_{12}\mathrm{W}^C_{13} \quad \text{in } \mathcal{UM}(\hat{C} \otimes C \otimes C).$$

Bicharacters

Lemma

A unitary $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ comes from a bicharacter $V \in \mathcal{UM}(\hat{C} \otimes A)$ (which is necessarily unique) if and only if

$$\mathbb{V}_{23}\mathbb{W}^C_{12} = \mathbb{W}^C_{12}\mathbb{V}_{13}\mathbb{V}_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_C \otimes \mathcal{H}_A),$$
$$\mathbb{W}^A_{23}\mathbb{V}_{12} = \mathbb{V}_{12}\mathbb{V}_{13}\mathbb{W}^A_{23} \qquad \text{in } \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A \otimes \mathcal{H}_A).$$

Modularity revisited

Definition [Sołtan-Woronowicz, 2001]

A multiplicative unitary $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ is said to be *modular* if there exist positive self-adjoint (may be unbounded) operators $\hat{Q}$ and Q on $\mathcal{H}$ and an operator $\widetilde{\mathbb{W}} \in \mathcal{U}(\bar{\mathcal{H}} \otimes \mathcal{H})$ such that

1. $\ker \hat{Q} = \ker Q = \{0\}$;
2. $\mathbb{W}(\hat{Q} \otimes Q)\mathbb{W}^* = \hat{Q} \otimes Q$;
3. $\forall\ \eta, \eta' \in \mathcal{H}$ and $\xi \in \mathcal{D}(Q^{-1})$, $\xi' \in \mathcal{D}(Q)$,

$$(\eta' \otimes \xi' \mid \mathbb{W} \mid \eta \otimes \xi) = (\bar{\eta} \otimes Q\xi' \mid \widetilde{\mathbb{W}} \mid \bar{\eta}' \otimes Q^{-1}\xi).$$

An example of bicharacter

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$$(\eta' \otimes \xi' \mid \mathbb{W} \mid \eta \otimes \xi) = (\bar{\eta} \otimes Q\xi' \mid \widetilde{\mathbb{W}} \mid \bar{\eta}' \otimes Q^{-1}\xi).$$

If $\mathbb{W}$ gives rise to the quantum group (C, Δ) then $\widetilde{\mathbb{W}}$ is a bicharacter from (C^{op}, Δ) to (C, Δ^{op}).

An important theorem

Theorem [Woronowicz, 2010]

Let $\mathcal{H}$ be a Hilbert space and let $\mathbb{W} \in \mathbb{B}(\mathcal{H} \otimes \mathcal{H})$ be a modular multiplicative unitary. If $a, b \in \mathbb{B}(\mathcal{H})$ satisfy $\mathbb{W}(a \otimes 1) = (1 \otimes b)\mathbb{W}$, then $a = b = \lambda 1$ for some $\lambda \in \mathbb{C}$. More generally, if $a, b \in \mathcal{M}(\mathbb{K}(\mathcal{H}) \otimes D)$ for some C*-algebra D satisfy $\mathbb{W}_{12} a_{13} = b_{23} \mathbb{W}_{12}$, then $a = b \in \mathbb{C} \cdot 1_{\mathcal{H}} \otimes \mathcal{M}(D)$.

An important theorem

Corollary [Kustermans, Vaes 2000]

Let (C, Δ_C) be a quantum group. If $c \in \mathcal{M}(C)$, then $\Delta_C(c) \in \mathcal{M}(C \otimes 1)$ or $\Delta_C(c) \in \mathcal{M}(1 \otimes C)$ if and only if $c \in \mathbb{C} \cdot 1$. More generally, if D is a C*-algebra and $c \in \mathcal{M}(C \otimes D)$, then $(\Delta_C \otimes \mathrm{id}_D)(c) \in \mathcal{M}(C \otimes 1 \otimes D)$ or $(\Delta_C \otimes \mathrm{id}_D)(c) \in \mathcal{M}(1 \otimes C \otimes D)$ if and only if $c \in \mathbb{C} \cdot 1 \otimes \mathcal{M}(D)$.

Properties of bicharacters I

Consider $\mathbb{G} = (C, \Delta_C)$, $\mathbb{H} = (A, \Delta_A)$ and $\mathbb{I} = (B, \Delta_B)$ are quantum groups.

- Given a bicharacter $V \in \mathcal{U}\mathcal{M}(\hat{C} \otimes A)$ we have:
  - $(R_{\hat{C}} \otimes R_A)V = V$.
  - $\hat{V} = \sigma(V^*) \in \mathcal{U}\mathcal{M}(A \otimes \hat{C})$ is the dual bicharacter.

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Properties of bicharacters II

- Given two bicharacters $\mathrm{V}^{C\to A} \in \mathcal{UM}(\hat{C}\otimes A)$ and $\mathrm{V}^{A\to B} \in \mathcal{UM}(\hat{A}\otimes B)$, there exists unique bicharacter $\mathrm{V}^{C\to B} \in \mathcal{UM}(\hat{C}\otimes B)$ satisfying

$$\mathbb{V}^{C\to B}_{13} = (\mathbb{V}^{C\to A}_{12})^{*}\mathbb{V}^{A\to B}_{23}\mathbb{V}^{C\to A}_{12}(\mathbb{V}^{A\to B}_{23})^{*}.$$

  We denote $\mathrm{V}^{C\to B} = \mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}$ as composition of $\mathrm{V}^{C\to A}$ and $\mathrm{V}^{A\to B}$.

- Identity bicharacter:

$$\mathrm{V}^{C\to A} = \mathrm{V}^{C\to A} * \mathrm{W}^{C} \text{ and } \mathrm{V}^{C\to A} = \mathrm{W}^{A} * \mathrm{V}^{C\to A}.$$

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Properties of bicharacters III

- Composition of bicharacters is associative:

$$(\mathrm{V}^{B\to D} * \mathrm{V}^{A\to B}) * \mathrm{V}^{C\to A} = \mathrm{V}^{B\to D} * (\mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}).$$

where $\mathrm{V}^{B\to D} \in \mathcal{U}\mathcal{M}(\hat{B}\otimes D)$ where $\mathbb{J} = (D, \Delta_D)$ is a quantum group.

- Compatibility with duality:

$$\widehat{\mathrm{V}^{C\to B}} = \widehat{\mathrm{V}^{A\to B}} * \widehat{\mathrm{V}^{C\to A}}$$

or equivalently

$$\widehat{\mathbb{V}_{13}^{C\to B}} = \widehat{\mathbb{V}_{12}^{A\to B}}^{*}\widehat{\mathbb{V}_{23}^{C\to A}}\widehat{\mathbb{V}_{12}^{A\to B}}\widehat{\mathbb{V}_{23}^{C\to A}}^{*}.$$

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Category of locally compact quantum groups

Proposition [Ng, 1997; Meyer, R., Woronowicz, 2011]

The composition of bicharacters is associative, and the multiplicative unitary W^{C} is an identity on C. Thus bicharacters with the above composition and locally compact quantum groups are the arrows and objects of a category. Duality is a contravariant functor acting on this category.

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Corepresentation and universal bialgebra of a quantum group

Definition

A *corepresentation* of $(\hat{C}, \Delta_{\hat{C}})$ on a C*-algebra D is a unitary multiplier $V \in \mathcal{UM}(\hat{C} \otimes D)$ that satisfies $(\Delta_{\hat{C}} \otimes \mathrm{id}_D)(V) = V_{23} V_{13}$.

Remark

Similarly *corepresentation* of (C, Δ_C) on a C*-algebra D is a unitary multiplier $V \in \mathcal{UM}(D \otimes C)$ that satisfies $(\mathrm{id}_D \otimes \Delta_C)(V) = V_{12} V_{13}$.

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Universal quantum group C^*-algebra

Proposition[Sołtan, Woronowicz, 2007]

- There exists a maximal corepresentation $\tilde{\mathcal{V}} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C)$ of (C, Δ_C) on a C*-algebra $\hat{C}^{\mathrm{u}}$ such that for any corepresentation $U \in \mathcal{UM}(D \otimes C)$ there exists a unique $\hat{\phi} \in \mathrm{Mor}(\hat{C}^{\mathrm{u}}, D)$ such that $(\hat{\phi} \otimes \mathrm{id}_C)\tilde{\mathcal{V}} = U$.
- There exists a unique $\Delta_{\hat{C}^{\mathrm{u}}} \in \mathrm{Mor}(\hat{C}^{\mathrm{u}}, \hat{C}^{\mathrm{u}} \otimes \hat{C}^{\mathrm{u}})$ such that:
  - $(\Delta_{\hat{C}^{\mathrm{u}}} \otimes \mathrm{id}_C)\tilde{\mathcal{V}} = \tilde{\mathcal{V}}_{23}\tilde{\mathcal{V}}_{13}$
  - $\Delta_{\hat{C}^{\mathrm{u}}}(\hat{C}^{\mathrm{u}})(1 \otimes \hat{C}^{\mathrm{u}})$ and $(\hat{C}^{\mathrm{u}} \otimes 1)\Delta_{\hat{C}^{\mathrm{u}}}$ are linearly dense in $(\hat{C}^{\mathrm{u}} \otimes \hat{C}^{\mathrm{u}})$.

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Universal C^*-bialgebras associated to a quantum group

Universal qunatum groups C*-algebra

$(\hat{C}^{\mathrm{u}}, \Delta_{\hat{C}^{\mathrm{u}}})$ is known as *quantum group* C*-*algebra* or the *universal dual* of (C, Δ) .

Corollary

There exists a maximal corepresentation $\mathcal{V} \in \mathcal{U}(\hat{C} \otimes C^{\mathrm{u}})$ of $(\hat{C}, \Delta_{\hat{C}})$ and C*-bialgebra $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$.

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Reducing morphisms

There exists two Hopf *-homomorphisms $\Lambda \in \mathrm{Mor}(C^{\mathrm{u}}, C)$ and $\hat{\Lambda} \in \mathrm{Mor}(\hat{C}^{\mathrm{u}}, \hat{C})$ such that

$$\begin{array}{ccc} C^{\mathrm{u}} & \xrightarrow{\Lambda} & C \\ \downarrow{\scriptstyle \Delta_{C^{\mathrm{u}}}} & & \downarrow{\scriptstyle \Delta_{C}} \\ C^{\mathrm{u}} \otimes C^{\mathrm{u}} & \xrightarrow[\Lambda\otimes\Lambda]{} & C \otimes C. \end{array} \qquad \begin{array}{ccc} \hat{C}^{\mathrm{u}} & \xrightarrow{\hat{\Lambda}} & \hat{C} \\ \downarrow{\scriptstyle \Delta_{\hat{C}^{\mathrm{u}}}} & & \downarrow{\scriptstyle \Delta_{\hat{C}}} \\ \hat{C}^{\mathrm{u}} \otimes \hat{C}^{\mathrm{u}} & \xrightarrow[\hat{\Lambda}\otimes\hat{\Lambda}]{} & \hat{C} \otimes \hat{C}. \end{array}$$

Preparation results for lifting of bicharacter

Results

- Let (A, Δ_A) be a C*-bialgebra. Bicharacters in $\mathcal{UM}(\hat{C} \otimes A)$ correspond bijectively to Hopf *-homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to (A, Δ_A).
- There is a unique bicharacter $\mathcal{X} \in \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes C^{\mathrm{u}})$ such that

$$\mathcal{V}_{23}\tilde{\mathcal{V}}_{12} = \tilde{\mathcal{V}}_{12}\mathcal{X}_{13}\mathcal{V}_{23} \qquad \text{in } \mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes \mathbb{K}(\mathcal{H}_C) \otimes C^{\mathrm{u}}).$$

  Moreover, $\mathcal{X}$ is universal in the following sense: $(\mathrm{id}_{\hat{C}^{\mathrm{u}}} \otimes \Lambda)\mathcal{X} = \tilde{\mathcal{V}}, (\hat{\Lambda} \otimes \mathrm{id}_{C^{\mathrm{u}}})\mathcal{X} = \mathcal{V}$ and $(\hat{\Lambda} \otimes \Lambda)\mathcal{X} = \mathrm{W}$.
- A bicharacter in $\mathcal{UM}(\hat{C} \otimes A)$ lifts uniquely to a bicharacter in $\mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes A^{\mathrm{u}})$ and hence to bicharacters in $\mathcal{UM}(\hat{C} \otimes A^{\mathrm{u}})$ and $\mathcal{UM}(\hat{C}^{\mathrm{u}} \otimes A)$.

Preparation results for lifting of bicharacter

Results

- Let (A, Δ_A) be a C*-bialgebra. Bicharacters in $\mathcal{U}\mathcal{M}(\hat{C} \otimes A)$ correspond bijectively to Hopf *-homomorphisms from $(C^{\mathrm{u}}, \Delta_{C^{\mathrm{u}}})$ to (A, Δ_A).
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Category of universal objects

Theorem [Ng, 1997; Meyer, R., Woronowicz, 2011]

There is an isomorphism between the categories of locally compact quantum groups with bicharacters from C to A and with Hopf *-homomorphisms $C^{\mathrm{u}} \to A^{\mathrm{u}}$ as morphisms $C \to A$, respectively.
The bicharacter associated to a Hopf *-homomorphism $\varphi\colon C^{\mathrm{u}} \to A^{\mathrm{u}}$ is $(\Lambda_{\hat{C}} \otimes \Lambda_A \varphi)(\mathcal{X}^C) \in \mathcal{UM}(\hat{C} \otimes A)$.
Furthermore, the duality on the level of bicharacters corresponds to the duality $\varphi \mapsto \hat{\varphi}$ on Hopf *-homomorphisms, where $\hat{\varphi}\colon \hat{A}^{\mathrm{u}} \to \hat{C}^{\mathrm{u}}$ is the unique Hopf *-homomorphism with
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Outline

History of quantum group morphisms/homomorphisms revisited

- 1997: Ng introduced equivalent notion of morphisms between basic multiplicative unitaries in terms of
  - Birepresentations.
  - Mutual coactions.
  - Hopf *-homomorphisms.
- 2001: Kustermans defined equivalent notion of morphisms of quantum groups in terms of
  - Hopf *-homomorphisms between universal quantum group.
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  - Hopf *-homomorphisms between universal quantum group.
  - Special coactions of von Neumann algebraic versions of quantum groups.

Right/Left coactions

Definition

A *right or left coaction* of (A, Δ_A) on a C^*-algebra C is a morphism $\alpha_R \colon C \to C \otimes A$ or $\alpha_L \colon C \to A \otimes C$ for which following diagram in the left or the right hand side commutes:

$$\begin{array}{ccc} C & \xrightarrow{\alpha_R} & C \otimes A \\ {\scriptstyle \alpha_R}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_A} \\ C \otimes A & \xrightarrow[\alpha_R \otimes \mathrm{id}_A]{} & C \otimes A \otimes A, \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\alpha_L} & A \otimes C \\ {\scriptstyle \alpha_L}\downarrow & & \downarrow{\scriptstyle \Delta_A \otimes \mathrm{id}_C} \\ A \otimes C & \xrightarrow[\mathrm{id}_A \otimes \alpha_L]{} & A \otimes A \otimes C. \end{array}$$

Right quantum group homomorphisms

Definition

A *right quantum group homomorphism* from (C, Δ_C) to (A, Δ_A) is a morphism $\Delta_R \colon C \to C \otimes A$ for which following two diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_R} & C \otimes A \\ \downarrow{\scriptstyle \Delta_R} & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_A} \\ C \otimes A & \xrightarrow[\Delta_R \otimes \mathrm{id}_A]{} & C \otimes A \otimes A. \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_R} & C \otimes A \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \Delta_C \otimes \mathrm{id}_A} \\ C \otimes C & \xrightarrow[\mathrm{id}_C \otimes \Delta_R]{} & C \otimes C \otimes A, \end{array}$$

Left quantum group homomorphisms

Definition

A *left quantum group homomorphism* from (C, Δ_C) to (A, Δ_A) is a morphism $\Delta_L\colon C \to A \otimes C$ for which following two diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ \downarrow{\scriptstyle \Delta_L} & & \downarrow{\scriptstyle \Delta_A \otimes \mathrm{id}_C} \\ A \otimes C & \xrightarrow[\mathrm{id}_A \otimes \Delta_L]{} & A \otimes A \otimes C. \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta_C} \\ C \otimes C & \xrightarrow[\Delta_L \otimes \mathrm{id}_C]{} & A \otimes C \otimes C, \end{array}$$

Right quantum group homomorphisms and bicharacters

Theorem [Meyer, R., Woronowicz, 2011]

For any right quantum group homomorphism $\Delta_R\colon C \to C \otimes A$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_R)(\mathrm{W}) = \mathrm{W}_{12} V_{13}.$$

This unitary is a bicharacter.

Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_R(x) := \mathbb{V}(x \otimes 1)\mathbb{V}^* \qquad \text{for all } x \in C$$

defines a right quantum group homomorphism from C to A. These two maps between bicharacters and quantum group homomorphisms are inverse to each other.

Left quantum group homomorphisms and bicharacters

Theorem [Meyer, R., Woronowicz, 2011]

For any left quantum group homomorphism $\Delta_L \colon C \to A \otimes C$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_L)(\mathrm{W}) = V_{12}\mathrm{W}_{13}.$$

This unitary is a bicharacter.

Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_L(x) := (R_A \otimes R_C)(\hat{\mathbb{V}}^*(1 \otimes R_C(x))\hat{\mathbb{V}}) \qquad \text{for all } x \in C$$

These two maps between bicharacters and quantum group homomorphisms are inverse to each other.

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Theorem [Meyer, R., Woronowicz, 2011]

For any left quantum group homomorphism $\Delta_L \colon C \to A \otimes C$, there is a unique unitary $V \in \mathcal{UM}(\hat{C} \otimes A)$ with

$$(\mathrm{id}_{\hat{C}} \otimes \Delta_L)(\mathrm{W}) = V_{12}\mathrm{W}_{13}.$$

This unitary is a bicharacter.

Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_L(x) := (R_A \otimes R_C)(\hat{\mathbb{V}}^*(1 \otimes R_C(x))\hat{\mathbb{V}}) \qquad \text{for all } x \in C$$

These two maps between bicharacters and left quantum group homomorphisms are inverse to each other.

Left quantum group homomorphisms and bicharacters

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This unitary is a bicharacter.
Conversely, let V be a bicharacter from C to A, and let $\mathbb{V} \in \mathcal{U}(\mathcal{H}_C \otimes \mathcal{H}_A)$ be the corresponding concrete bicharacter. Then

$$\Delta_L(x) := \sigma \circ (R_C \otimes R_A)(\mathbb{V}(R_C(x) \otimes 1)\mathbb{V}^*) \qquad \text{for all } x \in C$$

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Left quantum group homomorphisms and bicharacters

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$$\Delta_L(x) := \sigma \circ (R_C \otimes R_A)\Delta_R(R_C(x)) \qquad \text{for all } x \in C$$

These two maps between bicharacters and quantum group homomorphisms are inverse to each other.

Commutation relation between left and right homomorphisms

Lemma

Let $\Delta_L\colon C \to A \otimes C$ and $\Delta_R\colon C \to C \otimes B$ be a left and a right quantum group homomorphism. Then the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_L} & A \otimes C \\ \downarrow{\scriptstyle \Delta_R} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \Delta_R} \\ C \otimes B & \xrightarrow[\Delta_L \otimes \mathrm{id}_B]{} & A \otimes C \otimes B. \end{array}$$

Commutation relation between left and right homomorphisms

Lemma

Δ_L and Δ_R are associated to the same bicharacter $V \in \mathcal{UM}(\hat{C} \otimes A)$ if and only if the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ \downarrow{\scriptstyle \Delta_C} & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_L} \\ C \otimes C & \xrightarrow[\Delta_R \otimes \mathrm{id}_C]{} & C \otimes A \otimes C. \end{array}$$

Outline

Coaction category

Lemma

Right or left quantum group homomorphisms are injective and satisfies

$$\Delta_R(C)(1 \otimes A) \text{ is linearly dense in } C \otimes A$$

$$\Delta_L(C)(A \otimes 1) \text{ is linearly dense in } A \otimes C$$

Equivalently right and left quantum group homomorphisms are injective and continuous as coactions.

- Let $\mathfrak{C}^*\mathfrak{alg}(A)$ or $\mathfrak{C}^*\mathfrak{alg}(A, \Delta_A)$ denote the category of C^*-algebras with a continuous, injective A-coaction.
- A-equivariant morphisms as arrows in $\mathfrak{C}^*\mathfrak{alg}(A)$.

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Assumptions

- (A, Δ_A) and (B, Δ_B) be locally compact quantum groups.
- $\alpha\colon C \to C \otimes A$ be an injective and continuous right coaction of (A, Δ_A) on a C*-algebra C.
- $\Delta_R\colon A \to A \otimes B$ be a right quantum group homomorphism.
- $\mathfrak{For}\colon \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}$ be the functor that forgets the A-coaction.

Homomorphism as a functor between coaction categories

Theorem [Meyer, R., Woronowicz, 2011]

There is a unique continuous coaction γ of (B, Δ_B) on C such that the following diagram commutes:

$$\begin{array}{ccc} C & \xrightarrow{\quad\alpha\quad} & C \otimes A \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \Delta_R} \\ C \otimes B & \xrightarrow[\alpha \otimes \mathrm{id}_B]{} & C \otimes A \otimes B. \end{array}$$

This construction is a functor $F\colon \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}(B)$ with $\mathfrak{For} \circ F = \mathfrak{For}$ as any A-equivariant morphisms $D \to D'$ are also B-equivariant for $D, D' \in \mathfrak{C}^*\mathfrak{alg}A$. Conversely, any such functor is of this form for some right quantum group homomorphism Δ_R.

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Assumptions

- (A, Δ_A) and (B, Δ_B) be locally compact quantum groups.
- $\alpha\colon C \to C \otimes A$ be a right quantum group homomorphism where (C, Δ_C) is a quantum group.
- $\beta\colon A \to A \otimes B$ be another right quantum group homomorphism.
- $F_\alpha\colon \mathfrak{C}^*\mathfrak{alg}(C) \to \mathfrak{C}^*\mathfrak{alg}(A)$ and $F_\beta\colon \mathfrak{C}^*\mathfrak{alg}(A) \to \mathfrak{C}^*\mathfrak{alg}(B)$ be the associated functors.
- $\mathrm{V}^{C\to B} = \mathrm{V}^{A\to B} * \mathrm{V}^{C\to A}$.

Composition of right quantum group homomorphism

Proposition

There exists $\gamma\colon C \to C \otimes B$ which is the unique right quantum group homomorphism that makes the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C \otimes A \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle \mathrm{id}_C \otimes \beta} \\ C \otimes B & \xrightarrow[\alpha \otimes \mathrm{id}_B]{} & C \otimes A \otimes B. \end{array}$$

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Moreover, $\mathbb{V}^{C\to B}$ is the bicharacter associated to γ.

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Outline

Summary

- Multiplicative unitaries are the fundamental objects.
- Every modular/manageable multiplicative unitary $\mathbb{W} \in \mathcal{UM}(\hat{C} \otimes C)$ admits a unique lift to $\mathcal{X} \in \mathcal{UM}(\hat{C}^{u} \otimes C^{u})$. Hence they are *basic* in sense of Ng and hence the *birepresentations* (bicharacters in our terminology) are indeed the correct notion of homomorphisms between quantum groups.

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Summary

- Kustermans introduced the notion of homomorphisms between quantum groups (von Neumann algebraic setting) as Hopf*-homomorphisms between universal C^*-bialgebras which is equivalent to the bicharacters.
- Last but not least, bicharacters induces a functor between coaction categories via left/right quantum group homomorphism which is a new realization of homomorphisms between quantum groups.

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More details.....

http://arxiv.org/abs/1011.4284/v2

Thank you for your attention!