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Tammo tom Dieck
Georg-August-Universität

Corrections for the first printing

Page 7 +6: j is already assumed to be an inclusion. But the assertion is that J is an embedding.

Problem 1, second line: If gf is an embedding, then f is an embedding.

Page 11 +1,+2,+3 replace: $X_i \cap X_j$ in X_i and X_j coincide and these subspaces are closed in X_i and X_j. Give X the quotient topology with respect to the canonical map $\coprod X_j \to X$. Then the subspace topology of $X_j \subset X$ coincides with the given topology and X_j is closed in X. Similarly if "closed" is replaced by "open".

+11: (Erase): Hint: Use (1.3.3).

-15: …if each $x \in X$ …

Page 13 +1: Let X be a locally compact (non-compact) Hausdorff space. …
(The one-point compactification is also defined for a compact X, but then X is not dense in Y; it is then the topological sum $X + \{*\}$.

+10: …such that $H \cap V_x$ is closed in V_x.

+11 ff in (1.4.9) replace (1) by:
(1) A is locally closed in X if and only if $A = U \cap C$ with U open and C closed in X; one can take $C = \overline{A}$.

Page 15 A compact Hausdorff space is paracompact.

Page 16 -8: replace "subgroup" by "group".

Page 34 Line 2 after the list: $\mathbb{R}^0 = \{0\}$.

Page 36 (2.3.5) +1: A retraction $r\colon D^n \times I \to \ldots$.

Page 38 (2.4.3): Thus $\varphi\colon X \to Y^Z$ is continuous if $\varphi^\vee\colon X \times Y \to Z$ is continuous.

Page 39 +2: Replace $e_{Y,X}$ by $e_{Y,Z}$.

+9: …is a pointed map into $F^0(Y,Z)$ …

The symbol $e^0_{X,Y}$ has to be replaced 4 times by $e^0_{Y,Z}$ (after (2.4.8)).

Page 49 In the proof of (2.7.3) erase an additional bracket $\ldots \cdot (k * v))$.

Page 50 +2: Replace $h(z,t) = f(tz)$ by $h(z,t) = g(tz)/|g(tz)|$, since f was defined as a function on S^1.

Page 51 +9: The independence statement uses the rule $d(fg) = d(f)d(g)$ which appears in the exercises.

Page 52 -3: The fundamental group of the complement …

Page 89 Proof of (4.4.1) (2). $h_t(s) = (1-t)\min(2s,1) + ts$. Remark: $(u*k)(t) = u(\min(2s,1))$.

One should state the parameter invariance: Let $\alpha\colon I \to I$, $\alpha(0) = 0$, $\alpha(1) = 1$. Then $\Omega Y \to \Omega Y, u \mapsto u\alpha$ is pointed homotopic to the identity.

Page 90 4.5 +2,+3: $x \mapsto m(*, x)$, $x \mapsto m(x, *)$.

Page 104 Proof of (5.1.8): ... such that $h(b, 0) = \varphi J(b)$ for $b \in B$.

Since j has the HEP for Z, there exists $K_t \colon X \to Z$ such that $K_0 = \varphi F$ ....

... since both maps have the same composition with F and J.

Page 114 (5.4.2): replace $\diamond$ by $\square$.

Proof of (5.4.5): Then $\Psi \circ (\mathrm{id} \times T)$ factors over ...

Page 115 **6.** The following inclusions are cofibrations: $\{0\} \subset [0, \infty[$; $\{0\} \subset \mathbb{R}$; $[-1, 1] \subset \mathbb{R}$; $S^{n-1} \subset \mathbb{R}^n$; $D^n \subset \mathbb{R}^n$; $D^n_\pm \subset S^n$; $S^m \times \{0\} \subset S^n$ for $m < n$.

Page 123 In the formula for $H_t(s)$ replace u by u^-; or $u(t + 2(1 - s))$ by $u(2s - t - 1)$.

Page 124 Proof of (6.1.2): One should verify that adjunction yields a commutative diagram from the exact sequence of $(\Omega X, \Omega A)$ to the sequence of (X, A). For this purpose one should interchange the roles of I^k and I^{n-k} on page 123.

Page 126 -1: $T = I^n \times 0 \times I \cup \partial I^n \times I \times I \cup I^n \times I \times \partial I \subset I^{n+2}$

Page 128 +8: isomorphisms

Page 129 **9.** Let $* \in B \subset A \subset X$. Let $\pi_0(A, *) = 0$, $\pi_1(A, *) = 0$; then $\pi_1(X, *) \to \pi_1(X, A, *)$ is bijective. Let $\pi_0(X, *) = 0$, $\pi_1(X, *) = 0$; then $\partial \colon \pi_1(X, A, *) \to \pi_0(A, *)$ is bijective.

Page 133 (6.4.2): Suppose (Y_2, Y_0) is q-connected.

(6.4.3): For the second case: $i \geq 1$.

Page 135 +1: In the proof of (6.10.4) it is explained that Σ_* has the form p. 134,-1. Here we use the case $X = S(n)$ and $CX = D(n + 1)$.

+11: $\ldots \to \pi_1(S^1) \to \pi_1(S^3)$ of the

-10: ... are used to show that ...

Page 136 -10: The reference should be (2.3.2).

Page 137 Theorem (6.6.1) collects nine statements, one of which is the Brouwer fixed point theorem, and they are shown to be equivalent by elementary homotopy arguments. In our setting we have proved statement (3) of (6.6.1): The identity of S^{n-1} has degree one and is therefore not null homotopic.

Page 139 -2: Some statements of (6.6.1) have interesting different proofs. ... The proof of (3) does not need the full strength of the bijection $[S^n, S^n] \cong \mathbb{Z}$. It suffices to have a homotopy invariant which distinguishes the constant map from the identity. A standard argument for this uses the homology theory and the determination of the homology of spheres. ...

Page 144 +19: $A(k) = B(k) \cap A$

Page 148 -12: $n \geq 3$,

Page 150 -6: $(I^n, \partial I^n)$

-1: $\Phi(x, 1) \in Y_2$

Page 151 +6: ... and $\Psi^{-1}(Y \smallsetminus Y_2)$

+12: ... since $\Psi(\partial I^n \times I) \subset Y_1$

-6: Replace in the diagram A by Y_1.

Page 153 Remarks concerning (6.10.2): If $m = 1$ there is no assumption about A. For $m \geq 2$ the assumption is equivalent to $\pi_j(A, a) = 0$ for $0 \leq j \leq m-2$.

For $m = 1$ the theorem then asserts that $\pi_i(X/A, *) = 0$ for $1 \leq i \leq n-1$.

For $n = 1$ there is no assumption about (X, A). In that case $\pi_i(X, A, a) \to \pi_i(X/A, *)$ is bijective (surjective) for $1 \leq i \leq m-2$ $(i = m-1)$.

Page 154 +5: $f\colon A \to X$

(6.10.3) Remarks: For the second assertion one has to assume that f is a map between well-pointed spaces. In that case the canonical inclusion $A \to Z(f)$ into the pointed mapping cylinder is a cofibration, and one can apply the first part to this inclusion. For the proof consider the diagram

$$\begin{tikzcd}
A \times 0 \cup \{*\} \times I \cup A \times 1 \arrow[r] \arrow[d, "(1)"] & A \vee A \arrow[r] \arrow[d, "(2)"] & A \vee X \arrow[d, "(3)"] \\
A \times I \arrow[r] & A \times I/\{*\} \times I \arrow[r] & Z(f)
\end{tikzcd}$$

in which the squares are pushout diagrams. If A is well-pointed, (1) is a cofibration and (2), (3) are induced cofibration. If X is well-pointed, the canonical inclusion $A \to A \vee X$ is a cofibration.

Proof of (6.10.4): Here one has to use $CX = X \times I / X \times 0 \cup \{*\} \times I$. A map $g\colon (I^k, \partial I^k) \to (X, *)$ induces via $g \times \mathrm{id}$ a map $(I^{k+1}, \partial I^{k+1}, J^k) \to (CX, X, *)$.

The reference should be to Problem 6 (not 1).

Page 155 Proposition (6.10.6): $n \geq 2$.

Page 156 +9: $\pi_j(Y \times Z / Y \times *)$

(6.10.8): $\ldots \to \pi_k(Y \times S^n, Y \times *) \to \pi_k(Y \times S^n / Y \times *) \to \ldots$

Page 157 Problem 6. Apply (5.1.6) and (5.1.8) in order to see that i is a cofibration.

Problem 8. $A = \{0\} \cup \{n^{-1} \mid n \in \mathbb{N}\}$. $\pi_1(\Sigma A)$ is uncountable.

Page 159 +12: We introduced a naive form of spectra and use them ...

Page 160 -14: ... is a functor.

-12: erase the word "proper"

-7: $S^{(t)} = \mathbb{R}^t \cup \{\infty\}$

Page 161 +4: ... an element f of $\mathrm{ST}((X, n), (Y, m))$ ...

Page 162 Proposition (7.1.6): Replace by something discussed earlier; see (6.10.8).

Page 163 If f_k is a representing map then $-f_k$ is the inverse in the homotopy set, i.e., not well-defined as a map. The product structure uses the associativity of the $\wedge$-product, so one should work with compactly generated spaces.

Page 164 +5: $(X \times Y)/(X \times \{*\}) \cong X^+ \wedge Y$.

Page 166 -7: and $A \times 1 \times Y + X \times B \times 1 + A \times 1 \times B \times I \cup A \times I \times B \times 1$ is identified to a base point $\{*\}$.

Page 169 $A|B \times C|D = A \times C|B \times D$

-5*: Erase the symbol max: The set $f^{-1}D(t) = \{x \in X \mid \|fx\| \leq t\}$ is compact, ...

Page 171 (7.4.3) Lemma. ... $(x, y) \mapsto (\varphi(\|y\|) \cdot x, y)$ ...

Page 176 Erase $r \times 1$ in the first diagram. The retraction r is only used a little later.

Proof of (7.5.1): In order to show that $\beta(j \times 1)$ is homotopic to γ, after passage to the mapping cones, one has (as in the proof of 7.5.4) to use an excision, since $tx + (1-t)y$ is not contained in V for all elements in $V \times K \times I$.

Consider $\lambda\colon V \times K \times I \to \mathbb{R}^n\colon (x,y,t) \mapsto tx + (1-t)y$. The subset $\Delta_K \times I$ is mapped into K. Hence there exists an open neighbourhood L of Δ_K such that $\lambda(L \times I) \subset N$. Then consider the diagram

$$\begin{tikzcd}
(L, L \smallsetminus \Delta_K) \arrow[r, "\subset"] \arrow[dr, "{\beta,\gamma}"] & (V \times K, V \times K \smallsetminus \Delta_K) \arrow[d, "{\beta,\gamma}"] & (V \times K, (V \smallsetminus K) \times K) \arrow[l] \arrow[dl, "{\beta,\gamma}"] \\
& (V \times \mathbb{R}^n, V \times (\mathbb{R}^n \smallsetminus 0)) &
\end{tikzcd}$$

The left most β, γ are homotopic.

For the homotopy $(x,y) \mapsto (y, x - ty)$ one has to conclude from $x - ty \in \{0\}$ that $(x,y) \in D \times X$. Note that $D \supset X$ and D is convex.

Page 185 The decoration $\wedge A$ of an arrow has to be changed into $A \wedge -$.

Page 186 +4: Replace PE^* by PE^k.

-6: device

Page 187 +1: Erase the sentence "We met this problem already ..."; CW-complexes appear in the following chapter.

Page 207 Problem 8: Let $\varphi_0, \varphi_1\colon \coprod_j S_j^{n-1} \to A$ be ...

Page 209 +3: (1) is a special case of (2).

Page 210 Proof of (8.5.1): This is not directly an application of (6.4.2). Either refer to (6.10.1), or, better, reduce to open covering.

Page 211 Proof of (8.5.5): f maps $(X \times \partial I \cup B \times I)^n = X^n \times \partial I \cup B^{n-1} \times I$ into Y^n; hence is cellular on $X \times \partial I \cup B \times I$.

Page 212 +14,15: Change c_{-1}, c_1 into c_-, c_+.

-11: The characteristic map would be $(\Phi_j, \varphi_j)\colon (D^1, S^0) \to (X, X')$

-7 (above the diagram): Change "generated" into "generate".

The case $n \geq 2$: Apply this process to each path component.

Page 213 (8.6.4): Already proved in 6.10. The isomorphism also holds in a larger range. But the argument with cells may be more transparent.

Page 215 Proof of (8.6.8): $(\alpha_2)_*$

Page 249 +3 (diagram): $h_n(1 \times X, 1 \times A)$

Page 252 +7: Let U_2 be unbounded.

Page 254 +1,+2: Let (C, μ) be a comonoid in h-TOP^0, i.e., $\mu\colon C \to C \vee C$ is a pointed map such that the composition with the projection onto the summands is pointed homotopic to the identity.

Page 261 Add before the statement of Theorem (10.6.1): A proof of statement (5) in the following theorem follows from the more general theorem (10.6.3), since a map $S^n \to S^{n-1}$ is null homotopic and has therefore degree zero.

Page 263 Replace the sentence before theorem (10.6.3) by: The following theorem implies statement (5) of (10.6.1). See also (17.9.9) and 18.8 Problem 7.

Page 267 -5: Replace $\diamond$ by $\square$.

Page 268 (3): $h_*(IX_{01}, \partial IX_{01} \cup IA_{01})$

-11: $[3/4, 1]X_{01}$

Page 273 +6: $h_n(\partial IA \cup IB, \partial IA \cup IC)$

Page 274 Diagram (10.9.4): Erase a bracket in $h_n((A, B)$.

Page 279 +7: p^k (two times instead of p_k)

Page 280 In 11.2.3: $\mathrm{Ke}(a')$

Page 319 Lemma (13.1.4): $U = (f_j^{-1}\,]0, \infty[\;|\; j \in J)$

Theorem (13.1.5): Remove part (c) and say instead that the $s_{a,n}^{-1}\,]0, \infty[$ form a covering.

$g_j^{-1}[1/4, \infty[\subset f_j^{-1}\,]0, \infty[$

Page 320 +1: and $q_r(x) = 0$ for $r = 1$.

+10: ...of the previous lemma and (13.1.2).

Page 321 +15: $\max(0, \min_{i \in E} t_i(z) - \max_{j \notin E} t_j(z))$

+15: q_E instead of q_e.

Corollary (13.1.9): If $U(E) \cap U(F) \neq \emptyset$ and $|E| = |F|$, then $E = F$.

Replace U_n by $U\langle\, n\,\rangle$ and define $\tau_n \colon X \to [0, 1]$ as $\tau_n = \sum_{|E|=n} |E| q_E$. This is a continuous function with $\tau^{-1}\,]0, 1] = U\langle\, n\,\rangle$.

Page 322 -15: diagram which associates to a simplex

Page 325 -5: Replace V_j by B_j.

Page 331 -2: $v \colon U \to A$, $x \mapsto x \cdot f(x)^{-1}$.

Page 334 +10: ...bundle $q \colon X \to B$ for a ...

-13: "topologize"

Page 335 +4: ...yield a parametrized version.

Page 338 -1: diagram commutes

Page 339 +2: with (...) in the second summand, then ...

Page 359 +6: is called ***smooth*** or ***differentiable***
+11, replace with: A map $f \colon M \to N$ is ***smooth*** if it is continuous and for each $x \in M$ there exist charts (U, h, U') about x and (V, k, V') about $f(x)$ such that kfh^{-1} is smooth.

Page 360 With our previous conventions we should write $((V_k, h_k, B_k) \mid k \in \mathbb{N})$ at some places.
Proof of 15.1.2: $\ldots = \bigcup_{i=0}^{\infty} M_i$

Page 361 Proposition (15.1.3): Let M be a closed submanifold of N. In the proof one has to add $U_0 = N \smallsetminus M$ to the covering (U_j), the corresponding function f_0 is zero.

Page 362 +8 (Exercise 2) $q \colon E \to E/K$ (superfluous))
-6: $i_l i_k^{-1}$

Page 371 +13: $(D_i f_j(x) \mid m-n+1 \le i \le m, 1 \le j \le n)$
Problem 2: A is closed in B.

Page 374 Replace (15.6.1) by:

There exists a unique structure of a smooth manifold on TM such that the TU are open subsets and the φ_k are diffeomorphisms. The projection π_M carries the structure of an n-dimensional vector bundle for which the φ_k are bundle charts.

Replace (15.6.2) by:
Let $M \subset \mathbb{R}^q$ be a smooth n-dimensional submanifold. Then

$$TM = \{(x,v) \mid x \in M, v \in T_x(M)\} \subset M \times \mathbb{R}^q$$

is a smooth subbundle of the trivial bundle $M \times \mathbb{R}^q \to M$.

Proof. Let (U, φ, V) be a chart of $\mathbb{R}^q$ adapted to M. Then the diffeomorphism

$$U \times \mathbb{R}^q \to U \times \mathbb{R}^q, \quad (x,v) \mapsto (x, T_x\varphi(v))$$

restricted to $U \cap M$ is a bundle chart adapted to TM. □

Page 375 +9: $(g,v) \mapsto (Tl_g)v$

Page 377 +2: It embeds an open neighbourhood U of the zero section onto an open neighbourhood V of M in N.

(15.6.9): Let $t\colon E(\nu) \to N$ be a tubular map for a submanifold M with embedding $t\colon U \xrightarrow{\cong} V$ as above. There exists a smooth function $\varepsilon\colon M \to]0,\infty[$ such that

$$E\varepsilon(\nu) = \{y \in E(\nu)_x \mid \|y\| < \varepsilon(x)\} \subset U.$$

(Here $E(\nu)$ carries a smooth Riemannian metric, and $\|y\|$ is the corresponding norm.) The set $E_\varepsilon(\nu)$ is an open neighbourhood of the zero section. The assignment

$$h\colon E(\nu) \to E_\varepsilon(\nu), \quad y \mapsto \frac{\varepsilon(x)y}{\sqrt{\varepsilon(x)^2 + \|y\|^2}}, \quad y \in E(\nu)_x$$

is a fibrewise diffeomorphism and th is a tubular map which embeds $E(\nu)$ onto an open neighbourhood of M in N.

Replace (15.6.10) by:

(15.6.10) Proposition *The set*

$$N(M) = \{(x,v) \mid x \in M, v \in N_x M\} \subset M \times \mathbb{R}^q$$

together with the projection onto M is a smooth subbundle of the trivial bundle $M \times \mathbb{R}^q \to M$.

Remark. (15.6.2) and (15.6.10) should be combined and the proofs correspondingly combined.

Page 379 Proof of (15.6.14) +3: …Hence $D(U_a)$ and $U = \bigcup_{a \in A} D(U_a)$ is open in K and therefore $W = X \times X \smallsetminus (K \smallsetminus U)$ open in $X \times X$. By assumption, $A \times A$ is contained in W. …

Page 382 Proof of (15.7.8): $\mathbb{R}_+$.
The retraction r is obtained from a smooth embedding of M into some Euclidean space. Then one uses the tubular retraction for ∂M arising from this embedding. (A similar argument is used for (15.7.7).)

Page 383 +13: $f_x(y) = f(x)$

-9: $\delta(y) \geq 2\gamma(x)$

Page 384 +1: $\leq \frac{1}{2}\delta(y)$

First sentence in 15.9: ... and $g\colon B \to M$ ...

Page 385 15.9.1: $b = f(a)$, a is a regular point of $p \circ f$.
f transverse to $B \cap Y \Leftrightarrow f$ transverse to B in $f^{-1}(Y) \Leftrightarrow 0$ regular value of $p \circ f\colon f^{-1}(Y) \to Y \to \mathbb{R}^k$.
$(pf)^{-1}(0) = f^{-1}(Y) \cap f^{-1}(B) = f^{-1}(Y \cap B)$.

Page 386 -9: The proof uses the ...

Page 387 15.9.8: Replace C by a different letter, say D, since C has been used with different meaning. The homotopy is relative to $C = D$.

Page 388 +2: $T_x M \times \mathbb{R}^k$

Page 390 15.10.5: This is not directly an application of (15.10.3), since $D^k \times D^{n-k}$ has to be considered as a smooth manifold (15.10.2).

Page 409 -9: Let $\rho\colon k^*(-) \to l^*(-)$ be a natural ...

Page 412 first row in second diagram:
$h(IY, I(B \cup Y^0) \cup \partial IY)$

Page 419 +10: with a finitely generated free abelian group F.
Erase the last sentence of the proof

Page 420 +14: $\overset{(2)}{\cong} \mathrm{Hom}(\pi_n(K), A)$

Page 425 -9: There exist h-equivalences $\varphi_\pm\colon D^n_\pm \times X \to X_\pm$ over

Page 439 -4: ..., $j_B\colon (B, A \cap B) \to (X, A)$ the inclusion and ...

Page 442 (Unit element): There exists a unique element $1^* \in h^0(P)$ such that $1^* \frown 1 = 1$. Let $1^*_X \in h^0(X)$ be the image of 1^*. Then $1^*_X \frown x = x$ for $x \in h_n(X, B)$ (Exercise).

Page 443 First diagram third row: $\ldots \otimes h_{p+q}(\partial IX \cup IB)$

(18.2.2): Also $n = 0$; then the unit element axiom (4).

Page 447 Problem 4: The canonical class is obtained from a unit element $1 \in h_0(P)$ via suspension. Thus one should assume given this element.

Page 448 +3: i.e., $rj = \mathrm{id}(X)$.

Page 451 +1: Replace M by another letter, since already $M = \partial B$.

Page 453 +2*: diagrams

Page 464 Replace section 18.9
Let M be a closed connected n-manifold, oriented by a fundamental class $[M] \in H_n(M)$. For $M \times M$ we use the product orientation $[M \times M] = [M] \times [M]$. Let $d\colon M \to M \times M$ be the diagonal map. By naturality of the cap product and Poincaré duality there exists a unique element $\tau(M) \in H^n(M \times M)$ such that $\tau \cap [M \times M] = d_*[M]$. We call $d^*\tau(M) = e(M) \in H^n(M)$ the ***Euler class*** of the topological manifold M and $\langle e(M), [M] \rangle \in \mathbb{Z}$ its ***Euler number***. We now pass to coefficients in a field $\mathbb{K}$. The image of the fundamental class $[M] \in H_n(M; \mathbb{Z})$ in $H_n(M; \mathbb{K})$ is still denoted $[M]$, and similarly for $[M \times M]$.

Let $B = \{\alpha\}$ be a basis of $H^*(M)$ and $\{\alpha^0\}$ the dual basis in $H^*(M)$ with respect to the intersection form $\langle \alpha^0 \smile \beta, [M] \rangle = \delta_{\alpha\beta}$, $|\alpha^0| = n - |\alpha|$.

(0.0.1) Proposition. *The element $\tau(M) \in H^n(M \times M)$ is given by*

$$\tau(M) = \textstyle\sum_{\alpha \in B} (-1)^{|\alpha|} \alpha^0 \times \alpha \in H^n(M \times M).$$

A consequence is

$$e(M) = d^*\tau(M) = \textstyle\sum_{\alpha \in B} (-1)^{|\alpha|} \alpha^0 \smile \alpha.$$
$$\langle\, e(M), [M]\,\rangle = \textstyle\sum_\alpha (-1)^{|\alpha|} \langle\, \alpha^0 \smile \alpha, [M]\,\rangle = \sum_\alpha (-1)^{|\alpha|} = \chi(M).$$

Proof. The Künneth-isomorphism

$$H^*(M) \otimes H^*(M) \cong H^*(M \times M), \quad u \otimes v \mapsto u \times v$$

tells us that there exists a relation of the form $\tau = \sum_{\gamma,\delta \in B} A(\gamma,\delta)\gamma^0 \times \delta$. The following computations determine the coefficient $A(\gamma,\delta)$. Let α and β be basis elements of degree p. Then

$$\begin{aligned} &\langle\, (\alpha \times \beta^0) \smile \tau, [M \times M]\,\rangle = \langle\, \alpha \times \beta^0, \tau \frown [M \times M]\,\rangle \\ &= \langle\, \alpha \times \beta^0, d_*[M]\,\rangle = \langle\, d^*(\alpha \times \beta^0), [M]\,\rangle = \langle\, \alpha \smile \beta^0, [M]\,\rangle \\ &= (-1)^{p(n-p)} \langle\, \beta^0 \smile \alpha, [M]\,\rangle = (-1)^{p(n-p)} \delta_{\alpha\beta}. \end{aligned}$$

A second computation gives

$$\begin{aligned} &\langle\, (\alpha \times \beta^0) \smile \tau, [M \times M]\,\rangle = \langle\, (\alpha \times \beta^0) \smile \textstyle\sum A(\gamma,\delta)\gamma^0 \times \delta, [M \times M]\,\rangle \\ &= \textstyle\sum A(\gamma,\delta)(-1)^{|\beta^0||\gamma^0|} \langle\, (\alpha \smile \gamma^0) \times (\beta^0 \smile \delta), [M] \times [M]\,\rangle \\ &= \textstyle\sum A(\gamma,\delta)(-1)^{|\beta^0||\gamma^0|}(-1)^{n(|\beta^0|+|\delta|)} \langle\, \alpha \smile \gamma^0, [M]\,\rangle \langle\, \beta^0 \smile \delta, [M]\,\rangle. \end{aligned}$$

Only summands with $\gamma = \alpha$ and $\delta = \beta$ are non-zero. Thus this evaluation has the value $A(\alpha,\beta) \cdot (-1)^{pn}$ (collect the signs and compute modulo 2). We compare the two results and obtain $A(\alpha,\beta) = (-1)^p \delta_{\alpha\beta}$. □

The last equality in (0.0.1) holds by its proof in each field and therefore also in $\mathbb{Z}$.

We now relate the result to the previously defined Euler class and Euler number for a smooth manifold. Let $\nu\colon E(\nu) \to M$ be the normal bundle of the diagonal $d\colon M \to M \times M$ with disk- and sphere bundle $D(\nu)$ and $S(\nu)$ and tubular map $j\colon D(\nu) \to M \times M$. The fundamental class $[M \times M]$ induces a fundamental class $[D(\nu)] \in H_{2n}(D(\nu), S(\nu))$ via

$$H_{2n}(M \times M) \xrightarrow{\iota_*} H_{2n}(M \times M, M \times M \smallsetminus D) \xleftarrow[\cong]{j_*} H_{2n}(D(\nu), S(\nu)),$$

$\iota_*[M \times M] = z = j_*[D(\nu)]$. The diagram

$$\begin{tikzcd} H^n(M \times M) \arrow[d, "\frown [M \times M]"] & H^n(M \times M, M \times M \smallsetminus D) \arrow[l, "\iota^*"'] \arrow[r, "j^*", "\cong"'] \arrow[d, "\frown z"] & H^n(D(\nu), S(\nu)) \arrow[d, "\frown [D(\nu)]"] \\ H_n(M \times M) \arrow[r, "="] & H_n(M \times M) & H_n(D(\nu)) \arrow[l, "j_*"'] \end{tikzcd}$$

commutes (naturality of the cap-product). From the isomorphisms with the zero section $i\colon M \to D(\nu)$

$$H_n(M) \xrightarrow[\cong]{i_*} H_n(D(\nu)) \xleftarrow[\cong]{\frown [D(\nu)]} H^n(D(\nu), S(\nu))$$

we obtain an element $t(\nu)$ that satisfies $i_*[M] = t(\nu) \frown [D(\nu)]$. It is a generator and therefore a Thom class. Let $j^*t(M) = t(\nu)$. From the definitions we see that $d^*\tau$ is the image of $t(\nu)$ under $H^n(D(\nu), S(\nu)) \to H^n(D(\nu)) \xrightarrow{i^*} H^n(M)$, hence the Euler class $e(\nu)$ of ν. Now we use the fact that the normal bundle ν is isomorphic to the tangent bundle of M. Therefore $e(\nu)$ is an Euler class of M in this sense.

Problems

1. The projection $p\colon (M \times M, M \times M \smallsetminus D) \to M$ onto the first coordinate is locally trivial. The element $t(M) \in H^n(M \times M, M \times M \smallsetminus D)$ restricts to a generator in $H^n(x \times M, x \times (M \smallsetminus x))$ for each $x \in M$, hence is a Thom class in this sense.
2. There exists a tubular map $TM \to M \times M$ for the diagonal which is a map over M. For its construction one can use the so-called exponential map, see e.g. [**?**, §12].

Page 499 Proof of (20.1.8): Let

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{F} & \tilde{Y} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{f} & Y \end{array}$$

be commutative. Then p and q induce isomorphisms of π_n for $n \geq 2$. The spaces $\tilde{X}$ and $\tilde{Y}$ are simply connected CW-spaces. By (20.1.5) the map F is an h-equivalence. Hence f induces an isomorphisms of π_n for $n \geq 2$ and, by hypothesis, also for $n = 1$. Hence f is an h-equivalence.

Page 512 -3: $H_*(X; T) \cong H_*(S^n; T)$