Quantum groups and knot algebra

Tammo tom Dieck

Version ofMay 4, 2004

Contents

1 Tensor categories

1. Categories

We collect in this section some basic notation about categories.

A *category* $\mathcal{C}$ consists of the following data:

(1) A class (set) $\mathrm{Ob}(\mathcal{C})$ of *objects*.

(2) A set $\mathrm{Mor}(C,D)$ associated to each ordered pair (C,D) of objects C,D in $\mathcal{C}$. The elements of $\mathrm{Mor}(C,D)$ are called *morphisms* of the category *from C to D*. A morphism from C to D is denoted $f\colon C \to D$ and called *arrow* of the category. We call C the *source* and D the *range* of f. A morphism determines its sourse and range, i. e. $\mathrm{Mor}(C,D) \cap \mathrm{Mor}(E,F) \neq \emptyset$ implies $C=E$ and $D=F$.

(3) A morphism $\mathrm{id}(C) = \mathrm{id}_C = 1_C \in \mathrm{Mor}(C,C)$ for each object C, called *identity* of C. The identity of C is also denoted just by C or by 1.

(4) A map

$$\mathrm{Mor}(B,C) \times \mathrm{Mor}(A,B) \to \mathrm{Mor}(A,C), \quad (g,f) \mapsto g \circ f = gf$$

for each ordered tripel (A,B,C) of objects, called *composition* of morphisms.

These data are subject to the following axioms:

(5) The composition is associative, i. e. for morphisms $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ we have always $(hg)f = h(gf)$.

(6) For each morphism $f\colon A \to B$ we have $f = f \circ \mathrm{id}(A) = \mathrm{id}(B) \circ f$.

Instead of $\mathrm{Mor}(C,D)$ we also write $\mathrm{Mor}_{\mathcal{C}}(C,D)$ or $\mathcal{C}(C,D)$. Also the notation $\mathrm{Hom}(C,D)$ or $\mathrm{Hom}_{\mathcal{C}}(C,D)$ is used. A category is called *small* if its objects form a set.

Typical examples of categories are the category SET (objects) and maps (morphisms) or the category ABEL of abelian groups (objects) and homomorphisms (morphisms). We refer to these categories as the category of sets and the category of abelian groups.

A morphism $f\colon C \to D$ is called *isomorphism*, if there exists a morphism $g\colon D \to C$ such that $g \circ f = \mathrm{id}(C)$ and $f \circ g = \mathrm{id}(D)$. A morphism g with this property is uniquely determined by f and will be denoted f^{-1}. If there exists an isomorphism $f\colon C \to D$, then the objects C and D are called *isomorphic*. A morphism $f\colon C \to C$ is called *endomorphism* of C. An isomorphism $f\colon C \to C$ is called *automorphism* of C.

Occasionally it is useful to consider morphism sets with additional structure. Let $\mathfrak{K}$ be a commutative ring. A category is called $\mathfrak{K}$-*category*, if all morphism sets $\mathrm{Hom}(C,D)$ are left $\mathfrak{K}$-modules and if the composition $\mathrm{Hom}(C,D) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(B,D)$, $(f,g) \to fg$ is always $\mathfrak{K}$-bilinear. An example is the category $\mathfrak{K}$-Mod of left $\mathfrak{K}$-modules and $\mathfrak{K}$-linear maps.

We mention some methods to construct new categories from given ones.

(1.1) Let $\mathcal{C}$ be a category. The *dual category* $\mathcal{C}^\circ$ is obtained from $\mathcal{C}$ by reversing the arrows. This means: Both categories have the same objects. But we have $\mathcal{C}^\circ(C, D) = \mathcal{C}(D, C)$. The identities remain the same. Composition $*$ in $\mathcal{C}^\circ$ is defined by reversing the order: $f * g$ is defined if and only if $g \circ f$ is defined, and is the arrow which belongs to $g \circ f$ in $\mathcal{C}^\circ$. ♡

(1.2) Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ has as objects the pairs (C, D) of objects C in $\mathcal{C}$ and D in $\mathcal{D}$. The morphisms $(C_1, D_1) \to (C_2, D_2)$ are pairs of morphisms $f\colon C_1 \to C_2$, $g\colon D_1 \to D_2$. Composition is defined componentwise by the composition in the original categories. ♡

(1.3) Let $\mathcal{C}$ be a category. The category $P\mathcal{C}$ of arrows in $\mathcal{C}$ has as objects the morphisms of $\mathcal{C}$. A morphism from $f\colon C_1 \to C_2$ to $g\colon D_1 \to D_2$ is a pair of morphisms $\varphi_j\colon C_j \to D_j$ which satisfy $g\varphi_1 = \varphi_2 f$. ♡

(1.4) The objects of the category of endomorphisms $\mathrm{END}(\mathcal{C})$ are the endomorphisms $f\colon C \to C$ in $\mathcal{C}$. A morphism from f to $g\colon D \to D$ is a morphism $\varphi\colon C \to D$ which satisfies $g\varphi = \varphi f$. ♡

(1.5) Let B be an object in $\mathcal{C}$. A morphism $f\colon E \to B$ is called *object over* B. The category $\mathcal{C}_B$ of objects over B has as objects the objects over B. A morphism from $f\colon E \to B$ to $g\colon F \to B$ is a morphism $\varphi\colon E \to F$ such that $g\varphi = f$. Similarly for *objects* $f\colon B \to E$ *under* B. ♡

In general, one can define in a similar manner categories from diagrams of a fixed shape.

2. Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F\colon \mathcal{C} \to \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is an assignment which associates to each object C of $\mathcal{C}$ an object $F(C)$ of $\mathcal{D}$ and to each morphism $f\colon C \to D$ in $\mathcal{C}$ a morphism $F(f)\colon F(C) \to F(D)$ of $\mathcal{D}$. These data are subject to the following axioms:

(2.1) $$F(\mathrm{id}(C)) = \mathrm{id}(F(C)), \qquad F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $U\colon \mathcal{C} \to \mathcal{D}$ is an assignment which associates to each object C in $\mathcal{C}$ an object $U(C)$ in $\mathcal{D}$ and to each morphism $f\colon C \to D$ in $\mathcal{C}$ a morphism $U(f)\colon U(D) \to U(C)$ $\mathcal{D}$ such that:

(2.2) $$U(\mathrm{id}(C)) = \mathrm{id}(U(C)), \qquad U(g \circ f) = U(f) \circ U(g).$$

Functors are also called *covariant functors*.

An immediate consequence of (2.1) and (2.2) is:

(2.3) Proposition. *A (contravariant) functor maps isomorphisms to isomorphism.* □

A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ is essentially the same thing as a functor $\mathcal{C} \to \mathcal{D}^\circ$ into the dual categor or as a functor $\mathcal{C}^\circ \to \mathcal{D}$. A functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ from a product category is called a functor in two variables.

Each category produces functors from its morphism sets. Let D be an object of the category $\mathcal{C}$. The *contravariant* Hom-*functor*

$$\mathrm{Hom}(-, D) = \mathrm{Hom}(?, D)\colon \mathcal{C} \to \ \mathrm{SET}$$

associates to the object C the morphism set $\mathrm{Hom}(C, D)$ and to the morphism $\varphi\colon C_1 \to C_2$ the map

$$\mathrm{Hom}(\varphi, D)\colon \mathrm{Hom}(C_2, D) \to \mathrm{Hom}(C_1, D), \quad f \mapsto f \circ \varphi.$$

On easily checks the axioms of a functor. Similarly, for each object C we have the *covariant* Hom-*functor* $\mathrm{Hom}(C, -)\colon \mathcal{C} \to \ \mathrm{SET}$. Taken together, the Hom-functor is a functor

$$\mathrm{Hom}(-, -)\colon \mathcal{C}^\circ \times \mathcal{C} \to \mathrm{SET}$$

in two variables.

Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be functors. Their composition $G \circ F$ is the functor $\mathcal{A} \to \mathcal{C}$ defined as

(2.4) $$(G \circ F)(C) = G(F(C)), \qquad (G \circ F)(f) = G(F(f))$$

on objects C and morphisms f. We have the identity functor $\mathrm{Id}_\mathcal{C}\colon \mathcal{C} \to \mathcal{C}$ which is the identity on objects and morphisms. Composition of functors is assoviative. If $\mathcal{A}$ and $\mathcal{B}$ are small categories, then the functors from $\mathcal{A}$ to $\mathcal{B}$ form a set. Therefore small categories together with the functors between them form a category CAT.

The composition of two contravariant functors is defined similarly and yields a covariant functor. We can also compose co- and contravariant functors.

Since CAT is a category, we have the notion of an isomorphsm of (small) categories. It turns out that this notion for the comparison of categories is too rigid. We define instead the notion of equivalence of categories.

Let $F, G\colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\Phi\colon F \to G$ from F to G consists of a family $\Phi_C\colon F(C) \to G(C)$ of morphisms in $\mathcal{D}$, indexed by the objects C of $\mathcal{C}$, such that for each morphism $f\colon C \to D$ in $\mathcal{C}$ the equality

$$G(f) \circ \Phi_C = \Phi_D \circ F(f)\colon F(C) \to G(D)$$

holds. We call the Φ_C the *components* of the natural transformation. We often specify natural transformations by just writing their components. If all Φ_C are isomorphisms in $\mathcal{D}$, we call Φ a *natural isomorphism* and use the notation $\Phi\colon F \simeq G$. A natural transformation and a natural isomorphism between contravariant functors is defined similarly. The inverse morphisms Φ_C^{-1} of a natural isomorphism form a natural isomorphism.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *equivalence* of these categories, if there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \simeq \mathrm{Id}_\mathcal{C}$ and $FG \simeq \mathrm{Id}_\mathcal{D}$. Then also G is an equivalence of categories. Categories $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent*, if there exists an equivalence between them. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are equivalences of categories, then $G \circ F$ is an equivalence.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is called *essentially surjective*, if for each object W of $\mathcal{D}$ there exists an object V of $\mathcal{C}$ and an isomorphism $F(V) \cong W$. A functor F is called *faithful* (*fully faithful*), if for any two objects U, V of $\mathcal{C}$ the map $\mathrm{Hom}(U,V) \to \mathrm{Hom}(F(U), F(V))$, $f \mapsto F(f)$ is injective (bijective).

(2.5) Theorem. *A functor is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. Let $\Phi\colon GF \simeq \mathrm{Id}_{\mathcal{C}}$ and $\Psi\colon FG \simeq \mathrm{Id}_{\mathcal{D}}$ be natural equivalences. Suppose $Y \in \mathrm{Ob}(\mathcal{D})$ is given and let $X = G(Y)$. Then Ψ yields an isomorphism $F(X) = FG(Y) \cong Y$. Hence F is essentially surjective.

Let $f, g \in \mathrm{Hom}(U,V)$ be morphisms with the same image under F. Then we have $\Phi_V \circ GF(V) = f \circ \Phi_U$ and similarly for g. Since $GF(f) = GF(g)$, we conclude $f = g$. Hence F (and also G) is injective on Hom-sets.

Suppose $g\colon F(U) \to F(V)$ is given. W define $f\colon U \to V$ by

$$f = \Phi_V \circ G(g) \circ \Phi_U^{-1}.$$

There is a similar equality with $G(g)$ replaced by $GF(f)$. Since G is injective on Hom-sets, the equality $GF(f) = G(g)$ implies the equality $F(f) = g$. Hence F is surjective on Hom-sets and therefore fully faithful.

Conversely, assume that F is fully faithful and essentially surjective. For each object W of $\mathcal{D}$ we choose an object $G(W)$ in $\mathcal{C}$ and an isomorphism $\Psi_W\colon FG(W) \to W$. This is possible, since F is essentially surjective. Since F is fully faithful, there exists for each morphism $g\colon W \to W'$ in $\mathcal{D}$ a unique morphism $G(g)\colon G(W) \to G(W')$ such that

$$FG(g) = \Psi_{W'} \circ g \circ \Psi_W^{-1}.$$

One verifies that these choices define a functor $G\colon \mathcal{D} \to \mathcal{C}$. The Ψ_W yield a natural isomorphism $\Psi\colon FG \simeq \mathrm{Id}_{\mathcal{D}}$. We still have to define a natural isomorphism $\Phi\colon GF \simeq \mathrm{Id}_{\mathcal{C}}$. Let $\Phi_V\colon GF(V) \to V$ be the uniquely determined morphism such that $F\Phi_V = \Psi_{F(V)}$. One verifies that these data give a natural isomorphism. □

If $\Phi\colon F \to G$ and $\Psi\colon G \to H$ are natural transformations between functors $\mathcal{C} \to \mathcal{D}$, then

(2.6) $$(\Psi \circ \Phi)_C = \Psi_C \circ \Phi_C$$

defines a natural transformation $\Psi \circ \Phi\colon F \to H$. This composition of natural transformations is associative. We always have the identical natural transformation $\mathrm{Id}_F\colon F \to F$.

Let $\mathcal{C}$ and $\mathcal{D}$ be small categories. The *functor category* $[\mathcal{C}, \mathcal{D}]$ has as objects the functors $\mathcal{C} \to \mathcal{D}$, and the morphisms from F to G are the natural transformations $\Phi\colon F \to G$. Composition is defined by (2.6).

(2.7) Equivalent subcategories. A *subcategory* of a category $\mathcal{C}$ consists of a subset of objects and morphisms such that they form a category with the given composition. A subcategory is called *full*, if for any two objects its Hom-set is the same as in the larger category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is equivalent to $\mathcal{C}$ if each object of the large category is isomorphic to an object in the subcategory.♡

Let D and E be objects of a category and let $f\colon D \to E$ be a morphism. Composition with f yields a natural transformation $f_*\colon \mathrm{Hom}(?, D) \to \mathrm{Hom}(?, E)$ between Hom-functors by setting

$$f_{*C}\colon \mathrm{Hom}(C, D) \to \mathrm{Hom}(C, E), \quad g \mapsto f \circ g$$

and a natural transformation $f^*\colon \mathrm{Hom}(E, ?) \to \mathrm{Hom}(D, ?)$ by setting

$$f^*_C\colon \mathrm{Hom}(E, C) \to \mathrm{Hom}(D, C), \quad h \mapsto h \circ f.$$

The functors $\mathrm{Hom}(-, D)$ and $\mathrm{Hom}(-, E)$ are naturally isomorphic if and only if D and E are isomorphic.

(2.8) Yoneda-Lemma. Let $\mathcal{C}$ be a category. A natural transformation of the Hom-functor $\mathrm{Hom}(-, C)$ into a contravariant functor $G\colon \mathcal{C} \to \mathrm{SET}$ is determined by its value on $\mathrm{id}(C) \in G(C)$ and this value can be prescribed arbitrarily. ♡

Suppose F is fully faithful and $F(f)$ an isomorphism. Then f is an isomorphism.

3. Adjoint functors

From now on we don't use calligraphic letters for categories. Let $F\colon C \to D$ and $G\colon D \to C$ be functors. An *adjunction* $(\varepsilon, \eta)\colon F \dashv G$ is a pair of natural transformations

$$\varepsilon\colon FG \to \mathrm{id}_D, \qquad \eta\colon \mathrm{id}_C \to GF,$$

such that for all $V \in \mathrm{Ob}(V)$ and $W \in \mathrm{Ob}(D)$ the two morphisms

(3.1) $$G(W) \xrightarrow{\eta_{G(W)}} GFG(W) \xrightarrow{G(\varepsilon_W)} G(W)$$

(3.2) $$F(V) \xrightarrow{F(\eta_V)} FGF(V) \xrightarrow{\varepsilon_{F(V)}} F(V)$$

are the identity.

A natural transformation $\eta\colon \mathrm{id}_C \to GF$ yields a natural transformation between Hom-functors in two variables

$$\Phi_\eta\colon D(F(V), W) \to C(V, G(W))$$

as follows: Let $f\colon F(V) \to W$ be given. We consider $\Phi_\eta(f) = G(f) \circ \eta_V$. A natural transformation $\varepsilon\colon FG \to \mathrm{id}_D$ yields a natural transformation

$$\Psi_\varepsilon\colon C(V, G(W)) \to D(F(V), W),$$

if we assign to $g\colon V \to G(W)$ the morphism $\Psi_\varepsilon(g) = \varepsilon_W \circ F(g)$. If, conversely, $\Phi\colon D(F(V),W) \to C(V,G(W))$ is given, let $\eta_V^\Phi\colon V \to GF(V)$ be the image $\Phi(\mathrm{id}_{F(V)})$; and if Ψ is given, let $\varepsilon_W^\Phi\colon FG(W) \to W$ be the image $\Phi(\mathrm{id}_{G(W)})$. One verifies naturality.

(3.3) Proposition. *The assignments $\eta \mapsto \Phi_\eta$ and $\Phi \mapsto \eta^\Phi$ are inverse bijections between the natural transformations $\mathrm{id}_D \to GF$ and $C(F(?), ??) \to D(?, G(??))$. The assignments $\varepsilon \mapsto \Psi_\varepsilon$ and $\Psi \mapsto \varepsilon^\psi$ are inverse bijections between the natural transformations $FG \to \mathrm{id}_C$ and $C(?, G(??)) \to D(F(?), ??)$.*

The proof is a simple verification. For instance, $\eta_V^{\Phi_\eta}$ is the image of $\mathrm{id}_{F(V)}$ bei Φ_η, hence equal to $G(\mathrm{id}_{F(V)}) \circ \eta_V = \eta_V$ and therefore $\eta^{\Phi_\eta} = \eta$.

(3.4) Theorem. *Suppose given a natural transformation $\eta\colon \mathrm{id}_D \to GF$. Then Φ_η is a natural isomorphism if and only if there exists an adjunction $(\varepsilon, \eta)\colon F \dashv G$. In this case ε is uniquely determined by η. Suppose $\varepsilon\colon FG \to \mathrm{id}_C$ is given. Then Ψ_ε is a natural isomorphism if and only if there exists an adjunction $(\varepsilon, \eta)\colon F \dashv G$.*

Proof. Suppose there exists an adjunction $(\varepsilon, \eta)\colon F \dashv G$. We have the corresponding natural transformations Φ_η and Ψ_ε. From the properties of an adjunction we verify that Φ_η and Ψ_ε are inverse to each other. The image of $f\colon F(V) \to W$ under $\Psi_\varepsilon\Phi_\eta$ is $\varepsilon_W \circ FG(f) \circ F(\eta_V)$, and this equals $f \circ \varepsilon_{F(V)} \circ F(\eta_V) = f$ by naturality of ε and the properties of an adjunction. Similarly for $\Phi_\eta\Psi_\varepsilon$. Since Ψ_ε is inverse to Φ_η, we see that, by (3.1), ε is uniquely determined by η.

Conversely, assume that Φ_η is a natural isomorphism. To the inverse isomorphism belongs a ε and one verifies as above that (ε, η) is an adjunction. □

If there exists an adjunction $(\varepsilon, \eta)\colon F \dashv G$, we call F *left adjoint* to G and G *right adjoint* to F.

Let $(\varepsilon, \eta)\colon F \dashv G$ and $(\varepsilon', \eta')\colon F \dashv G'$ be adjunctions. We form the following *comparison morphisms*

$$\alpha = G'(\varepsilon_W) \circ \eta'_{G(W)}\colon G(W) \to G'(W), \quad \alpha' = G(\varepsilon'_W) \circ \eta_{G'(W)}\colon G'(W) \to G(W).$$

Then we have the following uniqueness theorem for adjunctions. We denote by $\alpha F\colon GF \to GF$ the natural transformation with components $\alpha_{F(V)}$ and by $F\alpha\colon FG \to FG$ the one with components $F\alpha_W$.

(3.5) Theorem. *The morphisms α and α' are inverse natural isomorphisms. We have $(\alpha F) \circ \eta = \eta'$ and $\varepsilon' \circ (F\alpha) = \varepsilon$.* □

4. Tensor categories

(4.1) Definition. A *tensor category* $(C, \otimes, I, a, r, l)$ consists of a category C, an object I of C, a functor $\otimes\colon C \times C \to C$ and a natural isomorphism (notation for objects)

$$\begin{aligned} a = a_{UVW} &: (U \otimes V) \otimes W \to U \otimes (V \otimes W) \\ r = r_V &: V \otimes I \to V \\ l = l_V &: I \otimes V \to V. \end{aligned}$$

The data are subject to the following axioms:
(4.2) given four objects U, V, W, X, the diagram

$$\begin{tikzcd} ((U \otimes V) \otimes W) \otimes X \arrow[r, "a"] \arrow[d, "a \otimes \mathrm{id}"] & (U \otimes V) \otimes (W \otimes X) \arrow[r, "a"] & U \otimes (V \otimes (W \otimes X)) \\ (U \otimes (V \otimes W)) \otimes X \arrow[rr, "a"] & & U \otimes ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id} \otimes a"'] \end{tikzcd}$$

is commutative.

(4.3) Given two objects U, V, the diagram

$$\begin{tikzcd} (U \otimes I) \otimes V \arrow[r, "a"] \arrow[d, "r \otimes \mathrm{id}"] & U \otimes (I \otimes V) \arrow[d, "\mathrm{id} \otimes l"] \\ U \otimes V \arrow[r, "\mathrm{id}"] & U \otimes V \end{tikzcd}$$

is commutative.
We call $\otimes$ the *tensor product*, a the *associator*, r the *right unit* and l the *left unit* of the tensor category. ♡

Tensor categories are also called *monoidal categories*. We explain the axioms. The functor $\otimes$ is a functor in two variables. The natural transformation a is one between functors

$$\otimes(\otimes \times \mathrm{id}),\ \otimes(\mathrm{id} \times \otimes)\colon C \times C \times C \to C,$$

hence involves three variables. Finally, r and l are natural transformations between functors $C \to C$. The object I is a neutral element for the tensor product. The isomorphisms a_{UVW} express the associativity of the tensor product. Axiom (4.1) is called the *pentagon* for U, V, W, X.

A tensor category is called *strict*, if a, r, and l are always identities.

In order to simplify notation we write the tensor product sometimes as juxtaposition of objects and morphisms, hence $AB = A \otimes B$ and $fg = f \otimes g$. In this case, in order to avoid ambiguity, one has to use a symbol for the composition of morphisms.

A variant of definition (4.1) concerns the R-tensor categories. Let R be a commutative ring and C an R-category, i. e. the Hom-sets of C are R-modules and composition is R-bilinear. A tensor category as in (4.1) is called *tensor R-category*, if the tensor product functor is an R-bilinear map

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(C, D) \to \mathrm{Hom}(A \otimes C, B \otimes D), \quad (f, g) \mapsto f \otimes g$$

on morphism sets.

Each tensor category has a *reversed category* $C^{rev} = (C, I, \otimes', a', r', l')$. Its data are given as follows:

$$\begin{aligned} A \otimes' B &:= B \otimes A \\ l' &:= r \\ r' &:= l \\ a' &:= a^{-1}. \end{aligned}$$

For the latter we note $(A \otimes' B) \otimes' C = C \otimes (B \otimes A)$.

(4.4) Note. *The reversed category is again a tensor category.* □

The basic example of a tensor category, from which the name is derived, is the category R-Mod of left modules over the commutative ring R with the usual tensor product of R-modules.

(4.5) Example. Let $\mathcal{A}$ be a category and $C = [\mathcal{A}, \mathcal{A}]$ the category of functors $\mathcal{A} \to \mathcal{A}$ (objects) and natural transformations between them (morphisms). Composition of functors defines a tensor product on the objects of C. Let $\alpha\colon F \to F'$ and $\beta\colon G \to G'$ be natural transformations. The natural transformation $\alpha \otimes \beta\colon F \otimes G \to F' \otimes G'$ is defined, for each object X of $\mathcal{A}$, by the diagram

$$\begin{tikzcd} FG(X) \arrow[r, "F(\beta_X)"] \arrow[d, "\alpha_{G(X)}"] \arrow[dr, "\alpha \otimes \beta"] & FG'(X) \arrow[d, "\alpha_{G'(X)}"] \\ F'G(X) \arrow[r, "F'(\beta_X)"] & F'G'(X). \end{tikzcd}$$

The diagram is commutative, since α is natural. One verifies the functoriality of the tensor product. The identical functor is the neutral element. This tensor category is strict. ♡

5. The neutral object

We derive some consequences of the axioms of a tensor category. It is advisable to draw diagrams for the chains of equalities in the following proofs.

(5.1) Theorem. *The following diagrams are commutative:*

$$\begin{tikzcd} (I \otimes U) \otimes V \arrow[r, "a"] \arrow[d, "l \otimes \mathrm{id}"] & I \otimes (U \otimes V) \arrow[d, "l"] \\ U \otimes V \arrow[r, "="] & U \otimes V \end{tikzcd} \qquad \begin{tikzcd} (U \otimes V) \otimes I \arrow[r, "a"] \arrow[d, "r"] & U \otimes (V \otimes I) \arrow[d, "\mathrm{id} \otimes r"] \\ U \otimes V \arrow[r, "="] & U \otimes V. \end{tikzcd}$$

PROOF. Consider the next diagram.

$$
\begin{tikzcd}
 & & I((IU)V) \arrow[dll, "1a"'] & \\
I(I(UV)) \arrow[d, "1l"] & (II)(UV) \arrow[l, "a"'] \arrow[d, "r(11)"] & ((II)U)V \arrow[l, "a"'] \arrow[r, "a1"] \arrow[d, "(r1)1"] & (I(IU))V \arrow[ul, "a"'] \arrow[d, "(1l)1"] \\
I(UV) & I(UV) \arrow[l, "="'] & (IU)V \arrow[l, "a"'] \arrow[r, "="] & (IU)V
\end{tikzcd}
$$

On top we have the pentagon for I, I, U, V. The squares are commutative, by naturality of a and axiom (4.2). We obtain, with $a^{-1} = \alpha$,

$$(*) \qquad \alpha \circ 1l = (1l)1 \circ \alpha \circ 1\alpha \colon I(I(UV)) \to (IU)V.$$

This yields the following chain of equalities between morphisms $I(I(UV)) \to UV$

$$
\begin{array}{rll}
 & l1 \circ \alpha \circ l & \text{naturality}\, l \\
= & l1 \circ l \circ 1\alpha & \text{naturality}\, l \\
= & l \circ 1(l1) \circ 1\alpha & \text{naturality}\, \alpha \\
= & l \circ a \circ (1l)1 \circ \alpha \circ 1\alpha & (*) \\
= & l \circ a \circ \alpha \circ 1l & \text{naturality}\, l \\
= & l \circ l &
\end{array}
$$

Since l is invertible, the claim about the left diagram follows. We obtain the right one by passing to the reversed category. □

(5.2) Proposition. *For each object V we have*

$$l_{I\otimes V} = \mathrm{id}_I \otimes l_V, \qquad r_{V\otimes I} = r_V \otimes \mathrm{id}_I.$$

PROOF. Naturality of l yields the following commutative diagram

$$
\begin{tikzcd}
I(IV) \arrow[r, "1_I l_V"] \arrow[d, "l_{IV}"] & IV \arrow[d, "l_V"] \\
IV \arrow[r, "l_V"] & V.
\end{tikzcd}
$$

Since l_V is an isomorphism, we obtain the first equality. Similarly for the second one. □

(5.3) Proposition. *The morphisms $r\colon I \otimes I \to I$ and $l\colon I \otimes I \to I$ are equal.*

PROOF. We have the following chain of morphisms $(II)I \to I$:

$$
\begin{array}{rll}
 & l \circ r1 & \text{axiom}\, (4.2) \\
= & l \circ 1l \circ a & \text{naturality}\, l \\
= & l \circ l \circ a & \text{theorem}\, (5.1) \\
= & l \circ l1. &
\end{array}
$$

This gives $r1 = l1$. This and twice the naturality of r implies

$$r \circ r = r \circ r1 = r \circ l1 = l \circ r\colon (II)I \to I.$$

Invertibility of r gives the desired result. □

(5.4) Theorem. *For any two morphisms $f, g\colon I \to I$ we have $f \circ g = g \circ f$ and $f \otimes g = g \otimes f = r_I^{-1} \circ (f \circ g) \circ r_I$.*

Proof. We have $f \otimes g = (f \otimes 1) \circ (1 \otimes g)$, since $\otimes$ is a functor. Naturality of r and l and (5.3) yields

$$f \otimes \mathrm{id} = r_I^{-1} \circ f \circ r_I, \quad \mathrm{id} \otimes g = r_I^{-1} \circ g \circ r_I.$$

This implies $f \otimes g = r^{-1} \circ (f \circ g) \circ r$. With $f \otimes g = (1 \otimes g) \circ (f \otimes 1)$ we obtain in a similar manner $f \otimes g = r^{-1} \circ (g \circ f) \circ r$. The remaining assertions follow easily. □

6. Tensor functors

(6.1) Definition. A *tensor functor* between tensor categories C and D consists of a functor $T\colon C \to D$ and natural isomorphisms $\varphi = \varphi_{A,B}\colon TA \otimes TB \to T(A \otimes B)$ and $i\colon T(I_C) \to I_D$ such that the following diagrams commute.

$$\begin{tikzcd}
(TA \otimes TB) \otimes TC \arrow[r, "a_{TA,TB,TC}"] \arrow[d, "\varphi \otimes 1"] & TA \otimes (TB \otimes TC) \arrow[d, "1 \otimes \varphi"] \\
T(A \otimes B) \otimes TC \arrow[d, "\varphi"] & TA \otimes T(B \otimes C) \arrow[d, "\varphi"] \\
T((A \otimes B) \otimes C) \arrow[r, "T(a_{A,B,C})"] & T(A \otimes (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
TI_C \otimes TV \arrow[r, "i \otimes 1"] \arrow[d, "\varphi"] & I_D \otimes TV \arrow[d, "l"] \\
T(I_C \otimes V) \arrow[r, "T(l)"] & TV
\end{tikzcd}
\qquad
\begin{tikzcd}
TV \otimes TI_C \arrow[r, "1 \otimes i"] \arrow[d, "\varphi"] & TV \otimes I_D \arrow[d, "r"] \\
T(V \otimes I_C) \arrow[r, "T(r)"] & TV
\end{tikzcd}$$

The tensor functor is called *strict* if i and $\varphi_{A,B}$ are always identities. A similar definition works for contravariant functors; we have to reverse the bottom arrow in the first diagram. By the way, we have not yet used that the φ are invertible. We talk about a *weak* tensor functor if we do not assume invertibility. ♡

(6.2) Definition. A natural *tensor transformation* $\Psi\colon F \to G$ between (weak) tensor functors (F, φ^F, i^F), $(G, \varphi^G, i^G)\colon C \to D$ is a natural transformation

$\Psi\colon F \to G$ such that the following diagrams commute:

$$\begin{tikzcd} FA \otimes FB \arrow[r, "\varphi^F"] \arrow[d, "\Psi_A \otimes \Psi_B"] & F(A \otimes B) \arrow[d, "\Psi_{A\otimes B}"] & F(I) \arrow[r, "\Psi_I"] \arrow[d, "i^F"] & G(I) \arrow[d, "i^G"] \\ GA \otimes GB \arrow[r, "\varphi^G"] & G(A \otimes B) & I \arrow[r, "="] & I. \end{tikzcd}$$

A natural *tensor isomorphism* is a natural tensor transformation which is a natural isomorphism. A *tensor equivalence* between tensor categories C, D is a tensor functor $F\colon C \to D$ such that there exists a tensor functor $F'\colon D \to C$ and natural tensor isomorphisms $F'F \simeq \mathrm{id}_C$ and $FF' \simeq \mathrm{id}_D$. ♡

Let $(T\colon C \to D, \varphi^T, i^T)$ and $(U\colon D \to E, \varphi^U, i^U)$ be tensor functors. We define their composition $(U \circ T, \varphi^{U\circ T}, i^{U\circ T})$ with the data

$$\varphi^{UT}_{A,B}\colon UT(A) \otimes UT(B) \xrightarrow{\varphi^U_{TA,TB}} U(TA \otimes TB) \xrightarrow{U(\varphi^T_{A,B})} UT(A \otimes B)$$

$$i^{UT}\colon UT(I_C) \xrightarrow{U(i^T)} U(I_D) \xrightarrow{i^U} I_E.$$

One verifies that this yields a tensor functor. Composition of tensor functors is associative.

7. Braided categories

Let $C = (C, \otimes, I, a, l, r)$ be a tensor category.

(7.1) Definition. A *screw* in C is a natural transformation in two variables

$$c_{VW} = c\colon V \otimes W \to W \otimes V.$$

A screw is called *coherent*, if the following four diagrams are commutative (notation again $VW = V \otimes W$).

$$\begin{tikzcd} (VU)W \arrow[d, "a"] & (UV)W \arrow[l, "c1"'] \arrow[r, "a"] & U(VW) \arrow[d, "c"] \\ V(UW) \arrow[r, "1c"] & V(WU) & (VW)U \arrow[l, "a"'] \end{tikzcd}$$

$$\begin{tikzcd} U(WV) \arrow[d, "a^{-1}"] & U(VW) \arrow[l, "1c"'] \arrow[r, "a^{-1}"] & (UV)W \arrow[d, "c"] \\ (UW)V \arrow[r, "c1"] & (WU)V & W(UV) \arrow[l, "a^{-1}"'] \end{tikzcd}$$

$$\begin{tikzcd} UI \arrow[r, "c"] \arrow[d, "r"] & IU \arrow[d, "l"] \\ U \arrow[r, "="] & U \end{tikzcd} \qquad \begin{tikzcd} IU \arrow[r, "c"] \arrow[d, "l"] & UI \arrow[d, "r"] \\ U \arrow[r, "="] & U \end{tikzcd}$$

In a strict category the first two diagrams can be expressed be the identities

$$c_{U,VW} = (1c_{U,W}) \circ (c_{U,V}1), \quad c_{UV,W} = (c_{U,W}1) \circ (1c_{V,W}).$$

A *braiding* for C is a coherent screw which is at the same time a natural isomorphism. In case of a braiding one can replace the second diagram by the first with c^{-1} in place of c. A tensor category together with a braiding is called a *braided category.* A screw is called *symmetry* if always $c_{W,V} \circ c_{V,W} = \mathrm{id}_{VW}$ holds. In this case the braided category is called *symmetric.* In the last two diagrams $c = c^{-1}$.♡

(7.2) Proposition. *If c is a braiding, then also c^{-1}.* □

(7.3) Proposition. *For a braiding the commutativity of the last two diagrams is a consequence of the other axioms.*

PROOF. We consider the following diagram.

$$\begin{tikzcd} & (MI)N \arrow[r, "a"] \arrow[dl, "c1"'] \arrow[d, "r1"] & M(IN) \arrow[dl, "1l"'] \arrow[dr, "c"] \arrow[d, phantom, "(1)"] & \\ (IM)N \arrow[r, "l1"] \arrow[dr, "a"'] & MN \arrow[r, "c"] & NM & (IN)M \arrow[l, "l1"'] \arrow[dl, "a"] \\ & I(MN) \arrow[r, "1c"] \arrow[u, "l"'] & I(NM) \arrow[u, "l"'] & \end{tikzcd}$$

We enumerate the areas of the diagram clockwise, beginning with (1). (6), (2), and (4) are commutative in any tensor category. (1) is the naturality of c. (3) is the naturality of l. The outer hexagon commutes because of coherence. Altogether we obtain from the triangle (5) the equality $c \circ (l1) \circ (c1) = c \circ (r1)$. Since c is invertible, (5) is commutative. This implies that the following diagram is commutative (apply (5) in case $N = I$).

$$\begin{tikzcd} IM \arrow[r, "r^{-1}"] & (IM)I \arrow[r, "l1"] & MI \arrow[r, "r"] & M \\ MI \arrow[r, "r^{-1}"] \arrow[u, "c"] & (MI)I \arrow[r, "r1"] \arrow[u, "c1"] & MI \arrow[r, "r"] \arrow[u, "="] & M \arrow[u, "="] \end{tikzcd}$$

The top row is, by naturality of r, equal to l; the bottom row is equal to r. This shows commutativity of the third diagram in the definition of coherence; similarly for the fourth one. □

Let C and D be tensor categories with braiding c and d. A tensor functor $T\colon C \to D$ is called a *braid functor*, if the diagram

$$\begin{tikzcd}
T(V \otimes W) \arrow[r, "T(c_{V,W})"] & T(W \otimes V) \\
TV \otimes TW \arrow[u, "\varphi"] \arrow[r, "d_{TV,TW}"] & TW \otimes TV \arrow[u, "\varphi"]
\end{tikzcd}$$

always commutes.

8. The Yang-Baxter relation. Twist

(8.1) Theorem. *For a coherent screw the following diagram is always commutative.*

(8.2)

$$\begin{tikzcd}
(MN)P \arrow[r, "a"] \arrow[d, "c1"] & M(NP) \arrow[d, "1c"] \\
(NM)P \arrow[d, "a"] & M(PN) \arrow[d, "a^{-1}"] \\
N(MP) \arrow[d, "1c"] & (MPN \arrow[d, "c1"] \\
N(PM) \arrow[d, "a^{-1}"] & (PM)N \arrow[d, "a"] \\
(NP)M \arrow[d, "c1"] & P(MN) \arrow[d, "1c"] \\
(PN)M \arrow[r, "a"] & P(NM)
\end{tikzcd}$$

In a strict category the commutativity can be written

$$(c_{NP}1) \circ (1c_{MP}) \circ (c_{MN}1) = (1c_{MN}) \circ (c_{MP}1) \circ (1c_{NP}).$$

This latter identity is called the hexagon property*.*

Proof. We insert the arrows $c\colon M(NP) \to (NP)P$ and $c\colon M(PN) \to (PN)M$. Then two coherence diagrams appear and a rhomb; the latter commutes by naturality of c. One could also insert the rhomb consisting of the morphisms $c\colon M(NP) \to (NP)M$ and $c\colon M(PN) \to (PN)M$. $\square$

If we consider the diagram (8.2) in the case $M = N = P$, then a morphism $c\colon MM \to MM$ which makes the diagram commutative is called a *Yang-Baxter morphism* for M. By (8.1), such morphisms are obtainable from a coherent screw. The relation for c which corresponds to the diagram (8.2) is called the *Yang-Baxter relation.*

(8.3) Theorem. *The identity of C together with a coherent screw is a tensor functor of C into the reversed category C^{rev}. A braiding is a tensor equivalence.*

Proof. The commutativity of the first diagram in section 6 comes from the diagram (8.2). The other diagrams are a consequence of the coherence. □

Let c and d be braidings of C. We consider them, together with the identity, as tensor functors $c, d\colon C \to C^{rev}$. A natural tensor transformation $\theta\colon c \to d$ consists of a natural transformation $\theta_X\colon X \to X$ such that

$$\begin{tikzcd} X \otimes Y \arrow[r, "c"] \arrow[d, "\theta_X \otimes \theta_Y"] & Y \otimes X \arrow[d, "\theta_{Y\otimes X}"] \\ X \otimes Y \arrow[r, "d"] & Y \otimes X \end{tikzcd}$$

is always commutative and such that $\theta_I = \mathrm{id}$. A natural tensor isomorphism $\theta\colon c^{-1} \to c$ is called *twist* for (C, c). In this case

(8.4) $$\theta_{X\otimes Y} = c_{Y,X}(\theta_Y \otimes \theta_X)c_{X,Y}.$$

By naturality of the twist, we obtain the following identities

$$\begin{aligned} \theta_{X\otimes Y} &= c_{Y,X}c_{X,Y}(\theta_X \otimes \theta_Y) \\ &= (\theta_X \otimes \theta_Y)c_{Y,X}c_{X,Y}. \end{aligned}$$

The term braiding is justified by the fact that a braiding immediately gives representations of the Artin braid groups. See chapter IV for this topic.

9. Universal braid categories

We extend a tensor category $C = (C, \otimes, I, a, r, l)$ in a universal manner into a braid category $Z(C)$.

(9.1) Objects of $Z(C)$.
Objects are pairs (V, c_V) consisting of an object V of C and a natural isomorphism (natural in the variable X)

$$c_{V,X}\colon V \otimes X \to X \otimes V,$$

such that the coherence diagram

$$\begin{tikzcd} (XV)Y \arrow[d, "a"] & (VX)Y \arrow[l, "c_{V,X}1"'] \arrow[r, "a"] & V(XY) \arrow[d, "c_{V,XY}"] \\ X(VY) \arrow[r, "1c_{V,Y}"] & X(YV) & (XY)V \arrow[l, "a"'] \end{tikzcd}$$

commutes for all X, Y. If the horizontal a are replaced by a^{-1}, we again obtain a commutative diagram, i. e. we can obtain $c_{V,XY}$ from $c_{V,X}$ and $c_{V,Y}$. In a strict category we have $c_{V,XY} = (1c_{V,Y}) \circ (c_{V,X}1)$. ♡

(9.2) Morphisms of $Z(C)$.
A morphism $f\colon (V, c_V) \to (W, c_W)$ is a morphism $f\colon V \to W$ in C such that the diagram

$$\begin{tikzcd}
VX \arrow[r, "c_{V,X}"] \arrow[d, "f1"] & XV \arrow[d, "1f"] \\
WX \arrow[r, "c_{W,X}"] & XW
\end{tikzcd}$$

commutes for all X. The identity of (V, c_V) is given by $\mathrm{id}(V)$. Composition in $Z(C)$ is defined by the composition in C. Thus f is an isomorphism in $Z(C)$ if and only if f is an isomorphism in C. ♡

(9.3) Neutral element of $Z(C)$.
The neutral element is (I, c_I), where c_I is defined as $r^{-1}l\colon IX \to X \to XI$. This yields an object in $Z(C)$. The naturality (9.2) of c_I follows from the naturality of l and r. The coherence diagram (9.1) takes in this case the following form:

$$\begin{tikzcd}
I(XY) \arrow[r, "l"] & XY & (XY)I \arrow[l, "r"'] \arrow[d, "a"] \\
(IX)Y \arrow[u, "a"] \arrow[r, "l1"] \arrow[d, "c_{I,X}1"] & XY \arrow[u, "="] & X(YI) \arrow[l, "1r"'] \\
(XI)Y \arrow[ur, "r1"'] \arrow[rr, "a"] & & X(IY) \arrow[ul, "1l"] \arrow[u, "1c_{I,Y}"']
\end{tikzcd}$$

Commutativityt follows from the axioms about r and l and from (5.1). ♡

(9.4) Tensor product in $Z(C)$.
We define $(V, c_V) \otimes (W, c_W) := (V \otimes W, c_{V \otimes W})$. This uses $c_{V \otimes W}$ which is defined by the following diagram:

$$\begin{tikzcd}
(VW)X \arrow[rrr, "c_{VW,X}"] \arrow[d, "a"] & & & X(VW) \\
V(WX) \arrow[r, "1c_{W,X}"] & V(XW) \arrow[r, "a^{-1}"] & (VX)W \arrow[r, "c_{V,X}1"] & (XV)W. \arrow[u, "a"]
\end{tikzcd}$$

In a strict category therefore $c_{VW,X} = (c_{V,X}1) \circ (1c_{W,X})$. One has to show that these data give an object f in $Z(C)$. By construction, c_{VW} is a natural isomorphism. It remains the coherence diagram; its structure is given (without associativities) in the following diagram.

$$
\begin{tikzcd}
\bullet \arrow[r, "11c_{W,Y}"] & \bullet \arrow[r, "c_{V,X}11"] \arrow[dr, "c_{V,XY}1"] & \bullet \arrow[d, "1c_{V,Y}1"] \\
A\bullet \arrow[u, "1c_{W,X}1"] \arrow[ur, "1c_{W,XY}"'] \arrow[rr, "c_{VW,XY}"] \arrow[d, "1c_{W,X}1"'] \arrow[dr, "c_{VW,X}1"] & (?) & \bullet B \\
\bullet \arrow[r, "c_{V,X}11"] & \bullet \arrow[r, "11c_{W,Y}"] \arrow[ur, "1c_{VW,Y}"'] & \bullet \arrow[u, "1c_{V,Y}1"']
\end{tikzcd}
$$

The bullets stand for permutations of tensor products of V, W, X, Y with certain of the five possible bracketings. In the top and bottom row we have the same permutations in the corners. The commutativity of (?) has to be shown. The other triangles are obtained by inserting the definitions (9.1) and (9.2). It has to be shown that the outer paths from A to B are the same morphisms. By naturality of a one can assume that we have the bracketing ()() in the corners. But then the horizontal top (bottom) arrows commute. In a strict category the diagram as such commutes. ♡

(9.5) The data a, r, l in $Z(C)$.
They are defined by the corresponding morphisms in C. One has to show that this yields morphisms in $Z(C)$.

In order to verify $l\colon (I, c_I) \otimes (V, c_V) \to (V, c_V)$ as morphism, we have to show that for each X the outer paths in the following diagram are equal.

$$
\begin{tikzcd}
(IV)X \arrow[d, "l1"] & \cong & I(VX) \arrow[r, "1c_{V,X}"] \arrow[d, "l"] & I(XV) \arrow[d, "l"] & \cong & (IX)V \arrow[r, "(r^{-1}\circ l)1"] \arrow[d, "l1"] & (XI)V \arrow[d, "r1"] & \cong & X(IV) \arrow[d, "1l"] \\
VX & = & VX \arrow[r, "c_{V,X}"] & XV & = & XV \arrow[r, "="] & XV & = & XV
\end{tikzcd}
$$

This uses only known properties of r and l. Similarly for r.

In order to see that a is a morphism, i. e. to see that

$$
\begin{tikzcd}
((UV)W)X \arrow[r, "c_{(UV)W,X}"] \arrow[d, "a1"] & X((UV)W) \arrow[d, "1a"] \\
(U(VW))X \arrow[r, "c_{U(VW),X}"] & X(U(VW))
\end{tikzcd}
$$

commutes, we insert definition (9.4) in the top and bottom row and obtain in each case

$$(c_{U,X}11) \circ (1c_{V,X}1) \circ (11c_{W,X}),$$

but with different bracketings between permutations of U, V, W, X. Naturality of a is used to show equality of the morphisms. ♡

(9.6) Proposition. *The data defined above make $Z(C)$ into a tensor category.*

Proof. Since a, r, and l in $Z(C)$ dare defined by the corresponding morphisms in C the defining diagrams (4.2) and (4.3) of a tensor category are again commutative. One has to show further that a, r, and l in $Z(C)$ are natural transformations. This follows from the corresponding statements in C. □

(9.7) Screw in $Z(C)$.
We associate to objects (V, c_V) and (W, c_W) in $Z(C)$ the morphism $c_{V,W}$ in C. We show that

$$c_{V,W}\colon (V, c_V) \otimes (W, c_W) \to (W, c_W) \otimes (V, c_V)$$

is a morphism in $Z(C)$. This means, according to (9.2): For each object X we have

$$1c_{V,W} \circ c_{VW,X} = c_{WV,X} \circ c_{V,W}1.$$

If we insert the definition (9.4), we are lead to a diagram of type (8.2), in which we have to show that one of the rhombs is commutative. This is done by inserting the other rhomb with (9.1). ♡

(9.8) Note. *The morphisms $c_{V,W}$ yield a braiding in $Z(C)$.*

Proof. We have to show that the screw is coherent and consists of isomorphisms. The coherence means that certain diagrams in $Z(C)$ commute. But the corresponding diagrams in C commute by (9.1) and (9.4). Since $c_{V,W}$ is an isomorphism in C, so it is in $Z(C)$. □

(9.9) Remark. We have a forgetful functor $\Pi\colon Z(C) \to C$, $(V, c_V) \mapsto V$. It is a strict tensor functor. ♡

10. Pairings and copairings

Let $C = (C, I, \otimes, a, r, l)$ be a tensor category. A *pairing* between objects (B, A) of C is a morphism $\varepsilon\colon B \otimes A \to I$ and a *copairing* is a morphism $\eta\colon I \to A \otimes B$. A pairing ε gives raise to a natural transformation in the variable Y

$$\Phi^\varepsilon\colon B \otimes (A \otimes Y) \cong (B \otimes A) \otimes Y \ rpf \varepsilon \otimes 1 \ I \otimes Y \ rpfl \ Y.$$

A copairing η yields a natural transformation

$$\Psi^\eta\colon Y \xrightarrow{l^{-1}} IY \xrightarrow{\eta 1} (AB)Y \cong A(BY).$$

A natural transformation $\Phi_Y\colon B(AY) \to Y$ is called *unital* if the following diagram commutes

$$\begin{tikzcd}
(B(AI))Y \arrow[r, "a"] \arrow[d, "\Phi_I 1_Y"] & B((AI)Y) \arrow[d, "1a"] \\
IY & B(A(IY)). \arrow[l, "\Phi_{IY}"']
\end{tikzcd}$$

A natural transformation $\Psi_Y\colon Y \to A(BY)$ is called unital if the following diagram commutes

$$\begin{tikzcd} IY \arrow[r, "\Psi_{IY}"] \arrow[d, "\Psi_I 1_Y"] & A(B(IY)) \\ (A(BI))Y \arrow[r, "a"] & A((BI)Y). \arrow[u, "1a"'] \end{tikzcd}$$

(10.1) Proposition. *The natural transformation Φ^ε associated to a pairing ε is unital. The natural transformation Ψ^η associated to a copairing η is unital.*

PROOF. We treat the case of a paring and display the relevant diagram for Φ^ε.

$$\begin{tikzcd} (B(AI))Y \arrow[d, "1a \circ a"] & ((BA)I)Y \arrow[l, "a1"'] \arrow[r, "\varepsilon 11"] \arrow[d, "a"] & (II)Y \arrow[r, "l1"] \arrow[d, "a"] & IY \arrow[d, "="] \\ B(A(IY)) & (BA)(IY) \arrow[l, "a"'] \arrow[r, "\varepsilon 11"] & I(IY) \arrow[r, "l"] & IY\ . \end{tikzcd}$$

The first square commutes by the pentagon axiom, the second by the naturality of a, and the third by an axiom about l. □

A natural transformation $\Phi\colon B(AY) \to Y$ yields a pairing

$$\varepsilon_\Phi\colon BA \xrightarrow{1r^{-1}} B(AI) \xrightarrow{\Phi} I$$

and a natural transformation $\Psi\colon Y \to A(BY)$ yields a copairing

$$\eta_\Psi\colon I \xrightarrow{\eta} A(BI) \xrightarrow{1l} AB.$$

(10.2) Proposition. *The assignments $\varepsilon \mapsto \Phi^\varepsilon$ and $\Phi \mapsto \varepsilon_\Phi$ are inverse bijections between parings $\varepsilon\colon BA \to I$ and unital natural transformations $\Phi\colon B(AY) \to Y$. The assignments $\eta \mapsto \Psi^\eta$ and $\Psi \mapsto \eta_\Psi$ are inverse bijections between copairings $\eta\colon I \to AB$ and natural transformations $\Psi\colon Y \to A(BY)$.*

PROOF. Let $\varepsilon\colon BA \to I$ be given. The following diagram shows $\varepsilon_{(\Phi^\varepsilon)} = \varepsilon$.

$$\begin{tikzcd} BA \arrow[r, "1r^{-1}"] \arrow[d, "="] & B(AI) \arrow[r, phantom, "\cong"] \arrow[d, "1r"] & (BA)I \arrow[r, "\varepsilon 1"] \arrow[d, "r"] & II \arrow[d, "r = l"] \\ BA \arrow[r, "="] & BA \arrow[r, phantom, "="] & BA \arrow[r, "\varepsilon"] & I \end{tikzcd}$$

Let $\Phi\colon B(AY) \to Y$ be a unital transformation. The following diagram shows

$\Phi^{(\varepsilon_\Phi)} = \Phi$.

$$\begin{tikzcd}
B(AY) & (BA)Y \arrow[l, "a"'] \arrow[r, "\varepsilon\Phi 1"] \arrow[d, "(1r^{-1})1"] & IY \\
& (B(AI))Y \arrow[r, "\Phi_I 1"] \arrow[d, "(1a)\circ a"] & IY \arrow[u, "="'] \arrow[d, "="] \\
& B(A(IY)) \arrow[r, "\Phi_{IY}"] \arrow[d, "11l"] & IY \arrow[d, "l"] \\
& B(AY) \arrow[r, "\Phi_Y"] & Y.
\end{tikzcd}$$

The square in the middle uses that Φ is unital. □

A pairing $\varepsilon\colon BA \to Y$ yields a natural transformation $C(X, AY) \to C(BX, Y)$. It maps $f\colon X \to AY$ to

$$BX \xrightarrow{1f} B(AY) \cong (BA)Y \xrightarrow{\varepsilon 1} IY \xrightarrow{l} Y.$$

Given such a natural transformation, the pairing is the image of 1_A under

$$C(A, A) \cong C(A, IA) \to C(BA, I).$$

A copairing yields a natural transformation $C(BX, Y) \to C(X, AY)$. It maps $f\colon BX \to Y$ to

$$X \xrightarrow{l^{-1}} IX \xrightarrow{\eta 1} (AB)X \cong A(BX) \xrightarrow{1f} AY.$$

The copairing is the image of 1_B under $C(B, B) \cong C(BI, B) \to C(I, AB)$.

A pairing $\varepsilon\colon BA \to I$ and a copairing $\eta\colon I \to AB$ are called a *left duality* for A if the following two morphisms are the identity

(10.3)
$$A \cong IA \xrightarrow{\eta 1} (AB)A \cong A(BA) \xrightarrow{1\eta} AI \cong A$$
$$B \cong BI \xrightarrow{1\eta} B(AB) \cong (BA)B \xrightarrow{\eta 1} IB \cong B.$$

We use the symbol

$$(\varepsilon, \eta)\colon B \dashv A$$

for such a left duality and call B a *dual object* of A.

We have defined in section 3 the notion of an adjunction $(\Phi, \Psi)\colon B\otimes ? \dashv A\otimes ?$.

(10.4) Proposition. *Suppose ε (η) is a pairing (copairing) of (B, A). Then $\Phi^\varepsilon, \Psi^\eta$ is an adjunction $B\otimes ? \to A\otimes ?$ if and only if (ε, η) is a left duality $A \dashv B$.*

Proof. An adjunction is characterized by the fact that the top row in the following diagram is the identity

$$\begin{tikzcd}
AY \arrow[rr, "\Psi^{\eta}_{AY}"] & & A(B(AY)) \arrow[rr, "1\Psi^{\varepsilon}_{Y}"] & & AY \\
I(AY) \arrow[u, "l"] \arrow[r, "\eta 1_{AY}"] & (AB)(AY) \arrow[ur, "a"] & & A((BA)Y) \arrow[ul, "1a"'] \arrow[r, "1\varepsilon 1"] & A(IY) \arrow[u, "1l"'] \\
(IA)Y \arrow[u, "a"] \arrow[r, "\eta"] & ((AB)A)Y \arrow[u, "a"] \arrow[rr, "a1"] & & A(BA))Y \arrow[u, "a"] \arrow[r, "1\varepsilon 1"] & (AI)Y).
\end{tikzcd}$$

(1) and (2) define the morphisms in the top row. (3) is a pentagon. The first map (call it α) in (10.3), tensored with Y, is the boundary path down-right-up in the diagram. Thus the top row is always the identity if an only if $\alpha 1_Y$ is always the identity.

Similarly for the second map in (10.3) and the second condition for an adjunction. □

(10.5) Theorem. *Let $(\varepsilon, \eta)\colon B \to A$ and $(\varepsilon', \eta')\colon B' \to A'$ be dualities. Let $f\colon A \to A'$ and $g\colon B' \to B$ be morphisms. The following equalities among morphisms are mutually equivalent:*

(1) $g\colon B' \cong B'I \xrightarrow{1\eta} B'(AB) \xrightarrow{1f1} B'(A'B) \cong (B'A')B \xrightarrow{\varepsilon' 1} IB \cong B$

(2) $f\colon A \cong IA \xrightarrow{\eta' 1} (A'B')A \xrightarrow{1g1} (A'B)A \cong A'(BA) \xrightarrow{1\varepsilon} A'I \cong A'$

(3) $\varepsilon' \circ 1f = \varepsilon \circ g1\colon B'A \to I$

(4) $1g \circ \eta' = f1 \circ \eta\colon I \to A'B.$

If f is the identity, then also g.

Proof. (3) $\Rightarrow$ (1). We consider the following diagram.

$$\begin{tikzcd}
B' \arrow[r] \arrow[d, "g"] & B'I \arrow[r, "1\eta"] & B'(AB) \arrow[r, phantom, "\cong"] & (B'A)B \arrow[r, "(1f)1"] \arrow[d, "(g1)1"] & (B'A)B \arrow[d, "\varepsilon' 1"] & \\
B \arrow[r] & BI \arrow[r, "1\eta"] & B(AB) \arrow[r, phantom, "\cong"] & (BA)B \arrow[r, "(\varepsilon 1)1"] & IB \arrow[r] & B
\end{tikzcd}$$

The right hand square uses (3). The bottom row is the identity, since (ε, η) is a duality.

(1) $\Rightarrow$ (3). For simplicity we consider a strict category. Then (1) is the commutative diagram

$$\begin{tikzcd}
B' \arrow[r, "1\eta"] \arrow[d, "g"] & B'AB \arrow[d, "1f1"] \\
B & B'A'B. \arrow[l, "\varepsilon' 1"]
\end{tikzcd}$$

We tensor it with A and augment it.

$$\begin{tikzcd}
B'A \arrow[r, "1\eta 1"] \arrow[d, "g1"] & B'ABA \arrow[r, "11\varepsilon"] \arrow[d, "1f11"] & B'A \arrow[d, "1f"] \\
BA \arrow[dr, "\varepsilon"'] & B'A'BA \arrow[l, "\varepsilon'"'] \arrow[r, "11\varepsilon"] & B'A' \arrow[dl, "\varepsilon'"] \\
 & B &
\end{tikzcd}$$

The top row is the identity, since (ε, η) is a duality. The outer part of the diagram is (3).

(2) $\Leftrightarrow$ (4) is proved similarly.

(1) $\Leftrightarrow$ (2). Again we work in a strict category. Suppose (1) holds. We tensor with A' from the left and with A from the right. This leads to the next diagram

$$\begin{tikzcd}
A \arrow[r, "\eta' 1"] & A'B'A' \arrow[r, "1\eta 1"] \arrow[d, "1g1"] & A'B'ABA \arrow[d, "11f11"] \\
A' & A'BA \arrow[l, "1\varepsilon"'] & A'B'A'BA. \arrow[l, "1\varepsilon' 1"']
\end{tikzcd}$$

Now we use the properties of a duality for (ε, η) and (ε', η') and conclude that the outer path from A to A' is f. □

The assignment $f \mapsto g$ of the previous theorem yields a bijection

(10.6) $$\alpha_\eta^{\varepsilon'} \colon C(A, A') \to C(B', B).$$

We use this notation, since g is obtained from f by using ε' and η. The inverse bijection uses ε and η'.

(10.7) Proposition. *The assignment* (10.6) *is contravariant functorial: If* $(\varepsilon'', \eta'') \colon B'' \dashv A''$ *is a further duality, then* $\alpha_\eta^{\varepsilon''}(f_2 \circ f_1) = \alpha_\eta^{\varepsilon'}(f_1) \circ \alpha_{\eta'}^{\varepsilon''}(f_2)$.

Proof. We apply (10.5.3) to the following diagram.

$$\begin{tikzcd}
B''A \arrow[r, "g_2 1"] \arrow[d, "1f_1"] & B'A \arrow[r, "g_1 1"] \arrow[d, "1f_1"] & BA \arrow[d, "\varepsilon"] \\
B''A' \arrow[r, "g_2 1"] \arrow[d, "1f_2"] & B'A' \arrow[r, "\varepsilon'"] \arrow[d, "\varepsilon'"] & I \arrow[d, "="] \\
B''A'' \arrow[r, "\varepsilon''"] & I \arrow[r, "="] & I.
\end{tikzcd}$$

The uniqueness statement of (10.5.3) says that $g_1 g_2$ belongs to $f_2 f_1$. □

A dual object of A is determined up to unique isomorphism. This is a consequence of the next proposition.

(10.8) Proposition. *Let* (ε, η): $B \dashv A$ *and* (ε', η'): $B' \dashv A$ *be dualities. Then there exists a unique isomorphism* $g : B \to B'$ *such that* $\varepsilon' \circ g1 = \varepsilon$ *and* $1g \circ \eta = \eta'$.

Proof. The conditions are just (3) and (4) of (10.5) applied to the identity of A. We have $g = \alpha^{\varepsilon}_{\eta'}(10._A)$. □

We call g in (10.8) a *comparison morphism*. If two objects have a dual object, then their tensor product has a dual object. More precisely:

(10.9) Proposition. *Let* (ε, η): $B \to A$ *and* (ε', η'): $B' \to A'$ *be dualities. Then the following pair of morphisms is a duality* $BB' \dashv A'A$:

$$(BB')(A'A) \cong B((B'A)A) \xrightarrow{1\varepsilon'1} B(IA) \cong BA \xrightarrow{\varepsilon}$$

$$I \xrightarrow{\eta} AB \cong A(IB) \xrightarrow{1\eta'1} A((A'B)B) \cong (AA')(B'B).$$

Proof. We obtain from (ε, η) and (ε', η') the following natural isomorphism

$$\begin{tikzcd}
C(X,(A'A)Y) \arrow[r, phantom, "\cong"] & C(X, A'(AY)) \arrow[d] \\
& C(B'X, AY) \arrow[d] \\
C((BB')X, Y) \arrow[r, phantom, "\cong"] & C(B(B'X), Y).
\end{tikzcd}$$

One verifies that the pairing of the proposition belongs to this isomorphism. Similarly for the inverse isomorphisms and the copairing. □

Tensor functor preserve dualities. We leave the verification of the following claim to the reader.

(10.10) Proposition. *Let* (T, φ, i) *be a tensor functor and* (ε, η): $B \dashv A$ *a duality. The morphisms* $(\tilde{\varepsilon}, \tilde{\eta})$: $TB \dashv TA$

$$\tilde{\varepsilon}\colon TB \otimes TA \xrightarrow{\varphi} T(B \otimes A) \xrightarrow{T(\varepsilon)} T(I_C) \xrightarrow{i} I_D$$

$$\tilde{\eta}\colon I_D \xrightarrow{i^{-1}} T(I_C) \xrightarrow{T(\eta)} T(A \otimes B) \xrightarrow{\varphi^{-1}} TA \otimes TB$$

are a duality. □

There is a parallel theory of right duality. It suffices to consider the reversed category.

In the next section we study the situation where each object has a given duality.

11. Duality

Let C be a tensor category. A *left duality* in C assigns to each object V another object V^* and a left duality

$$b_V \colon I \to V \otimes V^*, \qquad d_V \colon V^* \otimes V \to I.$$

A *right duality* in C assigns to each object V in C another object $V^\#$ and a right duality

$$a_V \colon I \to V^\# \otimes V, \qquad c_V \colon V \otimes V^\# \to I.$$

From a left duality we obtain a contravariant functor $* \colon C \to C$. It maps the object V to V^* and assigns to $f \colon V \to W$ the morphism $f^* \colon W^* \to V^*$ which is uniquely determined by the commutative diagram

$$\begin{tikzcd} W^*V \arrow[r, "f^*1"] \arrow[d, "1f"] & V^*V \arrow[d, "d_V"] \\ W^*W \arrow[r, "d_W"] & I. \end{tikzcd}$$

We also call this functor a *duality*. Similarly, a right duality yields a contravariant duality functor $\# \colon C \to C$.

We discuss the compatibility of duality and tensor products. In a strict category we define morphisms

$$\lambda_{V,W} \colon W^* \otimes V^* \to (V \otimes W)^*, \quad \mu_{V,W} \colon (V \otimes W)^* \to W^* \otimes V^*$$

by the commutative diagrams

$$\begin{tikzcd} W^*V^* \arrow[r, "1b_{VW}"] \arrow[d, "\lambda_{V,W}"] & W^*V^*(VW)(VW)^* \arrow[d, "1d_V11"] \\ (VW)^* & W^*W(VW)^* \arrow[l, "d_W1"'] \end{tikzcd}$$

$$\begin{tikzcd} (VW)^* \arrow[r, "1b_V"] \arrow[d, "\mu_{V,W}"] & (VW)^*VV^* \arrow[d, "11b_W1"] \\ W^*V^* & (VW)^*VWW^*V^*. \arrow[l, "d_{VW}11"'] \end{tikzcd}$$

In the general case we can say that $\lambda_{V,W}$ is the morphism which makes the following diagram commutative

$$\begin{tikzcd} (W^*V^*)(VW) \arrow[r, phantom, "\cong"] \arrow[d, "\lambda_{V,W}1"] & W^*(V^*(VW)) \arrow[r, phantom, "\cong"] & W^*((V^*V)W) \arrow[d, "1d_V"] \\ (VW)^*(VW) \arrow[d, "d_{VW}"] & & W^*(IW) \arrow[d, "1l"] \\ I & & W^*V. \arrow[ll, "d_W"'] \end{tikzcd}$$

We have already seen in the previous section that:

(1) Two pairings induce a pairing for the tensor product.
(2) Two dualities are related by a uniquely determined comparison morphism.

(11.1) Theorem. *The morphisms $\lambda_{V,W}$ and $\mu_{V;W}$ are inverse to each other. They constitute a natural transformation.*

Proof. The uniqueness of the comparison morphism shows that the λ and μ are inverse to each other. The naturality follows from the naturality of the comparison morphisms. □

We next show that the $\lambda_{V,W}$ are part of a contravariant tensor functor.

(11.2) Lemma. *The morphisms $r = l$: $II \to I$ and $r^{-1} = l^{-1}$: $I \to II$ are a duality.* □

As a consequence, we have a comparison morphism i: $I^* \to I$ between the duality of the lemma and the given duality. The commutative diagram

$$\begin{tikzcd}
I \arrow[r, "b_I"] \arrow[d, "="] & II^* \arrow[r, "l"] \arrow[d, "1i"] & I^* \arrow[d, "i"] \\
I \arrow[r, "r^{-1}"] & II \arrow[r, "l"] & I
\end{tikzcd}$$

shows that i equals lb_I: $I \to II^* \to I^*$. Similarly, the morphism $d_I r^{-1}$: $I^* \to I^*I \to I$ is inverse to i.

(11.3) Proposition. *The diagrams*

$$\begin{tikzcd}
V^*I^* \arrow[d, "\lambda_{I,V}"] & V^*I \arrow[l, "1i"'] \arrow[d, "r"] \\
(IV)^* & V^* \arrow[l, "l^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
I^*V^* \arrow[d, "\lambda_{V,I}"] & IV^* \arrow[l, "i1"'] \arrow[d, "l"] \\
(VI)^* & V^* \arrow[l, "r^*"']
\end{tikzcd}$$

are commutative □

(11.4) Proposition. *The diagram*

$$\begin{tikzcd}
(U^*V^*)W^* \arrow[r, "a"] \arrow[d, "\lambda_{U,V}1"] & U^*(V^*W^*) \arrow[d, "1\lambda_{V,W}"] \\
(VU)^*W^* \arrow[d, "\lambda_{VU,W}"] & U^*(WV)^* \arrow[d, "\lambda_{U,WV}"] \\
(W(VU))^* \arrow[r, "a^*"] & ((WV)U)^*
\end{tikzcd}$$

is commutative. □

Propositions (11.3) and (11.4) express the fact that $(*, \lambda, i)$ is a contravariant tensor functor. The uniqueness and the naturality of the comparison morphisms show that a left duality is unique up to natural isomorphism. The natural isomorphism is a tensor isomorphism.

The construction of Proposition (10.10) shows: Suppose $(F\colon C \to D, \varphi, i)$ is a tensor functor and C has a left duality. Then D has an induced left duality by (10.10). If D has a given left duality there exist comparison morphisms $\delta_V\colon F(V)^* \to F(V^*)$ which constitute a natural isomorphism

$$\delta\colon * \circ F \to F \circ *.$$

In this sense, a tensor functor is compatible with dualities.

The double dual of V is in general not the original object V. The inverse of a left dual is rather given by a right dual. We explain this and use the following data.

left dual	right dual
V^*	$V^{\#}$
$b\colon I \to VV^*$	$a\colon I \to V^{\#}V$
$d\colon V^*V \to I$	$c\colon VV^{\#} \to I$

With these data we define several morphisms (written in a strict category)

$$\begin{array}{ll}
\varphi_V^{\#*}\colon & V \xrightarrow{1b} VV^{\#}V^{\#*} \xrightarrow{c1} V^{\#*} \\
\psi_V^{\#*}\colon & V^{\#*} \xrightarrow{1a} V^{\#*}V^{\#} \xrightarrow{d1} V \\
\rho_V^{*\#}\colon & V \xrightarrow{a1} V^{*\#}V^*V \xrightarrow{1d} V^{*\#} \\
\sigma_V^{*\#}\colon & V^{*\#} \xrightarrow{b1} VV^*V^{*\#} \xrightarrow{1c} V.
\end{array}$$

(11.5) Theorem. *The pairs $\varphi_V^{\#*}, \psi_V^{\#*}$ and $\rho_V^{*\#}, \sigma_V^{+\#}$ are inverse pairs. The functors $\#*$ and $*\#$ are tensor functors with the structural data which result from the composition of the functors $*$ and $\#$. The morphisms $\varphi, \psi, \rho, \sigma$ yields tensor isomorphisms between these functors and the identity functor.*

Proof. The following diagram is used to show $\psi^{\#*}\varphi^{\#*} = \mathrm{id}$.

$$\begin{tikzcd}
V \arrow[r, "1b_{V^\#}"] \arrow[d, "1a_V"] & VV^{\#}V^{\#*} \arrow[r, "c_V1"] \arrow[d, "111a_V"] & V^{\#*} \arrow[d, "1a_V"] \\
VV^{\#}V \arrow[r, "1b_{V^\#}11"] \arrow[dr, "1"] & VV^{\#}V^{\#*}V^{\#}V \arrow[r, "c_V111"] \arrow[d, "11d_{V^\#}1"] & V^{\#*}V^{\#V} \arrow[d, "d_{V^\#}1"] \\
 & VV^{\#}V \arrow[r, "c_V1"] & V.
\end{tikzcd}$$

The left-bottom composition $V \to V$ is the identity by definition of right duality. $\square$

(11.6) Remark. Since $\#*$ and $*\#$ are tensor functors there exists a canonical isomorphism between $(*\#)*$ and $*(*\#)$. Make this explicit. Similarly for $(\#*)*$ and $*(\#*)$. $\heartsuit$

12. Duality and braiding

Let C be a tensor category with a given left duality $(*, b, d)$. Suppose C is braided with braiding morphisms $z_{V,W} \colon V \otimes W \to W \otimes V$.

(12.1) Theorem. *The morphisms*

$$a_V \colon I \xrightarrow{b_V} VV^* \xrightarrow{z^{-1}} V^*V$$
$$c_V \colon VV^* \xrightarrow{z} V^*V \xrightarrow{d_V} I$$

form a right duality for C. We obtain another right duality if we replace the braiding z by the braiding z^{-1}.

Proof. We have the following commutative diagram, written in a strict category.

$$\begin{tikzcd}
A \arrow[r, "1b_A"] \arrow[d, "="] & AAA^* \arrow[r, "(z_{AA^*}1) \circ (1z^{-1}_{A^*A})"] \arrow[d, "z_{A,AA^*}"] & AA^*A \arrow[r, "d_A1"] \arrow[d, "z_{A^*A,A}"] & A \arrow[d, "="] \\
A \arrow[r, "b_A1"] & AA^*A \arrow[r, "="] & AA^*A \arrow[r, "1d_A"] & A.
\end{tikzcd}$$

The first and third square commute by naturality of z. The square in the middle commutes by the Yang-Baxter relation; in order to see this, replace z_{A,AA^*} and $z_{A^*A,A}$ by the coherence diagrams. $\square$

The next result shows that we can construct from a braiding and a duality the inverse braiding.

(12.2) Theorem. *The morphism z^-_{VU}, defined by the next diagram, is an inverse of $z_{U,V}$.*

$$\begin{tikzcd}
VU \cong I(VU) \arrow[r, "b11"] \arrow[d, "z^-_{VU}"] & (UU^*)(VU) \cong U(U^*V)U \arrow[d, "1z_{U^*,V}1"] \\
UV \cong (UV)I & (UV)(U^*U) \cong U(VU^*)U \arrow[l, "11d"']
\end{tikzcd}$$

Proof. Consider the following diagram

$$\begin{tikzcd}
VU \arrow[r, "b11"] \arrow[d, "z1"] & (UU^*)VU \arrow[r, "1z1"] \arrow[d, "z1"] & UVU^*U \arrow[r, "11d"] \arrow[dl, "z11"] & UV \arrow[d, "z1"] \\
VU \arrow[r, "1b1"] & V(UU^*)U \arrow[rr, "11d"] & & VU
\end{tikzcd}$$

(1) and (2) are coherence diagrams. (3) and (4) are commutative by naturality of z. The bottom row is the identity. Similarly for the other composition. □

(12.3) Corollary. *A coherent screw in a category with left duality is a braiding.* □

(12.4) Theorem. *The following identity holds*

$$z_{U^*,V} = (d_U \otimes 1_{VU^*}) \circ (1_{U^*} \otimes z_{U,V}^{-1} \otimes 1_{U^*}) \circ (1_{U^*V} \otimes b_U).$$

Proof. This is proved by using a diagram as in the proof of (12.2). □

(12.5) Theorem. *The following two morphisms are equal*

$$V^*(UV) \xrightarrow{1z^{-1}} V^*(VU) \cong (V^*V)U \xrightarrow{d1} IU \cong U$$

$$V^*(UV) \cong (V^*U)V \xrightarrow{z1} (UV^*)V \cong U(V^*V) \xrightarrow{1d} UI \cong U.$$

Proof. Insert (12.4) into the second morphism. □

(12.6) Theorem. *The following diagram commutes*

$$\begin{tikzcd} MN \arrow[r, "b11"] \arrow[d, "z"] & NN^*MN \arrow[r, "1b111"] & NMM^*N^*MN \arrow[d, "11z11"] \\ NM & NNN^*N \arrow[l, "11d"'] & NMN^*M^*MN. \arrow[l, "111d1"'] \end{tikzcd}$$

13. Ribbon categories

We have already defined the notion of a twist δ for tensor category C, see section 8. If we have a tensor category with duality we require in addition the compatibility

(13.1) $$\delta_{V^*} = \delta_V^*.$$

A *ribbon category* is defined to be a tensor category with braiding, left duality and twist.

(13.2) Theorem. *In a ribbon category the following morphisms define a right duality*

$$a_V \colon I \xrightarrow{b_V} VV^* \xrightarrow{z_{V,V^*}} V^*V \xrightarrow{1\delta_V} V^*V$$

$$c_V \colon VV^* \xrightarrow{\delta_V 1} VV^* \xrightarrow{z_{V,V^*}} V^*V \xrightarrow{d_V} I.$$

Proof. From the previous section we have the right duality of Theorem (12.1); call its morphisms now $\tilde{a}$ and $\tilde{c}$. We show that the following diagrams commute

$$\begin{tikzcd} I \arrow[r, "="] \arrow[d, "a_V"] & I \arrow[d, "\tilde{a}_V"] \\ V^*V \arrow[r, "\delta_V^*"] & V^*V \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "="] & I \\ VV^* \arrow[u, "c_V"] \arrow[r, "1\delta_V^*"] & VV^*. \arrow[u, "\tilde{c}_V"'] \end{tikzcd}$$

The fact that $\tilde{a}, \tilde{c}$ is a right duality then implies that a, c is a right duality. For the first diagram we compute

$$(\delta_{V^*} \otimes \delta_V) z_{V,V^*} b_V = z_{V^*,V}^{-1} \delta_{V \otimes V^*} b_V = z_{V^*,V}^{-1} b_V \delta_I = z_{V^*,V}^{-1} b_V.$$

For the second diagram we consider the following diagram which uses (13.1)

$$\begin{tikzcd} VV^* \arrow[d, "z_{V,V^*}"] & VV^* \arrow[l, "\delta_V 1"'] \arrow[r, "1d_V^*"] \arrow[d, "z_{V,V^*}"] & VV^* \arrow[d, "z_{V,V^*}"] \\ V^*V \arrow[r, "1\delta_V"] \arrow[d, "d_V"] & V^*V & V^*V \arrow[l, "\delta_V^* 1"'] \arrow[d, "d_V"] \\ I \arrow[rr, "="] & & I \end{tikzcd}$$

The top squares commute by naturality. □

The fact that a, c is a right duality gives the following result:

(13.3) Theorem. *For each object V of a ribbon category*

$$\delta_V^{-2} = (d_V z_{V,V^*} \otimes 1)(1 \otimes z_{V,V^*} b_V).$$

Proof. We compute

$$\begin{aligned} \mathrm{id} &= (c_V \otimes 1)(1 \otimes a_V) \\ &= (d_V z_{V,V^*}(\delta_V \otimes 1) \otimes 1)((1 \otimes (1 \otimes \delta_V) z_{V,V^*} b_V) \\ &= \delta_V (d_V z_{V,V^*} \otimes 1)(1 \otimes z_{V,V^*} b_V) \delta_V. \end{aligned}$$

The first two equalities are definitions. The third one holds by naturality of the tensor product. □

By naturality of z, the result of the previous Theorem gives the identity

$$\delta_V^{-2} = (d_V \otimes 1)(1_{V^*} \otimes z_{V,V}^{-1})(z_{V,V^*} b_V \otimes 1_V).$$

We can use the right duality a, c in order to associate to $f \colon V \to W$ a dual morphism $f^\# \colon W^\# \to V^\#$.

(13.4) Theorem. *The left dual f^* and the right dual $f^\#$ coincide.*

PROOF. The morphism $f^\#$ is characterized by $(f^\# 1) \circ a_V = (1f) \circ a_V$. The morphism f^* by a similar equality with the b-morphisms. If we apply $(\delta 1) \circ z$ and $(1\delta) \circ z$ to this characterization of f^*, we obtain $(f^*1 \circ a_V = 1f \circ a_V$. We conclude $f^\# = f^*$. $\square$

(13.5) Theorem. *The morphisms*

$$I \xrightarrow{b_V} VV^* \xrightarrow{f1} VV^* \xrightarrow{c_V} I$$

$$I \xrightarrow{a_V} V^*V \xrightarrow{1f} V^*V \xrightarrow{d_V} I$$

coincide for each endomorphism $f\colon V \to V$.

PROOF. We insert the definition of a_V and c_V and use the naturality of z and δ. $\square$

We call the morphism of the previous Theorem

$$\mathrm{Tr}(f) := c_V \circ (f \otimes 1) \circ b_V \in \mathrm{End}(I)$$

the *trace* (or *quantum trace*) of the endomorphism f.

(13.6) Theorem. *The quantum trace has the following properties:*

(1) $\mathrm{Tr}(fg) = \mathrm{Tr}(gf)$
(2) $\mathrm{Tr}(f) = \mathrm{Tr}(f^*)$
(3) $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f)\mathrm{Tr}(g)$.

14. Tensor module categories

Let $\mathfrak{A} = (\mathfrak{A}, \otimes, I, a, r, l)$ be a tensor category. In this notation, $\otimes$ is the tensor product functor, I the neutral object, $a\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ the associativity isomorphism, $l\colon I \otimes X \to X$ the left unit and $r\colon X \otimes I \to X$ the right unit isomorphism. For the axioms of a tensor category (pentagon, triangle) see section 4. Let $\mathfrak{B}$ be another category.

We use functors $*\colon \mathfrak{B} \times \mathfrak{A} \to \mathfrak{B}$ in two variables and denote them by $(Y, X) \mapsto Y * X$ and $(f, g) \mapsto f * g$ on objects and morphisms. We shall also use the $\otimes$-sign instead of $*$ and call the functor $*$ a tensor product.

(14.1) Definition. A *right action* $(*, \alpha, \rho)$ of $\mathfrak{A}$ on $\mathfrak{B}$ consists of a functor

$$*\colon \mathfrak{B} \times \mathfrak{A} \to \mathfrak{B},$$

a natural isomorphism in three variables

$$\alpha = \alpha_{U,V,W}\colon (U * V) * W \to U * (V \otimes W), \qquad U \in \mathrm{Ob}(\mathfrak{B}), V, W \in \mathrm{Ob}(\mathfrak{A}),$$

and a natural isomorphism

$$\rho = \rho_X \colon X * I \to X, \qquad X \in \mathrm{Ob}(\mathfrak{B}),$$

such that the following axioms hold (pentagon (14.2), triangle (14.3)):
(14.2) Given four objects $U \in \mathrm{Ob}(\mathfrak{B})$,$V, W, X \in \mathrm{Ob}(\mathfrak{A})$, the diagram

$$\begin{tikzcd}
((U * V) * W) * X \arrow[r, "\alpha"] \arrow[d, "\alpha \otimes \mathrm{id}"] & (U * V) * (W \otimes X) \arrow[r, "\alpha"] & U * (V \otimes (W \otimes X)) \\
(U * (V \otimes W)) * X \arrow[rr, "\alpha"] & & U * ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id} \otimes a"']
\end{tikzcd}$$

is commutative.

(14.3) Given two objects $U \in \mathrm{Ob}(\mathfrak{B})$, $V \in \mathrm{Ob}(\mathfrak{A})$, the diagram

$$\begin{tikzcd}
(U * I) * V \arrow[r, "\alpha"] \arrow[d, "\rho_U * \mathrm{id}"] & U * (I \otimes V) \arrow[d, "\mathrm{id} \otimes l_V"] \\
U * V \arrow[r, "\mathrm{id}"] & U * V
\end{tikzcd}$$

is commutative.
The action is called *strict*, if $\mathfrak{A}$ is a strict tensor category and α and ρ are the identity.

A category $\mathfrak{B}$ together with a right action of $\mathfrak{A}$ on $\mathfrak{B}$ is called a right *tensor module category* over $\mathfrak{A}$, or *right* $\mathfrak{A}$*-module* for short. The tensor module is called strict, if the action is strict. ♡

There is, of course, an analogous definition of a left action. But note that in this case α changes brackets from right to left. Either one uses the same convention for a or one has to work with a^{-1}. This remark is relevant for the next example.

(14.4) Example. Let $\mathfrak{A}^{rev} = (\mathfrak{A}, I, \otimes', a', r', l')$ denote the reversed tensor category of $\mathfrak{A}$, see (4.3). If $(*, \alpha, \rho)$ is a right action of $\mathfrak{A}$ on $\mathfrak{B}$, the assignments

$$\begin{aligned} U' *' X &= X * U \\ \alpha'_{U,V,X} &= \alpha_{X,V,U} \\ \lambda'_X &= \rho_X \end{aligned}$$

yield a left action $(*', \alpha', \lambda')$ of $\mathfrak{A}^{rev}$ on $\mathfrak{B}$. ♡

(14.5) Example. A tensor category acts on itself by $* = \otimes$, $\alpha = a$, and $\rho = r$.♡

(14.6) Example. Suppose $\mathcal{C}$ and $\mathcal{D}$ are (small) categories. Let $\mathfrak{A} = [\mathcal{D}, \mathcal{D}]$ and $\mathfrak{B} = [\mathcal{D}, \mathcal{C}]$ denote the functor categories. Composition of functors defines a map

$$* \colon \mathrm{Ob}[\mathcal{D}, \mathcal{C}] \times \mathrm{Ob}[\mathcal{D}, \mathcal{D}] \to \mathrm{Ob}[\mathcal{D}, \mathcal{C}].$$

Let $\varphi\colon F \to F'$ and $\psi\colon G \to G'$ be morphisms in $\mathfrak{B}$ and $\mathfrak{A}$, respectively. Then $\varphi * \psi\colon F * G \to F' * G'$ is the natural transformation

$$FG(X) \xrightarrow{F(\varphi_X)} FG'(X) \xrightarrow{\psi_{G'(X)}} F'G'(X).$$

The case $\mathcal{C} = \mathcal{D}$ of this construction makes $\mathfrak{A}$ into a strict tensor category and then $*$ defines a strict right action of $\mathfrak{A}$ on $\mathfrak{B}$. ♡

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be tensor categories and $T = (T, \varphi, i)\colon \mathfrak{A} \to \mathfrak{A}'$ a tensor functor, consisting of a functor $T\colon \mathfrak{A} \to \mathfrak{A}'$, a natural isomorphism $\varphi_{A,B}\colon TA \otimes TB \to T(A \otimes B)$, and an isomorphism $i\colon T(I_C) \to T_D$, see section 6. Let $\mathfrak{B}$ be a right $\mathfrak{A}$-module and $\mathfrak{B}'$ a right $\mathfrak{A}'$-module.

(14.7) Definition. A *T-tensor module functor* $(U, \omega)\colon \mathfrak{B} \to \mathfrak{B}'$ consists of a functor $U\colon \mathfrak{B} \to \mathfrak{B}'$ and a natural isomorphism $\omega\colon U(X) * T(A) \to U(X * A)$ such that the diagrams

$$\begin{tikzcd}
(UA * TB) * TC \arrow[r, "\alpha_{UX,TB,TC}"] \arrow[d, "\omega \otimes 1"] & UA * (TB \otimes TC) \arrow[d, "1 \otimes \varphi"] \\
U(A * B) * TC \arrow[d, "\omega"] & UA * T(B \otimes C) \arrow[d, "\omega"] \\
U((A * B) * C) \arrow[r, "U(\alpha_{A,B,C})"] & U(A * (B \otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
U(V) * TI_C \arrow[r, "1 \otimes i"] \arrow[d, "\omega"] & U(V) * I_D \arrow[d, "\rho'"] \\
U(V * I_C) \arrow[r, "U(\rho)"] & U(V)
\end{tikzcd}$$

are commutative. ♡

(14.8) Example. If $\mathfrak{A}$ and $\mathfrak{A}'$ act on itself as in (14.5), then (T, φ) is also a T-tensor module functor for these actions. ♡

15. Categories with cylinder braiding

We work with the following data:

(15.1) Definition. An *action pair* $(\mathfrak{B}, \mathfrak{A})$ consists of

(1) $(\mathfrak{A}, z)$ is a braided tensor category. The braiding z consists of natural isomorphisms $z_{X,Y}\colon X \otimes Y \to Y \otimes X$.

(2) $(\mathfrak{B}, *, \alpha, \rho)$ is a right $\mathfrak{A}$-module.

(3) $\mathfrak{A}$ is a subcategory of $\mathfrak{B}$ with $\mathrm{Ob}\mathfrak{A} = \mathrm{Ob}\mathfrak{B}$.

(4) $*, \alpha, \rho$ restrict to $\otimes, a, r$ on $\mathfrak{A} \times \mathfrak{A}$. ♡

A $\mathfrak{B}$-*endomorphism* t of $\mathfrak{A}$ is a family of morphisms $t_X \in \mathrm{Mor}_{\mathfrak{B}}(X, X)$ such that for $f \in \mathrm{Mor}_{\mathfrak{A}}(X, Y)$ the naturality $t_Y f = f t_X$ holds.

(15.2) Definition. A *cylinder twist* for the action pair $(\mathfrak{B}, \mathfrak{A})$ consists of a $\mathfrak{B}$-automorphism t of $\mathfrak{A}$ such that for each pair X, Y of objects the following relations hold:

(15.3)
$$z_{Y,X}(t_Y \otimes 1_X) z_{X,Y}(t_X \otimes 1_Y) = (t_X \otimes 1_Y) z_{Y,X}(t_Y \otimes 1_X) z_{X,Y}$$

(15.4)
$$(t_X \otimes 1_Y) z_{Y,X}(t_Y \otimes 1_X) z_{X,Y} = t_{X \otimes Y}.$$

The first equation is called the *four braid relation* for X, Y. An action pair together with a cylinder twist t is called a *tensor pair* with *cylinder braiding.* ♡

Recall the notion of a left duality (b, d) in a tensor category $\mathfrak{A}$, see section 11.

(15.5) Definition. Suppose that $\mathfrak{A}$ is provided with a left duality (b, d). A cylinder twist is *compatible with the left duality* if the following holds:

(15.6)
$$d_X(t_{X^*} \otimes 1) z_{X,X^*}(t_X \otimes 1) z_{X^*,X} = d_X$$

(15.7)
$$z_{X^*,X}(t_{X^*} \otimes 1) z_{X,X^*}(t_X \otimes 1) b_X = b_X.$$

Note that these relations involve terms which appear in (15.2). Thus, by (15.2), we could require instead

(15.8)
$$d_X\, t_{X^* \otimes X} = d_X, \qquad t_{X \otimes X^*}\, b_X = b_X.$$

Similarly, compatibility with a right duality (a, c) is defined by the relations

(15.9)
$$c_X\, t_{X \otimes X^\#} = c_X, \qquad t_{X^\# \otimes X}\, a_X = a_X.$$

Compatibility of t with the neutral object is the relation:

(15.10)
$$t_I = \mathrm{id}.$$

(15.11) Note. *If t is compatible with the neutral object, then also with duality.*

PROOF. Since $b_X \colon I \to X \otimes X^*$ is a morphism in $\mathfrak{A}$ and t a $\mathfrak{B}$-automorphism, we have $t_{X \otimes X^*} b_X = b_X t_I = b_X$, i. e. (15.9) holds. Similarly for (15.8), (15.10), (15.11). □

(15.12) Proposition. *The relation $(t_X \otimes 1) z_{Y,X}(t_Y \otimes 1_X) z_{X,Y} = t_{X \otimes Y}$ implies the four braid relation.*

PROOF. Since $z_{X \otimes Y}$ is a morphism of $\mathfrak{A}$ we have the naturality of t

$$t_{Y \otimes X} z_{X,Y} = z_{X,Y} t_{X \otimes Y}.$$

We compose both sides of the hypothesis (15.14) from the left with $z_{X,Y}$ and from the right with $z^{-1}_{X,Y}$, use the naturality of t, and obtain

$$t_{Y\otimes X} = z_{X,Y}(t_X \otimes 1)z_{Y,X}(t_Y \otimes 1).$$

We interchange X and Y in this relation and compare it with the hypothesis. The four braid relation drops out. □

16. Rooted structures

Let $(\mathfrak{B}, \mathfrak{A})$ be an action pair.

(16.1) Definition. A *left duality* for $(\mathfrak{B}, \mathfrak{A})$ consists of

(1) A left duality (b, d) for $\mathfrak{A}$.

(2) A pair of morphisms in $\mathfrak{B}$

$$\beta_X \colon I \to X^*, \qquad \delta_X \colon X \to I$$

for each object X in $\mathfrak{A}$.

These data are assumed to satisfy the following axioms:

(16.2) $$d_X(\beta_X \otimes 1_X) = \delta_X$$

(16.3) $$(\delta_X \otimes 1_{X^*})b_X = \beta_X$$

(16.4) $$\beta_{X\otimes Y} = (\beta_X \otimes 1_Y)\beta_Y$$

(16.5) $$\delta_{X\otimes Y} = \delta_Y(\delta_X \otimes 1_Y).$$

We call δ_X a *rooting* and β_X a *corooting* of X. ♡

There are similar axioms for a right duality $(a, c, \alpha \colon I \to X, \gamma \colon X^* \to I)$.

(16.6) Definition. A left duality for $(\mathfrak{B}, \mathfrak{A})$ *is compatible with a cylinder braiding* if the following axioms hold:

(16.7) $$\delta_X t_X = \delta_X$$

(16.8) $$t_X \beta_X = \beta_X$$

(16.9) $$z_{Y,X}(t_Y \otimes 1)z_{X,Y}(\beta_X \otimes 1) = (\beta_X \otimes 1)t_Y$$

(16.10) $$(\delta_X \otimes 1)z_{Y,X}t_Y \otimes 1)z_{X,Y} = t_Y(\delta_X \otimes 1).$$

We apply $t_X \otimes 1$ to the left hand side of (16.9) and obtain with (16.8) and the axiom about a cylinder braiding the relation

(16.11) $$t_{X\otimes Y}(\beta_X \otimes 1) = (\beta_X \otimes 1)t_Y.$$

In the presence of the other axioms, this is equivalent to (16.9).

Similarly if we apply $t_X \otimes 1$ to the right hand side of (16.10), we obtain

(16.12) $$t_Y(\delta_X \otimes 1) = (\delta_X \otimes) t_{X \otimes Y}$$

as an equivalent for (16.10).

If (a, c) is a right duality, we replace the axioms (16.2) and (16.3) by

(16.13) $$c_X(\alpha_X \otimes 1_{X^*}) = \gamma_X$$

(16.14) $$(\gamma_X \otimes 1_X) a_X = \alpha_X.$$

If the category has a left duality and a twist, we can take in this case the associated right duality (see section 13).

2 Hopf Algebras

1. Hopf algebras

We fix a commutative ring $\mathfrak{K}$ and work in the category $\mathfrak{K}$-Mod of left $\mathfrak{K}$-modules. The tensor product of $\mathfrak{K}$-modules M and N is denoted by $M \otimes N$. The associativity

(1.1) $$a\colon (M \otimes N) \otimes P \to M \otimes (N \otimes P), \ \ (m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$$

and the natural isomorphisms (left and right unit object)

(1.2) $$\begin{array}{ll} l\colon \mathfrak{K} \otimes M \to M, & k \otimes m \mapsto km \\ r\colon M \otimes \mathfrak{K} \to M, & m \otimes k \mapsto km \end{array}$$

will be treated as an identity. The natural isomorphism

(1.3) $$\tau\colon M \otimes N \to N \otimes M, \ \ m \otimes n \mapsto n \otimes m$$

expresses the commutativity of the tensor product. These data make $\mathfrak{K}$-Mod into a tensor category.

An *algebra* (A, m, e) in $\mathfrak{K}$-Mod consists of a $\mathfrak{K}$-module A and linear maps $m\colon A \otimes A \to A$ (multiplication), $e\colon \mathfrak{K} \to A$ (unit) such that the following axioms hold: $m(m \otimes 1) = m(1 \otimes m)$ (associativity), and $m(e \otimes 1) = l$, $m(1 \otimes e) = r$. The algebra is called commutative if $m\tau = m$. Usually we write $m(a \otimes b) = a \cdot b = ab$.

A *coalgebra* (C, μ, ε) in $\mathfrak{K}$-Mod consists of a $\mathfrak{K}$-module C and linear maps $\mu\colon C \to C \otimes C$ (comultiplication), $\varepsilon\colon C \to \mathfrak{K}$ (counit) such that the following axiom hold: $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$ (coassociativity), and $(\varepsilon \otimes 1)\mu = l^{-1}$, $(1 \otimes \varepsilon)\mu = r^{-1}$. The coalgebra is called cocommutative if $\tau\mu = \mu$.

We use the following symbolic notation for the comultiplication (μ-convention, Sweedler convention): $\mu(x) = \sum_x x_1 \otimes x_2$. (The $\sum$-sign or its index may also be omitted.) The counit axiom then reads $\sum \varepsilon(x_1)x_2 = x = \sum x_1 \varepsilon(x_2)$. Repeated application of μ leads to notations like $(\mu \otimes 1)\mu(x) = \sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes x_2 \otimes x_3$. Another example is the relation $\sum \varepsilon(a_1)\varepsilon(a_3)a_2 = a$.

A homomorphism of algebras $\varphi\colon (A, m, e) \to (A', m', e')$ is a linear map $\varphi\colon A \to A'$ such that $\varphi m = m(\varphi \otimes \varphi)$ and $e' = \varphi e$. A homomorphism of coalgebras $\psi\colon (C, \mu, \varepsilon) \to (C', \mu', \varepsilon')$ is a linear map $\psi\colon C \to C'$ such that $(\psi \otimes \psi)\mu = \mu'\psi$ and $\varepsilon'\psi = \varepsilon$.

The tensor product of algebras (A_i, m_i, e_i) is the algebra (A, m, e) with $A = A_1 \otimes A_2$ and $m = (m_1 \otimes m_2)(1 \otimes \tau \otimes 1)$ and $e = e_1 \otimes e_2\colon \mathfrak{K} \cong \mathfrak{K} \otimes \mathfrak{K} \to A_1 \otimes A_2$. The multiplication m is determined by $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2$ without categorical notation. The tensor product of coalgebras $(C_i, \mu_i, \varepsilon_i)$ is the coalgebra (C, μ, ε) with $C = C_1 \otimes C_2$ and $\mu = (1 \otimes \tau \otimes 1)(\mu_1 \otimes \mu_2)$ and $\varepsilon = \varepsilon_1 \varepsilon_2\colon C_1 \otimes C_2 \to \mathfrak{K} \otimes \mathfrak{K} \cong \mathfrak{K}$.

A *bialgebra* $(H, m, e, \mu, \varepsilon)$ is an algebra (H, m, e) and a coalgebra (H, μ, ε) such that μ and ε are homomorphisms of algebras. (Here $H \otimes H$ carries the tensor product structure which was defined in the preceding paragraph.) The equality $\mu m = (m \otimes m)(1 \otimes \tau \otimes 1)(\mu \otimes \mu)$ expresses the fact that μ is compatible with multiplication. The same equality says that m is compatible with comultiplication. This and a similar interpretation of the identities $\mathrm{id} = \varepsilon e$, $\mu e = (e \otimes e)\mu$, $m(\varepsilon \otimes \varepsilon) = \varepsilon m$ is used to show that a bialgebra can, equivalently, be defined by requiring that m and e are homomorphisms of coalgebras. A homomorphism of bialgebras is a $\mathfrak{K}$-linear map which is at the same time a homomorphism of the underlying coalgebras. Similarly for the tensor product of bialgebras.

Let (C, μ, ε) be a coalgebra. Let $C^* = \mathrm{Hom}(C, \mathfrak{K})$ denote the dual module. The data

$$m\colon C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\mu^*} C^*$$

and $e\colon \mathfrak{K} \cong \mathfrak{K}^* \xrightarrow{\varepsilon^*} C^*$ define the *dual algebra* (C^*, m, e) of the coalgebra. (The definition of m uses $C^* \otimes C^* \to (C \otimes C)^*$, $\varphi \otimes \psi \mapsto (c \otimes d \mapsto \varphi(c)\psi(d))$. This map is an isomorphism if C is a finitely generated, projective $\mathfrak{K}$-module.)

Let (A, m, e) be an algebra with A a finitely generated, projective $\mathfrak{K}$-module. The data

$$\mu\colon A^* \xrightarrow{m^*} (A \otimes A)^* \cong A^* \otimes A^*$$

and $\varepsilon\colon A^* \xrightarrow{e^*} \mathfrak{K}^* \cong \mathfrak{K}$ define the *dual coalgebra* (A^*, μ, ε) of the algebra.

(1.4) Proposition. *Let C be a coalgebra and A an algebra. Then* $\mathrm{Hom}(C, A)$ *carries the structure of an algebra with product $\alpha * \beta = m(\alpha \otimes \beta)\mu$, for $\alpha, \beta \in$* $\mathrm{Hom}(C, A)$*, and unit $e\varepsilon$.*

Proof. The map $(\alpha, \beta) \mapsto \alpha \otimes \beta$ is bilinear by construction. The (co-)associativity of m and μ is used to verify that $*$ is associative. The unit and counit axioms yield

$$\alpha * (e\varepsilon) = m(\alpha \otimes e\varepsilon)\mu = m(1 \otimes e)(\alpha \otimes 1)(1 \otimes \varepsilon)\mu = \alpha.$$

Hence $e\varepsilon$ is a right unit. □

The product in (1.4) is called *convolution*. A simple verification yields the following naturality properties of the convolution.

(1.5) Proposition. *Suppose $\gamma\colon A_1 \to A$ is a homomorphism of algebras and $\delta\colon C_1 \to C_2$ a homomorphism of coalgebras. Then*

$$\gamma_*\colon \mathrm{Hom}(C, A_1) \to \mathrm{Hom}(C, A_2), \quad \alpha \mapsto \gamma\alpha$$

$$\delta^*\colon \mathrm{Hom}(C_2, A) \to \mathrm{Hom}(C_1, A), \quad \alpha \mapsto \alpha\delta$$

are homomorphisms of algebras. □

(1.6) Example. Let D be a Hopf algebra and A a commutative algebra. The convolution product induces on the set $A\mathrm{Hom}(D, A)$ of algebra homomorphisms $D \to A$ the structure of a group. ♡

An *antipode* for a bialgebra H is an $s \in \mathrm{Hom}(H,H)$ such that s is a twosided inverse of $\mathrm{id}(H) \in \mathrm{Hom}(H,H)$ in the convolution algebra. A bialgebra H is called *Hopf algebra* if it has an antipode. The *antipode axiom* reads in Sweedler notation

(1.7) $$\sum s(x_1)x_2 = e\varepsilon(x) = \sum x_1 s(x_2).$$

As an exercise in Sweedler notation verify $\sum a_1 \otimes s(a_2)a_3 = a \otimes 1$ and $\sum a_1 \otimes s(a_2)a_3a_4 = \sum a_1 \otimes a_2$.

(1.8) The Group Algebra. Let G be a group and $\mathfrak{K}G$ the group algebra. The $\mathfrak{K}$-module $\mathfrak{K}G$ is the free $\mathfrak{K}$-module on the set G, and the multiplication $\mathfrak{K}G \otimes \mathfrak{K}G \cong \mathfrak{K}(G \times G) \to \mathfrak{K}G$ is the linear extension of the group multiplication. This algebra becomes a Hopf algebra, if we define the comultiplication by $\mu(g) = g \otimes g$ for $g \in G$, the counit by $\varepsilon(g) = 1$, and the antipode by $s(g) = g^{-1}$.

Let G be a finite group and $\mathcal{O}(G)$ the $\mathfrak{K}$-algebra of all maps $G \to \mathfrak{K}$ with pointwise addition and multiplication. Identify $\mathcal{O}(G \times G)$ with $\mathcal{O}(G) \otimes \mathcal{O}(G)$. Show that the group multiplication m induces a comultiplication $\mu = m^*\colon \mathcal{O}(G) \to \mathcal{O}(G \times G)$. The data $\varepsilon(f) = f(1)$ and $s(f)(g) = f(g^{-1})$ complete $\mathcal{O}(G)$ to a Hopf algebra. Evaluation at $g \in G$ defines an algebra homomorphism $\mathcal{O}(G) \to \mathfrak{K}$. Show that G is canonically isomorphic to the group $A\mathrm{Hom}(\mathcal{O}(G), \mathfrak{K})$ of (1.6).

An element g in a Hopf algebra H is called *group-like* if $\mu(g) = g \otimes g$ and $\varepsilon(g) = 1$. The set of group like elements in H is a group under multiplication. The inverse of g is $s(g)$. ♡

(1.9) Proposition. *Let H be a Hopf algebra with antipode s. Then s is an antihomomorphism of algebras and coalgebras, i. e. $s(xy) = s(y)s(x)$, $se = e$, $\varepsilon s = s$, $\tau(s \otimes s)\mu = \mu s$. If H is commutative or cocommutative, then $s^2 = \mathrm{id}$.*

Proof. We use the convolution algebra $X = \mathrm{Hom}(H \otimes H, H)$ and consider the elements $\alpha = m$, $\beta = m(s \otimes s)\tau$, $\gamma = sm$. We want to show $\beta = \gamma$. This is implied by $\gamma * \alpha = 1_X = \alpha * \beta$. (Here 1_X is the unit element of the algebra X.) The definition of the convolution and of the comultiplication in $H \otimes H$ yields $(\gamma * \alpha)(x \otimes y) = \sum_{x,y} s(x_1y_1)x_2y_2$. Since μ is a homomorphism of algebras, we have $\mu(xy) = \sum_{x,y} x_1y_1 \otimes x_2y_2$. Hence $(\gamma * \alpha)(x \otimes y) = m(s \otimes 1)\mu(xy) = e\varepsilon(xy) = e(\varepsilon(x)\varepsilon(y)) = 1_X(x \otimes y)$. Thus we have shown $\gamma * \alpha = 1_X$. On the other hand $(\alpha * \beta)(x \otimes y) = \sum_{xy} x_1y_1s(y_2)s(x_2) = \sum_x x_1(e\varepsilon(y))s(x_2) = e(\varepsilon(x)\varepsilon(y)) = 1_X(x \otimes y)$. We have for the unit element $1 = e(1) = e\varepsilon(1)$ of H

$$1 = e\varepsilon(1) = m(s \otimes \mathrm{id})\mu(1) = m(s \otimes \mathrm{id})(1 \otimes 1) = s(1) \cdot 1 = s(1),$$

and therefore $se = e$. A dual proof shows that s is a homomorphism of coalgebras.

Let H be commutative. Since s is an antihomomorphism, it is a homomorphism, and we obtain

$$s * s^2 = m(s \otimes s)(1 \otimes s)\mu = sm(1 \otimes s)\mu = se\varepsilon = e\varepsilon.$$

This implies $s^2 = \mathrm{id}$. A dual proof works in case H is cocommutative. □

(1.10) Proposition. *Let H_1 and H_2 be Hopf algebras and $\alpha\colon H_1 \to H_2$ a homomorphism of bialgebras. Then α commutes with the antipodes.*

Proof. Let s_i be the antipode of H_i. Proposition (1.5) yields

$$1 = \alpha_*(1) = \alpha_*(\mathrm{id} * s_1) = \alpha * \alpha s_1, \quad 1 = \alpha^*(1) = \alpha^*(s_2 * \mathrm{id}) = s_2\alpha * \alpha,$$

and this implies the equality of αs_1 and $s_2\alpha$, since both elements are $*$-inverse to α. □

Let $(H, m, e, \mu, \varepsilon) = H$ be a bialgebra. Then (*coopposite*) $(H, m, e, \tau\mu, \varepsilon) =$ (*opposite*)${}^0H$, $(H, m\tau, e, \mu, \varepsilon) = H^0$, and $(H, m\tau, e, \tau\mu, \varepsilon) = {}^0H^0$ are bialgebras. If s is an invertible antipode of H, then s^{-1}, s^{-1}, s are antipodes of H^0, ${}^0H$, ${}^0H^0$. An antipode of H is a homomorphism $H \to {}^0H^0$ of Hopf algebras.

(1.11) Proposition. *Let (H, s) be a Hopf algebra. The following assertions are equivalent:*

(1) *${}^0H$ is a Hopf algebra with antipode u.*

(2) *H^0 is a Hopf algebra with antipode t.*

(3) *s is an isomorphism.*

If one of these statements holds, then $st = ts = \mathrm{id}$ and $t = u$.

Proof. (2) ⇒ (3). We consider the following diagramm

$$\begin{tikzcd} H \otimes H \arrow[r, "1 \otimes t"] & H \otimes H \arrow[r, "s \otimes s"] \arrow[d, "m\tau"] & H \otimes H \arrow[d, "m"] \\ H \arrow[u, "\mu"] \arrow[r, "e\varepsilon"] & H \arrow[r, "s"] & H. \end{tikzcd}$$

The square on the left commutes, since t is an antipode. By (1.9), the square on the right is commutative. Thus $s * st = e\varepsilon$. Similarly, we see $st * s = e\varepsilon$, and hence $st = \mathrm{id}$. We interchange the squares and deduce $ts = \mathrm{id}$.

(3) ⇒ (2). Let t be the inverse of s. We want to show $m\tau(1 \otimes t)\mu = e\varepsilon$. It suffices to show $sm\tau(1 \otimes t)\mu = e\varepsilon$. We use $sm\tau = m(s \otimes s)$ and $st = \mathrm{id}$ and obtain

$$sm\tau(1 \otimes t)\mu = m(s \otimes s)(1 \otimes t)\mu = m(s \otimes s)\mu = e\varepsilon.$$

In a similar manner one shows (1) ⇒ (3). □

Let $(A, m, e, \mu, \varepsilon)$ be a Hopf algebra in which A is a finitely generated, projective $\mathfrak{K}$-module. The dual coalgebra and the dual algebra together yield the *dual Hopf algebra* $(A^*, \mu^*, \varepsilon^*, m^*, e^*)$. If s is the antipode of A, then s^* is the antipode of A^*. Suppose $(e_j \mid j \in J)$ is a $\mathfrak{K}$-basis of A. Write, with summation convention sum over an upper-lower index,

$$m(e_i \otimes e_j) = m_{ij}^s e_s, \quad \mu(e_s) = \mu_s^{ij} e_i \otimes e_j$$

$$e(1) = \varepsilon^j e_j, \quad \varepsilon(e_j) = \varepsilon_j, \quad s(e_j) = s_j^i e_i.$$

The unit, counit, and antipode axiom then take the following form

$$\varepsilon^j m_{ij}^s = \delta_k^s = m_{kj}^s \varepsilon^j$$

$$\varepsilon_i \mu_s^{ij} = \delta_s^j = \mu_s^{ji} \varepsilon_i$$

$$\mu_s^{ij} s_i^k m_{kj}^t = \varepsilon^t \varepsilon_s = \mu_s^{ji} s_i^k m_{jk}^t.$$

In the dual Hopf algebra $m^*(e^i \otimes e^j) = \mu_s^{ij} e^s$ and $\mu^*(e^s) = m_{ij}^s e^i \otimes e^j$, in terms of the dual basis $(e^j \mid j \in J)$. Write out formulas for the (co-)associativity.

Let (C, μ, ε) be a coalgebra and $I \subset C$ a submodule such that $\varepsilon(I) = 0$ and $\mu(I) \subset I \otimes C + C \otimes I$. Then there is a unique coalgebra structure on C/I such that the quotient map $p\colon C \to C/I$ is a homomorphism of coalgebras. If, moreover, $(C, m, e, \mu, \varepsilon)$ is a bialgebra and also I a twosided ideal of (C, m, e), then C/I carries the structure of a bialgebra, and p is a homomorphism of bialgebras. If s is an antipode and $s(I) \subset I$, then s induces an antipode on C/I. A twosided ideal I with $\mu(I) \subset I \otimes C + C \otimes I$, $\varepsilon(I)$ and $s(I) \subset I$ is called a *Hopf ideal.*

An element x of a bialgebra H is called *primitive*, if $\mu(x) = x \otimes 1 + 1 \otimes x$. Let $P(H) \subset H$ be the $\mathfrak{K}$-module of the primitive elements of H. The bracket $(x, y) \mapsto [x, y] = xy - yx$ defines the structure of a Lie algebra on $P(H)$. The inclusion $P(H) \subset H$ yields, by the universal property of the universal enveloping algebra, a homomorphism $\iota\colon U(P(H)) \to H$. For cocommutative Hopf algebras over a field of characteristic zero with an additional technical condition, ι is an isomorphism ([**??**], p. 110).

(1.12) Note. Let G be a compact Lie group with multiplication $h\colon G \times G \to G$. Consider homology $H_*(-)$ with rational coefficients. We use the Künneth-formula. The multiplication h yields on $H_*(G)$ a multiplication

$$H_*(G) \otimes H_*(G) \cong H_*(G \times G) \xrightarrow{h_*} H_*(G).$$

The diagonal $d\colon G \to G \times G$, $g \mapsto (g, g)$ yields a comultiplication

$$\Delta\colon H_*(G) \xrightarrow{d_*} H_*(G \times G) \cong H_*(G) \otimes H_*(G).$$

This situation was studied by Heinz Hopf [**??**]. The letter Δ for the comultiplication (and even the term "diagonal") has its origin in this topological context. In this text, Latin-Greek duality is the preferred notation. For background on Hopf algebras see [**??**], [**??**], [**??**], [**??**]. ♡

(1.13) Example. Let $\mathfrak{K}$ be a field of characteristic $p > 0$. Let $A = \mathfrak{K}[x]/(x^p)$. The following data define a Hopf algebra structure on A: $\mu(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$, $s(x) = -x$. ♡

(1.14) Example. A coalgebra structure on the algebra of formal powers series $\mathfrak{K}[[x]]$ is, by definition, a (continuous) homomorphism $\mu\colon \mathfrak{K}[[x]] \to \mathfrak{K}[[x_1, x_2]]$ with $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$ and $\varepsilon(x) = 0$. Here $\mathfrak{K}[[x_1, x_2]]$ is interpreted as a completed tensor product $\mathfrak{K}[[x_1]]\hat{\otimes}\mathfrak{K}[[x_2]]$. Then μ is given by the power series $\mu(x) = F(x_1, x_2)$ with the properties

$$F(x,0) = 0 = F(0,x), \quad F(F(x,y),z) = F(x,F(y,z)).$$

Such power series F are called *formal groups laws.* See [**??**], Ch. VII. ♡

(1.15) Example. Consider the coalgebra $C = \mathfrak{K}[x]$ with $\mu(x) = x \otimes 1 + 1 \otimes x$. Assume that $\mathfrak{K}$ is a field of characteristic zero. The dual algebra C^* can be identitied with the algebra of formal power series $k[[y]]$. The element x^r is dual to $r!y^r$. ♡

(1.16) Example. Let A be an algebra and C a coalgebra. Suppose A and C are finitely generated, projective $\mathfrak{K}$-modules. Dually to (1.4) one can define on $\mathrm{Hom}(A, C)$ the structure of a coalgebra with counit $\varphi \mapsto \varepsilon\varphi e$. ♡

(1.17) Example. Let H be a bialgebra and $s\colon H \to H$ an antihomomorphism of algebras. Then the set of x for which the antipode axiom (1.7) holds is a subalgebra. ♡

2. Modules and Comodules

Let (A, m, e) be an algebra in $\mathfrak{K}$-Mod. An A-module (M, m_M) consists of a $\mathfrak{K}$-module M and a $\mathfrak{K}$-linear map $m_M\colon A \otimes M \to M$, the action of A on M, such that $m_M(m \otimes 1) = m_M(1 \otimes m_M)$ and $m_M(e \otimes 1) = l$. We usually write $m_M(a \otimes m) = a \cdot m = am$. Then the axioms read $(a_1a_2)m = a_1(a_2m)$ and $1 \cdot m = m$.

Let (C, μ, ε) be a coalgebra. A C-comodule (M, μ_M) consists of a $\mathfrak{K}$-module M and a linear map $\mu_M\colon M \to C \otimes M$, the coaction of C on M, such that $(\mu \otimes 1)\mu_M = (1 \otimes \mu_M)\mu_M$ and $(\varepsilon \otimes 1)\mu_M = l^{-1}$. For comodules we use the Sweedler convention with upper indices $\mu_M(x) = \sum x^1 \otimes x^2$. The axioms then read $\sum (x^1)_1 \otimes (x^1)_2 \otimes x^2 = \sum x^1 \otimes x^{11} \otimes x^{22}$ and $l(\sum \varepsilon(x^1) \otimes x^2) = x$.

The objects defined above should be called left modules (comodules). It should be clear, how right modules (comodules) are defined.

If M and N are A-modules, then a $\mathfrak{K}$-linear map $f\colon M \to N$ is called A-linear if $f(a \cdot m) = a \cdot f(m)$ holds for $a \in A$ and $m \in M$. If M and N are C-comodules, then a $\mathfrak{K}$-linear map $f\colon M \to N$ is called C-colinear provided $(1 \otimes \mu)\mu_M = \mu_N f$. In this way, we obtain the category A-Mod of left A-modules and C-Com of left C-comodules. We write Mod-A and Com-C for the categories of right (co-) modules.

(2.1) Example. Let $C = \mathfrak{K}[x_{ij} \mid 1 \le i, j \le n]$. This becomes a bialgebra with comultiplication $\mu(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$ and counit $\varepsilon(x_{ij}) = \delta_{ij}$. The $\mathfrak{K}$-module V with basis $v_1, \ldots, v_n$ is a C-comodule with structure map $\mu_V\colon V \to C \otimes V$, $v_j \mapsto \sum_k x_{jk} \otimes v_k$. Let $M_n(\mathfrak{K})$ be the algebra of (n, n)-matrices over $\mathfrak{K}$ and let M^* be the dual $\mathfrak{K}$-module. Let $x_{ij} \in M^*$ be the function which maps a matrix to its (i,j)-entry. Then the dual μ of the matrix multiplication satisfies $\mu(x_{ij}) = \sum_k x_{ik} \otimes x_{kj}$. Dualize the standard $M_n(\mathfrak{K})$-module $\mathfrak{K}^n$. ♡

(2.2) Example. Let G be a group and $H = \mathfrak{K}G$ the group algebra. Suppose $V = \bigoplus_{g\in G} V_g$ is a G-graded $\mathfrak{K}$-module. Set $\mu_V\colon V \to H \otimes V$, $v_g \mapsto g \otimes v_g$ for $v_g \in V_g$. Then V becomes an H-comodule. Any H-comodule arises this way:

Let $\mu_V\colon \to \mathfrak{K}G \otimes V$ be a comodule structure. Set $\mu_V(v) = \sum_g g \otimes p_g(v)$. The comodule axioms yield $p_g p_h = \delta_{gh} p_g$ and $\sum_g p_g = \mathrm{id}_V$. Hence $V = \bigoplus_g V_g$, with V_g the image of p_g, and $\mu_V\colon V_g \to g \otimes V_g$, $v_g \mapsto g \otimes v_g$. ♡

The tensor product $M \otimes_{\mathfrak{K}} N = M \otimes N$ of A-modules M and N is in general not an A-module in a natural way. However, if $(A, m, e, \mu, \varepsilon)$ is a bialgebra, then an A-module structure $m_{M\otimes N}$ on $M \otimes N$ is defined by $a \cdot (x \otimes y) = \sum a_1 x \otimes a_2 y$ for $a \in A$, $x \otimes y \in M \otimes N$. The counit $\varepsilon\colon A \to \mathfrak{K}$ makes $\mathfrak{K}$ into an A-module. Under this tensor product, the natural maps (1.1) and (1.2) are A-linear. Dually, given a bialgebra A, we can define the tensor product of A-comodules by $\mu_{M\otimes N} = (m \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1)(\mu_M \otimes \mu_N)$. In this case, (1.1) and (1.2) are A-colinear. These structures make A-Mod and A-Com into a tensor category. Each structure of a tensor category on A-Mod with underlying data (1.1) – (1.3) arises from a bialgebra structure on the algebra A; the A-module structure on $A \otimes A$ yields a linear map $\mu\colon A \to A \otimes A$ by $a \cdot (1 \otimes 1) = \mu(a)$.

One can define formally algebras and coalgebras in tensor categories. In our case, this leads to the following definitions.

Suppose H is a bialgebra. An algebra in H-Mod or an (H-module)-algebra is an algebra (A, m, e) such that the structure maps m and e are morphisms in H-mod, i. e. are H-linear. (This assumes the H-module structures on $A \otimes A$ and $\mathfrak{K}$ defined above.) In terms of element this means: For $x \in H$ and $a, b \in A$

(2.3)
$$x \cdot (ab) = \sum (x_1 \cdot a)(x_2 \cdot b), \quad x \cdot e(k) = e(\tilde{\varepsilon}(x)k).$$

Here $\tilde{\varepsilon}$ is the counit of H.

A coalgebra in H-Mod or an (H-module)-coalgebra is a coalgebra (C, μ, ε) where the structure maps μ and ε are morphisms in H-Mod. In terms of elements this means: For $x \in H$, $c \in C$

(2.4)
$$\mu(x \cdot c) = \sum x_1 \cdot c_1 \otimes x_2 \cdot c_2, \quad \varepsilon(x \cdot a) = \tilde{\varepsilon}(x)\varepsilon(a).$$

In a similar manner one defines algebras and coalgebras in H-Com. Finally, combining these notions, one defines bialgebras in H-Mod and H-Com.

(2.5) Example. Let K be a field and $L|K$ a Galois extension with finite Galois group G. Then L is a KG-module and a K-algebra. These data make L into a (KG-module)-algebra. ♡

(2.6) Example. Let $K[D]$ be the polynomial ring with comultiplication $\mu(D) = D \otimes 1 + 1 \otimes D$. The algebra $K[x]$ becomes a $K[D]$-module, if D^i acts as the formal differential operator $\left(\frac{d}{dx}\right)^i$. The Leibniz rule says that $K[x]$ is a ($K[D]$-module)-algebra. ♡

(2.7) Example. Let B be a bialgebra in H-Mod. Let s be an antipode. Then s is H-linear. Hence B is a Hopf algebra in H-Mod. ♡

(2.8) Example. Let H be a bialgebra, A an algebra, and $\mu_A\colon A \to H \otimes A$ an H-comodule structure. Then (A, μ_A) is an (H-comodule)-algebra if and only if μ_A is a homomorphism of algebras. ♡

(2.9) Example. Let H be a bialgebra, A an algebra and an H-module. If (2.3) holds for $x = y, z$, then also for $x = yz$. Hence it suffices to verify (2.3) for algebra generators of H. ♡

(2.10) Example. Let C be a coalgebra and $\mu_M\colon M \to C \otimes M$ a comodule structure. Let $p\colon C^* \otimes C \to \mathfrak{K}$ denote the evaluation $(\varphi, c) \mapsto \varphi(c)$. The map

$$C^* \otimes M \xrightarrow{1 \otimes \mu_M} C^* \otimes C \otimes M \xrightarrow{p \otimes 1} \mathfrak{K} \otimes M = M, \quad \varphi \otimes m \mapsto \varphi \cdot m$$

satisfies $\psi\varphi \cdot m = \varphi \cdot (\psi \cdot m)$ and defines the structure of a right C^*-module structure on M.

The comultiplication $\mu\colon C \to C\otimes C$ makes C into a left and right C-comodule. By

$$\varphi \dashv c = \sum \varphi(c_2)c_1, \quad c \vdash \varphi = \sum \varphi(c_1)c_2.$$

we obtain corresponding left and right actions of C^* on C. ♡

(2.11) Example. A group acts on itself by conjugation $(g, h) \mapsto ghg^{-1}$. A generalization for Hopf algebras H is the *left* and *right adjoint action* of H on itself:

$$\begin{array}{rcccl} h \cdot_l k & = & (ad_l h)(k) & = & \sum h_1 k\, s(h_2) \\ k \cdot_r h & = & (ad_r h)(k) & = & \sum s(h_1) k h_2. \end{array}$$

Verify $ab \cdot_l k = a \cdot_l (b \cdot_l k)$ and similarly for $\cdot_r$.

There are dual notions which make H into a comodule over itself, the *left* and *right adjoint coaction* ρ_l and ρ_r:

$$\begin{array}{ll} \rho_l\colon & H \to H \otimes H, \quad h \mapsto \sum h_1 s(h_3) \otimes h_2 \\ \rho_r\colon & H \to H \otimes H, \quad h \mapsto \sum h_2 \otimes s(h_1) h_3. \end{array}$$

Verify that ρ_l is a left comodule structure. ♡

(2.12) Example. Let B be a coalgebra which is free as a $\mathfrak{K}$-module with basis $(b_j \mid j \in J)$. Let $\mu_M\colon M \to M\otimes B$ be a right comodule structure and set $\mu_M(x) = \sum_j p_j(x) \otimes b_j$. This yields linear maps $p_j\colon M \to M$. Write $\mu(b_k) = \sum_{r,s} \mu_k^{rs} b_r \otimes b_s$ for the comultiplication. The p_j have the following properties:

(1) For each $x \in M$ only a finite number of $p_j(x) \neq 0$.

(2) $p_i p_j = \sum_k \mu_k^{ij} p_k$.

(3) $\sum_j \varepsilon(b_j) p_j = \mathrm{id}(M)$.

Conversely, given linear operators with these properties, then they define a comodule structure on M. ♡

3. Dual modules

Let A be a $\mathfrak{K}$-algebra. The opposite algebra A^0 has multiplication $a_1 \otimes a_2 \mapsto a_2 a_1$. Let M be a left A-module and $M^* = \mathrm{Hom}_{\mathfrak{K}}(M, \mathfrak{K})$ the dual module. The map

$$A^0 \times M^* \to M^*, \quad (a, \varphi) \mapsto a \cdot \varphi, \quad (a \cdot \varphi)(x) = \varphi(ax)$$

is a left A^0-module structure. Let $s\colon A \to A^0$ be a homomorphism (= antihomomorphism $A \to A$), then M^* becomes a left A-module via $(a \cdot \varphi)(x) = \varphi(s(a)x)$. Denote this module for the moment by $D_s(M)$. An A-linear map $\varphi\colon M \to N$ induces an A-linear map $D_s(\varphi) = \varphi^*\colon D_s(N) \to D_s(M)$. We obtain a contravariant functor $D_s\colon A$-Mod $\to A$-Mod which is the ordinary duality functor on the underlying $\mathfrak{K}$-modules.

Let $\kappa\colon M \to M^{**}$, $m \mapsto (\varphi \mapsto \varphi(m))$ be the canonical map into the bidual. The morphism $\kappa\colon M \to D_s^2(M)$ is in general not A-linear. Denote left multiplication by $a \in A$ just by a again. Then the diagram

(3.1)
$$\begin{tikzcd} M \arrow[r, "\kappa"] \arrow[d, "s^2(a)"'] & D_s^2(M) \arrow[d, "a"] \\ M \arrow[r, "\kappa"] & D_s^2(M) \end{tikzcd}$$

is commutative. Thus, if s is an involution, then κ is A-linear. Suppose $u \in A$ is a unit such that

(3.2) $$s^2(a) = uau^{-1}, \quad a \in A.$$

Then $\kappa \circ u\colon M \to D^2(M)$ is A-linear, and we obtain a natural transformation $\mathrm{id} \to D_s^2$.

(3.3) Proposition. *Let A be a bialgebra, $s\colon A \to A$ an antihomomorphism, and M an A-module which has a finite $\mathfrak{K}$-basis. The evaluation $e_M\colon D_s(M) \otimes M \to \mathfrak{K}$ is A-linear, if one half of the antipode axiom holds: $e\varepsilon(a) = \sum s(a_1)a_2$.*

PROOF. We compute

$$e_M(a \cdot (\varphi \otimes x)) = \sum e_M(a_1 \cdot \varphi \otimes a_2 x) = \sum (a_1 \cdot \varphi)(a_2 x) = \varphi(\sum s(a_1)a_2 x)$$

and

$$a \cdot e_M(\varphi \otimes x) = a \cdot \varphi(x) = \varepsilon(a)\varphi(x) = \varphi(e\varepsilon(a)x).$$

Both value are equal for all φ, x if and only if $e\varepsilon(a) = \sum s(a_1)a_2$. □

For the other part of the antipode axiom, see example (3.8). The last proposition shows that tensor products and duality with suitable properties require a Hopf algebra. We therefore develop duality theory from this point of view for a Hopf algebra $A = (A, m, e, \mu, \varepsilon)$ with antipode s.

Let M and N be A-modules. Then $\mathrm{Hom}_{\mathfrak{K}}(M, N)$ is an $A \otimes A^0$-module via

$$((a \otimes b) \cdot \varphi)(x) = a\varphi(bx).$$

If φ is already A-linear, then $(ab \otimes c) \cdot \varphi = (a \otimes bc) \cdot \varphi$. The homomorphism of algebras $(1 \otimes s)\mu \colon A \to A \otimes A^0$ makes $\mathrm{Hom}(M,N)$ into an A-module. We denote it by $\mathfrak{H}(M,N)$. The A-action then reads

$$(a \cdot \varphi)(x) = \sum a_1 \varphi(s(a_2)x).$$

By the counit axiom, this specializes to $\mathfrak{H}(M,\mathfrak{K}) = D_s(M) = M^*$. The assignment $(M,N) \mapsto \mathfrak{H}(M,N)$ is a functor A-Mod $\times$ A-Mod $\to$ A-Mod, called *formal* Hom-*functor*. It differs from $\mathrm{Hom}_A(M,N)$, but in our case $\mathrm{Hom}_A(M,N) \subset \mathfrak{H}(M,N)$. This subspace can be characterized as an eigenspace of the character ε:

(3.4) Proposition. *The map $\varphi \in \mathrm{Hom}(M,N)$ is A-linear if and only if $a \cdot \varphi = \varepsilon(a)\varphi$ for all $a \in A$.*

Proof. Let φ be A-linear. Then

$$(a \cdot \varphi)(x) = \sigma a_1 \varphi(s(a_2)x) = \sum a_1 s(a_2)\varphi(x) = \varepsilon(a)\varphi(x).$$

Conversely, if this equality holds, then

$$\varphi(ax) = \sum \varphi(\varepsilon(a_1)a_2 x) = \sum \varepsilon(a_1)\varphi(a_2 x) = \sum a_1 \varphi(s(a_2)a_3 x).$$

We use $\sum a_1 \otimes s(a_2)a_3 = a \otimes 1$, and see that the right most sum equals $a\varphi(x)$.□

As an example, consider the group algebra $A = \mathfrak{K}G$. Then $(g \cdot \varphi)(x) = g\varphi(g^{-1}x)$, and $\mathrm{Hom}_G(M,N)$ is the G-fixed point set of this G-action on $\mathrm{Hom}(M,N)$.

(3.5) Theorem. *The following canonical morphisms are A-linear:*

(1) *The evaluation* $e_{M,N} \colon \mathfrak{H}(M,N) \otimes M \to N$, $\varphi \otimes x \to \varphi(x)$.

(2) $\Theta \colon N \otimes M^* \to \mathfrak{H}(M,N)$, $x \otimes \varphi \mapsto \varphi(?)x$.

(3) *The composition of morphisms* $\circ \colon \mathfrak{H}(N,P) \otimes \mathfrak{H}(M,N) \to \mathfrak{H}(M,P)$.

(4) *The adjunction morphism* $\Lambda \colon \mathfrak{H}(M \otimes N, P) \to \mathfrak{H}(M, \mathfrak{H}(N,P))$.

(5) $T \colon M^* \otimes N^* \to (N \otimes M)^*$, $\varphi \otimes \psi \mapsto (y \otimes x \mapsto \varphi(x)\psi(y))$.

Proof. (1) is based on the computation

$$e_{M,N}(a \cdot (\varphi \otimes x)) = e_{M,N}(\sum (a_1 \cdot \varphi) \otimes a_2 x) = \sum a_1 \varphi(s(a_2)a_3 x).$$

We use $\sum a_1 \otimes s(a_2)a_3 = a \otimes 1$ to see that this equals $a\varphi(x) = ae_{M,N}(\varphi \otimes x)$.
(2) We use the definitions

$$\Theta(a \cdot (x \otimes \varphi)) = \Theta(\sum a_1 x \otimes a_2 \cdot \varphi) = \sum \varphi(s(a_2)?)a_1 x$$

$$a \cdot \Theta(x \otimes \varphi) = a \cdot \varphi(?)x = \sum a_1 \varphi(s(a_2)?)x.$$

Both expressions are equal, since multiplication by a_1 is $\mathfrak{K}$-linear.
(3) We compute

$$a \cdot (g \otimes f) = \sum a_1 \cdot g \otimes a_2 \cdot f = \sum a_1 g(s(a_2)?) \otimes a_3 f(s(a_4)?).$$

Under composition, this yields $\sum a_1 g(s(a_2)a_3 f(s(a_4)?))$. We use the identity $\sum a_1 \otimes s(a_2)a_3 \otimes a_4 = \sum a_1 \otimes 1 \otimes a_2$ and see that the sum equals $\sum a_1 g(f(sa_2)?) = a \cdot gf$.
(4) We use that s is an antihomomorphism of coalgebras, i. e. the relation $\mu s = \tau(s \otimes s)\mu$. To begin with, Λ is an isomorphism of $A \otimes A^0 \otimes A^0$-modules with actions

$$\begin{array}{rcll} ((a \otimes b \otimes c) \cdot \varphi)(x \otimes y) & = & a\varphi(bx \otimes cy) & \text{left} \\ ((a \otimes b \otimes c) \cdot \psi)(x)(y) & = & a(\psi(bx)(cy)) & \text{right.} \end{array}$$

The A-module structure on the left side uses $(1 \otimes \mu s)\mu$, on the right side

$$((1 \otimes s)\mu \otimes 1)(1 \otimes s)\mu = (1 \otimes \tau)(1 \otimes s \otimes s)(\mu \otimes 1)\mu.$$

Coassociativity of μ and antihomomorphy of s yield the equality.
(5) The definitions give

$$T(a \cdot (\varphi \otimes \psi))(y \otimes x) = \sum \varphi(s(a_1)x)\psi(s(a_2)y).$$

We use $\mu s = \tau(s \otimes s)\mu$ and see that this equals $(a \cdot T(\varphi \otimes \psi))(y \otimes x)$. □

Let V be a free left A-module with finite basis (e_i) and dual basis (e^i) for V^*. Then the map

$$b_V \colon \mathfrak{K} \to V \otimes V^*, \quad 1 \mapsto \sum_i e_i \otimes e^i$$

is A-linear. Denote the evaluation by $d_V \colon V^* \otimes V$, $\varphi \otimes v \mapsto \varphi(v)$. Then we have:

(3.6) Proposition. *The b_V, d_V are a left duality on the category of finitely generated free A-modules.* □

If the antipode s has an inverse $t = s^{-1}$, we define another dual module $V^{\#} = \mathrm{Hom}_{\mathfrak{K}}(M, \mathfrak{K})$ with action $a \cdot \varphi)(m) = \varphi(t(a) \cdot m)$. In this case we define $c_V \colon V \otimes V^{\#} \to \mathfrak{K}$, $v \otimes \varphi \mapsto \varphi(v)$ and in the finitely generated free case $a_V \colon \mathfrak{K} \to V^{\#} \otimes V$, $1 \to \sum_i e^i \otimes e_i$. One verifies that the maps a_V and c_V are A-linear.

(3.7) Proposition. *The a_V, c_V are a right duality on the category of finitely generated free A-modules.* □

We have, on general grounds, a canonical isomorphism $\varphi_V^{\#*} \colon V \to V^{\#*}$, see I.11. In our case this is the canonical map κ into the double dual $D_t D_s V$.

(3.8) Example. Assume the hypothesis of (3.3). Let (e_i) be a basis of M and (e^i) be the dual basis of M^*. Then $\mathfrak{K} \to M \otimes D_s(M)$, $1 \mapsto \sum e_i \otimes e^i$ is A-linear if and only if $e\varepsilon(a) = \sum a_1 s(a_2)$. ♡

(3.9) Example. The canonical map $M \to \mathfrak{H}(\mathfrak{K}, M)$, $m \mapsto (k \mapsto km)$ is A-linear. ♡

(3.10) Example. Let A be cocommutative. Then the dualization $\mathfrak{H}(M, N) \to \mathfrak{H}(N^*, M^*)$, $\varphi \mapsto \varphi^*$ is A-linear. ♡

(3.11) Example. Let A be cocommutative. Then the tautological map

$$\mathfrak{H}(M_1, N_1) \otimes \mathfrak{H}(M_2, N_2) \to \mathfrak{H}(M_1 \otimes M_2, N_1 \otimes N_2), \quad \varphi \otimes \psi \mapsto \varphi \otimes \psi$$

is A-linear. ♡

4. The finite dual

Let A be a Hopf algebra over a field $\mathfrak{K}$. We use finite dimensional A-modules to construct the finite dual of A. The algebra A acts on the left and right on the dual vector space A^* by $(a \cdot f)(x) = f(xa)$ and $(f \cdot a)(x) = f(ax)$. These actions commute.

(4.1) Proposition. *The following assertions about $f \in A^*$ are equivalent:*
(1) *f vanishes on a right ideal of finite codimension.*
(2) *f vanishes on a left ideal of finite codimension.*
(3) *f vanishes on a twosided ideal of finite codimension.*
(4) *$f \cdot A = \{f \cdot a \mid a \in A\}$ is finite dimensional.*
(5) *$A \cdot f$ is finite dimensional.*
(6) *$A \cdot f \cdot A$ is finite dimensional.*
(7) *$m^*(f) \in A^* \otimes A^*$.*

Proof. (1) $\Leftrightarrow$ (4). Let $I \subset A$ be a right ideal with $\dim A/I < \infty$ and $f(I) = 0$. If $a \equiv b \bmod I$, say $a = b + c$ with $c \in I$, then $(f \cdot a)(x) = f(ax) = f(bx + cx) = f(bx) = (f \cdot b)(x)$. Hence $f \cdot A$ is finite dimensional. Let $f \cdot A$ be finite dimensional. Then $I = \{a \in A \mid f \cdot a = 0\}$ is a right ideal of finite codimension and $f(I) = 0$.
(2) $\Leftrightarrow$ (5) and (3) $\Leftrightarrow$ (6) similarly. (3) $\Rightarrow$ (1), (2) is clear.
(2) $\Rightarrow$ (3). Let $L \subset A$ be a left ideal of finite codimension with $f(L) = 0$. Then A/L is a left A-module. We have the homomorphism of algebras

$$A \to \mathrm{End}_{\mathfrak{K}}(A/L), \quad a \mapsto l_a, \quad l_a(x) = ax$$

with kernel a twosided ideal I of finite codimension and contained in L.
(1) $\Leftrightarrow$ (3) similarly.
(5) $\Leftrightarrow$ (7). Let $g_1, \ldots, g_n$ be a basis of $A \cdot f$. We expand $a \cdot f$ in this basis, say $a \cdot f = \sum h_i(a) g_i$. The h_i are then elements of A^*. By definition of m^*, we have for $a, b \in A$

$$m^*(f)(b \otimes a) = f(ba) = (a \cdot f)(b) = \sum h_i(a) g_i(b) = (\sum g_i \otimes h_i)(b \otimes a),$$

and therefore $m^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$. Conversely, if $m^*(f) = \sum g_i \otimes h_i \in A^* \otimes A^*$, then $a \cdot f = \sum h_i(a) g_i$, and $A \cdot f$ is therefore contained in the subspace generated by $g_1, \ldots, g_n$. □

Let $A^{\mathrm{fin}} = \{f \in A^* \mid m^*(f) \subset A^* \otimes A^*\}$. This is the pre-image of $A^* \otimes A^* \subset (A \otimes A)^*$ under m^* and therefore a subspace.

(4.2) Proposition. *We have $m^*(A^{\mathrm{fin}}) \subset A^{\mathrm{fin}} \otimes A^{\mathrm{fin}}$. Therefore $(A^{\mathrm{fin}}, m^*, e^*)$ is a coalgebra.*

PROOF. Write $m^*(f) = \sum g_i \otimes h_i$ with (g_i) a basis of $A \cdot f$. We claim: $g_i, h_i \in A^{\mathrm{fin}}$. Since $g_i \in A \cdot f$ we have $A \cdot g_i \subset A \cdot f$ and $g_i \in A^{\mathrm{fin}}$ by (4.1). Choose $a_j \in A$ with $g_i(a_j) = \delta_{ij}$. Then $f \cdot a_j = \sum g_i(a_j) h_j = h_j$. Hence $h_j \in f \cdot A$ and $h_j \in A^{\mathrm{fin}}$ by (4.1). □

(4.3) Proposition. *Let $A = (A, m, e, \mu, \varepsilon)$ be a bialgebra. Then the data $(A^{\mathrm{fin}}, \mu^*, \varepsilon^*, m^*, e^*)$ define a bialgebra. If s is an antipode for A, then s^* is an antipode for A^{fin}.*

PROOF. We have the algebra $(A^*, \mu^*, \varepsilon^*)$ and by (4.2) the coalgebra $(A^{\mathrm{fin}}, m^*, e^*)$. We show that A^{fin} is a subalgebra of A^*. Let $f, g \in A^{\mathrm{fin}}$. Then $A \cdot f$ and $A \cdot g$ are finite dimensional. We have

$$\begin{aligned} (a \cdot fg)(x) &= fg(xa) = \textstyle\sum f((xa)_1) g((xa)_2) \\ &= \textstyle\sum (a_1 \cdot f)(x_1)(a_2 \cdot f)(x_2) \\ &= \textstyle\sum ((a_1 \cdot f)(a_2 \cdot f))(x). \end{aligned}$$

The relation $a \cdot fg = \sum (a_1 \cdot f)(a_2 \cdot f)$ shows that $a \cdot fg$ is contained in the subspace generated by $(A \cdot f)(A \cdot g)$ which is finite dimensional. Also $\varepsilon \in A^{\mathrm{fin}}$, since ε vanishes on an ideal of codimension one. The axioms of a bialgebra are checked by dualizing those of A.

Let s be an antipode. We show $s^*(A^{\mathrm{fin}}) \subset A^{\mathrm{fin}}$. We compute $(a \cdot s^*(f)) = s^*(f)(xa) = f(s(xa)) = f(s(a)s(x)) = (f \cdot s(a))(s(x)) = s^*(f \cdot s(a))(x)$. This shows $A \cdot s^*(f) \subset s^*(f \cdot A)$. By (4.1), $s^*(f) \subset A^{\mathrm{fin}}$. The antipode axiom for s^* is checked by dualizing the one for s. □

The coalgebras (bialgebras) constructed in (4.2) and (4.3) are called *finite duals*. The displayed formula in the last proof shows that A^{fin} is an A-module algebra.

(4.4) Proposition. *Let V be a finite dimensional A-module with basis (v_j). Write $a \cdot v_i = \sum_j x_{ij}(a) v_j$. Then $x_{ij} \in A^*$. Elements of A^* which are linear combinations of functions x_{ij} arising from various finite dimensional modules are called* representative functions *of A. The algebra A^{fin} is the subspace of representative functions of A^*.*

5. Pairings

Let A and B denote Hopf algebras over $\mathfrak{K}$. A *pairing* between A, B is a bilinear map $\langle -, - \rangle \colon A \times B \to \mathfrak{K}$ with the following properties: For $x, y \in A$ and $u, v \in B$

$$\begin{array}{lrcl} (1) & \langle xy, u \rangle & = & \langle x \otimes y,\ \mu(u) \rangle \\ (2) & \langle x, uv \rangle & = & \langle \mu(x), u \otimes v \rangle \\ (3) & \langle 1, u \rangle & = & \varepsilon(u) \\ (4) & \langle x, 1 \rangle & = & \varepsilon(x). \end{array}$$

The bilinear form $\langle -,- \rangle$ on $A \times B$ induces a bilinear form on $A \otimes A \times B \otimes B$ by $\langle x \otimes y,\ u \otimes v \rangle = \langle x,u \rangle \langle y,v \rangle$. This is used in (1) and (2). With the μ-convention axiom (1) reads $\langle xy,u \rangle = \sum \langle x,u_1 \rangle \langle y,u_2 \rangle$. A pairing is called a *duality* between A, B, if $\langle x,u \rangle = 0$ for all $u \in B$ implies $x = 0$, and $\langle x,u \rangle = 0$ for all $x \in A$ implies $u = 0$.

Let $p\colon A \otimes B \to \mathfrak{K}$ be the linear map which corresponds to the bilinear map $\langle -,- \rangle\colon A \times B \to \mathfrak{K}$. The axioms (1) and (3) of a pairing amount to the following commutative diagrams

$$\begin{tikzcd}
A \otimes A \otimes B \arrow[r, "1 \otimes 1 \otimes \mu"] \arrow[d, "m \otimes 1"] & A \otimes A \otimes B \otimes B \arrow[d, "1 \otimes \tau \otimes 1"] \\
A \otimes B \arrow[d, "p"] & A \otimes B \otimes A \otimes B \arrow[d, "p \otimes p"] \\
\mathfrak{K} \arrow[r, "="] & \mathfrak{K}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathfrak{K} \otimes B \arrow[r, "1 \otimes \varepsilon"] \arrow[d, "e \otimes 1"] & \mathfrak{K} \otimes \mathfrak{K} \arrow[d, "\cong"] \\
A \otimes B \arrow[r, "p"] & \mathfrak{K}.
\end{tikzcd}$$

Similarly for (2) and (4).

(5.1) Remark. Let $\langle -,- \rangle\colon A \times B \to \mathfrak{K}$ be a pairing. The coalgebra B has a dual algebra B^*. Axiom (1) of a pairing is equivalent to the statement that $\alpha\colon A \to B^*$, $x \mapsto \langle x,- \rangle$ is compatible with multiplications, and axiom (3) to the statement that α is compatible with units. Axiom (2) is equivalent to the commutativity of the following diagram

$$\begin{tikzcd}
A \arrow[rr, "\mu"] \arrow[d, "\alpha"] & & A \otimes A \arrow[d, "\alpha \otimes \alpha"] \\
B^* \arrow[r, "m^*"] & (B \otimes B)^* & B^* \otimes B^*. \arrow[l]
\end{tikzcd}$$

Hence axioms (2) and (4) say that α is a homomorphism of the algebra A into the finite dual of B. We call the pairing a *strict duality* if this map and the analogous homomorphism of B into the finite dual of A are isomorphisms. ♡

(5.2) Example. Let A be a finite dimensional Hopf algebra over the field $\mathfrak{K}$ and A^* the dual Hopf algebra. Then evaluation $A^* \times A \to \mathfrak{K}$, $(\varphi, a) \mapsto \varphi(a)$ is a pairing and a strict duality between Hopf algebras. ♡

(5.3) Example. Let $\mathfrak{K}[x]$ be the polynomial algebra with $\mu(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$. There exists a unique pairing $\langle -,- \rangle\colon \mathfrak{K}[x] \times \mathfrak{K}[x] \to \mathfrak{K}$ such that $\langle x,x \rangle = 1$. Show $\langle x^m, x^n \rangle = \delta_{m,n} n!$. ♡

(5.4) Proposition. *Let $\langle -,- \rangle$ be a pairing between Hopf algebras A, B with antipodes s_A, s_B. Then $\langle s_A x, u \rangle = \langle x, s_B u \rangle$.*

PROOF. This is a consequence of the fact that a homomorphism of bialgebras is compatible with antipodes. Another proof follows from the following computation. □

(5.5) Proposition. *The elements $p \circ (s \otimes 1)$ and $p \circ (1 \otimes s)$ are both convolution inverse to p in $(A \otimes B)^*$.* □

(5.6) Proposition. *Let A, B be Hopf algebras and $\langle -,- \rangle \colon A \times B \to \mathfrak{K}$ a bilinear form such that $\langle xy, u \rangle = \langle x \otimes y, \mu(u) \rangle$ for all $x, y \in A$ and $u \in B$. If $\langle x, uv \rangle = \langle \mu(x), u \otimes v \rangle$ holds for $x = a$ and $x = b$ and all u, v, then also for $x = ab$ and all u, v. If $\langle x, 1 \rangle = \varepsilon(x)$ holds for $x = a$, $x = b$, then also for $x = ab$.*

PROOF. We compute

$$\begin{aligned} \langle ab, uv \rangle &= \langle a \otimes b,\ \mu(uv) \rangle \\ &= \sum \langle a \otimes b,\ u_1 v_1 \otimes u_2 v_2 \rangle \\ &= \sum \langle a, u_1 v_1 \rangle \langle b, u_2 v_2 \rangle \\ &= \sum \langle a_1, u_1 \rangle \langle a_2, v_1 \rangle \langle b_1, u_2 \rangle \langle b_2, v_2 \rangle. \end{aligned}$$

The first equality uses hypothesis (1), the second the homomorphism of algebras μ, the third the definition of $\langle -,- \rangle$ on tensor products, the fourth the hypothesis about a and b. On the other hand,

$$\langle \mu(ab), u \otimes v \rangle = \sum \langle a_1 b_1, u \rangle \langle a_2 b_2, v \rangle = \sum \langle a_1, u_1 \rangle \langle b_1, u_2 \rangle \langle a_2, v_1 \rangle \langle a_2, v_2 \rangle.$$

The first equality uses $\mu(ab) = \sum a_1 b_1 \otimes a_2 b_2$ and the definition of $\langle -,- \rangle$, the second the hypothesis (1).

Finally, we have $\langle ab, 1 \rangle = \langle a \otimes b,\ \mu(1) \rangle = \langle a, 1 \rangle \langle b, 1 \rangle = \varepsilon(a)\varepsilon(b) = \varepsilon(ab)$. □

(5.7) Corollary. *If the bilinear form $\langle -,- \rangle \colon A \times B \to \mathfrak{K}$ satisfies (1) and (3), then it suffices to verify (2) and (4) for algebra generators of A.* □

Let $\langle -,- \rangle \colon A \times B \to \mathfrak{K}$ be a pairing. The left radical J_A is the kernel of the adjoint mapping $\alpha \colon A \to B^*$, $x \mapsto \langle x, - \rangle$, hence a two-sided ideal. Similarly, for the right radical $J_B \subset B$. Let $\alpha' \colon A \otimes A \to (B \otimes B)^*$ denote the adjoint mapping of the pairing $A \otimes A \times B \otimes B \to \mathfrak{K}$ induced by $\langle -,- \rangle$. We have a factorisation

$$\alpha' \colon A \otimes A \xrightarrow{\alpha \otimes \alpha} B^* \otimes B^* \xrightarrow{T} (B \otimes B)^*$$

with the map $T \colon \varphi \otimes \psi \mapsto (b \otimes c \mapsto \varphi(b)\psi(c))$. If $\mathfrak{K}$ is a field, then T is injective, and therefore the kernel of α' is $J_A \otimes A + A \otimes J_A$. For $x \in J_A$ we have $\langle \mu(x), u \otimes v \rangle = \langle x, uv \rangle$ for all $u, v \in B$. We see that $\mu(x)$ is contained in the kernel of α'. Together with exercise 4 we see that J_A is a Hopf ideal. Thus, if $\mathfrak{K}$ is a field, we have an induced duality pairing $\langle -,- \rangle \colon A/J_A \times B/J_B \to \mathfrak{K}$. The ideals J_A, J_B are Hopf ideals, if A, B are Hopf algebras.

(5.8) Proposition. *For $x \in A$, $y \in B$, and t the inverse of s we have*

$$\sum \langle x_1, y_1 \rangle \langle x_2, s(y_2) \rangle = \varepsilon(x)\varepsilon(y), \quad \sum \langle x_1, y_2 \rangle \langle x_2, t(y_1) \rangle = \varepsilon(x)\varepsilon(y).$$

Proof. We compute

$$\begin{aligned} \sum \langle x_1, y_1 \rangle \langle x_2, s(y_2) \rangle &= \sum \langle \mu(x), (1 \otimes s)\mu(y) \rangle \\ &= \langle x, m(1 \otimes s)\mu(y) \rangle = \langle x, \varepsilon(y) \rangle = \varepsilon(x)\varepsilon(y). \end{aligned}$$

A similar computation yields the second equality; it uses that t is the antipode for the coopposite algebra. □

It is useful to have a notion which is dual to that of a pairing. Let $r\colon \mathfrak{K} \to B \otimes A$ be a linear map. This map is determined by the element

$$r(1) = R = \sum_i b_i \otimes a^i \in B \otimes A.$$

We define associated elements

$$\begin{aligned} R_{13}^B &= \sum b_i \otimes 1 \otimes a^i \in B \otimes B \otimes A \\ R_{23} &= \sum 1 \otimes b_i \otimes a^i \in B \otimes B \otimes A \\ R_{13}^A &= \sum b_i \otimes 1 \otimes a^i \in B \otimes A \otimes A \\ R_{12} &= \sum b_i \otimes a^i \otimes 1 \in B \otimes A \otimes A. \end{aligned}$$

The map $r\colon \mathfrak{K} \to B \otimes A$ is called a *copairing* between Hopf algebras B, A, if it has the following properties:

$$\begin{array}{lrcl} (1) & (\mu \otimes 1)R &=& R_{13}^B R_{23} \\ (2) & (1 \otimes \mu)R &=& R_{12} R_{13}^A \\ (3) & (\varepsilon \otimes 1)R &=& 1 \\ (4) & (1 \otimes \varepsilon)R &=& 1. \end{array}$$

Write these equations in diagram form. Then (1) and (3) are given by diagrams which are dual to the diagrams for (1) and (3) of a pairing. (Dual: reversal of arrows, interchange m and μ, e and ε.)

(5.9) Proposition. *Let $p\colon A \otimes B \to \mathfrak{K}$ be a pairing. It induces four covariant functors from comodules to modules, and interchanges A, B and left, right. These functors are defined on objects in the following way:*

(1) *If $M \in$ Com-B, then $M \in A$-Mod with structure $(a, m) \mapsto \sum \langle a, m^2 \rangle m^1$.*
(2) *If $M \in B$-Com, then $M \in$ Mod-A with structure $(m, a) \mapsto \sum \langle a, m^1 \rangle m^2$.*
(3) *If $M \in A$-Com, then $M \in$ Mod-B with structure $(m, b) \mapsto \sum \langle m^1, b \rangle m^2$.*
(4) *If $M \in$ Com-A, then $M \in B$-Mod with structure $(b, m) \mapsto \sum \langle m^2, b \rangle m^1$.*

The functors are tensor functors.

Proof. It suffices to consider case (1). Let $a, b \in A$. Then

$$\begin{aligned} ab \cdot m &= \sum \langle ab, m^2 \rangle m^1 \\ &= \sum \langle a \otimes b, (m^2)_1 \otimes (m^2)_2 \rangle m^1 \\ &= \sum \langle a, (m^2)_1 \rangle \langle b, (m^2)_2 \rangle m^1 \\ &= \sum \langle a, m^2 \rangle \langle b, m^3 \rangle m^1 \\ &= \sum \langle b, m^2 \rangle \langle a, m^{12} \rangle m^{11} \\ &= a \cdot \sum \langle b, m^2 \rangle m^1 \\ &= a \cdot (b \cdot m). \end{aligned}$$

Suppose $M, N \in \text{Com-}B$. Let $M^\vee$, $N^\vee$ be the corresponding A-modules. We want to show $(M \otimes N)^\vee = M^\vee \otimes N^\vee$; on the left we have to use the tnesor product of comodules and on the right the tensor product of modules. The a-action on $x \otimes y \in (M \otimes N)^\vee$ is defined by $\sum \langle a, (x \otimes y)^2 \rangle (x \otimes y)^1$. By definition of the tensor product of comodules $\sum (x \otimes y)^1 \otimes (x \otimes y)^2 = \sum x^1 \otimes y^1 \otimes x^2 y^2$. Hence the definig sum equals

$$\begin{aligned} \sum \langle a, x^2 y^2 \rangle x^1 \otimes y^1 &= \sum \langle a_1, x^2 \rangle \langle a_2, y^2 \rangle x^1 \otimes y^1 \\ &= a_1 \cdot x \otimes a_2 \cdot y \\ &= a \cdot (x \otimes y), \end{aligned}$$

the latter in $M^\vee \otimes N^\vee$. □

(5.10) Proposition. *Let $r \colon \mathfrak{K} \to B \otimes A$, $1 \mapsto \sum b_i \otimes a^i$ be a copairing. It induces four covariant functors from modules to comodules and interchanges A, B and left, right. These functors are defined on objects in the following way:*

(1) *If $M \in \text{Mod-}A$, then $M \in B\text{-Com}$ with structure $m \mapsto \sum b_i \otimes m a^i$.*
(2) *If $M \in A\text{-Mod}$, then $M \in \text{Com-}B$ with structure $m \mapsto \sum a^i m \otimes b_i$.*
(3) *If $M \in \text{Mod-}B$, then $M \in A\text{-Com}$ with structure $m \mapsto \sum a^i \otimes m b_i$.*
(4) *If $M \in B\text{-Mod}$, then $M \in \text{Com-}A$ with structure $m \mapsto \sum b_i m \otimes a^i$.*

The functors are tensor functors.

As an exercise, the reader may treat (5.9 and (5.10) in categorical form with commutative diagrams.

(5.11) Example. Let $R \in B \otimes A$ be the element of a copairing. The antipode axiom yields the chain $(1 \otimes m)(1 \otimes 1 \otimes 1)(1 \otimes \mu)R = (1 \otimes e\varepsilon)R = (1 \otimes e)(1 \otimes \varepsilon)R = 1 \otimes 1$. On the other hand $(1 \otimes \mu)R = R_{12}R_{13} = \sum_{i,j} b_i b_j \otimes a^i \otimes a^j$ yields for the same element $\sum b_i b_j \otimes a^i s(a^j) = R \cdot (1 \otimes s)R$. Similarly $(s \otimes 1)R \cdot R = 1 \otimes 1$. Hence R is invertible in $B \otimes A$ with inverse $R^{-1} = (s \otimes 1)R = (1 \otimes s)R$. This example is the dual to (5.5). ♡

(5.12) Example. Let A and B denote $\mathfrak{K}$-modules. Linear maps $r\colon \mathfrak{K} \to B \otimes A$ and $p\colon A \otimes B \to \mathfrak{K}$ are called *dual maps*, provided

$$A \xrightarrow{1 \otimes r} A \otimes B \otimes A \xrightarrow{p \otimes 1} A$$
$$B \xrightarrow{r \otimes 1} B \otimes A \otimes B \xrightarrow{1 \otimes p} B$$

are identities. Suppose A and B have finite bases $(a^i \mid i \in I)$ and $(b_i \mid i \in I)$ with $p(a^i \otimes b_j) = \delta^i_j$. Then r with $r(1) = \sum b_i \otimes a^i$ is dual to p, and p is a pairing if and only if r is a copairing. Suppose that p and r are dual in this strong sense. Then the functors in (5.9) are equivalences, with inverse functor the appropriate one of (5.10). ♡

(5.13) Example. Let A be a Hopf algebra and $R \in A \otimes A$ an invertible element. Define $R_{12}, R_{13}, R_{23} \in A \otimes A \otimes A$ by inserting a 1 at the place of the missing index. Set $\nu(x) = R \cdot \mu(x) \cdot R^{-1}$. Then ν is coassociative, provided

$$R_{12} \cdot (\mu \otimes 1)R = R_{23} \cdot (1 \otimes \mu)R.$$

This equality is satisfied, if

$$(\mu \otimes 1)R = R_{13}R_{23}, \quad (1 \otimes \mu)R = R_{13}R_{12}, \quad R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

hold. Suppose, in addition, that $(s \otimes 1)R = R^{-1} = (1 \otimes s^{-1})R$ and $z = m(s^2 \otimes 1)R$ is invertible. Then $(A, m, e, \nu, \varepsilon, z^{-1}sz)$ is a Hopf algebra. ♡

(5.14) Example. Let G be a group. The pairings $\langle -, - \rangle\colon \mathfrak{K}(G) \times \mathfrak{K}(G) \to \mathfrak{K}$ correspond via $\langle g, h \rangle = \alpha(g, h)$ to bi-characters $\alpha\colon G \times G \to \mathfrak{K}$, i. e. $\alpha(g, uv) = \alpha(g, u)\alpha(g, v)$, and similarly for the other variable.

Let G be finite abelian. Is the Hopf algebra $\mathfrak{K}G$ always (or in certain cases) isomorphic to its dual? ♡

6. The quantum double

Let A and B be Hopf algebras with invertible antipode s. We set $t = s^{-1}$. We denote by ${}^0A = (A, m, e, \tau\mu, \varepsilon, t)$ the Hopf algebra with the coopposite antipode ${}^0\mu = \tau\mu$. We assume given a pairing $\langle -, - \rangle\colon A \times B \to \mathfrak{K}$. The Hopf algebra D in the next theorem is the quantum double of Drinfeld.

(6.1) Theorem. *There exists a unique structure of a Hopf algebra on ${}^0A \otimes B = D$ with the following properties:*

(1) *The coalgebra structure on ${}^0A \otimes B$ is the tensor product of the coalgebras ${}^0A$ and B.*

(2) *The multiplication $\diamond$ is given by the following formula*

$$(a \otimes b) \diamond (c \otimes d) = \sum_{b,c} \langle c_3, b_1 \rangle \langle c_1, t(b_3) \rangle ac_2 \otimes b_2 d.$$

The formulation of the theorem uses the μ-convention for the Hopf algebras A and B, i. e. $(\mu \otimes 1)\mu(b) = \sum b_1 \otimes b_2 \otimes b_3$ and $(\mu \otimes 1)\mu(c) = \sum c_1 \otimes c_2 \otimes c_3$. We verify that ${}^0A = {}^0A \otimes 1$ and $B = 1 \otimes B$ are subalgebras: $(a \otimes 1) \diamond (b \otimes 1) = \sum \langle c_3, 1 \rangle \langle c_1, 1 \rangle a c_2 \otimes 1 = \sum \varepsilon(c_1)\varepsilon(c_3) a c_2 \otimes 1 = ac \otimes 1$, and similarly for B. Also $(a \otimes 1) \diamond (1 \otimes d) = a \otimes d$.

We can also express the ordinary product in terms of the $\diamond$-product.

(6.2) Proposition. *For $c \in A$ and $b \in B$*

$$\sum_{b,c} \langle c_1, b_3 \rangle \langle c_3, t(b_1) \rangle (1 \otimes b_2) \diamond (c_2 \otimes 1) = c \otimes b.$$

Proof. Insert the definition of the $\diamond$-product, apply (5.8), and use the counit axiom. □

Proof of (6.1). We verify that the product is associative. The definition of the product yields for $((a \otimes b) \diamond (c \otimes d)) \diamond (e \otimes f)$ the sum[1]

$$\sum_{b,c,b_2,d,e} \langle c_3 b_1 \rangle \langle c_1, t(b_3) \rangle \langle e_3, (b_2 d)_1 \rangle \langle e_1, t((b_2 d)_3) \rangle a c_2 e_2 \otimes (b_2 d)_2 f.$$

We have $(b_2 d)_j = b_{2j} d_j$. We use the μ-convention to replace $b_1, b_{21}, b_{22}, b_{23}, b_3$ by b_1, b_2, b_3, b_4, b_5 and $e_{11}, e_{12}, e_2, e_{31}, e_{32}$ by e_1, e_2, e_3, e_4, e_5. The definition of the product yields for the bracketing $(a \otimes b) \diamond ((c \otimes d) \diamond (e \otimes f))$ the sum

$$\sum_{b,c,e_2,d,e} \langle e_3, d_1 \rangle \langle e_1, t(d_3) \rangle \langle (ce_2)_3, b_1 \rangle \langle (ce_2)_1, t(b_3) \rangle a (ce_2)_2 \otimes b_2 d_2 f.$$

Here we use the following replacements: $e_1, e_{21}, e_{22}, e_{23}, e_3$ by e_1, e_2, e_3, e_4, e_5 and $b_{11}, b_{12}, b_2, b_{31}, b_{32}$ by b_1, b_2, b_3, b_4, b_5. Having done this, we see that both sums are equal.

A verification with the counit axiom shows that $1 \otimes 1$ is a unit element. It is a little tedious to verify that the comultiplication in ${}^0A \otimes B$ is a homomorphism of algebras. Recall that ${}^0A$ has a comultiplication ${}^0\mu$. The μ-convention, the definition of the product in $D \otimes D$, and the definition of $\diamond$ yields for $\mu(a \otimes b) \diamond \mu(b \otimes c)$ the expression

$$\sum (\langle c_{23}, b_{11} \rangle \langle c_{21}, t(b_{13}) \rangle a_2 c_{22} \otimes b_{12} d_1) \otimes (\langle c_{13}, b_{21} \rangle \langle c_{11}, t(b_{23}) \rangle a_1 c_{12} \otimes b_{22} d_2).$$

We replace $b_{11}, b_{12}, b_{13}, b_{21}, b_{22}, b_{23}$ by b_1 to b_6, and similarly for c. Then we obtain

$$\sum_{a,b,c,d} \langle c_6, b_1 \rangle \langle c_4, t(b_3) \rangle \langle c_3, b_4 \rangle \langle c_1, t(b_6) \rangle a_2 c_5 \otimes b_2 d_1 \otimes a_1 c_2 \otimes b_5 d_2.$$

We apply the identity (5.8) to the part $\langle c_3, b_4 \rangle \langle c_4, t(b_3) \rangle$. Reindexing, according to the μ-convention, yields the sum

$$\sum_{a,b,c,d} \varepsilon(b_3)\varepsilon(c_3) \langle c_5, b_1 \rangle \langle c_1, t(b_5) \rangle a_2 c_4 \otimes b_2 d_1 \otimes a_1 c_2 \otimes b_4 d_2.$$

[1] In the following computation, the letter e has nothing to do with the unit map.

We apply the counit axiom to $\varepsilon(b_3)b_4$ and $\varepsilon(c_3)c_4$, and arrive at the sum

$$\sum_{a,b,c,d} \langle c_4, b_1 \rangle \langle c_1, t(b_4) \rangle a_2 c_3 \otimes b_2 d_1 \otimes a_1 c_2 \otimes b_3 d_2.$$

If we expand $\mu((a \otimes b) \diamond (c \otimes d))$ according to the definitions, we obtain the same sum without any rewriting.

We leave to the reader the verification that $\varepsilon \otimes \varepsilon$ is a homomorphism of algebras.

If D has an antipode S, then S coincides with the given antipodes on the Hopf subalgebras ${}^0A$ and B (1.10). Since S is an antihomomorphism, we must have $S((a \otimes 1) \diamond (1 \otimes b)) = (1 \otimes s(b)) \diamond (t(a) \otimes 1)$. We show that this formula defines an antihomomorphism; hence, by (1.17), S is an antipode. The definitions yield

$$S(c \otimes d) \diamond S(a \otimes b) = (1 \otimes s(d)) \diamond (t(c) \otimes s(b)) \diamond (t(a) \otimes 1)$$

and

$$S((a \otimes b) \diamond (c \otimes d)) = \sum \langle c_3, b_1 \rangle \langle c_1, t(b_3) \rangle (1 \otimes s(d)) \diamond (1 \otimes s(b_2)) \diamond (t(c_2) \otimes 1) \diamond (t(a) \otimes 1).$$

We use (6.2) in the sum for $t(c), s(b)$ in place of c, b and use (5.8) in order to see that both values are equal. The antipode S is invertible. The inverse T is given by $T(a \otimes b) = (1 \otimes t(b)) \diamond (s(a) \otimes 1)$. □

Suppose B is a finite dimensional Hopf algebra over the field $\mathfrak{K}$ and $A = B^*$ the dual Hopf algebra. Then (5.1) and the evaluation pairing (5.2) yields the quantum double of B. In general, we call $D = D(A, B)$ the *quantum double* of A, B with respect to the given pairing. In the case $A = B^*$, the $\diamond$-product can be written

(6.3) $$(1 \otimes b) \diamond (f \otimes 1) = \sum f(t(b_3) ? b_1) \otimes b_2.$$

Here $f(u?v)\colon x \mapsto f(uxv)$ for $f \in B^*$.

7. Yetter-Drinfeld modules

Let M carry a left B-module and a right B-comodule structure. Suppose the compatibility condition

(7.1) $$\sum b_1 m^1 \otimes b_2 m^2 = \sum (b_2 m)^1 \otimes (b_2 m)^2 b_1$$

holds for $b \in B$ and $m \in M$. Then B is called a *Yetter-Drinfeld module* of type ${}_BYD^B$ (upper index: comodule; lower index: module). The pairing induces a left A-module structure on M, see (3.4).

(7.2) Proposition. *Let M be a Yetter-Drinfeld module of type ${}_BYD^B$ with induced left A-module structure. Then the map*

$$A \otimes B \otimes M \to M, \quad a \otimes b \otimes m \mapsto (a \otimes b) \cdot m := a \cdot (b \cdot m)$$

makes M into a left module over the quantum double.

PROOF. We have to show

$$((1 \otimes b) \diamond (a \otimes 1)) \cdot m = b \cdot (a \cdot m).$$

The right hand side is, by definition, equal to $\langle a, m^2 \rangle bm^1$ (we skip the $\sum$ sign). We apply the definitions to the left hand side and obtain

$$\langle a_3, b_1 \rangle \langle a_2, t(b_3) \rangle \langle a_2, (b_2 m)^2 \rangle (b_2 m)^1.$$

By the pairing axioms, this equals $\langle a, t(b_3)(b_2 m)^2 b_1 \rangle (b_2 m)^1$. We rewrite the μ-convention $1, 2, 3 \to 11, 12, 2$ and apply (7.1). We obtain

$$\langle a, t(b_2) b_{12} m^2 \rangle b_{11} m^1 = \langle a, \varepsilon(b_2) m^2 \rangle b_1 m^1 = \langle a, m^2 \rangle bm^1,$$

the value of the right hand side. □

In the case when the pairing is a strict duality, we can use (5.9) and (5.10) to switch between modules and comodules. Therefore, in this case, the left $D(A, B)$-modules correspond bijectively to ${}_B YD^B$-modules (equivalence of categories).

(7.3) Example. The set of $b \in B$ such that (7.1) holds for all $m \in M$ is a subalgebra of B. ♡

(7.4) Example. Let G be a group and $B = \mathfrak{K}G$. By (2.??), a right B-comodule is a G-graded $\mathfrak{K}$-module $V = \oplus_{g \in G} V_g$. Suppose V is also a B-module. Then (7.1) means the following: The action of $h \in G$ maps V_g into $V_{ghg^{-1}}$. ♡

8. Braiding

A bialgebra A gives raise to a tensor product of A-modules (section 2). But the twist map (1.3) τ: $M \otimes N \to N \otimes M$ is in general not A-linear. A braiding will be a remedy of this defect. A subcategory $\mathfrak{M}$ of A-Mod is called a *tensor category*, if it is closed under tensor products and contains $\mathfrak{K}$. A *braiding* for a tensor category $\mathfrak{M}$ is a natural isomorphism

$$c_{M,N} \colon M \otimes N \to N \otimes M$$

(natural in $M, N \in \mathfrak{M}$) such that

(8.1)
$$\begin{aligned} c_{U,V \otimes W} &= (1 \otimes c_{U,W})(c_{U,V} \otimes 1) \\ c_{U \otimes V,W} &= (c_{W,V} \otimes 1)(1 \otimes c_{V,W}) \end{aligned}$$

for all objects $U, V, W \in \mathfrak{M}$. An invertible element $R = \sum_r a_r \otimes b_r \in A \otimes A$ with the property

(8.2)
$$R \cdot \mu(a) = \tau\mu(a) \cdot R$$

for all $a \in R$ is called a *screw* for A. A verification from the definitions shows that a screw defines an A-linear isomorphism

(8.3) $$c_{M,N}\colon M \otimes N \to N \otimes M, \quad x \otimes y \mapsto \sum b_r y \otimes a_r x = \tau(R \cdot (x \otimes y)),$$

natural in the two variables M, N. We use the elements of $A \otimes A \otimes A$

$$R_{12} = \sum a_r \otimes b_r \otimes 1, \quad R_{13} = \sum a_r \otimes 1 \otimes b_r, \quad R_{23} = \sum 1 \otimes a_r \otimes b_r.$$

A screw is called *coherent*, provided

(8.4) $$(\mu \otimes 1)R = R_{13}R_{23}, \quad (1 \otimes \mu)R = R_{13}R_{12}.$$

We verify from the definitions:

(8.5) Proposition. *A screw R yields a braiding for A-Mod if and only if it is coherent. We call it the R-braiding.* □

The natural isomorphisms (1.2) should be compatible with the braiding, i. e. we ask for the relations $l \circ c_{M,\mathfrak{K}} = r$, $r \circ c_{\mathfrak{K},M} = l$. These relations hold for an R-braiding, provided

(8.6) $$l(\varepsilon \otimes 1)R = 1 = r(1 \otimes \varepsilon)R,$$

and this can be written in the form $\sum \varepsilon(a_r)b_r = 1 = \sum a_r \varepsilon(b_r)$.

(8.7) Proposition. *Suppose R is invertible and satisfies* (8.5)*. Then* (8.6) *holds.*

Proof. We use the counit axiom and $(\mu \otimes 1)R = R_{13}R_{23}$ in the following computation:

$$\begin{aligned} R &= (l(\varepsilon \otimes 1)\mu \otimes 1)R \\ &= (l \otimes 1)(\varepsilon \otimes 1 \otimes 1)(\mu \otimes 1)R \\ &= (l \otimes 1)(\varepsilon \otimes 1 \otimes 1)R_{13}R_{23} \\ &= \sum_{i,j} \varepsilon(a_i)a_j \otimes b_i b_j \\ &= \sum_j a_j \otimes (\sum_i \varepsilon(a_i)b_i)b_j \\ &= (1 \otimes l(\varepsilon \otimes 1)R) \cdot R. \end{aligned}$$

Since R is invertible, we obtain the first equality of (8.6). Similarly for the second one. □

(8.8) Proposition. *Let A be a Hopf algebra with antipode s. Suppose $R \in A \otimes A$ satisfies $(\mu \otimes 1)R = R_{13}R_{23}$ and $l(\varepsilon \otimes 1)R = 1$. Then $(s \otimes 1)R$ is an inverse of R.*

Proof. The antipode axiom and $l(\varepsilon \otimes 1)R = 1$ yield

$$(m \otimes 1)(s \otimes 1 \otimes 1)(\mu \otimes 1)R = (e\varepsilon \otimes 1)R = 1 \otimes 1.$$

We insert $(\mu \otimes 1)R = R_{13}R_{23}$ and obtain

$$1 \otimes 1 = (m \otimes 1)(s \otimes 1 \otimes 1)R_{13}R_{23} = \sum_{i,j} s(a_i)a_j \otimes b_i b_j = (s \otimes 1)R \cdot R.$$

If we start with $(m \otimes 1)(1 \otimes s \otimes 1)(\mu \otimes 1)R = 1 \otimes 1$, we obtain by a similar computation $R \cdot (s \otimes 1)R = 1 \otimes 1$. □

A coherent screw $R \in A \otimes A$ is called a *universal R-matrix* for A. A Hopf algebra A with invertible antipode together with a universal R-matrix is called a *braid algebra* or a *quasi-triangular Hopf algebra* (Drinfeld [**??**]).

The identity

(8.9) $$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

is called *Yang-Baxter relation*[2] for $R \in A \otimes A$.

(8.10) Proposition. *The relations $R \cdot \mu = \tau\mu \cdot R$ and $R_{13}R_{23} = (\mu \otimes 1)R$ imply the Yang-Baxter relation for R.*

Proof. This is shown by the following computation:

$$R_{12}R_{13}R_{23} = R_{12} \cdot (\mu \otimes 1)R = (\tau\mu \otimes 1)R \cdot R_{12} = (\tau \otimes 1)(\mu \otimes 1)R \cdot R_{12} = R_{23}R_{13}R_{12}.$$

Let $R = \sum a_r \otimes b_r \in A \otimes A$, and define $c_{M,N}\colon M \otimes N \to N \otimes M$ as in (8.3). Then R satisfies the Yang-Baxter equation (8.9) if and only if the diagrams

(8.11)
$$\begin{tikzcd}
N \otimes M \otimes P \arrow[r, "c_{23}"] & N \otimes P \otimes M \arrow[d, "c_{12}"] \\
M \otimes N \otimes P \arrow[u, "c_{12}"] \arrow[d, "c_{23}"] & P \otimes N \otimes M \\
M \otimes P \otimes N \arrow[r, "c_{12}"] & P \otimes M \otimes N \arrow[u, "c_{23}"]
\end{tikzcd}$$

are always commutative. (The index indicates the factors which are switched.) An important special case is $M = N = P$. Denote $c_{M,M}$ by $X\colon M \otimes M \to M \otimes M$. Then (8.11) gives the identity

(8.12) $$(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X).$$

An endomorphism X of $M \otimes M$ which satisfies (8.12) is called a *Yang-Baxter operator*.

(8.13) Example. A Yang-Baxter operator defines a representation of the Artin braid group $B(n)$ on the n-fold tensor power $M^{\otimes n}$. The *Artin braid group* $Z(n)$

[2] If we use notations like $R^1 = R_{23}$, then this relation reads $R^1 R^2 R^3 = R^3 R^2 R^1$.

on n strings has generators $g_1, \dots, g_{n-1}$ and relations

$$\begin{array}{ll} g_i g_j = g_j g_i & \text{for } |i-j| \geq 2 \\ g_i g_j g_i = g_j g_i g_j & \text{for } |i-j| = 1. \end{array}$$

If $X\colon M \otimes M \to M \otimes M$ satisfies (8.10), we set $X_i\colon M^{\otimes n} \to M^{\otimes n}$, with X acting on factors $(i, i+1)$, and $\mathrm{id}(M)$ on the remaining factors. The assignment $g_i \mapsto X_i$ defines a representation of $Z(n)$ on $M^{\otimes n}$. ♡

(8.14) Example. Let H be a bialgebra and $R = \sum a_r \otimes b_r \in M \otimes M$. Consider the linear maps

$$\begin{array}{rcl} \lambda\colon H^* \to H, & \alpha \mapsto & \sum \alpha(a_r) b_r \\ \lambda'\colon H^* \to H, & \alpha \mapsto & \sum a_r \alpha(b_r). \end{array}$$

If R satisfies (8.5) and (8.6), then λ is a homomorphism and λ' an antihomomorphism of algebras. Suppose H is a finitely generated, projective $\mathfrak{K}$-module. Then λ' is a homomorphism and λ an antihomomorphism of coalgebras. ♡

(8.15) Example. If R is an invertible screw, then also $(\tau R)^{-1} = \tau(R^{-1})$. Moreover, τR is an invertible screw for the coopposite algebra with antipode $\tau\mu$. If R yields a braiding c, then τR^{-1} yields the inverse braiding. ♡

9. Quantum double and R-matrix

The notion of an R-matrix has many applications in topology, physics and other fields. The construction of the quantum double will now be motivated by its relation to R-matrices.

We assume given Hopf algebras A and B and a pairing $\langle -,- \rangle\colon A \times B \to \mathfrak{K}$ as in section 6. In order to avoid technicalities, we assume that A and B are finitely generated free modules with dual basis (a^i) of A and (b_i) of B with respect to the pairing, i. e. $\langle a^i, b_j \rangle = \delta^i_j$. We think of

$$R = 1 \otimes b_i \otimes a^i \otimes 1 \in D \otimes D = A \otimes B \otimes A \otimes B.$$

We form R_{12}, R_{23}, R_{23} in $D \otimes D \otimes D$. The missing index indicates the position of $1 \otimes 1 \in D = A \otimes B$. We assume given a Hopf algebra structure on ${}^0\!A \otimes B = D$ with the following properties:

(1) ${}^0\!A = {}^0\!A \otimes 1$ and $1 \otimes B = B$ are Hopf subalgebras.

(2) The comultiplication Δ in D is the canonical one of the tensor product ${}^0\!A \otimes B$.

(3) The product $\diamond$ in D satisfies $(a \otimes 1) \diamond (1 \otimes b) = a \otimes b$.

These data suffice to rewrite the properties of the associated copairing (see ??) in the following form (note that we use the comultiplication ${}^0\mu$ in ${}^0\!A$):

(9.1) Proposition. *The element R has the following properties:*

$$\begin{array}{lrcl} (1) & (\Delta\otimes 1)R & = & R_{13}R_{23} \\ (2) & (1\otimes\Delta)R & = & R_{13}R_{12} \\ (3) & (\varepsilon\otimes 1)R & = & 1\otimes 1 \\ (4) & (1\otimes\varepsilon)R & = & 1\otimes 1 \\ (5) & (s\otimes 1)R & = & R^{-1} \\ (6) & (1\otimes s^{-1})R & = & R^{-1}. \end{array}$$

(9.2) Theorem. *Let D carry the structure of a Hopf algebra with properties* (1) – (3) *above. Then the following are equivalent:*

(1) $R\cdot\Delta(x) = \tau\Delta(x)\cdot R$.
(2) *R satisfies the Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$.*
(3) *The product $\diamond$ is the one of* (6.1).

PROOF. We use the summation convention: Summation over an upper-lower index. The Yang-Baxter equation $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ in $D\otimes D\otimes D$ then reads

$$1\otimes b_s b_i\otimes a^s\otimes b_j\otimes a^i a^j\otimes 1 = 1\otimes b_j b_s\otimes(1\otimes b_i)\diamond(a^s\otimes 1)\otimes a^i a^j\otimes 1.$$

We skip the right and left most $1's$ in the notation. We apply $s^{-1}\otimes 1$ to the equality $(s\otimes 1)R\cdot R = s(b_j)b_k\otimes a^j a^k = 1\otimes 1$ and obtain $s^{-1}(b_k)b_j\otimes a^j a^k = 1\otimes 1$. We use this in the Yang-Baxter equation and obtain

$$s^{-1}(b_k)b_s b_i\otimes a^s\otimes b_j\otimes a^i a^j a^k = b_s\otimes(1\otimes b_i)\diamond(a^s\otimes 1)\otimes a^i.$$

This is equivalent to the Yang-Baxter equation, since an analogous computation will give it back. We fix b_s and a^i and compute the $\diamond$-product by comparing coefficients. We set in terms of a basis

$$a^i a^j a^k = \mu_t^{ijk}a_t, \quad b_l b_s b_i = m_{lsi}^r b_r, \quad s^{-1}(b_k) = \sigma_k^l b_l.$$

The Yang-Baxter equation is seen to be equivalent to

$$(1\otimes b_t)\diamond(a^r\otimes 1) = \sigma_k^l m_{lsi}^r \mu_t^{ijk} a^s\otimes b_j.$$

We have to verify that this is the multiplication rule of (6.1). This follows from the definitions, if we use

$$(\mu\otimes 1)\mu(a^r) = m_{lsi}^r a^l\otimes a^s\otimes a^i, \quad (\mu\otimes 1)\mu(b_t) = \mu_t^{ijk} b_i\otimes b_j\otimes b_k.$$

But these equations only express the rule $\langle\, (\mu\otimes 1)\mu(x), y_1\otimes y_2\otimes y_3\,\rangle = \langle\, x, y_1 y_2 y_3\,\rangle$ in terms of a basis. This finishes the equivalence of (2) and (3). In ordern to show the equivalence of (1) and (2), it suffices to verify (1) for basis elements x. The relation

$$R\cdot\Delta(1\otimes e_t) = \tau\Delta(1\otimes e_t)\cdot R$$

holds for all t if and only both sides are equal when tensored with $\otimes(e^t\otimes 1)$ and summed over t. But this summation leads exactly to the Yang-Baxter equation. Similarly for $x = e^t\otimes 1$. □

10. Duality and braiding

We now discuss the relation between braiding and duality and explain what the general considerations of chapter one mean in the case of modules.

We start with the left duality b_V, d_V given by evaluation an coevaluation, see section three. From the braiding $z_{MN}\colon M \otimes N \to N \otimes M$ we obtain a right duality $a_V = z^{-1} b_V\colon \mathfrak{K} \to V^* \otimes V$ and $c_V = d_V z\colon V \otimes V^* \to \mathfrak{K}$. Let $u = \sum s(b_r) a_r$ and $\tilde{u} = \sum d_r s(c_r)$ with $R^{-1} = \sum c_r \otimes d_r$. Then one computes

$$c_v(v \otimes \varphi) = \varphi(u \cdot v), \quad a_V(1) = \sum e^i \otimes \tilde{u} e_i.$$

In the latter case we use a finitely generated module V with basis (e_i) and dual basis (e^i) of V^*. The fact that a_V, c_V is a right duality means that $u\tilde{u}$ and $\tilde{u}u$ act on V as identity. If one uses instead the braiding z^{-1} in order to define a_V, c_V, then one has to use the elements $\sum s(c_r) d_r$ for c_V and $\sum a_r s(b_r)$ for a_V.

If we use the dualities above, then the canonical isomorphism $\varphi_V^{\#*}\colon V \to V^{\#*}$ is the map $v \mapsto \kappa(uv)$, if κ denotes the canonical isomorphism into the double dual. If we use κ as an identification, then $V^{\#*}$ becomes V with actions of $a \in A$ as multiplication by $s^2(a)$. We now verify certain identities in the universal example A itself; this is then independent of the use of finitely generated free modules.

(10.1) Theorem. *Suppose $R = \sum a_r \otimes b_r \in A \otimes A$ is an element with inverse $\sum c_r \otimes d_r$ which satisfies $R \cdot \mu = \tau\mu \cdot r$. Set $u = \sum s(b_r) a_r$. Then u is invertible and $u^{-1} = \sum s^{-1}(d_r) c_r$. Moreover $s^2(a) = uau^{-1}$ for all $a \in A$.*

Proof. We prove the equality $ux = s^2(x)u$ for $x \in A$. This does not use invertibility of s and R. We set as usual $(\mu \otimes 1)\mu(x) = \sum x_1 \otimes x_2 \otimes x_3$. Then

$$(R \otimes 1)(\sum x_1 \otimes x_2 \otimes x_3) = (\sum x_1 \otimes x_2 \otimes x_3)(R \otimes 1).$$

We apply $1 \otimes s \otimes s^2$ to this equation, interchange the first and third factor, and multiply. This gives

$$\sum s^2(x_3) s(b_i x_2) a_i x_1 = \sum s^2(x_3) s(x_1) b_i) x_2 a_i.$$

This can also be written in the following form

$$\sum s(x_2 s(x_3) \cdot s(b_i a_i x_1 = \sum s^2(x_3) \cdot s(b_i) \cdot s(x_1) x_2 a_i.$$

The antipode axiom and the counit axiom yield

$$\sum s(x_1) x_2 \otimes x_3 = 1 \otimes x.$$

We apply $1 \otimes s^2$ to this equation and use in the identiy above; then the right hand side becomes $s^2(x)u$. In a similar manner we use $\sum x_1 \otimes x_2 s(x_3)$ and see that the left hand side is ux.

We set $= \sum s^{-1}(d_r) a_r$. We know already $us^{-1}(d_j) = s(d_j)u$. This implies

$$uv = u \sum s^{-1}(d_j) c_j = \sum s(d_j) u c_j = \sum s(b_k d_j) a_k c_j.$$

By definition $\sum a_k c_j \otimes b_k d_j = 1 \otimes 1$; we apply $m(1 \otimes s)\tau$ and obtain $\sum s(b_k d_j) a_k c_j = 1$. We use this and see $uv = 1$ and then $s^2(v)u = uv = 1$. Hence u has a right and left inverse. $\square$

Consider the following elements

$$\begin{array}{ll} u_1 = \sum s(b_i)a_i, & v_1 = \sum s^{-1}(d_j)c_j \\ u_2 = \sum s(c_j)d_j, & v_2 = \sum s^{-1}(a_i)b_i \\ u_3 = \sum b_i s^{-1}(a_i), & v_3 = \sum d_j s(c_j) \\ u_4 = \sum c_j s^{-1}(d_j) & v_4 = \sum a_i s(b_i). \end{array}$$

(10.2) Theorem. *The elements displayed above have the following properties:*

(1) *u_i is inverse to v_i.*

(2) *$s^2(x) = u_i x u_i^{-1}$.*

(3) *$s(u_1)^{-1} = u_2$, $s(u_3)^{-1} = u_4$.*

(4) *The elements u_i commute with each other and $u_i u_j^{-1}$ is containded in the center of A.*

If $(s \otimes s)R = R$, then $u_1 = u_3$, $u_2 = u_4$. If $\tau R = R^{-1}$, then $u_1 = u_2$.

Proof. We know already that u_1 is inverse to v_1. Moreover $s^2(x) = u_1 x u_1^{-1}$. In particular $s^2(u_1) = u_1$. By construction

$$s(v_2)u_1, \quad s(v_1) = u_2, \quad s(u_4) = v_3, \quad s(u_3) = v_4.$$

From (1) we thus obtain (3). Hence u_2, v_2 are invertible and

$$u_2 v_2 = s(v_1)s^{-1}(u_1) = s(v_1)s(u_1) = s(u_1 v_1) = 1.$$

We calculate

$$u_2 x u_2^{-1} = s(v_1)xs(v_1^{-1}) = s(v_1^{-1}s^{-1}(x)v_1) = s(u_1 s^{-1} u_1) = s^2(x).$$

We set $x = u_1$ and see $u_1 u_2 = u_2 i_1$, and $u_1 x u_1^{-1} = u_2 x u_2^{-1}$ shows that $u_1 u_2^{-1}$ is contained in the center.

We now consider the oppposite algebra with antipode s^{-1}. This yields similar assertions about u_3, v_3, u_4, v_4. The remaining assertion follow easily. $\square$

11. Cobraiding

Braidings on the category of comodules can be defined by suitable linear forms $\rho\colon A \otimes A \to \mathfrak{K}$. The multiplication by R is replaced by a multiplication with ρ. For comodules U and V this is defined to be

$$l^\rho = (\rho \otimes 1 \otimes 1)(1 \otimes \tau \otimes 1)(\mu_V \otimes \mu_U)\colon V \otimes U \to V \otimes U.$$

In terms of elements this reads

(11.1) $$c_{U,V}\colon U \otimes V \to V \otimes U, \quad x \otimes y \mapsto \sum \rho(y^1 \otimes x^1) y^2 \otimes x^2.$$

If ρ is invertible, i. e. if there exists ρ' with $\rho * \rho' = \rho' * \rho = \varepsilon$, then $c^{\rho}_{U,V}$ is an isomorphism.

Let C be a coalgebra, A an algebra, M a C-comodule, and N an A-module. Then $\mathrm{Hom}(M,N)$ becomes a module over the convolution algebra via

$$\mathrm{Hom}(C,A) \otimes \mathrm{Hom}(M,N) \to \mathrm{Hom}(C \otimes M, A \otimes N) \to \mathrm{Hom}(M,N), \rho \otimes h \mapsto g * h.$$

The first map is the tautological map, the second is composition with $\mu_M \colon M \to C \otimes M$ and $m_N \colon A \otimes N \to N$. In particular, if N is a $\mathfrak{K}$-module, then $\mathrm{Hom}(M,N)$ is a module over the dual algebra C^*. In terms of elements we have

$$(\rho * h)(m) = \sum \rho(m^1) h(m^2).$$

In this way $\mathrm{Hom}(A \otimes A, A)$ is a left (and similarly a right) module over the convolution algebra $(A \otimes A)^*$. The relation in $\mathrm{Hom}(A \otimes A, A)$

(11.2) $$\rho * m = m\tau * \rho$$

has the following explicit meaning:

(11.3) $$\sum \rho(v_1 \otimes u_1) v_2 u_2 = \sum \rho(v_2 \otimes u_2) u_1 v_1.$$

One verifies from the definitions:

(11.4) Proposition. *Suppose ρ satisfies* (9.2). *Then $c_{M,N}$ is a morphism of comodules.* □

Let $\varepsilon_{ij} \colon A \otimes A \otimes A \to A \otimes A$ be the map which maps the factor of the missing index by ε and the other factors identically. Set $\rho_{ij} = \rho \circ \varepsilon_{ij}$. The identities in $(A \otimes A \otimes A)^*$

(11.5) $$\rho(m \otimes 1) = \rho_{13} * \rho_{23}, \quad \rho(1 \otimes m) = \rho_{13} * \rho_{12}$$

are in terms of elements

(11.6)
$$\begin{aligned} \rho(xy \otimes z) &= \sum \rho(x^1 \otimes z^1) \varepsilon(y^1) \varepsilon(x^2) \rho(y^2 \otimes x^2) = \sum \rho(x \otimes z^1) \rho(y \otimes z^2) \\ \rho(x \otimes yz) &= \sum \rho(x^1 \otimes z^1) \varepsilon(y^1) \varepsilon(z^2) \rho(x^2 \otimes y^2) = \sum \rho(x^1 \otimes z) \rho(x^2 \otimes y). \end{aligned}$$

(11.7) Proposition. *The equalities* (8.1) *hold for the morphisms $c_{M,N}$ in* (9.1) *provided ρ satisfies* (9.5). □

We call $\rho \in (A \otimes A)^*$ a *braid form* for A if ρ is invertible and satisfies (9.2) and (9.5). A *cobraided bialgebra* is a pair (A, ρ) consisting of a bialgebra A and a braid form ρ for A. For a cobraided algebra the morphisms (9.2) define a braiding on A-Com.

12. The FRT-construction

The construction of Faddeev, Reshetikhin, and Takhtadjian (FRT-constuction) associates a cobraided algebra $A = A(V,X), \rho$ to each Yang-Baxter operator $X \colon V \otimes V \to V \otimes V$ on a finitely generated free $\mathfrak{K}$-module V. We describe this construction.

Suppose V has a basis $v_1, \dots, v_n$. Let $\tilde{A}$ be the free algebra on $\mathrm{Hom}(V,V)$. We use the model

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}(V^{\otimes n}, V^{\otimes n}) = \tilde{A}.$$

The multiplication in $\tilde{A}$ is given by the tautological identification $E_k \otimes E_l \cong E_{k+l}$, $f \otimes g \mapsto f \otimes g$, with $E_k = \mathrm{Hom}(V^{\otimes k}, V^{\otimes k})$. The canonical basis $T_i^j \colon v_k \mapsto \delta_{i,k} v_j$ of E_1 induces the basis

$$T_i^j = T_{i_1}^{j_1} \otimes \cdots \otimes T_{i_k}^{j_k}$$

of E_k, with multi-index notation $i = (i_1, \dots, i_k)$, $j = (j_1, \dots, j_k)$. For $k = 0$ we identify $E_0 = \mathfrak{K}$ and $T_\emptyset^\emptyset = 1$. The following data make $\tilde{A}$ into a bialgebra: The comultiplication in $\tilde{A}$ is given by $\mu(T_i^j) = \sum_k T_i^k \otimes T_k^j$ and the counit by $\varepsilon(T_i^j) = \delta_i^j$.

Let now $X \colon V \otimes V \to V \otimes V$ be a Yang-Baxter operator. This operator induces a tensor functor from the braid category ZA into $\mathfrak{K}$-Mod, as explained in ??. In particular we have for $k, l \in \mathbb{N}_0$ morphisms

$$X_{k,l} \colon V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes l} \otimes V^{\otimes k}$$

which satisfy the braiding relations

(12.1) $$X_{k+l,m} = (X_{k,m} \otimes 1)(1 \otimes X_{l,m}), \quad X_{k,l+m} = (1 \otimes X_{k,m})(X_{k,n} \otimes 1).$$

The $X_{k,l}$ are the identities for $k = 0$ or $l = 0$. Suppose $X = X_{k,l}$ has the matrix expression

$$X(v_i \otimes v_j) = \sum_{a,b} X_{ij}^{ab} v_a \otimes v_b$$

in multi-index notation $v_i = v_{i_1} \otimes \cdots \otimes v_{i_k}$, if $i = (i_1, \dots, i_k)$.

We define a linear form

$$\tilde{f} \colon \tilde{A} \to \mathfrak{K}, \quad T_i^j \mapsto F_i^j.$$

(12.2) Proposition. *The linear form $\tilde{\rho}$ satisfies* (9.5).

Proof. This is a restatement of (9.8). □

The $\mathfrak{K}$-module V is a canonical $\tilde{A}$ via

(12.3) $$V \to \tilde{A} \otimes V, \quad v_i \mapsto \sum_j T_i^j \otimes v_j.$$

Similarly, $V^{\otimes k}$ is an $\tilde{A}$-module by the same formula in multi-index notation.

The morphism $X\colon V\otimes V\to V\otimes V$ is not in general a morphism of $\tilde{A}$-modules, i. e. (9.3) is not yet satisfied. The equality (9.3) means that the elements

(12.4)
$$C^{kl}_{ij}=\sum_{\alpha,\beta}X^{\alpha\beta}_{ij}T^{kl}_{\alpha\beta}-\sum_{\alpha\beta}T^{\alpha\beta}_{ij}X^{kl}_{\alpha\beta}$$

are zero. Let $I\subset\tilde{A}$ denote the twosided ideal generated by the C^{kl}_{ij}. One verifies

$$\mu(C^{kl}_{ij})=\sum_{\alpha\beta}C^{\alpha\beta}_{ij}\otimes T^{kl}_{\alpha\beta}+\sum_{\alpha\beta}T^{\alpha\beta}_{ij}\otimes C^{kl}_{\alpha\beta},\quad \varepsilon(C^{kl}_{ij}=0.$$

This implies $\mu(I)\subset I\otimes\tilde{A}+\tilde{A}\otimes I$ and $\varepsilon(I)=0$. Hence we can form the factor bialgebra $A=\tilde{A}/I$.

(12.5) Proposition. *The form $\tilde{\rho}$ factors over $A\otimes A$ and induces a braid form $\rho\colon A\otimes A\to\mathfrak{K}$.*

Proof. □

13. Cylinder braiding

In this section we describe the formalism of categories with cylinder braiding for modules and comodules. For the categorical background see I.14.

Let (A,μ,ε) be a bialgebra over the commutative ring $\mathfrak{K}$ with universal R-matrix $R=\sum_r a_r\otimes b_r\in A\otimes A$, comultiplication μ and counit ε. Let $\mathfrak{A}$ denote the category of left A-modules and A-linear maps and $\mathfrak{B}$ the category of left A-modules and $\mathfrak{K}$-linear maps. The tensor product over $\mathfrak{K}$ induces functors

$$\otimes\colon\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A},\qquad *\colon\mathfrak{B}\times\mathfrak{A}\to\mathfrak{B}.$$

The R-matrix induces in the usual way a braiding $x\otimes y\mapsto\sum_r b_r y\otimes a_r x$ on $\mathfrak{A}$, as explained in section 8. With the unit object $I=\mathfrak{K}$ we obtain an action pair $\mathfrak{B},\mathfrak{A}$ in the sense of I(14.??).

(13.1) Theorem. *The cylinder twists for $\mathfrak{B},\mathfrak{A}$ correspond bijectively to the invertible elements $v\in A$ such that*

(13.2)
$$\mu(v)=(v\otimes 1)\cdot\tau R\cdot(1\otimes v)\cdot R.$$

Proof. Let v be given. We define for an A-module X a $\mathfrak{K}$-linear map by

$$t_X\colon X\to X\quad x\mapsto vx.$$

We verify that these maps yield a cylinder twist.

Conversely, suppose a cylinder twist is given. Set $v=t_A(1)$. We claim that v satisfies I(??). We consider A and $A\otimes A$ as left A-modules in a canonical way. Since μ is a homomorphism of algebras, μ is in particular A-linear; therefore we have $t_{A\otimes A}\mu = \mu t_A$. Since right multiplication by $a \in A$ is a homomorphism of left A-modules, we have $t_A(a) = va$. Now we compute

$$\begin{aligned} \mu(v) &= \mu t_A(1) \\ &= t_{A\otimes A}(1\otimes 1) \\ &= (t_A \otimes 1) z_{A,A} (t_A \otimes 1) z_{A,A} (1 \otimes 1) \\ &= \sum_{r,s} v b_s a_r \otimes a_s v b_r \\ &= (v \otimes 1)\cdot \tau R \cdot (1 \otimes v)\cdot R. \end{aligned}$$

In order to see that v is invertible ?? □

We call an invertible element $v \in A$ which satisfies (10.??) a *universal cylinder twist* for the braided algebra (A, R).

If a ribbon algebra is defined as in [**??**, p. 361], then the element θ^{-1} loc. cit. is a cylinder twist in the sense above. ??

Suppose in addition that A is a Hopf algebra with antipode s. Recall that we have the dual module $M^* = \mathrm{Hom}_{\mathfrak{K}}(M, \mathfrak{K})$ with A-action $(a \cdot \varphi)(x) = \varphi(s(a)x)$ and duality map the evaluation

$$d_M \colon M^* \otimes M \to \mathfrak{K}, \quad \varphi \otimes x \mapsto \varphi(x).$$

For a free, finitely generated module M with basis (e_i) and dual basis (e^i) we have the coevaluation

$$b_M \colon \mathfrak{K} \to M \otimes M^*, \quad 1 \mapsto \sum_i e_i \otimes e^i.$$

These data yield a left duality on the category of finitely generated free A-modules. For this category we have:

(13.3) Proposition. *Suppose $\varepsilon(v) = 1$. Then the cylinder twist is compatible with dualities.*

Proof. This is a consequence of (??). We can also make the following computations. We obtain from the definitions $d_M t_{M^*\otimes M}(\varphi \otimes x) = d_M(\sum v_1 x \otimes v_2 \varphi) = \varphi(\sum s(v_1) v_2 x) = \varepsilon(x)\varphi(x)$; the last equality by the antipode axiom. Another computation shows $t_{M\otimes M^*} b_M(1) = \varepsilon(v) b_M(1)$. □

We now turn to the dual situation of cobraided algebras and comodules.

14. Cylinder forms

Let $A = (A, m, e, \mu, \varepsilon)$ be a bialgebra over the commutative ring $\mathfrak{K}$ with multiplication m, unit e, comultiplication μ, and counit ε. Let $r \colon A \otimes A \to \mathfrak{K}$ be a linear form. We associate to A-comodules M, N a $\mathfrak{K}$-linear map

$$z_{M,N} \colon M \otimes N \to N \otimes M, \quad x \otimes y \mapsto \sum r(y^1 \otimes x^1) y^2 \otimes x^2,$$

where we have used the Sweedler notation $x \mapsto \sum x^1 \otimes x^2$ for a left A-comodule structure $\mu_M\colon M \to A \otimes M$ on M. We call r a *braid form* on A, if the $z_{M,N}$ yield a braiding on the tensor category A-Com of left A-comodules. We refer to section 9 for the properties of r which make it into a braid form.

Let (C, μ, ε) be a coalgebra. We use the Sweedler notation like $\mu(a) = \sum a_1 \otimes a_2$ and $(\mu \otimes 1)\mu(a) = \mu_2(a) = \sum a_1 \otimes a_2 \otimes a_3 = \sum a_{11} \otimes a_{12} \otimes a_2$ for the comultiplication. The multiplication in the dual algebra C^* is denoted as convolution: If $f, g \in C^*$ are $\mathfrak{K}$-linear forms on C, then the convolution $f * g$ is the form defined by $a \mapsto \sum f(a_1)g(a_2)$. The unit element of the algebra C^* is ε. Therefore g is a (convolution) inverse of f, if $f * g = g * f = \varepsilon$. We apply this formalism to the coalgebras A and $A \otimes A$. If f and g are linear forms on A, we denote by $f \otimes g$ the linear form on $A \otimes A$ defined by $a \otimes b \mapsto f(a)g(b)$. The twist on $A \otimes A$ is $\tau(a \otimes b) = b \otimes a$.

Here is the main definition of this paper. Let (A, r) be a bialgebra with braid form r. A linear form $f\colon A \to \mathfrak{K}$ is called a *cylinder form* for (A, r), if it is convolution invertible and satisfies

(14.1) $$f \circ m = (f \otimes \varepsilon) * r\tau * (\varepsilon \otimes f) * r = r\tau * (\varepsilon \otimes f) * r * (f \otimes \varepsilon).$$

In terms of elements and Sweedler notation (11.1) assumes the following form:

(14.2) For any two elements $a, b \in A$ the identities

$$f(ab) = \sum f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3) = \sum r(b_1 \otimes a_1) f(b_2) r(a_2 \otimes b_3) f(a_3)$$

hold.

A cylinder form (in fact any linear form) yields for each A-comodule M a $\mathfrak{K}$-linear endomorphism

$$t_M\colon M \to M, \quad x \mapsto \sum f(x^1)x^2.$$

If $\varphi\colon M \to N$ is a morphism of comodules, then $\varphi \circ t_M = t_N \circ \varphi$. Since t_M is in general not a morphism of comodules we express this fact by saying: The t_M constitute a *weak* endomorphism of the identity functor of A-Com. We call t_M the *cylinder twist* on M. The axiom (11.1) for a cylinder form has the following consequence.

(14.3) Proposition. *The linear map t_M is invertible. For any two comodules M, N the identities*

$$t_{M\otimes N} = z_{N,M}(t_N \otimes 1_M) z_{M,N}(t_M \otimes 1_N) = (t_M \otimes 1_N) z_{N,M}(t_N \otimes 1_M) z_{M,N}$$

hold.

Proof. Let g be a convolution inverse of f. Set $s_M\colon M \to M$, $x \mapsto \sum g(x^1)x^2$. Then

$$s_M t_M(x) = \sum f(x^1) g(x^{21}) x^{22} = \sum \varepsilon(x^1) x^2 = x,$$

by the definition of the convolution inverse and the counit axiom. Hence s_M is inverse to t_M.

In order to verify the second equality, we insert the definitions and see that the second map is

$$x \otimes y \mapsto \sum f(x^1) r(y^1 \otimes x^{21}) f(y^{21}) r(y^{221} \otimes x^{221}) y^{222} \otimes x^{222}$$

and the third map

$$x \otimes y \mapsto \sum r(y^1 \otimes x^1) f(y^{21}) r(y^{21} \otimes x^{221}) f(x^{221}) y^{222} \otimes x^{222}.$$

The coassociativity of the comodule structure yields a rewriting of the form

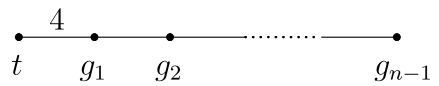
$$\sum y^1 \otimes y^{21} \otimes y^{221} \otimes y^{222} = \sum (y^1)_1 \otimes (y^1)_2 \otimes (y^1)_3 \otimes y^2$$

and similarly for x. We now apply (11.2) in the case $(a,b) = (x^1, y^1)$.

By definition of the comodule structure of $M \otimes N$, the map $t_{M \otimes N}$ has the form $x \otimes y \mapsto \sum f(x^1 y^1) x^2 \otimes y^2$. Again we use (11.2) for $(a,b) = (x^1, y^1)$ and obtain the first equality of (11.3). □

15. Tensor representations of braid groups

The braid group ZB_n associated to the Coxeter graph B_n



B_n

with n vertices has generators $t, g_1, \ldots, g_{n-1}$ and relations:

(15.1)
$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & i > 1 \\ g_ig_j & = & g_jg_i & |i-j| \geq 2 \\ g_ig_jg_i & = & g_jg_ig_j & |i-j| = 1. \end{array}$$

We recall: The group ZB_n is the group of braids with n strings in the cylinder $(\mathbb{C} \setminus 0) \times [0,1]$ from $\{1, \ldots, n\} \times 0$ to $\{1, \ldots, n\} \times 1$. This topological interpretation is the reason for using the cylinder terminology. For the relation between the root system B_n and ZB_n see [**??**].

Let V be a $\mathfrak{K}$-module. Suppose $X \colon V \otimes V \to V \otimes V$ and $F \colon V \to V$ are $\mathfrak{K}$-linear automorphisms with the following properties:

(1) X is a Yang-Baxter operator, i. e. satisfies the equation

$$(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X)$$

on $V \otimes V \otimes V$.

(2) With $Y = F \otimes 1_V$, the *four braid relation* $YXYX = XYXY$ is satisfied.

If (1) and (2) hold, we call (X, F) a *four braid pair.* For the construction of four braid pairs associated to standard R-matrices see [**??**]. For a geometric interpretation of (2) in terms of symmetric braids with 4 strings see [**??**].

Given a four braid pair (X, F), we obtain a tensor representation of ZB_n on the n-fold tensor power $V^{\otimes n}$ of V by setting:

(15.2)
$$\begin{array}{rcl} t & \mapsto & F \otimes 1 \otimes \cdots \otimes 1 \\ g_i & \mapsto & X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \end{array}$$

The X in X_i acts on the factors i and $i+1$.

These representations give raise to further operators, if we apply them to special elements in the braid groups. We set

$$t(1) = t, \quad t(j) = g_{j-1}g_{j-2}\cdots g_1 t g_1 g_2 \cdots g_{j-1}, \quad t_n = t(1)t(2)\cdots t(n)$$

$$g(j) = g_j g_{j+1} \cdots g_{j+n-1}, \quad x_{m,n} = g(m)g(m-1)\cdots g(1).$$

The elements $t(j)$ pairwise commute. We denote by $T_n\colon V^{\otimes n} \to V^{\otimes n}$ and $X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ the operators induced by t_n and $x_{m,n}$, respectively.

(15.3) Proposition. *The following identities hold*

$$T_{m+n} = X_{n,m}(T_n \otimes 1)X_{m,n}(T_m \otimes 1) = (T_m \otimes 1)X_{n,m}(T_n \otimes 1)X_{m,n}.$$

Proof. We use some fact about Coxeter groups [**??**, CH. IV, §1]. If we adjoin the relations $t^2 = 1$ and $g_j^2 = 1$ to (2.1) we obtain the Coxeter group CB_n. The element t_n is given as a product of n^2 generators t, g_j. The uniquely determined element of CB_n has length n^2 and is equal to t_n. The element $x_{n,m}t_n x_{m,n}t_m$ of CB_{m+n} has length $(m+n)^2$ and therefore equals t_{m+n} in CB_{m+n}. By a fundamental fact about braid groups [**??**, CH. IV, §1.5, Prop. 5], the corresponding elements in the braid group are equal. We now apply the tensor representation and obtain the first equality in (2.3). □

For later use we record:

(15.4) Proposition. *The element t_n is contained in the center of ZB_n.* □

16. Cylinder forms from four braid pairs

Let V be a free $\mathfrak{K}$-module with basis $v_1, \ldots, v_n$. Associated to a Yang-Baxter operator $X\colon V \otimes V \to V \otimes V$ is a bialgebra $A = A(V, X)$ with braid form r, obtained via the FRT-construction (see section 9 for the construction of A and r). We show that a four braid pair (X, F) induces a canonical cylinder form on (A, r).

Recall that A is a quotient of a free algebra $\tilde{A}$. We use the model

$$\bigoplus_{n=0}^{\infty} \mathrm{Hom}(V^{\otimes n}, V^{\otimes n}) = \tilde{A}.$$

The multiplication in $\tilde{A}$ is given by the canonical identification $E_k \otimes E_l \cong E_{k+l}, f \otimes g \mapsto f \otimes g$, with $E_k = \mathrm{Hom}(V^{\otimes k}, V^{\otimes k})$. The canonical basis $T_i^j\colon v_k \mapsto \delta_{i,k} v_j$ of E_1 induces the basis

$$T_i^j = T_{i_1}^{j_1} \otimes \cdots \otimes T_{i_k}^{j_k}$$

of E_k, with multi-index notation $i = (i_1, \ldots, i_k)$, $j = (j_1, \ldots, j_k)$. The comultiplication in $\tilde{A}$ is given by $\mu(T_i^j) = \sum_k T_i^k \otimes T_k^j$ and the counit by $\varepsilon(T_i^j) = \delta_i^j$.

In section 2 we defined an operator $T_k \in E_k$ from a given four braid pair (X, F). We express T_k in terms of our basis

$$T_k(v_i) = \sum_j F_i^j v_j,$$

again using multi-index notation $v_i = v_{i_1} \otimes \cdots \otimes v_{i_k}$, if $i = (i_1, \ldots, i_k)$. We define a linear form

$$\tilde{f}\colon \tilde{A} \to \mathfrak{K}, \quad T_i^j \mapsto F_i^j.$$

(16.1) Theorem. *The linear form $\tilde{f}$ factors over the quotient map $\tilde{A} \to A$ and induces a cylinder form f for (A, r).*

Proof. Suppose the operator $X = X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ has the form $X(v_i \otimes v_j) = \sum_{ab} X_{ij}^{ab} v_a \otimes v_b$. We define a form $\tilde{r}\colon \tilde{A} \otimes \tilde{A} \to \mathfrak{K}$ by

$$\tilde{r}\colon E_k \otimes E_l \to \mathfrak{K}, \quad T_i^a \otimes T_j^b \mapsto X_{ji}^{ab}.$$

The form $\tilde{r}$ factors over the quotient $A \otimes A$ and induces r.

Claim: The forms $\tilde{r}$ and $\tilde{f}$ satisfy (11.1) and (11.2). Proof of the claim. In the proof we use the summation convention: summation over an upper-lower index. Then we can write $\mu_2(T_i^c) = T_i^k \otimes T_k^a \otimes T_a^c$ and $\mu_2(T_j^d) = T_j^l \otimes T_l^b \otimes T_b^d$. The equality (11.2) amounts to

$$F_{ij}^{cd} = F_i^k X_{kj}^{la} F_l^b X_{ba}^{cd} = X_{ij}^{lk} F_l^b X_{bk}^{ad} F_a^c.$$

These equations are also a translation of (2.3) into matrix form. This finishes the proof of the claim.

We have to show that $\tilde{f}$ maps the kernel I of the projection $\tilde{A} \to A$ to zero. But this is a consequence of (11.2), applied in the case $b = 1$, since one of the terms a_1, a_2, a_3 is contained in I and $\tilde{r}$ is the zero map on $I \otimes \tilde{A}$ and $\tilde{A} \otimes I$.

It remains to show that f is convolution invertible. The pair (X^{-1}, F^{-1}) is a four braid pair. Let $\bar{r}$ and $\bar{f}$ be the induced operators on $\tilde{A}$. Then $\tilde{f} * \bar{f} = \varepsilon = \bar{f} * \tilde{f}$ on $\tilde{A}$, and (11.2) holds for $(\bar{f}, \bar{r})$ in place of (f, r). The Yang-Baxter operator X^{-1} defines the same quotient A of $\tilde{A}$ as X. Hence the kernel ideal obtained from X^{-1} equals I, and therefore $\bar{f}(I) = 0$. □

We have the comodule $V \to A \otimes V$, $v_i \mapsto \sum_j T_i^j \otimes v_j$, and similarly for $V^{\otimes k}$ in multi-index notation. By construction we have:

(16.2) Proposition. *The cylinder form f induces on $V^{\otimes k}$ the cylinder twist $t_{V^{\otimes k}} = T_k$.* □

17. The four braid relation and R-matrices

We state in this section our main results about R-matrices.

Let W be a module over the integral domain $\mathfrak{K}$. We study automorphisms X and Y of W which satisfy the *four braid relation*

(17.1) $$XYXY = YXYX.$$

We are particularly interested in the following case:

(1) The automorphism X is an R-matrix, also called Yang-Baxter operator; this means: $W = V \otimes V$ for a $\mathfrak{K}$-module V and X satisfies the Yang-Baxter equation

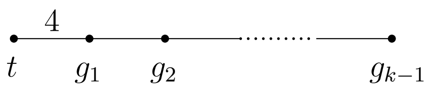
(17.2) $$(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X)$$

on $V \otimes V \otimes V$.

(2) The automorphism Y has the form $F \otimes 1$ for an automorphism F of V.

If (1) and (2) hold, we call (X, F) a *four braid pair.*

The interest in this case comes from the representation theory of braid groups. Recall that the braid group ZB_k associated to the Coxeter graph B_k



B_k

with k vertices has generators $t, g_1, \ldots, g_{k-1}$ and relations:

(17.3) $$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & i > 1 \\ g_ig_j & = & g_jg_i & |i-j| \geq 2 \\ g_ig_jg_i & = & g_jg_ig_j & |i-j| = 1. \end{array}$$

Given automorphisms F and X as above, we obtain a tensor representation of ZB_k on the k-fold tensor power $V^{\otimes k}$ of V by setting:

(17.4) $$\begin{array}{rcl} t & \mapsto & F \otimes 1 \otimes \cdots \otimes 1 \\ g_i & \mapsto & X_{(i)} = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \end{array}$$

The X in $X_{(i)}$ acts on the factors i and $i+1$.

After this preparation we state the main results about R-matrices. Let V denote a free $\mathfrak{K}$-module with basis $v_1, \ldots, v_n$. We write $v_{ij} = v_i \otimes v_j$ and use the lexicographical ordering of this basis when we display matrices.

Let $q \in \mathfrak{K}^*$ be a unit. The standard R-matrix $X_n(q) = X_n$ associated to the root system A_{n-1} is the linear map

(17.5) $$X_n v_{ij} = \begin{cases} qv_{ij} & i = j \\ v_{ji} & i > j \\ v_{ji} + \delta v_{ij} & i < j, \end{cases}$$

where $1 \le i, j \le n$ and $\delta = q - q^{-1}$. (See e. g. [**??**, p. 171] for an elementary verification of (1.2).)

Let $F\colon V \to V$ be an automorphism of the following form

(17.6)
$$F(v_j) = \begin{cases} \beta v_{n+1-j} & \text{if} \quad 2j < n+1 \\ a v_j & \text{if} \quad 2j = n+1 \\ w v_j + \beta' v_{n+1-j} & \text{if} \quad 2j > n+1. \end{cases}$$

The a-term does not appear for even n. We set $z = \beta\beta'$.

(17.7) Theorem. *Suppose F has the form* (1.6) *and X is given as in* (1.5)*. Then (X, F) is a four braid pair in the case $n = 2k$. If $n = 2k-1$, then (X, F) is a four braid pair if and only if $a^2 = aw + z$.*

Next we consider the standard R-matrices related to the root systems B_n and C_n. See [**??**] and [**??**] for the use of these matrices in knot and tangle theory.

The matrix $X_n = X_n(B)$ describes an automorphism of $V \otimes V$ with $\dim V = 2n+1 = m$ in the lexicographical basis v_{ij}, $1 \le i, j \le 2n+1 = m$. If $i+j \ne m+1$, then X_n coincides with a matrix of A-type, as specified in (1.5). The subspace of $V \otimes V$ generated by the v_{ij} for $i + j = m + 1$ is invariant under X_n. The corresponding matrix block will be denoted by Z_n. We describe Z_n inductively. Again we use $\delta = q - q^{-1}$ and set $p = q^{1/2}$. We let Z_0 denote the unit matrix of size 1. The matrix Z_n is a symmetric matrix with central matrix Z_{n-1}, i. e. we adjoin to Z_{n-1} new rows and columns in the positions 1 and $2n+1$. The $(2n+1)$st row is $(q^{-1}, 0, \dots, 0)$. The first row is

$$-\delta(q^{-(2n-1)} - 1, q^{-(2n-2)}, \dots, q^{-n}, p^{-(2n-1)}, q^{-n+1}, \dots, q^{-1}, 1) + (0, \dots, 0, q).$$

Let now F_n denote a $(2n+1, 2n+1)$-matrix as in (1.6) with $w = p^{-1} - p$, $a = -p$, $\beta = \beta' = 1$ and set $Y_n = F_n \otimes 1$.

(17.8) Theorem. *The matrices $X_n(B)$ and Y_n are a four braid pair.*

We now consider the R-matrices $X_n(C) = X'_n$ which act on $V \otimes V$ with $\dim V = 2n = m$. Again, the v_{ij} for $i+j \ne 2n+1$ are mapped as for $X_n(B)$. The subspace of $V \otimes V$ generated by the v_{ij} with $i+j = 2n+1$ is invariant under X'_n and the corresponding matrix block Z'_n is defined inductively, beginning with

$$Z'_1 = \begin{pmatrix} \delta(1+q^{-2}) & q^{-1} \\ q^{-1} & 0 \end{pmatrix}.$$

The matrix Z'_n is a symmetric matrix with central matrix Z'_{n-1}. Its $2n$-th row is $(q^{-1}, 0, \dots, 0)$. The first row is

$$\delta(1 + q^{-2n}, q^{-2n+1}, \dots, q^{-n-1}, -q^{-n+1}, \dots, -q^{-1}, -1) + (0, \dots, 0, q).$$

Let now F'_n be a $(2n, 2n)$-matrix as in (1.6) with $\beta = \beta'$ and set $Y'_n = F'_n \otimes 1$.

(17.9) Theorem. *The matrices $X_n(C)$ and Y'_n are a four braid pair.*

The fundamental four braid pairs of the theorems above induce, by general formalism of quantum groups and braided tensor categories, further such pairs on integrable modules over the quantum groups related to A, B, and C (see [**??**]).

18. R-matrices of type A_n

This section contains the proof of Theorem (1.7).

We begin with a prototype computation which is used later on several occasions. We think of X and Y as given by $(2,2)$-block-matrices of the following type:

(18.1)
$$X = \begin{pmatrix} Z & 0 \\ 0 & qI \end{pmatrix}, \qquad Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here $q \in \mathfrak{K}^*$ (the units of $\mathfrak{K}$) and I is a unit matrix. The matrices Z, A (and qI, D) are square matrices of the same size, respectively. A computation of the four braid relation in block form yields:

(18.2) Proposition. *The four braid relation* (1.1) *holds for the matrices* (2.1) *if and only if the following equalities hold:*
(I) $ZAZA + qZBC = AZAZ + qBCZ$
(II) $Z(AZB + qBD) = q(AZB + qBD)$
(III) $(CZA + qDC)Z = q(CZA + qDC)$. □

Equation (II) means that the columns of $AZB+qBD$ are eigenvectors of Z for the eigenvalue q (if they are nonzero). Equation (III) has a similar interpretation for the row vectors of $CZA + qDC$ (or consider the transpose).

We now turn our attention to R-matrices. We begin with the simplest nontrivial R-matrix $X = X_2(q)$ of type (1.5)

(18.3)
$$X = \begin{pmatrix} q & & & \\ & \delta & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix} \qquad \delta = q - q^{-1} \neq 0$$

and look for matrices
$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
such that (X, F) is a four braid pair.

(18.4) Proposition. *Suppose F is not a multiple of the identity. Then (X,F) is a four braid pair if and only if $a = 0$.*

Proof. We reorder the basis $v_{12}, v_{21}, v_{11}, v_{22}$. Then X has the form (2.1) with
$$Z = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix},$$
and Y has the form (2.1) with
$$A = D = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad B = C = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

We check the conditions (2.2). Equation (I) holds if and only if $a^2\delta = ad\delta$. Hence either $a = 0$ or $a = d$. We compute $AZB + qBD$ to be

$$\begin{pmatrix} ac & ab\delta + qbd \\ qac & bd \end{pmatrix}.$$

Since $(q,1)^t$ is the eigenvector for the eigenvalue q of Z, we must have, by (II), that $ac = 0$ and $ab = 0$. In case $a \neq 0$, we arrive at a multiple of the identity. If $a = 0$, then (II), and dually (III), are satisfied. □

Proof of Theorem (1.7). For the proof we need a bit of organization. We take advantage of the fact that X and F have many zeros and repetitions. We have two involutions σ and τ on the set of indices $J = \{(i,j) \mid 1 \le i,j \le n\}$, namely $\sigma(i,j) = (n+1-i,j)$ and $\tau(i,j) = (j,i)$. Since $\sigma\tau\sigma\tau = \tau\sigma\tau\sigma$, they formally generate the dihedral group D_8 of order 8. We decompose J into the orbits under this D_8-action. We have to consider 4 orbit types. Set $n+1-i = i'$.

Let $n = 2k$. Then we have orbits of type $(i,i),(i',i),(i,i'),(i',i')$ of length 4. This is the orbit of (i,j) if $i = j$ or $i = j'$. If $i \neq j,j'$, then the orbit of (i,j) has length 8.

Let $n = 2k-1$. Then we have the fixed point (k,k). There is another orbit type of length 4, namely $(k,j),(j,k),(j',k),(k,j')$ for $j \neq m$. The subspace spanned by an orbit is invariant under X and Y. Therefore it suffices to verify the four braid relation on these subspaces. The matrices involved depend only on the isomorphism type of the orbit. Hence we need only consider the cases $n = 3$ and $n = 4$.

We present some details of the computation.

Let $n = 3$. We consider the subspace generated by $v_{12}, v_{21}, v_{23}, v_{32}$. The corresponding matrices have the following form:

$$X = \begin{pmatrix} \delta & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \delta & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 & \beta' \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ \beta & 0 & 0 & w \end{pmatrix}.$$

We compute the product XY and its square

$$\begin{pmatrix} 0 & a & 0 & \delta\beta' \\ 0 & 0 & 0 & \beta' \\ \beta & 0 & a\delta & w \\ 0 & 0 & a & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & \delta a\beta' & a\beta' \\ 0 & 0 & a\beta' & 0 \\ \delta a\beta & a\beta & \delta^2 a^2 + aw & \delta z + \delta aw \\ a\beta & 0 & \delta a^2 & aw \end{pmatrix}.$$

If we transpose the latter matrix and interchange β' and β we obtain $(YX)^2$. On the other hand, we obtain by this procedure the same matrix if and only if a satisfies $a^2 = aw + z$.

Let $n = 4$. We use the subspace generated by $v_{12}, v_{21}, v_{13}, v_{31}, v_{24}, v_{42}, v_{34}, v_{43}$. Then X has the block diagonal matrix

$$X = \begin{pmatrix} H & & & \\ & H & & \\ & & H & \\ & & & H \end{pmatrix}, \quad H = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}$$

and Y the matrix

$$Y = \left(\begin{array}{cc|cc|cc|cc} & & 0 & 0 & 0 & \beta' & & \\ & & 0 & \beta' & 0 & 0 & & \\ \hline 0 & 0 & 0 & 0 & & & 0 & \beta' \\ 0 & \beta & 0 & w & & & 0 & 0 \\ \hline 0 & 0 & & & 0 & 0 & \beta' & 0 \\ \beta & 0 & & & 0 & w & 0 & 0 \\ \hline & & 0 & 0 & \beta & 0 & w & 0 \\ & & \beta & 0 & 0 & 0 & 0 & w \end{array}\right).$$

Empty places contain, as always, a zero. These data yield the product:

$$XY = \left(\begin{array}{cc|cc|cc|cc} & & 0 & \beta' & 0 & \delta\beta' & & \\ & & 0 & 0 & 0 & \beta' & & \\ \hline 0 & \beta & 0 & w & & & 0 & \delta\beta' \\ 0 & 0 & 0 & 0 & & & 0 & \beta' \\ \hline \beta & 0 & & & 0 & w & \delta\beta' & 0 \\ 0 & 0 & & & 0 & 0 & \beta' & 0 \\ \hline & & \beta & 0 & \delta\beta & 0 & w\delta & w \\ & & 0 & 0 & \beta & 0 & w & 0 \end{array}\right)$$

and its square

$$(XY)^2 = \begin{pmatrix} 0 & 0 & 0 & \beta'\beta' H \\ 0 & 0 & \beta\beta' H & \beta' w H \\ 0 & \beta\beta' H & z\delta H & \beta' w(\delta H + I) \\ \beta\beta H & \beta w H & \beta w(\delta H + I) & z\delta H + w^2(\delta H + I) \end{pmatrix}.$$

This matrix does not change if we transpose it and interchange β' and β. This finishes the case $n = 4$. □

(18.5) Remarks. The matrix Y satisfies the equation $Y^2 = wY + z$. A similar result as (1.7) holds for the slightly more general R-matrices in [**??**, p. 71]. Suppose $q + q^{-1}$ is invertible in $\mathfrak{K}$. Then the eigenspace $S^2(V)$ of X for the eigenvalue q has the basis $v_{ij} + q^{-1} v_{ji}$ for $i < j$ and v_{ii}; and the eigenspace $\wedge^2(V)$ for the eigenvalue $-q^{-1}$ has the basis $v_{ij} - q v_{ji}$, $i < j$. ♡

19. R-matrices of type B_n and C_n

This section contains the proof of Theorems (1.8) and (1.9).

We need information about the eigenspaces of Z_n. We set $e_j = v_{j,m+1-j}$ and $m = 2n+1$. The eigenvectors in (3.1) are linearly independent. They form a basis if we assume that $(q+q^{-1})(q-q^{-m+1})(q^{-1}+q^{-m+1})$ is invertible in $\mathfrak{K}$. There are analogous assumptions in (3.2).

(19.1) Proposition. *The matrix Z_n has eigenvalues $q, -q^{-1}, q^{-m+1}$.*

(1) *The q-eigenspace has the basis*

$$\begin{aligned} z_j &= qe_j + q^{-1}e_{m+1-j} - e_{j+1} - e_{m+1-(j+1)}, \qquad 1 \le j \le n-1 \\ z_n &= qe_n - (p+p^{-1})e_{n+1} + q^{-1}e_{n+2}. \end{aligned}$$

(2) *The $(-q^{-1})$-eigenspace has the basis*

$$\begin{aligned} y_j &= (e_j - e_{m+1-j}) - (q^{-1}e_{j+1} - qe_{m+1-(j+1)}), \qquad 1 \le j \le n-1 \\ y_n &= e_n + (p-p^{-1})e_{n+1} - e_{n+2}. \end{aligned}$$

(3) *An eigenvector for q^{-m+1} is*

$$(1, q, \ldots, q^{n-1}, p^{2n-1}, q^n, \ldots, q^{2n-1}).$$

Proof. We prove (1) by induction on n. The case $n = 1$ is a simple verification. For the induction step it remains to check:

(1) z_1 is an eigenvector of Z_n.

(2) The scalar product of the first row of Z_n with $z_2, \ldots, z_n$ is zero.

For (1), we compute the scalar product of z_1 with the first row to be

$$\delta(1-q^{-2n+1})q + \delta q^{-2n+2} + \delta q^{-1} + q^{-1}q^{-1} = q^2,$$

and with the second row to be

$$-\delta q^{-2n+2}q - \delta(1-q^{-2n+3}) - q^{-1} = -q.$$

These values are correct. The scalar product with rows 3 to m gives trivially the correct result. For (2), we compute the scalar product of the first row of Z_n with z_2 to be

$$-\delta q^{-2n+2}q + \delta q^{-2n+3} + \delta q^{-2} - \delta q^{-2} = 0$$

and similarly for $z_3, \ldots, z_{n-1}$. For z_n we have

$$-\delta q^{-n}q + \delta(p+p^{-1})p^{-2n+1} - \delta q^{-n+1}q^{-1} = 0.$$

The verification for the other eigenspaces is similar. □

Proof of Theorem (1.8). As before, we decompose $X_n(B)$ and Y_n into suitable blocks. The subspace W of $V \otimes V$ generated by the v_{ii}, $v_{i,m+1-i}$ for $1 \le i \le m$

is invariant under X and Y. The remaining basis elements generate a subspace where the four braid relation is satisfied by the results of section 2. We order the basis of W as follows:

$$v_{1,m}, v_{2,m-1}, \dots, v_{m,1}, v_{11}, \dots, v_{mm}.$$

We assume that $v_{n+1,n+1}$ occurs among the $v_{j,m+1-j}$. In that case, we are in the formal situation of (2.2) with $Z = Z_n$ and diagonal matrices $A = \mathrm{Dia}(0, \dots, 0, -p, w, \dots, w)$, $D = \mathrm{Dia}(0, \dots, 0, w, \dots, w)$ with w appearing n times in A and D. Moreover $B^t = C$ and

$$B = \left(\begin{array}{c|c} 0 & J \\ \hline 0 & 0 \\ \hline J & 0 \end{array}\right), \qquad J = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

i. e. J is the co-unit matrix. We write Z in block form

$$Z = \begin{pmatrix} \alpha & b & c \\ d & 1 & 0 \\ f & 0 & 0 \end{pmatrix}$$

with diagonal block $\alpha, 1, 0$ of size $n, 1, n$. Then we compute

$$AZB + qBD = \begin{pmatrix} 0 & qwJ \\ 0 & -pdJ \\ 0 & wfJ \end{pmatrix}.$$

From the structure of d and f one easily verifies the following: Let $S_1, \dots, S_n$ be the first n columns of $AZB + qBD$ from right to left. Then the column vectors are related to the eigenvectors of the q-eigenspace of Z as follows:

$$wz_j = S_j - qS_{j+1}, \quad \text{for} \quad 1 \le j \le n-1, \qquad wz_n = S_n.$$

Hence the S_j are eigenvectors. This implies (2.2, II), and (2.2, III) follows by transposition.

A similar computation with the block matrices shows that $AZA + qBC$ is q times the unit matrix. This implies (2.2, I). $\square$

We also need the eigenspace structure of Z'_n.

(19.2) Proposition. *Z'_n has eigenvalues q, $-q^{-1}$, $-q^{-m-1}$.*

(1) *The q-eigenspace has the basis*

$$\begin{aligned} z'_j &= qe_j + q^{-1}e_{m+1-j} - e_{j+1} - e_{m+1-(j+1)}, \qquad 1 \le j \le n-1 \\ z'_n &= qe_n + q^{-1}e_{n+1}. \end{aligned}$$

(2) *The $(-q^{-1})$-eigenspace has the basis*

$$y'_j = (e_j - e_{m+1-j}) - (q^{-1}e_{j+1} - qe_{m+1-(j+1)}), \qquad 1 \le j \le n-1.$$

(3) *An eigenvector for the eigenvalue $-q^{-m-1}$ is*

$$(1, q, \ldots, q^{n-1}, -q^{n+1}, \ldots, -q^{2n}).$$

Proof. The proof of (3.2) is by induction on n as for (3.1). □

Proof of Theorem (1.9). We use the same method as for Theorem (1.8). We have $A = D = \mathrm{Dia}(0, \ldots, 0, w, \ldots, w)$ with w appearing n times and $B = C = \beta J$. We write Z' in block form

$$Z' = \begin{pmatrix} \alpha & b \\ c & 0 \end{pmatrix}$$

with blocks of size n and compute

$$AZ'B + qBD = \beta w \begin{pmatrix} 0 & qJ \\ 0 & cJ \end{pmatrix}.$$

One verifies that the non-zero columns $S'_1, \ldots, S'_n$ of $AZ'B + qBD$ from right to left have the form $S'_j - q^{-1}S'_{j+1} = \beta w z'_j$ for $1 \le j \le n-1$ and $S'_n = \beta w z'_n$. Therefore (2.2, II and III) hold.

The matrix $AZA + qBC$ is again q times the unit matrix. Therefore (2.2, I) holds. □

3 The Quantum Group SL_2

1. The Hopf algebra U

We define the basic example of a quantum group. We fix an invertible parameter $v \in \mathfrak{K}$ and an integer $t \geq 1$. The algebra $U(t;v)$ is the associative algebra with 1 over $\mathfrak{K}$ with generators K, K^{-1}, E, F and relations

(1.1) Relations of the algebra $U(t;v)$.

$$KK^{-1} = K^{-1}K = 1$$

$$KE = v^2 EK, \quad KF = v^{-2} FK$$

$$[E,F] := EF - FE = \frac{K^t - K^{-t}}{v^t - v^{-t}}.$$

It is assumed that $v^t - v^{-t} \in \mathfrak{K}$ is a unit, in particular $v^{2t} \neq 1$. We write $U = U(1;v)$. The following data make $U(t;v)$ into a Hopf algebra with invertible antipode s:

(1.2) $$\mu(K) = K \otimes K, \; \mu(E) = E \otimes 1 + K^t \otimes E, \; \mu(F) = F \otimes K^{-t} + 1 \otimes F$$

(1.3) $$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0$$

(1.4) $$s(K) = K^{-1}, \quad s(E) = -K^{-t}E, \quad s(F) = -FK^t.$$

The meaning of this statement is the following: We specify (anti-)homomorphisms of algebras μ, ε, s by (1.2) – (1.4). One has to verify that these are well defined, i. e. compatible with (1.1). Then one has to show that μ is coassociative, i. e. the homomorphisms of algebras $(\mu \otimes 1)\mu$ and $(1 \otimes \mu)\mu$ coincide. It suffices to check this on the generators E, F, K. Finally, one has to check the counit axiom and the antipode axiom. Again it suffices to check these on algebra generators. All these verifications are straightforward. The inverse of s is given by

$$s^{-1}(K) = K^{-1}, \quad s^{-1}(E) = -EK^{-t}, \quad s^{-1}(F) = -K^t F.$$

There is a second comultiplication which makes U into a Hopf algebra, see exercise 1.

In order to work with the algebra U we need commutator relations. We write

(1.5) $$[m;v] = [m] = \frac{v^m - v^{-m}}{v - v^{-1}}, \qquad m \in \mathbb{Z}$$

(1.6) $$[m,K;v] = [m,K] = \frac{v^m K - v^{-m}K^{-1}}{v - v^{-1}}, \qquad m \in \mathbb{Z}.$$

If we think of an indeterminate v, then $[m] \in \mathbb{Z}[v, v^{-1}]$. Hence we can specialize to any $v \in \mathfrak{K}$. The following identities are proved by induction on m:

(1.7)
$$\begin{aligned} [E, F^m] &= [m]F^{m-1}[-m+1, K] = [m][m-1, K]F^{m-1} \\ [E^m, F] &= [m][-m+1, K]E^{m-1} = [m]E^{m-1}[m-1, K]. \end{aligned}$$

We give the proof for the first one. We use

$$[E^m, F^{n+1}] = [E^m, F^n]F + F^n[E^m, F].$$

The case $m = 1$ in (1.7) is the defining relation (1.1) for $[E, F]$. The induction step is

$$\begin{aligned} [E, F^{m+1}] &= [m]F^{m-1}[-m+1, K]F + F^m[0, K] \\ &= F^m([m][-m+1, K] + [0, K]). \end{aligned}$$

We now use $[m]v^{-m+1} + [1] = [m+1]v^{-m}$ in order to rewrite the bracket as $[m+1][-m, K]$.

The relations (1.7) for the algebra $U(t; v)$ are obtained by using v^t, K^t instead of v, K in (1.7).

We now state the general commutator rule. This uses the following notations.

(1.8)
$$[n; v]! = [n]! = [1][2] \dots [n]$$

(1.9)
$$[n, j, K; v] = [n, j, K] = \frac{[n, K][n-1, K] \dots [n-j+1, K]}{[j]!}.$$

(1.9) is a generalized binomial coefficient. If we formally replace K by 1, we obtain $[n, j; v]$. There is a generalized Pascal formula (exercise 3). It can be used to show inductively that $[n, j; v] \in \mathbb{Z}[v, v^{-1}]$. We assume that the $[m]!$ are invertible in $\mathfrak{K}$ and define divided powers

(1.10)
$$E^{(m)} = \frac{E^m}{[m]!}, \qquad F^{(m)} = \frac{F^m}{[m]!}.$$

Now we can state:

(1.11) Proposition. *The following commutator relations hold in U*

$$[E^{(m)}, F^{(n)}] = \sum_{j=1}^{\min(m,n)} F^{(n-j)}[-m-n+2j, j, K]E^{(m-j)}.$$

A simple verification with (1.1) yields:

(1.12) Proposition. *The element*

$$C = EF + \frac{v^{-t}K^t + v^tK^{-t}}{(v^t - v^{-t})^2} = FE + \frac{v^tK^t + v^{-t}K^{-t}}{(v^t - v^{-t})^2}$$

is contained in the center of the algebra $U(t; v)$. □

We call C the *Casimir element* of $U(t;v)$.

(1.13) Exercises and supplements.

1. The following data make $U(t;v)$ into a Hopf algebra:

$$\bar{\mu}(K) = K \otimes K,\ \bar{\mu}(E) = E \otimes 1 + K^{-t} \otimes E,\ \bar{\mu}(F) = F \otimes K^t + 1 \otimes F$$

$$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0$$
$$\bar{s}(K) = K^{-1}, \quad \bar{s}(E) = -K^t E, \quad \bar{s}(F) = -FK^{-t}.$$

2. The assignment $\omega(E) = F$, $\omega(F) = E$, $\omega(K) = K^{-1}$ defines an automorphism of the algebra U and an antiautomorphism of the associated coalgebra. Apply ω to the first identity in (1.7) to obtain the second one.

3. Verify the Pascal formula

$$[n+1, s, K] = [s+1][n, s, K] - [n-s, K][n, s-1, K].$$

4. Verify (1.11) by induction on m. Use the Pascal formula of the preceding exercise. For $U(t;v)$, one has to replace v, K again by v^t, K^t.

5. The algebra U carries a $\mathbb{Z}$-grading: Assign the degree $1, -1, 0$ to E, F, K, respectively.

2. Integrable U-modules

We consider the algebra $U = U(t;v)$ over a field $\mathfrak{K}$ and assume that v is not a root of unity (this is called the *generic* case). A U-module M is called *split* if the $\mathfrak{K}$-vector space M is a direct sum $M = \bigoplus_{n \in \mathbb{Z}} M^n$ and K acts on M^n as multiplication by v^n (eigenspace). Since v is not a root of unity, the powers v^n, $n \in \mathbb{Z}$, are pairwise different; therefore the definition of split is meaningful. (Later we shall allow more general eigenvalues.) The relations (1.1) imply for a split module:

(2.1) Note. *$E(M^n) \subset M^{n+2}$, $F(M^n) \subset M^{n-2}$ and $EF - FE\colon M^n \to M^n$ is multiplication by $[n; v^t]$. Conversely, operators E and F on a $\mathbb{Z}$-graded vector space $\bigoplus M^n = M$ define on M the structure of a split U-module if the statements of the first sentence hold.* □

A split module M is called *integrable*, if the operators E and F are *locally nilpotent*, i. e. for each $x \in M$ there exists $n \in \mathbb{N}$ such that $E^n x = 0 = F^n x$. The Casimir element C from (1.12) acts on $x \in M^n$ as

$$C(x) = EF(x) + c_n \cdot x = FE(x) + d_n \cdot x$$

with

$$c_n = \frac{v^{t(n-1)} + v^{-t(n-1)}}{(v^t - v^{-t})^2} = d_{-n}, \quad n \in \mathbb{Z}.$$

(2.2) Theorem. *Let $n \in \mathbb{Z}$, $m \in \mathbb{N}$. Let M be an integrable U-module.*

(1) *The subspace* $M^n(m) = \{x \in M^n \mid E^m x = 0\}$ *is* C*-stable, and* C *satisfies*

$$(C - d_n)(C - d_{n+2}) \cdots (C - d_{n+2m}) = 0$$

on this subspace.

(2) *The subspace* $M^n[m] = \{x \in M^n \mid F^m x = 0\}$ *is* C*-stable, and* C *satisfies*

$$(C - c_n)(C - c_{n-2}) \cdots (C - c_{n-2m}) = 0$$

on this subspace.

(3) M^n *is the direct sum of* C*-eigenspaces. If* $n \geq 0$*, the eigenvalues are contained in* $\{d_n, d_{n+2}, d_{n+4}, \ldots\}$*. If* $n \leq 0$*, the eigenvalues are contained in* $\{d_{-n}, d_{-n+2}, d_{-n+4}, \ldots\}$*.*

Proof. (1) Since C lies in the center of U, the subspaces $M^n(m)$ are C-stable. The identity is proved by induction on m. For $x \in M^n(0)$ we have $C(x) = FE(x) + d_n x = d_n x$ by definition of C, hence $(C - d_n)x = 0$. For $x \in M^n(m)$ we have $Ex \in M^{n+2}(m-1)$. By induction,

$$(C - d_{n+2}) \cdots (C - d_{n+2m})Ex = 0.$$

We apply F to this element, use $FE = C - d_n$ on M^n and the fact that C is central.

(2) is proved similarly.

(3) Let $n \geq 0$. The elements d_n, $n \in \mathbb{N}_0$ are pairwise different. Hence the product $(C - d_n) \cdots (C - d_{n+2m})$ is the minimal polynomial of the operator C on $M^n(m)$, and it consists of different linear factors. By linear algebra, $M^n(m)$ is the direct sum of the C-eigenspaces. Since $M^n = \bigcup_{m \geq 0} M^n(m)$, also M^n is the direct sum of C-eigenspaces. Similarly for $n \leq 0$. $\square$

(2.3) Corollary. *An integrable* U*-module* M *is the direct sum of subspaces* $M(m)$ *such that* C *acts on* $M(m)$ *as multiplication by* d_m*,* $m \in \mathbb{N}_0$*.* $\square$

(2.4) Theorem. *Let* M *be an integrable* U*-module such that* C *acts as* d_m*,* $m \in \mathbb{N}_0$*.*

(1) *If* $M^n \neq 0$*, then* $n \in \{-m, -m+2, \ldots, m-2, m\}$*.*

(2) *Let* $n, n+2 \in \{-m, \ldots, m\}$*. Then* $FE\colon M^n \to M^n$ *and* $EF\colon M^{n+2} \to M^{n+2}$ *are multiplication by* $d_m - d_n$*. In particular,* $E\colon M^n \to M^{n+2}$*, and* $F\colon M^{n+2} \to M^n$ *are isomorphisms.*

Proof. (1) By (2.2), the eigenvalue d_m of C is contained in $\{d_{n+2k} \mid k \in \mathbb{N}_0\}$ or in $\{d_{-n+2k} \mid k \in \mathbb{N}_0\}$. Hence either $0 \leq m = 2k + n$, $n = m - 2k$, $n \geq 0$ or $0 \leq m = -n + 2k$, $n = -m + 2k$, $n \leq 0$.

(2) Let $x \in M^n$. Then $FEx = (C - d_n)x = (d_m - d_n)x$. Let $y \in M^{n+2}$. Then $EFy = (C - c_{n+2})y = (d_m - c_{n+2})y = (d_m - d_n)y$. Since $m \geq n+2$, we have $d_m - d_n \neq 0$. $\square$

We now consider also more general modules M. If $0 \neq x \in M$ and $Kx = \lambda x$, then x is called a *weight vector* of *weight* λ. A weight vector x is called *primitive* (or a heighest weight vector), if $Ex = 0$. The subspace $M(\lambda)$ spanned by the weight vectors of weight λ is called the *weight space* of weight λ. A module is called *split* if it is the direct sum of its weight spaces.

We construct a universal module which is generated by a primitive vector of weight $0 \neq \lambda \in \mathfrak{K}$. It is called the *Verma module* $V(\lambda)$. We start with the free $\mathfrak{K}$-module with basis y_j, $j \in \mathbb{N}_0$ $(y_{-1} = 0)$ and action

$$\textbf{(2.5)} \qquad \begin{aligned} K^{\pm 1} y_n &= \lambda^{\pm} v^{\mp 2n} y_n \\ F y_n &= y_{n+1} \\ E y_n &= [n; v^t][-n+1, \lambda^t; v^t] y_{n-1}. \end{aligned}$$

We recall the notation

$$[n; v^t] = \frac{v^{nt} - v^{-nt}}{v^t - v^{-t}}, \quad [m, \lambda^t; v^t] = \frac{v^{tm}\lambda^t - v^{-tm}\lambda^{-t}}{v^t - v^{-t}}.$$

For the proof, we try to define a module structure with $Ey_n = a_n y_{n-1}$ in (2.5) and show that a necessary and sufficient condition for a_n is the given value. The relations $KE = v^2 EK$ and $KF = v^{-2} FK$ are satisfied by construction. The relation $EF - FK = (K^t - K^{-t})(v^t - v^{-t})^{-1}$ is satisfied if and only if

$$a_{n+1} - a_n = \frac{\lambda^t v^{-2nt} - \lambda^{-t} v^{2n}}{v^t - v^{-t}}.$$

If this holds, then $a_{n+1} = \sum_{j=0}^{n} (a_{j+1} - a_j)$ is seen to have the stated value (and conversely).

(2.6) Proposition. *Let M be a U-module which contains a primitive vector x of weight λ. Then there exists a unique U-linear $f\colon V(\lambda) \to M$ with $f(y_0) = x$.*

Proof. Define a $\mathfrak{K}$-linear map $f\colon V(\lambda) \to M$ by $f(y_n) = F^n x$. Then $f(Fy_n) = Ff(y_n)$ by construction. Also $f(Ky_n) = Kf(y_n)$, since $F^n x$ has weight λv^{-2n}. Finally, $Ef(y_n) = EF^n x = [E, F^n]x = [n; v^t]F^{n-1}[-n+1, K^t; v^t]x = a_n F^{n-1} x = a_n f(y_{n-1} = f(Ey_n)$. (We have used (1.7) and the fact that x is primitive.) Hence f is also compatible with E. □

The Verma module $V(\lambda)$ can have proper submodules. If $Ey_{n+1} = 0$, then the span of $\{y_m \mid m > n\}$ is a submodule. The relation $Ey_{n+1} = 0$ is equivalent to $a_{n+1} = 0$, i. e. equivalent to

$$\textbf{(2.7)} \qquad (v^{(n+1)t} - v^{-(n+1)t})(v^{-nt}\lambda^t - v^{tn}\lambda^{-t}) = 0,$$

since $v^t - v^{-t}$ is assumed to be invertible. In the generic case we thus have:

$$\textbf{(2.8)} \qquad a_{n+1} = 0 \quad \Leftrightarrow \quad \lambda = \eta v^n, \ \eta^{2t} = 1.$$

In this case, we have the factor module $V_{n,\eta}$ of dimension $n+1$ of $V(\lambda)$ with basis $y_0, \dots, y_n$ and operators

(2.9)
$$\begin{array}{rcl} Ky_j &=& \eta v^{n-2j} v_j \\ Fy_j &=& y_{j+1}, \qquad Fy_n = 0 \\ Ey_j &=& \eta^t [j; v^t][n-j+1; v^t] y_{j-1}, \quad Ey_0 = 0. \end{array}$$

(2.10) Lemma. *Let M be an integrable module and $0 \neq y \in M(\lambda)$. Then $\lambda = \eta v^n$ for some $n \in \mathbb{Z}$ and $\eta^{2t} = 1$. If y is primitive, then $n \in \mathbb{N}_0$.*

PROOF. Choose $k \in \mathbb{N}_0$ such that $x = E^k y \neq 0$ and $Ex = 0$. Then $x \in M(\lambda v^{2k})$. There exists $n \in \mathbb{N}_0$ such that $F^{n+1}x = 0$ but $F^n x \neq 0$. Let $j \colon V(\lambda v^{2k}) \to M$ be the morphism with $f(y_0) = x$. Then $a_{n+1} F^n x = a_{n+1} f(y_n) = f(Ey_{n+1}) = f(EF^{n+1} y_0) = EF^{n+1} x = 0$. Hence $a_{n+1} = 0$, and we can apply (2.8): $\lambda v^{2k} = \eta v^n$ for some $n \in \mathbb{N}_0$ and $\eta^{2t} = 1$. □

(2.11) Theorem. *A split, finite dimensional, simple U-module is isomorphic to a module $V_{n,\eta}$. The modules $V_{n,\eta}$ are simple. An isomorphism $V_{n,\eta} \cong V_{m,\gamma}$ implies $n = m$ and $\eta = \gamma$.*

PROOF. Let M be split and finite dimensional. If $0 \neq x \in M(\lambda)$, then $E^n x \in M(\lambda v^{2n})$ if $E^n x \neq 0$. Since v is not a root of unity, the λv^{2n} are pairwise different. Since M has finite dimension, there exists $n \in \mathbb{N}_0$ such that $E^n x \neq 0$ but $E^{n+1} x = 0$, i. e. M contains primitive vectors.

Let $x \in M$ be primitive. Suppose $F^{n+1} x = 0$ and $F^n x \neq 0$. The vectors $F^j x$, $0 \leq j \leq n$, have eigenvalues ηv^{n-2j} (by the proof of (2.10)) and are therefore linearly independent. The universal map $f \colon V(\eta v^n) \to M$ with $f(y_0) = x$ factors over the quotient $V_{n,\eta}$ and induces $\bar{f} \colon V_{n,\eta} \to M$. Since M is simple, the map $\bar{f}$ is surjective. The image contains the elements $F^j x$ and has therefore at least dimension $n+1$. Hence $\bar{f}$ is an isomorphism.

Let $U \neq 0$ be a submodule of $V_{n,\eta}$. We write $0 \neq y \in U$ as a linear combination of the weight vectors y_j. From the structure of $V_{n,\eta}$ we see that for some $k \in \mathbb{N}_0$ the vector $E^k y$ is primitive. A primitive vector of $V_{n,\eta}$ is a scalar multiple of y_0. Hence $y_0 \in U$, and $U = V_{n,\eta}$, since y_0 generates $V_{n,\eta}$. From the uniqueness of primitive vectors we see that the isomorphism type of $V_{n,\eta}$ determines (n, η). □

(2.12) Corollary. *The modules $V_{0,\eta}$ are the one-dimensional U-modules.* □

(2.13) Theorem. *Let M be a U-module. The following are equivalent:*
(1) *M is integrable and split.*
(2) *M is a direct sum of split, finite dimensional, simple modules.*

PROOF. (2) ⇒ (1). A direct sum of split modules is split. A direct sum of modules with locally nilpotent action of E and F has again this property.

(1) ⇒ (2). From (2.10) we know that weights have the form ηv^n, $n \in \mathbb{Z}$, $\eta^{2t} = 1$. The group of weights is therefore $\Gamma = \{\eta v^n \mid \eta^{2t} = 1, n \in \mathbb{Z}\}$. Since $EM(\lambda) \subset M(\lambda v^2)$, $FM(\lambda) \subset M(\lambda v^{-2})$, we can decompose M into the direct sum of modules M_j, where M_j contains only weights from a single coset $j \in \Gamma/\Gamma_0$, $\Gamma_0 = \{v^{2n} \mid n \in \mathbb{Z}\}$. Hence it suffices to decompose a module M_j. Each coset Γ/Γ_0 has associated to it a unique $2t$-th root of unity η. We call it the *type* of the coset and of the associated module M_j. We leave it as an exercise to verify that

$M \otimes V_{0,\gamma}$ has type $\eta\gamma$ if M has type η. Thus it suffices to study modules of type 1. Then we are in the situation which we investigated in (2.2) – (2.4). A module M of type 1 is the direct sum of submodules $M(m)$ such that the Casimir operator C acts on $M(m)$ as multiplication by d_m. From (2.4) we see that $M(m)$ is generated by primitive vectors of weight v^m. The submodule generated by a single primitive vector of weight v^m is isomorphic to $V_{m,1}$. Thus $M(m)$ is the sum of simple modules isomorphic to $V_{m,1}$, and hence, by general theory of semi-simple modules, the direct sum of such modules. □

The *quantum plane* P is the $\mathfrak{K}$-algebra generated by x, y with relation $yx = vxy$. In the next theorem we deal with $U = U(1; v)$.

(2.14) Theorem. *There exists a unique structure of a (U-module)-algebra on P with the following properties:*

$$Ex = y,\ Ey = 0,\ Fx = 0,\ Fy = x,\ Kx = v^{-1}x,\ Ky = vy.$$

Proof. The algebra P is the free $\mathfrak{K}$-module with basis $(x^r y^s \mid r, s \in \mathbb{N}_0)$. We define linear maps $E, F, K \colon P \to P$ by

(2.15)
$$\begin{array}{rcl} E(x^r y^s) & = & [r]x^{r-1}y^{s+1} \\ F(x^r y^s) & = & [s]x^{r+1}y^{s-1} \\ K(x^r y^s) & = & v^{s-r}x^r y^s. \end{array}$$

We set $x^{-1} = 0$ and $y^{-1} = 0$. One verifies that these maps satisfy the relations (1.1); they define therefore the structure of a U-module on P. The operators E, F, K yield the structure of a U-algebra if and only if the following relations hold:

(2.16)
$$\begin{array}{rcl} E(u \cdot v) & = & Eu \cdot v + Ku \cdot Ev \\ F(u \cdot v) & = & Fu \cdot K^{-1}v + u \cdot Fv \\ K(u \cdot v) & = & Ku \cdot Kv. \end{array}$$

They are verified from (2.15).

The rules (2.16) and the initial conditions of the theorem yield inductively (2.15), whence the uniqueness. □

Let $P_n \subset P$ denote the $\mathfrak{K}$-submodule with basis $\{r^r y^s \mid r + s = n\}$. Then P_n is U-stable; the resulting module is isomorphic to the simple module $V_{n,1}$.

The Clebsch-Gordan decomposition is the isomorphism of $U = U(1; v)$-modules

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|}.$$

In order to verify it, observe that the V_k are determined by the weights of their primitive vectors. Therefore one has to find in $V_n \otimes V_m$ a primitive vector of weight v^{n+m-2j}. For dimensional reasons we must then have an isomorphism as claimed.

With basis vectors $x_j = x_j^{(n)} \in V_n$ we set

$$X^{n+m-2p} = \sum_{j=0}^{p} \alpha_j x_j^{(n)} \otimes x_{p-j}^{(m)}.$$

We apply $\mu(E) = E \otimes 1 + K \otimes E$ and obtain

$$\sum_{j=0}^{p} \alpha_j \left([n-j+1] x_{j-1}^{(n)} \otimes x_{p-j}^{(m)} + v^{n-2j}[m-p+j+1] x_j^{(n)} \otimes x_{p-j+1}^{(m)} \right).$$

This is zero if and only if

$$\alpha_{j+1}[n-j+2] + \alpha_j v^{n-2j}[m+p+j+1] = 0.$$

We set $\alpha_0 = 1$ and determine recursively

$$X^{n+m-2p} = \sum_{j=0}^{p} (-1)^j v^{jn-j(j-1)} \frac{[m-p+j]![n-j]!}{[m-p]![n]!} x_j^{(n)} \otimes x_{p-j}^{(m)}$$

as a weight vector of the required type.

(2.17) Exercises and supplements.

1. The tensor product of split modules is split. More precisely: $M(\lambda) \otimes N(\gamma) \subset (M \otimes N)(\lambda\gamma)$.

2. The tensor product of integrable modules is integrable.

3. If M has type η, then $M \otimes V_{0,\gamma}$ has type $\eta\gamma$.

4. Suppose $\mathfrak{K}$ is algebraically closed. Then a finite dimensional U-module is split and integrable.

5. The elements $K^a F^b E^c$, $a \in \mathbb{Z}$, $b, c \in \mathbb{N}_0$ form a $\mathfrak{K}$-basis of $U(t;v)$. (Similarly, $E^b K^a F^c$ or $F^c K^a E^b$.) From the relations (1.1) it is easily verified that these elements generate the algebra. To show linear independence, apply a linear relation to suitable modules $V_{n,1}$. Deduce the general case from the generic one by specialization. A basis of this type is called a *Poincaré-Birkhoff-Witt basis* for U.

6. The center of the algebra U is, in the generic case, the polynomial algebra over $\mathfrak{K}$ in the Casimir element C. Use the basis $F^c K^a E^b$ and determine first the elements which commute with K. Show by induction that $C^n = F^n E^n + \sum_0^{n-1} F^j Q_j E^j$ with $Q_j \in \mathfrak{K}[K, K^{-1}]$.

7. The quantum plane is a faithful U-module. In order to deal with the algebras $U(t;v)$ one has to use the quantum plane $P(t) = \mathfrak{K}\{x,y\}/(yx - v^t xy)$ and $[r;v^t]$ etc. in (2.15).

8. A primitive vector in an integrable module generates an irreducible summand.

9. Suppose λ does not have the form ηv^n, $n \in \mathbb{N}_0$, $\eta^{2t} = 1$. Then the Verma module $V(\lambda)$ is simple. **10.** It is sometimes useful to use a different normalisation of the basis vectors in the module $V_n = V_{n,1}$, namely as follows

$$\begin{array}{rclrcl} Kx_j &=& \eta v^{n-2j} x_j & & & \\ Fx_j &=& [j+1; v^t] x_{j+1}, & Fx_n &=& 0 \\ Ex_j &=& \eta^t [n-j+1; v^t] x_{j-1}, & Ex_0 &=& 0. \end{array}$$

An isomorphism to the previously considered presentation is obtained by setting

$$x_j = \frac{1}{[j; v^t]!} y_j.$$

3. The algebra B and its pairing

We use the Hopf algebra $B(s, c^t)$. As an algebra over $\mathfrak{K}$ it is generated by K, K^{-1}, E with relations $KK^{-1} = K^{-1}K = 1$ and $KE = c^t EK$. We use the elementary fact that $(E^a K^\alpha \mid a \in \mathbb{N}_0, \alpha \in \mathbb{Z})$ is a $\mathfrak{K}$-basis of this algebra. The other data of this Hopf algebra are

$$\mu(K) = K \otimes K, \quad \mu(E) = E \otimes 1 + K^s \otimes E$$

$$\varepsilon(K) = 1, \quad \varepsilon(E) = 0$$
$$s(K) = K^{-1}, \quad s(E) = -K^{-s}E.$$

Here $s, t \in \mathbb{N}$, and $c \in \mathfrak{K}$ is an invertible parameter.

(3.1) Theorem. *There exists a unique pairing* $\langle -, - \rangle \colon B(s, c^t) \times B(t, c^s) \to \mathfrak{K}$ *with the following properties:*

(1) $\langle E, E \rangle = \pi$
(2) $\langle E, K \rangle = \langle K, E \rangle = 0$
(3) $\langle K, K \rangle = c.$

Here $0 \neq b \in \mathfrak{K}$ *is an arbitrary parameter. The pairing assumes the values*

$$\langle E^a K^\alpha, E^b K^\beta \rangle = \delta_{a,b} \pi^a c^{\alpha\beta} (a; c^{st})!.$$

Proof. We define linear forms $\Phi(E)$ and $\Phi(K^{\pm 1})$ in the dual $B(t, c^s)^*$ by

$$\Phi(E)(E^a K^\alpha) = \delta_{1,a}\pi, \quad \Phi(K^\pm)(E^a K^\alpha) = \delta_{0,a} c^{\pm\alpha}.$$

We verify that the assignment $E \mapsto \Phi(E)$, $K^{\pm 1} \mapsto \Phi(K^{\pm 1})$ defines a homomorphism of algebras $B(s, c^t) \to B(t, c^s)^*$.

We check the compatibility with the relation $KE = c^t EK$. We compute

$$\begin{aligned} \Phi(K)\Phi(E)(E^a K^\alpha) &= (\Phi(K) \otimes \Phi(E))(\mu(E)^a \mu(K)^\alpha) \\ &= (\Phi(K) \otimes \Phi(E)(\sum_{j=0}^{a} (a, j; c^{st}) E^j K^{t(a-j)+\alpha} \otimes E^{a-j} K^\alpha). \end{aligned}$$

The first equality is the definition of the product in B^* as the dual of μ. The second equality uses the binomial formula (A.1). By the definition of $\Phi(K)$ and $\Phi(E)$, a summand can contribute to the result only if $j = 0$ and $a - j = 1$. The result is the value $c^{t+\alpha}\pi$. A similar computation for $\Phi(E)\Phi(K)$ leads to $j = 1$, $a - j = 0$, and the value $c^\alpha \pi$.

The compatibility with $KK^{-1} = K^{-1}K = 1$ is simpler and left to the reader.

We now define a bilinear form by $\langle x, y \rangle = \Phi(x)(y)$. Then the axioms (1) and (3) of a pairing hold, see I.3.

Next we determine the values of the pairing. For this purpose we compute the product in the dual algebra $B^* = B(t, c^s)^*$. Let $(x_{a,\alpha} \mid a \in \mathbb{N}_0, \alpha \in \mathbb{Z})$ be the algebraic dual basis to $(E^a K^\alpha \mid a \in \mathbb{N}_0, \alpha \in \mathbb{Z})$. Elements in B^* can be written as formal linear combinations $\sum \lambda_{a,\alpha} x_{a,\alpha}$. We set $x_{k,l} \cdot x_{p,q} = \sum a_{m,n} x_{m,n}$. The coefficient $a_{m,n}$ is the coefficient of $E^k K^l \otimes E^p K^q$ in $\mu(E^m K^n)$. We have

$$\mu(E^m K^n) = \sum_{j=0}^{m} (m, j; c^{st}) E^j K^{t(m-j)+n} \otimes E^{m-j} K^n,$$

and therefore

$$k = j, \quad l = t(m-j) + n, \quad p = m - j, \quad q = n$$

$$m = p + k, \quad n = q, \quad l = tp + q.$$

We thus obtain

$$x_{k,l} x_{p,q} = \delta_{l,tp+q} (p + k, k; c^{sk}) x_{p+k,q}.$$

This simplifies for the elements $y_{k,l} = (k; c^{st})! x_{k,l}$ to

(3.2)
$$y_{k,l} y_{p,q} = \delta_{l,tp+q} y_{p+k,q}.$$

We have $\Phi(E) = b \sum_m y_{1,m}$ and $\Phi(K) = \sum_m c^m y_{0,m}$. Hence we consider in general the elements

$$Y(k, \alpha) = \sum_m c^{\alpha m} y_{k,m}.$$

From (3.2) we obtain

(3.3)
$$Y(k, \alpha) Y(l, \beta) = c^{\alpha l} Y(k + l, \alpha + \beta),$$

and in particular $Y(k, \alpha) = Y(1, 0)^k Y(0, 1)^\alpha$. The unit element of B^* is $Y(0, 0)$. We see that

$$\Phi(E)^r \Phi(K)^\alpha = \pi^r Y(r, \alpha).$$

This equality is equivalent to the statement about the values of the pairing (3.1). We see that the pairing is formally symmetric, and therefore also the axioms (2) and (4) of a pairing hold. □

We determine formally the comultiplication ν in $B^* = B(s, c^t)^*$. This is thought of as a map into a suitable completed tensor product $B^* \hat{\otimes} B^*$. Additively, the latter consists of all formal linear combinations of the elements $x_{a,b} \otimes x_{c,d}$. We set

$$\nu(x_{m,n}) = \sum \alpha_{ab,cd} x_{a,b} \otimes x_{c,d}.$$

The coefficient $\alpha_{ab,cd}$ is the coefficient of $E^m K^n$ in the product $E^a K^b E^c K^d = c^{bct} E^{a+c} K^{b+d}$. Hence

$$\nu(x_{m,n}) = \sum_{a+c=m, b+d=n} c^{bct} x_{a,b} \otimes x_{c,d},$$

and, in general, ν is extended linearly. Observe that a and c can only assume non-negative values.

(3.4) Theorem. *Let $\mathfrak{K}$ be a field. The pairing* (3.1) *induces an isomorphism of $B(s,c^t)$ with the finite dual of $B(t,c^s)$.*

Proof. As always for a pairing, the map $\Phi\colon x \mapsto \langle x, - \rangle$ is a homomorphism from $B(s,c^t)$ into the finite dual of $B(t,c^s)$. Suppose $y = \sum \lambda_{a,\alpha} E^a K^\alpha$ is contained in the kernel of Φ. Then

$$\langle y, E^b K^\beta \rangle = \sum_\alpha \lambda_{b,\alpha} \pi^b c^{\alpha\beta} (b : c^{st})! = 0,$$

and therefore $\sum_\alpha \lambda_{b,\alpha} c^{\alpha\beta} = 0$. If this holds for arbitrary β, then $\lambda_{b,\alpha} = 0$ (determinant of Vandermonde). Hence Φ is injective.

Let $V \subset B(t,c^s)$ be a subspace generated by a finite number of elements $E^a K^\alpha$. Again an argument with the determinant of Vandermonde shows that any linear form on V can be realized by an element in the image of Φ.

Suppose W is a finite dimensional $B(t,c^s)$-module. Then E and K satisfy a polynomial equation on W. Therefore $B(t,c^s)$, modulo the kernel of W, is generated by a finite number of elements $E^a K^\alpha$. Hence we can realize the representative functions of W by the image of Φ. □

(3.5) Exercises and supplements.

1. Show that $B(s,c)$ has the $\mathfrak{K}$-basis $(E^a K^\alpha \mid a \in \mathbb{N}_0, \alpha \in \mathbb{Z})$.

2. Show the uniqueness of the pairing (3.1). For this purpose, derive from (3.1.1) – (3.1.3) the values of $\langle E, E^a K^\alpha \rangle$ and $\langle K, E^a K^\alpha \rangle$ which were used in the definition of $\Phi(E)$ and $\Phi(K)$.

4. The algebra U as a quantum double

The pairing of the previous section yields a quantum double. We show that this quantum double is almost the algebra U. We start with the pairing $B(s,c^t) \times B(t,c^s) \to \mathfrak{K}$ and use the following elements in the quantum double $\tilde{U}(s,t;c) = \tilde{U} = B(s,c^t) \otimes B(t,c^s)$

$$E = E \otimes 1, \quad K = K \otimes 1, \quad F = 1 \otimes E, \quad L = 1 \otimes K.$$

(4.1) Lemma. *The following relations hold in $\tilde{U}$:*

$$KE = c^t EK,\ KF = c^{-t} FK,\ LE = c^{-s} EL,\ LF = c^s FL,\ EF - FE = \pi(L^t - K^s).$$

Moreover, K and L commute.

Proof. We use the letter T for the inverse of the antipode. The first and the fourth relation are clear from the definition of the B-algebra. The other relations require a computation with the $\diamond$-product.

$$\begin{aligned}(1\otimes K)\diamond(E\otimes 1) &= \langle 1,K\rangle\langle E,T(K)\rangle 1\otimes K\\ &+\langle 1,K\rangle\langle K^s,T(K)\rangle E\otimes K\\ &+\langle E,K\rangle\langle K^s,T(K)\rangle K^s\otimes K.\end{aligned}$$

Only the second summand yields a contribution; the definition of the pairing and $E\otimes K=(E\otimes 1)\diamond(1\otimes K)$ give the third relation. The second is proved similarly.

In order to compute $FE=(1\otimes E)\diamond(E\otimes 1)$ via $\sum\langle E_3,E_1\rangle\langle E_1,T(E_3)\rangle E_2\otimes E_2$ one has to consider nine summands. We display only those which are non-zero by the general properties of the pairing:

$$\begin{aligned}(1\otimes E)\diamond(E\otimes 1) &= \langle 1,K^t\rangle\langle E,T(E)\rangle 1\otimes K^t\\ &+\langle 1,K^t\rangle\langle K^s,T(1)\rangle E\otimes E\\ &+\langle E,E\rangle\langle K^s,T(1)\rangle K^s\otimes 1.\end{aligned}$$

The result is the last relation of the lemma. □

From the construction we see that $\tilde{U}$ has the $\mathfrak{K}$-basis $(E^aK^mL^nF^b \mid a,b\in \mathbb{N}_0,\ m,n\in\mathbb{Z})$. The algebra U' with generators E,F,K,K^{-1},L,L^{-1} and relations (4.1) together with the obvious ones is generated by the same set. The canonical map $U'\to\tilde{U}$ is therefore an isomorphism. We thus have a description of $\tilde{U}$ by generators and relations.

The quantum double is related in the following manner to an algebra U. Let $h=(s,t)$ denote the greatest common divisor of s and t. Write $rh=st$, $t=ht(1)$, $s=hs(1)$. Set $c^h=v^2$ and $\pi=(v^{-r}-v^r)^{-1}$. Define $U(r;v)$ with generators E,F,M,M^{-1}. We obtain a homomorphism $\rho\colon \tilde{U}(s,t;c)\to U(r;v)$ by the assignment $E\mapsto E$, $F\mapsto F$, $K\mapsto M^{t(1)}$, $L\mapsto M^{-s(1)}$. This is a surjective homomorphism of Hopf algebras and the kernel is the ideal generated by $K^{s(1)}-L^{-t(1)}$.

We discuss Yetter-Drinfeld modules for $B=B(s,c^t)$. We begin with a description of right B-comodules. Let $\mu_M\colon M\to M\otimes B$ be a comodule structure. We define linear operators $p_{a,\alpha}\colon M\to M$ by

$$\mu_M(x)=\sum_{a,\alpha}p_{a,\alpha}(x)\otimes E^aK^\alpha.$$

The axioms of a comodule are satisfied if and only if

$$\textbf{(4.2)}\qquad p_{b,\beta}p_{a,\alpha}=\delta_{sa+\alpha,\beta}(a+b,b;c^{st})p_{a+b,\alpha}$$

$$\textbf{(4.3)}\qquad \sum_\alpha p_{0,\alpha}=\mathrm{id}(M).$$

Since $p_{0,\beta}p_{0,\alpha}=\delta_{\alpha,\beta}p_{0,\alpha}$, the $p_{0,\alpha}$ are orthogonal idempotens with sum 1. If M_α denotes the image of $p_{0,\alpha}$, then M is the direct sum of the M_α. From (4.2) we obtain

$$p_{0,sa+\alpha}p_{a,\alpha}=p_{a,\alpha},\quad p_{a,\alpha}p_{0,\beta}=\delta_{\alpha,\beta}p_{a,\alpha}.$$

Hence $p_{a,\alpha}$ can be viewed as an operator $M_\alpha \to M_{sa+\alpha}$ which is zero on M_β for $\beta \neq \alpha$. Since $p_{1,\alpha}^a = (a)!p_{a,\alpha}$, everything is determined by the $p_{1,\alpha}$ in the generic case. Therefore the comodule structures on M correspond to direct sum decompositions $M = \bigoplus_\alpha M_\alpha$ together with linear maps $p_{1,\alpha}\colon M_\alpha \to M_{s+\alpha}$ (in the generic case).

We now turn our attention to structures of Yetter-Drinfeld modules on M. We use the description of the comodule structure as above. The condition I(4.4) for $b = E$ is satisfied if and only if

(4.4) $$Ep_{a,\alpha} + K^s p_{a-1,\alpha} = c^{\alpha t} p_{a-1,\alpha} + p_{a,\alpha-s}E.$$

We call the Yetter-Drinfeld module *split* if K acts on M_α as multiplication by $c^{-\alpha}$ and $E(M_\alpha) \subset M_{\alpha-s}$. For a split module, (4.4) is satisfied for $a = 1$, if we require the commutator relation on M_α

$$Ep_{1,\alpha} - p_{1,\alpha-s}E = c^{\alpha t} - c^{-\alpha s}.$$

In the standard case $s = t$, $c^t = v^2$, which is related to U, this is in accordance with $F = (v^t - v^{-t})^{-1} \sum_\alpha p_{1,\alpha}$, and this single relation implies the other relations (4.4) for $a > 1$.

5. The R-matrix for U

In this section we make the category of integrable U-modules into a braided tensor category. Since U is infinite dimensional, a universal R-matrix is only obtainable in a suitably completed tensor product of U with itself. It is easier to define the R-matrix as an operator on integrable modules. For such operators we use the following conventions: If κ is an operator, then $\mu(\kappa)$ is the operator on tensor products defined by $\mu(\kappa)_{M,N} = \kappa_{M\otimes N}$. Similarly, if Θ is an operator on tensor products, then $(\mu \otimes 1)\Theta$ is the operator which acts on $M \otimes N \otimes P$ as $\Theta_{M\otimes N,P}$. If $x \in U$, and l_x is left translation by x, then $\mu(l_x) = l_{\mu(x)}$ so that this terminology is compatible with ordinary comultiplication. Recall that integrable U-modules are direct sums $M = \bigoplus M^m$ such that K acts on M^m as multiplication by v^m; and the operators E and F are locally nilpotent.

We start with the construction of a modified R-matrix. We have the Hopf algebra $U = U(t,v)$ and the Hopf algebra $\bar{U} = \bar{U}(t,v)$ which have the same algebra structure but different comultiplications μ and $\bar{\mu}$, see II.1. The comultiplication $\bar{\mu}$ leads to a different tensor product $M\bar{\otimes}N$ of U-modules M and N. Is there a natural isomorphism $\Theta_{M,N}\colon M\bar{\otimes}N \to M \otimes N$? In the universal case $M = N = U$ we are then looking for an element Θ such that

(5.1) $$\mu(x)\Theta = \Theta\bar{\mu}(x)$$

holds for all $x \in U$. We set $w = v^t$.

(5.2) Proposition. *The following series formally satisfies* (5.1)

$$\Theta = \sum_{n\geq 0}(-1)^n w^{-n(n-1)/2}\frac{(w-w^{-1})^n}{[n;w]!}F^n\otimes E^n.$$

The inverse $\bar\Theta$ of Θ is obtained by replacing w by w^{-1} in the series.

Proof. If (5.1) is satisfied for x and y, then also for xy. It therefore suffices to consider $x = E, F, K$. We set $\Theta = \sum_n a_n F^n \otimes E^n$. Then (5.1) yields for $x = E$

$$\sum a_n EF^n \otimes E^n + \sum a_n K^t F^n \otimes E^{n+1} = \sum a_n F^n E \otimes E^n + \sum a_n F^n K^{-t}\otimes E^{n+1}.$$

Since $[E, F^n] = [n; w][n-1, K^t; w]F^{n-1}$, by (1.7), this leads to

$$\sum a_n[n;w][n-1,K^t;w]F^{n-1}\otimes E^n = \sum a_n(w^{-2n}K^{-t}-K)F^n\otimes E^{n+1}.$$

We compare coefficients and obtain

$$a_n = -\frac{w-w^{-1}}{[n;w]}w^{-(n-1)}a_{n-1}.$$

A similar computation can be carried through for $x = F$. The element $\mu(K) = K\otimes K$ commutes with $F^n \otimes E^n$.

If $\bar\Theta$ is obtained from Θ by replacing w by w^{-1}, then $\bar\Theta\Theta = 1$ amounts to the identity (9.13) between binomial coefficients. □

We can view the formal series Θ as an operator on tensor products $M\bar\otimes N$ of integrable modules, since only finitely many summands act non-trivially on a given element. The formal computations in the proof of (5.2) are therefore valid when viewed as operator identities on integrable modules. The operator Θ yields an isomorphism

$$\Theta_{M,N}\colon M\bar\otimes N \to M\otimes N$$

for integrable modules M and N; it is U-linear by (5.1) and natural in M and N by construction.

Let $a\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{Z}$ be a function. We define a homogeneous operator $\kappa = \kappa_a$ on integrable modules. It acts on $M^m\otimes N^n$ as multiplication by $v^{a(m,n)}$. The operator κ is in general not U-linear. Suppose that for $x\in U$

(5.3) $$\kappa\cdot\bar\mu(x) = \tau\mu(x)\cdot\kappa$$

holds as an operator identity. Then $R = \kappa\bar\Theta$ satisfies the basic identity $R\cdot\mu(x) = \tau\mu(x)\cdot R$ of a universal R-matrix.

(5.4) Proposition. *The homogeneous operator κ which acts on $M^m\otimes N^n$ as multiplication with $v^{a(m,n)}$ satisfies* (5.3) *for all $x\in U$ (viewed as operator on integrable modules) if*

$$a(m+2,n) = a(m,n)+tn, \quad a(m,n+2) = a(m,n)+tm.$$

PROOF. It suffices to consider $x = E, F$. For E and $x \otimes y \in M^m \otimes N^n$ we obtain

$$\begin{aligned} \kappa\bar{\mu}(E)(x \otimes y) &= \kappa(E \otimes 1 + K^{-t} \otimes E)(x \otimes y) \\ &= \kappa(Ex \otimes y + v^{-mt} x \otimes Ey) \\ &= v^{a(m+n)} Ex \otimes y + v^{-mt} v^{a(m,n+2)} x \otimes Ey \end{aligned}$$

and

$$\tau\mu(E)\kappa(x \otimes y) = v^{a(m,n)}(1 \otimes E + E \otimes K^t)(x \otimes y) = v^{a(m,n)}(x \otimes Ey + v^{tn} Ex \otimes y).$$

We compare coefficients and see the claim. □

We use the choice $a(m, n) = mnt/2$. This requires to have $v^{1/2} \in \mathfrak{K}$. So let us assume this. The recursion formula (5.4) determines a up to the initial conditions $a(\varepsilon_1, \varepsilon_2)$, $\varepsilon_j \in \{0, 1\}$; hence a choice of a is possible which does not use $v^{1/2}$.

(5.5) Proposition. *The operator κ satisfies the following relations on the tensor products of integrable modules*

$$\kappa(E \otimes 1) = (E \otimes K^t)\kappa, \qquad \kappa(1 \otimes E) = (K^t \otimes E)\kappa,$$

$$\kappa(F \otimes 1) = (F \otimes K^{-t})\kappa, \qquad \kappa(1 \otimes F) = (K^{-t} \otimes F)\kappa.$$

It commutes with the K-elements. □

This proposition can also be used to show $R \cdot \mu = \tau\mu \cdot R$ for $R = \kappa\bar{\Theta}$.

The operator $R = \kappa\bar{\Theta}$ yields a screw on the category of integrable U-modules, i. e. a natural isomorphism

(5.6) $$\tau \circ R := c_{M,N} \colon M \otimes N \to N \otimes M$$

for integrable U-modules M and N. We show that this is a braiding. In order to prove this, we verify some identities for the operator Θ. Let Θ_{12}, Θ_{23} denote the series which are obtained from Θ by replacing $F^n \otimes E^n$ with $F^n \otimes E^n \otimes 1$, $1 \otimes F^n \otimes E^n$, respectively. We use the abbreviation

(5.7) $$E_n = (-1)^n w^{-n(n-1)/2} \frac{(w - w^{-1})^n}{[n; w]!} F^n.$$

Then $\Theta = \sum E_n \otimes E^n$.

(5.8) Proposition. *The following identities hold for all $n \in \mathbb{N}_0$*

$$(\mu \otimes 1)(E_n \otimes E^n) = \sum_{i+j=n} (1 \otimes E_i \otimes E^i)(1 \otimes K^{tj} \otimes 1)(E_j \otimes 1 \otimes E^j)$$

$$(1 \otimes \mu)(E_n \otimes E^n) = \sum_{i+j=n} (E_i \otimes E^i \otimes 1)(1 \otimes K^{tj} \otimes 1)(E_j \otimes 1 \otimes E^j).$$

PROOF. Use the binomial formula for $\mu(E)^n$ and verify $E_i E_j = (n, i; w^2) E_{i+j}$. □

We use notations like $\Theta_{1,K^t,3} = \sum E_n \otimes K^{tn} \otimes E^n$. Then we can condense (5.6) into the following identities

(5.9)
$$\begin{array}{rcl} (\mu \otimes 1)\Theta & = & \Theta_{23}\Theta_{1,K^{-t},3} \\ (\mu \otimes 1)\Theta & = & \Theta_{12}\Theta_{1,K^t,3} \\ (1 \otimes \mu)\bar{\Theta} & = & \bar{\Theta}_{1,K^{-t},3}\bar{\Theta}_{23} \\ (1 \otimes \mu)\bar{\Theta} & = & \bar{\Theta}_{1,K^t,3}\bar{\Theta}_{12}. \end{array}$$

We write κ_{13} for the operator which acts on $M^m \otimes N^n \otimes P^p$ as multiplication by $w^{mp/2}$, and similarly for κ_{12} and κ_{23}. Then the following relations hold between operators on integrable modules:

(5.10)
$$\kappa_{23}\bar{\Theta}_{1,K^{-t},3} = \bar{\Theta}_{13}\kappa_{23}$$

(5.11)
$$\kappa_{12}\bar{\Theta}_{1,K^t,3} = \bar{\Theta}_{13}\kappa_{12}$$

(5.12)
$$(\mu \otimes 1)\kappa = \kappa_{13}\kappa_{23}, \quad (1 \otimes \mu)\kappa = \kappa_{13}\kappa_{12}.$$

(5.13) Theorem. *The operators* (5.6) $c_{M,N}$ *are a braiding on the category of integrable U-modules.*

PROOF. We have to verify the operator identities I(5.4). We have $(\mu \otimes 1)R = (\mu \otimes 1)\kappa\bar{\Theta} = \kappa_{13}\kappa_{23}\bar{\Theta}_{1,K^{-t},3}\bar{\Theta}_{23}$ and $R_{13}R_{23} = \kappa_{13}\bar{\Theta}_{13}\kappa_{23}\Theta_{23}$. The equality of these terms boils down to $\kappa_{23}\bar{\Theta}_{1,K^{-t},3} = \bar{\Theta}_{13}\kappa_{23}$, and this is (5.8). Similarly for the other operator identity in I(5.4). □

(5.14) Example. Let V denote the irreducible $U = U(1,v)$-module with basis x_0, x_1 and action

$$Ex_0 = 0,\ Ex_1 = x_0,\ Fx_0 = x_1,\ Fx_1 = 0,\ Kx_0 = vx_0,\ Kx_1 = v^{-1}x_1.$$

Use the basis $x_0 \otimes x_0, x_0 \otimes x_1, x_1 \otimes x_0, x_1 \otimes x_1$ in $V \otimes V$. Then $\bar{\Theta}$ is the operator $1 + (v - v^{-1})F \otimes E$, since $F^2 = E^2 = 0$. The operator κ is the diagonal operator $\mathrm{Dia}(p, p^{-1}, p^{-1}, p)$, $p = v^{1/2}$. Therefore R has the matrix

$$\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^{-1} & 0 & 0 \\ 0 & p - p^{-3} & p^{-1} & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

and $c_{V,V}$ has the matrix

$$p^{-1}X = p^{-1}\begin{pmatrix} v & & & \\ & v - v^{-1} & 1 & \\ & 1 & 0 & \\ & & & v \end{pmatrix}.$$

The matrix X is the simplest non-trivial matrix which satisfies the Yang-Baxter equation I(5.10). It has the eigenvalues v, of multiplicity 3, and $-v^{-1}$, and the minimal polynomial $(X - v)(X + v^{-1})$. ♡

Consider the irreducible $U(1,v) = U$-module V_n with basis $x_j^{(n)}$, $0 \le j \le n$, such that $Fx_j^{(n)} = [j+1]x_{j+1}^{(n)}$. We have the primitive vector in $V_n \otimes V_m$

$$X^{n,m,p} = X^p = \sum_{j=0}^{p} (-1)^j v^{j(n-j+1)} [n-j]![m-p+j]! x_j \otimes x_{p-j}$$

for $0 \le p \le \min(m,n)$. These vectors determine the Clebsch-Gordan decomposition $V_n \otimes V_m = \bigoplus_p V_{n+m-2p}$. Let $\bar{X}^p$ denote the vector which is obtained from X^p by the substitution $v \mapsto v^{-1}$. (Note that $[k]!$ is invariant under this substitution.)

(5.15) Proposition. $\bar{\Theta} X^p = \bar{X}^p$.

Proof. From the definition of X^p and $\bar{\Theta}$ we obtain

$$\bar{\Theta} X^p = \sum_{t=0}^{p} (-1)^t [n-t]![m-p+t]! \beta_t x_t \otimes x_{p-t}$$

with

$$\beta_t = v^{t(n-t+1)} \sum_k (-1)^k v^{-k(k+1)/2 + k(2t-n+1)} (v - v^{-1})^k [k]! \begin{bmatrix} n+k-t \\ k \end{bmatrix} \begin{bmatrix} t \\ k \end{bmatrix}.$$

We insert (A.17) $\begin{bmatrix} n+k-t \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} t-n-1 \\ k \end{bmatrix}$, apply (A.20), and obtain $\beta_t = v^{-t(n-t+1)}$. □

We apply the preceding proposition to compute the action of the braiding on the tensor product $V_n \otimes V_m$. We denote by $\hat{R}$ the braiding map which is multiplication by R followed by the standard twist map τ. The basic information is the next result.

(5.16) Proposition. $\hat{R} X^{n,m,p} = (-1)^p v^{nm/2 - p(n+m-p+1)} X^{m,n,p}$.

Proof. We apply κ to $\bar{\Theta} X^p = \bar{X}^p$, interchange the factors, and replace the summation index j by $p-j$. Then the result drops out. □

(5.17) Corollary. *The vectors $X^p \in V_m \otimes V_m$ are eigenvectors of $\hat{R}$ with eigenvalue $\lambda_{m,p} = (-1)^p v^{m^2/2 - 2pm + p(p-1)}$.* □

The operator $\hat{R}\colon V_m \otimes V_m \to V_m \otimes V_m$ is U-linear. It is multiplication by the scalar $\lambda_{m,p}$ on the irreducible summand V_{m+m-2p}. Hence the Clebsch-Gordan decomposition is the eigenspace decomposition of the quantized interchange map (the braiding) $\hat{R}$. This is a remarkable difference to the classical case. The ordinary interchange operator has eigenvalues ± 1 and cannot see the Clebsch-Gordan decomposition.

We compute the $\hat{R}$-matrices in $V_n \otimes V_n$. We write $\hat{R}(x_i \otimes x_j) = \sum_{\mu,\nu} a_{ij}^{\mu\nu} x_\mu \otimes x_\nu$. Then

$$a_{ij}^{\mu\nu} = v^{k(k-1)/2 + (n-2\mu)(n-2\nu)/2} \frac{(v - v^{-1})^k}{[k]!} [i+1] \cdots [\nu][n-j+1] \cdots [n-\mu]$$

with $k = \nu - i = j - \mu \geq 0$, and zero otherwise. It turns out that this matrix is in general not symmetric. Therefore we choose another basis z_j of V_n, defined by $x_i = v^{-i(n-i-1)/2}\sqrt{\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]}z_i$. We set $\hat{R}(z_i \otimes z_j) = \sum b_{ij}^{\mu\nu} z_\mu \otimes z_\nu$ and compute

$$b_{ij}^{\mu\nu} = v^{n^2/2+(\mu+j)(\nu+i)/2-n(\mu+\nu)}(v^{1/2} - v^{-3/2})^k \sqrt{\begin{bmatrix} \nu \\ k \end{bmatrix} \begin{bmatrix} j \\ k \end{bmatrix} \begin{bmatrix} n-i \\ k \end{bmatrix} \begin{bmatrix} n-\mu \\ k \end{bmatrix}}.$$

As an example, we obtain for $n = 2$ the following matrix $\hat{R}$.

$$\left[\begin{array}{ccc|ccc|ccc}
q & & & & & & & & \\
 & \delta & & 1 & & & & & \\
 & & \mu & & \lambda & & q^{-1} & & \\
\hline
 & 1 & & 0 & & & & & \\
 & & \lambda & & 1 & & & & \\
 & & & & & \delta & & 1 & \\
\hline
 & & q^{-1} & & & & 0 & & \\
 & & & & & 1 & & 0 & \\
 & & & & & & & & q
\end{array}\right]$$

It uses

$$q = v^2, \quad \delta = q - q^{-1}, \quad \mu = \delta(1 - q^{-1}), \quad \lambda = v^{-1}\delta.$$

(5.18) Exercises and supplements.

1. Show $(\mu \otimes 1)\Theta \cdot \Theta_{12} = (1 \otimes \mu)\Theta \cdot \Theta_{23}$.

2. Verify (5.8), (5.9), (5.10), (5.11), (5.12).

3. Verify $E_i E_j = (n, i; w^2)E_{i+j}$.

4. The vector $\sum_{j=0}^p v^{-j(m-p+j)} x_j \otimes x_{p-j}$ is an eigenvector of $\hat{R}$ on $V_m \otimes V_m$ with eigenvalue $v^{m^2/2}$. Show first that this vector equals $F^{(p)}(x_0 \otimes x_0)$.

5. The p-block B_p is the subspace of $V_m \otimes V_m$ spanned by the $x_i \otimes x_j$ with $i+j = p$. The p-block is $\hat{R}$-stable and decomposes into one-dimensional eigenspaces. The eigenvalue $v^{m^2/2-r(2m+1-r)}$ appears in the p-block for $r \leq p \leq 2m - r$.

6. The quantum Weyl group

Let $e = \pm 1$. We consider the automorphisms T_e of $U = U(t, v)$ defined by

$$T_e E = -K^{et} F, \quad T_e F = -EK^{-et}, \quad T_e K = K^{-1}.$$

The inverse of T_e is the homomorphism $T_{-e}^{\#}$ given by

$$T_{-e}^{\#}(E) = -FK^{-et}, \quad T_{-e}^{\#}(F) = -K^{et}E, \quad T_{-e}^{\#}(K) = K^{-1}.$$

These automorphisms are related to the comultiplications μ and $\bar{\mu}$.

(6.1) Proposition. *The homomorphisms*

$$T_1, T_1^{\#} \colon \bar{U} \to U, \quad T_{-1}, T_{-1}^{\#} \colon U \to \bar{U}$$

are isomorphisms of Hopf algebras.

PROOF. One verifies easily relations like $\mu T_1(E) = (T_1 \otimes T_1)\bar{\mu}(E)$. □

Recall that we have the operator Θ which satisfies $\mu(x)\cdot\Theta = \Theta\cdot\bar{\mu}(x)$. Together with (6.1) this implies

$$\Theta \cdot \bar{\mu} T_{-1}(x) = \mu T_{-1}(x) \cdot \Theta. \tag{6.2}$$

A similar relation holds for $T_{-1}^{\#}$. We now construct related automorphisms $T_e, T_e^{\#} \colon M \to M$ for integrable U-modules M. Recall that such a module is the direct sum $M = \bigoplus_{m \geq 0} M(m)$, where the Casimir operator C acts on $M(m)$ as multiplication by s_m. The operator $T_e \colon M \to M$ is the sum of operators $T_e \colon M(m) \to M(m)$. We set, with $w = v^t$,

$$T_e(x) = (-1)^r w^{er(s+1)} \frac{F^s E^r}{[r+s; w]!}(x) \tag{6.3}$$

$$T_e^{\#}(x) = (-1)^s w^{-es(r+1)} \frac{F^s E^r}{[r+s; w]!}(x) \tag{6.4}$$

for $x \in M(r+s)^{s-r}$. Then $T_e \colon M(m)^k \to M(m)^{-k}$. This defines the operators T_e and $T_e^{\#}$ uniquely on integrable modules.

(6.5) Proposition. *For $u \in U$ and $x \in M$ we have $T_e(ux) = T_e(u)T_e(x)$ and similarly for $T_e^{\#}$.*

PROOF. For the proof of (6.5) we interprete the claim in a different manner. It suffices to consider the case $t = 1$. Let $P = \mathfrak{K}\{x, y\}/(yx - vxy)$ denote the quantum plane. Then T_e and $T_e^{\#}$ are the $\mathfrak{K}$-linear isomorphisms on P given by

$$T_e(x^r y^s) = (-1)^r v^{er(s+1)} x^s y^r, \quad T_e^{\#}(x^r y^s) = (-1)^s v^{-es(r+1)} x^s y^r. \tag{6.6}$$

The assertion of (6.5), for T_e, is equivalent to the commutativity of the diagramm

$$\begin{tikzcd} P \arrow[r, "l_u"] \arrow[d, "T_e"] & P \arrow[d, "T_e"] \\ P \arrow[r, "l_{T_e u}"] & P, \end{tikzcd} \tag{6.7}$$

where l_u denotes the left translation by $u \in U$. It suffices to verify (6.7) for algebra generators of U, and this is straightforward. □

We now compare the operators T and $T \otimes T$ on tensor products.

(6.8) Proposition. $(T_1 \otimes T_1)\bar{\Theta} = T_1$.

PROOF. It suffices to verify the identity on tensor products $V_n \otimes V_m$ of the irreducible modules V_n, since T_1 is compatible with direct sums of U-modules. We first check that the primitive vectors $X^p = X^{n,m,p}$ have the same image under both maps. We use the basis $x_j^{(n)} = x_j$ of V_n as before. The basis element $x_j^{(n)}$ corresponds to $\begin{bmatrix} n \\ j \end{bmatrix} x^j y^{n-j}$ in the quantum plane. Therefore, by (6.6),

$$T_1 x_j^{(n)} = (-1)^j v^{j(n-j+1)} x_{n-j}^{(n)}.$$

We have already computed $\bar{\Theta} X^p$ in (5.15). Altogether, we see that $(T_1 \otimes T_1)\bar{\Theta} X^p$ is the vector

$$\sum_{j=0}^{p} (-1)^{p-j} [n-j]![m-p+j]! v^{(p-j)(m-p+j+1)} x_{n-j}^{(n)} \otimes x_{m-p-j}^{(m)}.$$

This is a vector of weight v^{2p-n-m} and annihilated by F. These properties show that $(T_1 \otimes T_1)\bar{\Theta} X^p$ is contained in V_{n+m-2p} and determine it uniquely up to a scalar multiple. Therefore, $T_1 X^p$ is a scalar multiple of $(T_1 \otimes T_1)\bar{\Theta} X^p$. In order to determine which multiple, we compute the coefficient λ of $x_n \otimes x_{m-p}$ in $T_1 X^p$. By the definition of X^p and the binomial formula for $\mu(F)^{n+m-2p}$, we see that $[n+m-2p]\lambda$ equals

$$\sum_{j=0}^{p} (-1)^j v^j [n-j]![m-p+j]! \begin{bmatrix} n+m-2p \\ n-j \end{bmatrix} [j+1] \cdots [n][p-j+1] \cdots [m-p].$$

Grouping the factors differently yields

$$\lambda = [n]![m-p]! \sum_{j=0}^{p} (-1)^j v^j \begin{bmatrix} m-p+j \\ j \end{bmatrix} \begin{bmatrix} m-p \\ p-j \end{bmatrix}.$$

We rewrite $(-1)^j \begin{bmatrix} m-p+j \\ j \end{bmatrix} = \begin{bmatrix} p-m-1 \\ j \end{bmatrix}$ and

$$j = p(m-p+1) + (p-m-1)(p-j) - j(m-p).$$

Then (A.19) shows that the sum equals $v^{p(m-p+1)} \begin{bmatrix} -1 \\ p \end{bmatrix} = (-1)^p v^{p(m-p+1)}$. This proves our assertion about $T_1 X^p$.

We now verify: If $(T_1 \otimes T_1)\bar{\Theta}(x) = T_1(x)$ holds for x, then also for Fx. Since vectors of the type $F^k X$ for primitive X generate $V_n \otimes V_m$, we conclude that the identity (6.8) holds on all of $V_n \otimes V_m$. We use (6.1) and (6.5) in the following verification. Let $x \in M \otimes N$ and suppose $(T_1 \otimes T_1)\bar{\Theta}(x) = T_1(x)$. Then

$$\begin{aligned} T_1(F \cdot x) &= T_1(F) \cdot T_1(x) = \mu T_1(F) \star T_1(x) \\ &= (T_1 \otimes T_1)\bar{\mu}(F) \star (T_1 \otimes T_1)\bar{\Theta}(x) \\ &= (T_1 \otimes T_1)(\bar{\mu}(F) \star \bar{\Theta}(x)) \\ &= (T_1 \otimes T_1)(\bar{\Theta}\mu(F) \star x) \\ &= (T_1 \otimes T_1)\bar{\Theta}(F \cdot x). \end{aligned}$$

(We have denoted the action of $U \otimes U$ on $M \otimes N$ by $\star$ and the action of U by a dot.)

We extend the algebra U by the operator T. The extended algebra $U^\flat$ is additively the tensor product $U \otimes \mathfrak{K}[T, T^{-1}]$. The multiplication is defined by

$$(u_1 \otimes T^k) \cdot (u_2 \otimes T^l) = u_1 T^k(u_2) \otimes T^{k+l},$$

where T is one of the operators T_e, $T_e^\#$. Proposition (6.5) says that $U^\flat$ acts on integrable modules by

$$(u \otimes T^k)(x) = uT^k(x),$$

We extend the Hopf algebra structure of U to $U^\flat$ in the following manner. The extension uses the operator $\Theta = \sum E_n \otimes E^n$. We set

$$\mu(1 \otimes T) = \sum_n E_n \otimes T \otimes E^n \otimes T.$$

We have to check that this is compatible with the relation $(1 \otimes T)(u \otimes 1) = Tu \otimes T$. We compute

$$\mu(1 \otimes T)\mu(u \otimes 1) = \sum E_n T(u_1) \otimes T \otimes E^n T(u_2) \otimes T$$

$$\mu(Tu \otimes T) = \sum (Tu)_1 E_n \otimes T \otimes (Tu)_2 E^n \otimes T.$$

Equality of these expressions amounts to $\mu T(u) \cdot \Theta = \Theta \cdot (T \otimes T)\mu(x)$, and this holds, by (6.2), for $T = T_{-1}$ and $T = T_{-1}^\#$. For the following considerations we fix $T = T_{-1}^\#$. Then we view μ as a homomorphism of algebras from $U^\flat$ into a suitably completed tensor product of $U^\flat \otimes U^\flat$. The coassociativity $(\mu \otimes 1)\mu(1 \otimes T) = (1 \otimes \mu)\mu(1 \otimes T)$ is equivalent to the identity $(\mu \otimes 1)\Theta \cdot \Theta_{12} = (1 \otimes \mu)\Theta \cdot \Theta_{23}$; for this, compare section 5.

We have two $U^\flat$-module structures on the tensor product $M \otimes N$ of integrable modules: The $U^\flat$-extension of the U-module structure, and the one coming from the comultiplication in $U^\flat$. These structures conincide, as the following computation shows.

$$\begin{aligned}
(1 \otimes T) \cdot (x \otimes y) &= \sum (E_n \otimes T)x \otimes (E^n \otimes T)y \\
&= \sum E_n T_{-1}^\#(x) \otimes E^n T_{-1}^\#(y) \\
&= \Theta \cdot (T_{-1}^\# x \otimes T_{-1}^\# y) \\
&= T_{-1}^\#(x \otimes y).
\end{aligned}$$

For the last equality we have used (6.8).

The antipode axiom for the element T is more complicated. We need the operator $m(s \otimes 1)\Theta = \Omega$. This is the operator

(6.9) $$\Omega = \sum_{n \geq 0} v^{-3n(n-1)/2} \frac{(v - v^{-1})^n}{[n]!} F^n K^n E^n;$$

it is well defined on integrable modules.

(6.10) Proposition. *The operator Ω satisfies*

$$K^{-1}E\Omega = K\Omega E, \quad \Omega F = FK\Omega K.$$

PROOF. We write $\Theta = \sum a_r \otimes b_r$. Then (5.1) gives

$$\sum Ea_r \otimes b_r + \sum Ka_r \otimes Eb_r = \sum a_r E \otimes b_r + \sum a_r K^{-1} \otimes b_r E,$$

and application of $m(s \otimes 1)$ yields

$$\sum s(a_r)s(E)b_r + \sum s(a_r)K^{-1}Eb_r = \sum s(E)s(a_r)b_r + \sum Ks(a_r)b_r E.$$

Because of $s(E) = -K^{-1}E$, the first two summands cancel and the remaining ones yield the desired equality. Similarly for F. Finally, Ω commutes with K. □

(6.11) Proposition. *The operator Ω is invertible. Its inverse is given by*

$$\Omega^{-1} = \sum_{n\geq 0}(-1)^n v^{-n(n-1)/2}\frac{(v-v^{-1})^n}{[n]!}F^n K^{-n}E^n.$$

PROOF. We write $\bar{\Theta} = \sum \bar{E}_n \otimes E^n$. We apply $m(s \otimes 1)$ to $\bar{\Theta}\Theta = 1$ and obtain

$$1 = \sum s(\bar{E}_m)s(E_n)E^n E^m = \sum s(\bar{E}_m)\Omega E^m = \sum s(\bar{E}_m)K^{-2m}E^m\Omega.$$

We insert the value for $s(\bar{E}_m)$ and obtain the result. □

The antipode axiom for $1 \otimes T$ assumes the following form

$$1 \otimes 1 = e\varepsilon(1 \otimes T) = m(s \otimes 1)\mu(1 \otimes T) = m(\sum s(E_n \otimes T) \otimes (E^n \otimes T)).$$

Since s is an antihomomorphism, $s(E_n \otimes T) = s(1 \otimes T) \cdot (s(E_n) \otimes 1)$. Suppose $s(1 \otimes T)$ has the form $\alpha \otimes T^{-1}$. Then $(\alpha \otimes T^{-1}) \cdot (s(E_n) \otimes 1) = \alpha T^{-1}s(E_n) \otimes T^{-1}$, and altogether,

$$\begin{aligned} m(s \otimes 1)\mu(1 \otimes T) &= \sum m(\alpha T^{-1}s(E_n) \otimes T^{-1}) \otimes (E^n \otimes T)) \\ &= \alpha T^{-1}(\sum s(E_n)E^n) \otimes 1 \\ &= \alpha T^{-1}(\Omega) \otimes 1. \end{aligned}$$

Therefore, the antipode axiom holds, if we set $\alpha = T^{-1}(\Omega^{-1})$. We compute

$$\alpha = \sum (-1)^n v^{-n(n-1)/2}\frac{(v-v^{-1})^n}{[n]!}E^n K^n F^n.$$

(6.12) Proposition. *There exists a homogeneous operator ξ, which acts on M^m as multiplication by $v^G(m)$, such that $\Omega \circ \xi$ is U-linear.*

PROOF. It suffices to verify compatibility with E and F. We use proposition (6.10). For $x \in M^m$ we have

$$\Omega\xi Ex = v^{G(\lambda+2)}\Omega Ex = v^{G(\lambda+2)-\lambda-2}K\Omega Ex$$
$$E\Omega\xi x = v^{G(\lambda)}E\Omega x = v^{G(\lambda)+\lambda+2}K^{-1}E\Omega x.$$

Equality holds, if

$$G(\lambda+2) = G(\lambda) + 2(\lambda+2).$$

A function with this property is $G(m) = \frac{1}{2}m(m+2)$. □

7. The example $SL_q(2)$

We illustrate the theory with the quantum group associated to SL_2. For simplicity we work over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$.

Let V be a two-dimensional $\mathfrak{K}$-module with basis v_1, v_2. In terms of the basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$ the matrix

(7.1)
$$X = q^{-1/2} \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}$$

defines a Yang-Baxter operator. The FRT-construction associates to X the algebra A generated by $a, b, c, d = T_1^1, T_1^2, T_2^1, T_2^2$ with relations

$$\begin{aligned} ab &= qba & bd &= qdb \\ ac &= qca & cd &= qdc \\ bc &= cb & & \\ ad - da &= (q-q^{-1})bc. & & \end{aligned}$$

The matrix

(7.2)
$$F = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix}$$

yields a four braid pair (X, F) for arbitrary parameters with invertible $\alpha\beta$ (see [**??**], also for an n-dimensional generalization). The quantum plane $P = \mathfrak{K}\{x, y\}/(xy - qyx)$ is a left A-comodule via the map $\mu_P \colon P \to A \otimes P$ given by

$$\mu_P(x^i y^j) = \sum_{r=0}^{i} \sum_{s=0}^{j} q^{-s(i+j-r-s)-r(i-r)} \begin{bmatrix} i \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s}$$

where $\begin{bmatrix} i \\ r \end{bmatrix}$ is a q-binomial coefficient

$$\begin{bmatrix} i \\ r \end{bmatrix} = \frac{[i]!}{[r]![i-r]!}, \quad [i]! = [1][2]\cdots[i], \quad [i] = \frac{q^i - q^{-i}}{q - q^{-1}}$$

(compare [**??**, IV], where different conventions are used).

The operator $T_2 = (F \otimes 1)X(F \otimes 1)X$ on $V \otimes V$ has the matrix (with $\delta = q - q^{-1}$)

$$\begin{pmatrix} 0 & 0 & 0 & \beta^2 \\ 0 & \alpha\beta\delta & \alpha\beta & q\beta\theta \\ 0 & \alpha\beta & 0 & \beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix} = \begin{pmatrix} F_{11}^{11} & F_{12}^{11} & F_{21}^{11} & F_{22}^{11} \\ F_{11}^{12} & F_{12}^{12} & F_{21}^{12} & F_{22}^{12} \\ F_{11}^{21} & F_{12}^{21} & F_{21}^{21} & F_{22}^{21} \\ F_{11}^{22} & F_{12}^{22} & F_{21}^{22} & F_{22}^{22} \end{pmatrix}$$

with respect to the basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$. This is also the matrix of values of the cylinder form f

$$f \begin{pmatrix} aa & ac & ca & cc \\ ab & ad & cb & cd \\ ba & bc & da & dc \\ bb & bd & db & dd \end{pmatrix}.$$

Let $\det_q = ad - qbc$ be the quantum determinant. It is a group-like central element of A. The quotient of A by the ideal generated by $\det_q$ is the Hopf algebra $SL_q(2)$.

(7.3) Proposition. *The form f has the value $-q^{-1}\alpha\beta$ on $\det_q$. If $-q^{-1}\alpha\beta = 1$, then f factors over $SL_q(2)$.*

PROOF. The stated value of $f(\det_q)$ is computed from the data above. We use the fact

$$r(x \otimes \det_q) = r(\det_q \otimes x) = \varepsilon(x),$$

see [**??**, p. 195]. From (1.2) we obtain for $a \in A$ and $b = \det_q$

$$\begin{aligned} f(ab) &= \sum f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3) \\ &= \sum f(a_1) \varepsilon(a_2) f(\det_q) \varepsilon(a_3) \\ &= f(a), \end{aligned}$$

by the assumption $f(\det_q) = 1$ and the counit axiom. □

We consider the subspace $W = V_2$ of the quantum plane generated by x^2, xy, y^2. We have

$$\begin{aligned} \mu_P(x^2) &= b^2 \otimes y^2 + (1+q^{-2}) ab \otimes xy + a^2 \otimes x^2 \\ \mu_P(xy) &= bd \otimes y^2 + (ad + q^{-1}bc) \otimes xy + ac \otimes x^2 \\ \mu_P(y^2) &= d^2 \otimes y^2 + (1+q^{-2}) cd \otimes xy + c^2 \otimes x^2. \end{aligned}$$

This yields the following matrix for t_W in the basis x^2, xy, y^2

$$\begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & (q+q^{-1})\beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In the Clebsch-Gordan decomposition $V \otimes V = V_2 \oplus V_0$ the subspace V_0 (the trivial irreducible module) is spanned by $u = v_2 \otimes v_1 - q^{-1} v_1 \otimes v_2$. This is the eigenvector of X with eigenvalue $-q^{-3/2}$. It is mapped by T_2 to $-q^{-1}\alpha\beta u$. If we require this to be the identity we must have $\alpha\beta = -q$. We already obtained this condition by considering the quantum determinant.

The matrix of t_W in the basis $w_1 = x^2, w_2 = \sqrt{1+q^{-2}}xy, w_3 = y^2$ is

(7.4)
$$F_2 = \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & \sqrt{1+q^2}\beta\theta \\ \alpha^2 & \sqrt{1+q^2}\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In the case $\alpha = \beta$ it is symmetric.

The R-matrix X on $W \otimes W$ in the lexicographic basis $w_i \otimes w_j$ with $w_1 = x^2, w_2 = \sqrt{1+q^{-2}}xy, w_3 = y^2$ has the form

(7.5)
$$X_2 = \left[\begin{array}{ccc|ccc|ccc}
q^2 & & & & & & & & \\
 & \delta^* & & 1 & & & & & \\
 & & \mu & & \lambda & & q^{-2} & & \\
\hline
 & 1 & & 0 & & & & & \\
 & & \lambda & & 1 & & & & \\
 & & & & & \delta^* & & 1 & \\
\hline
 & & q^{-2} & & & & 0 & & \\
 & & & & & 1 & & 0 & \\
 & & & & & & & & q^2
\end{array}\right]$$

It uses $\delta^* = q^2 - q^{-2}$, $\quad \mu = \delta^*(1-q^{-2})$, $\quad \lambda = q^{-1}\delta^*$. By construction, (X_2, F_2) is a four braid pair.

There arises the problem to compute t_W on irreducible comodules W. We treat instead the more familiar dual situation of modules over the quantized universal enveloping algebra.

8. The cylinder braiding for U-modules

The construction of the cylinder form is the simplest method to produce a universal operator for the cylinder twist. In order to compute the cylinder twist explicitly we pass to the dual situation of the quantized universal enveloping algebra U. One can formally dualize comodules to modules and thus obtain a cylinder braiding for suitable classes of U-modules from the results of the previous sections. But we start rather from scratch.

We work with the Hopf algebra $U = U_q(sl_2)$ as in [**??**]. It is the the associative algebra over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$ generated[3] by K, K^{-1}, E, F with relations $KK^{-1} = K^{-1}K = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, $EF - FE = (K-K^{-1})/(q-q^{-1})$, comultiplication $\mu(K) = K \otimes K$, $\mu(E) = E \otimes 1 + K \otimes E$, $\mu(F) = F \otimes K^{-1} + 1 \otimes F$, and counit $\varepsilon(K) = 1$, $\varepsilon(E) = \varepsilon(F) = 0$. A left U-module M is called *integrable* if the following holds:

(1) $M = \bigoplus M^n$ is the direct sum of weight spaces M^n on which K acts as multiplication by q^n, $n \in \mathbb{Z}$.

(2) E and F are locally nilpotent on M.

Let U-Int denote the category of integrable U-modules and U-linear maps. (It would be sufficient to consider only finite dimensional such modules.) An integrable U-module M is semi-simple: It has a unique isotypical decomposition $M = \bigoplus_{n \geq 0} M(n)$, and $M(n)$ is isomorphic to a direct sum of copies of the irreducible module V_n. The module V_n has a $\mathfrak{K}$-basis $x_0, x_1, \dots, x_n$ with $F(x_i) = [i+1]x_{i+1}$, $E(x_i) = [n-i+1]x_{i-1}$, $x_{-1} = 0$, $x_{n+1} = 0$; moreover,

[3] There is another use of the letter F. It has nothing to do with the 2×2-matrix F in (5.2).

$x_i \in V_n^{n-2i}$. The category of integrable U-modules is braided. The braiding is induced by the universal R-matrix $R = \kappa \circ \Psi$ with

(8.1)
$$\Psi = \sum_{n \geq 0} q^{n(n-1)/2} \frac{(q - q^{-1})^n}{[n]!} F^n \otimes E^n$$

and $\kappa = q^{H \otimes H/2}$. Note that Ψ is a well-defined operator on integrable U-modules. (This operator is called $\bar{\Theta}$ in [**??**, section 4.1] and L_i' in [**??**, p. 46].) The operator κ acts on $M^m \otimes N^n$ as multiplication by $q^{mn/2}$. If we view H as the operator $H\colon M^m \to M^m$, $x \mapsto mx$, then $q^{H \otimes H/2}$ is a suggestive notation for κ. The braiding $z_{M,N}\colon M \otimes N \to N \otimes M$ is $\tau \circ R$, i. e. action of R followed by the interchange operator $\tau\colon x \otimes y \mapsto y \otimes x$.

A four braid pair (X, F) on the vector space V yields a tensor representation of ZB_n on $V^{\otimes n}$. We start with the standard four braid pair (5.1), (5.2) on the two-dimensional U-module $V = V_1$. Let $T_n\colon V^{\otimes n} \to V^{\otimes n}$ be the associated cylinder twist, as defined in section 2. By the Clebsch-Gordan decomposition, V_n is contained with multiplicity 1 in $V^{\otimes n}$. Similarly, $V_{m+n} \subset V_m \otimes V_n$ with multiplicity one [**??**, VII.7].

(8.2) Lemma. *There exists a projection operator $e_n\colon V^{\otimes n} \to V^{\otimes n}$ with image V_n which commutes with T_n.*

Proof. Let H_n be the Hecke algebra over $\mathfrak{K}$ generated by $x_1, \ldots, x_{n-1}$ with braid relations $x_i x_j x_i = x_j x_i x_j$ for $|i-j| = 1$ and $x_j x_i = x_i x_j$ for $|i-j| > 1$ and quadratic relations $(x_i + 1)(x_i - q^2) = 0$. Since X satisfies $(X - q^{1/2})(X + q^{-3/2}) = 0$, we obtain from the action of $ZA_{n-1} \subset ZB_n$ on $V^{\otimes n}$ an action of H_n if we let x_i act as $q^{3/2} g_i$. Since T_n comes from a central element of ZB_n, see (2.4), the H_n-action commutes with T_n. It is well known that there exists an idempotent $e_n \in H_n$ such that $e_n V^{\otimes n} = V_n$ (quantized Schur-Weyl duality). This fact implies the assertion of the Lemma. □

(8.3) Corollary. *The subspace $V_n \subset V^{\otimes n}$ is T_n-stable.* □

A similar proof shows that all summands in the isotypical decomposition of $V^{\otimes n}$ are T_n-stable.

We denote by τ_n the restriction of T_n to V_n. We have the induced operator $\tau_{m,n} = z_{n,m}(\tau_n \otimes 1) z_{m,n}(\tau_m \otimes 1)$ on $V_m \otimes V_n$, where $z_{m,n}$ denotes the braiding on $V_m \otimes V_n$.

(8.4) Lemma. *The subspace $V_{m+n} \subset V_m \otimes V_n$ is $\tau_{m,n}$-stable. The induced morphism equals τ_{m+n}.*

Proof. Consider $V_m \otimes V_n \subset V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes(m+n)}$. The projection operator $e_m \otimes e_n$ is again obtained from the action of a certain element in the Hecke algebra H_{m+n}. Hence $V_m \otimes V_n$ is T_{m+n}-stable and the action on the subspace V_{m+n} is τ_{m+n}. We now use the equality (2.3)

$$T_{m+n} = X_{n,m}(T_n \otimes 1) X_{m,n}(T_m \otimes 1).$$

The essential fact is that $X_{m,n}$ is the braiding on $V^{\otimes m} \otimes V^{\otimes n}$. It induces, by naturality of the braiding, the braiding $z_{m,n}$ on $V_m \otimes V_n$. □

Let $A(n) = (\alpha_i^j(n))$ be the matrix of τ_n with respect to $x_0, \dots, x_n$. In the next theorem we derive a recursive description of $A(n)$. We need more notation to state it. Define inductively polynomials γ_k by $\gamma_{-1} = 0$, $\gamma_0 = 1$ and, for $k > 0$,

(8.5)
$$\alpha\gamma_{k+1} = q^k \theta \gamma_k + \beta q^{k-1} \delta[k] \gamma_{k-1}.$$

Here $\gamma_k = \gamma_k(\theta, q, \alpha, \beta)$ is a polynomial in θ with coefficients in $\mathbb{Z}[q, q^{-1}, \alpha^{-1}, \beta]$ and $\delta = q - q^{-1}$. Let $D(n)$ denote the codiagonal matrix with $\alpha^k \beta^{n-k} q^{k(n-k)}$ in the k-th row and $(n-k)$-th column and zeros otherwise. (We enumerate rows and columns from 0 to n.) Let $B(n)$ be the upper triangular matrix

(8.6)
$$B(n) = \begin{pmatrix} \gamma_0 & \begin{bmatrix} n \\ 1 \end{bmatrix}\gamma_1 & \begin{bmatrix} n \\ 2 \end{bmatrix}\gamma_2 & \cdots & \gamma_n \\ & \gamma_0 & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}\gamma_1 & \cdots & \gamma_{n-1} \\ & & \cdots & \cdots & \cdots \\ & & & \gamma_0 & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}.$$

Thus the $(n-k)$-th row of $B(n)$ is

$$0, \dots, 0, \begin{bmatrix} k \\ 0 \end{bmatrix}\gamma_0, \begin{bmatrix} k \\ 1 \end{bmatrix}\gamma_1, \begin{bmatrix} k \\ 2 \end{bmatrix}\gamma_2, \dots, \begin{bmatrix} k \\ k-1 \end{bmatrix}\gamma_{k-1}, \begin{bmatrix} k \\ k \end{bmatrix}\gamma_k.$$

(8.7) Theorem. *The matrix $A(n)$ is equal to the product $D(n)B(n)$.*

Proof. The proof is by induction on n. We first compute the matrix of $\tau_{n,1}$ on $V_n \otimes V_1$ and then restrict to V_{n+1}. In order to display the matrix of $\tau_{n,1}$ we use the basis

$$x_0 \otimes x_0, \dots, x_n \otimes x_0, x_0 \otimes x_1, \dots, x_n \otimes x_1.$$

The matrix of $\tau_{n,1}$ has the block form

$$\begin{pmatrix} 0 & \beta A(n) \\ \alpha A(n) & A'(n) \end{pmatrix}.$$

The matrix $A'(n)$ is obtained from $A(n)$ in the following manner: Let $\alpha_0, \dots, \alpha_n$ denote the columns of $A(n)$ and $\beta_0, \dots, \beta_n$ the columns of $A'(n)$. We claim that

$$\beta_i = \alpha q^{2i-n} \theta \alpha_i + \beta q^{2i-n-1} \delta[n-i+1] \alpha_{i-1} + \alpha \delta [i+1] \alpha_{i+1},$$

with $\alpha_{-1} = \alpha_{n+1} = 0$.

Recall that $\tau_{n,1} = (\tau_n \otimes 1) z_{1,n} (\tau_1 \otimes 1) z_{n,1}$. The universal R-matrix has in our case the simple form

$$R = \kappa \circ (1 + (q - q^{-1}) F \otimes E).$$

For the convenience of the reader we display the four steps in the calculation of $\tau_{n,1}$, separately for $x_i \otimes x_0$ and $x_i \otimes x_1$.

$$\begin{aligned} x_i \otimes x_0 &\mapsto q^{(n-2i)/2} x_0 \otimes x_i \\ &\mapsto \alpha q^{(n-2i)/2} x_1 \otimes x_i \\ &\mapsto \alpha x_i \otimes x_1 \\ &\mapsto \sum_j \alpha \alpha_i^j x_j \otimes x_0. \end{aligned}$$

$$\begin{aligned} x_i \otimes x_1 &\mapsto q^{-(n-2i)/2} x_1 \otimes x_i + \delta[i+1] q^{(n-2i-2)/2} x_0 \otimes x_{i+1} \\ &\mapsto q^{-(n-2i)/2} (\beta x_0 + \theta x_1) \otimes x_i + \alpha \delta [i+1] q^{(n-2i-2)/2} x_1 \otimes x_{i+1} \\ &\mapsto \beta x_i \otimes x_0 + \beta q^{-n+2i-1} \delta[n-i+1] x_{i-1} \otimes x_1 \\ &\quad + q^{2i-n} \theta x_i \otimes x_1 + \alpha \delta[i+1] x_{i+1} \otimes x_1 \\ &\mapsto \sum_j \alpha_i^j x_i \otimes x_0 + \sum_j \beta q^{2i-n+1} \delta[n-i+1] \alpha_{i-1}^j x_j \otimes x_1 \\ &\quad + \sum_j q^{2i-n} \theta \alpha_i^j x_j \otimes x_1 + \sum_j \alpha \delta [i+1] \alpha_{i+1}^j x_j \otimes x_1. \end{aligned}$$

This proves the claim about the matrix for $\tau_{n,1}$.

We now use the following fact about the Clebsch-Gordan decomposition (it is easily verified in our case, but see e. g. [**??**, VII.7] for more general results): In the Clebsch-Gordan decomposition $V_n \otimes V_1 = V_{n+1} \oplus V_{n-1}$ a basis of V_{n+1} is given by

$$y_j = \frac{F^j}{[j]!}(x_0 \otimes x_0) = q^{-j} x_j \otimes x_0 + x_{j-1} \otimes x_1.$$

We apply $\tau_{n,1}$ to the y_j. Since there are no overlaps between the coordinates of the y_j, we can directly write $\tau_{n,1}(y_j)$ as a linear combination of the y_k.

We assume inductively that $A(n)$ has bottom-right triangular form with codiagonal as specified by $D(n)$. Then $A'(n)$ has a nonzero line one step above the codiagonal and is bottom-right triangular otherwise. From the results so far we see that the columns of $A(n+1)$, enumerated from 0 to $n+1$, are obtained inductively as follows: The 0-th row is $(0, \dots, 0, \beta^{n+1})$. Below this 0-th row the j-th column, $0 \le j \le n+1$, has the form

(8.8) $$\alpha q^j \alpha_j + q^{2j-n-2} \theta \alpha_{j-1} + \beta q^{2j-n-3} \delta[n-j+2] \alpha_{j-2}.$$

From this recursive formula one derives immediately that the codiagonal of $A(n)$ is given by $D(n)$.

Finally, we prove by induction that $A(n)$ is as claimed. The element in row k and column $n-k+j$ equals

$$\alpha^k \beta^{n-k} q^{k(n-k)} \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j.$$

For $n = 1$, we have defined τ_1 as $A(1)$. For the induction step we use (6.8) in order to determine the element of $A(n)$ in column $n-k+j$ and row $k+1$. The assertion is then equivalent to the following identity:

$$\begin{aligned} & \alpha^k\beta^{n-k}q^{k(n-k)}\left(\alpha\begin{bmatrix}k\\ j\end{bmatrix}\gamma_j + q^{n-2k+2j-2}\theta\begin{bmatrix}k\\ j-1\end{bmatrix}\gamma_{j-1}\right.\\ & \left. +\beta q^{n-2k+2j-3}\delta[k-j+2]\begin{bmatrix}k\\ j-2\end{bmatrix}\gamma_{j-2}\right)\\ = \; & \alpha^{k+1}\beta^{n-k}q^{(n-k)(k+1)}\begin{bmatrix}k+1\\ j\end{bmatrix}\gamma_j. \end{aligned}$$

We cancel α-, β-, and q-factors, use the Pascal formula

(8.9)
$$\begin{bmatrix}a+1\\ b\end{bmatrix} = q^b\begin{bmatrix}a\\ b\end{bmatrix} + q^{-a+b-1}\begin{bmatrix}a\\ b-1\end{bmatrix}$$

and the identity

$$\delta[k-j+2]\begin{bmatrix}k\\ j-2\end{bmatrix} = \begin{bmatrix}k\\ j-1\end{bmatrix}[j-1]$$

and see that the identity in question is equivalent to the recursion formula (6.5) defining the γ-polynomials. □

We formulate the main result of this section in a different way. First we note that it was not essential to work with the function field $\mathfrak{K}$. In fact $\mathfrak{K}$ could be any commutative ring and q, α, β suitable parameters in it. We think of θ as being an indeterminate.

Let $L(\alpha, \beta)$ be the operator on integrable U-modules which acts on V_n as

$$x_j \mapsto \alpha^{n-j}\beta^j q^{j(n-j)}x_{n-j}.$$

Let

(8.10)
$$T(\alpha,\beta) = \sum_{k=0}^{\infty}\gamma_k\frac{E^k}{[k]!}$$

which is well-defined as an operator on integrable U-modules. Then (6.7) can be expressed as follows:

(8.11) Theorem. *The operator $t(\alpha,\beta) = L(\alpha,\beta)\circ T(\alpha,\beta)$ acts on V_n as τ_n.* □

In section 8 we give another derivation of this operator from the universal point of view.

One can develop a parallel theory by starting with the four braid pair (X^{-1}, F^{-1}). This leads to matrices which are top-left triangular. By computing the inverse of (5.1) and (5.2) we see that, in the case $(\alpha,\beta) = (1,1)$, We have to replace (q,θ) by $(q^{-1}, -\theta)$.

The following proposition may be occasionally useful. Introduce a new basis $u_0, \ldots, u_n$ in V_n by

$$x_i = q^{-i(n-i)/2}\sqrt{\begin{bmatrix} n \\ i \end{bmatrix}}\, u_i.$$

Then a little computation shows:

(8.12) Proposition. *Suppose $\alpha = \beta$. With respect to the basis (u_i) the R-matrix and the matrix for τ_n are symmetric.* □

9. The γ-polynomials

For later use we derive some identies for the γ-polynomials of the previous section. A basic one comes from the compatibility of the cylinder twist with tensor products. Again we use $\delta = q - q^{-1}$. We give two proofs for (7.1).

(9.1) Theorem. *The γ-polynomials satisfy the product formula*

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} \alpha^{-k}\beta^k q^{mn-k(k+1)/2}\delta^k [k]! \begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}\gamma_{m-k}\gamma_{n-k}.$$

Proof. For the proof we consider $\tau_{m,n} = (\tau_m \otimes 1)z_{n,m}(\tau_n \otimes 1)z_{m,n}$ on $V_m \otimes V_n$ and compute the coefficient of $x_m \otimes x_n$ in $\tau_{m,n}(x_m \otimes x_n)$. The reason for this is that $x_m \otimes x_n$ is the F-primitive vector (lowest weight vector) of the summand $V_{m+n} \subset V_m \otimes V_n$ in the standard basis. Hence the coefficient in question is $\alpha^{m+n}_{m+n}(m+n)$ which, by (6.7), equals $\alpha^{m+n}\gamma_{m+n}$.

From the form of the universal R-matrix we see directly $z_{m,n}(x_m \otimes x_n) = q^{mn/2}x_n \otimes x_m$. If we write

$$z_{n,m}(x_j \otimes x_m) = \sum_{u,v} r^{uv}_{jm} x_u \otimes x_v,$$

then

$$\tau_{m,n}(x_m \otimes x_n) = \sum_{j,k,u,v} q^{mn/2}\alpha^j_n(n) r^{uv}_{jm}\alpha^k_u(m) x_k \otimes x_v.$$

We need the coefficient for $(k,v) = (m,n)$, hence have the formal identity

$$\alpha^{m+n}\gamma_{m+n} = \sum_{j,u} q^{mn/2}\alpha^j_n(n) r^{un}_{jm}\alpha^m_u(m).$$

The universal R-matrix yields

$$z_{n,m}(x_j \otimes x_m) = \sum_{k \geq 0} v^{\bullet}\delta^k [j+1]\cdots[j+k] x_{j+k} \otimes x_{m-k}$$

with $\bullet = k(k-1)/2 + (n-2j-2k)(2k-m)/2$. Moreover

$$\alpha^{n-k}_n(n) = \alpha^{n-k}\beta^k q^{(n-k)k}\gamma_{n-k}, \quad \alpha^m_{m-k}(m) = \alpha^m \begin{bmatrix} m \\ k \end{bmatrix}\gamma_{m-k}.$$

We insert these values into the formal equation for $\alpha^{m+n}\gamma_{m+n}$ and obtain the desired result. $\square$

The dependence of γ_k on the parameters α and β is not very essential. Define inductively polynomials γ'_k in θ over $\mathbb{Z}[q, q^{-1}]$ by $\gamma'_{-1} = 0, \gamma'_0 = 1$ and, for $k \geq 1$,

$$\gamma'_{k+1} = q^k \theta \gamma'_k + q^{k-1}\delta[k]\gamma'_{k-1},$$

i. e. $\gamma'_k(\theta, q) = \gamma_k(\theta, q, 1, 1)$. A simple rewriting of the recursion formulas yields then the identity

$$\textbf{(9.2)} \qquad \gamma_k(\theta, q, \alpha, \beta) = \gamma'_k\left(\frac{\theta}{\sqrt{\alpha\beta}}, q\right)\left(\frac{\beta}{\alpha}\right)^{k/2}.$$

Note that γ'_k contains only powers θ^l with $l \equiv k \bmod 2$.

We normalize the γ' to monic polynomials $\beta_k(\theta) = q^{-k(k-1)/2}\gamma'_k(\theta)$. The new polynomials satisfy the recursion relation

$$\textbf{(9.3)} \qquad \beta_{k+1} = \theta\beta_k + (1 - q^{-2k})\beta_{k-1}.$$

In order to find an explicit expression for the β_k, we introduce a new variable ρ via the quadratic relation $\theta = \rho - \rho^{-1}$.

(9.4) Proposition. *The polynomials β satisfy the following identity*

$$\beta_n(\rho - \rho^{-1}) = \sum_{j=0}^{n} (-1)^j q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \rho^{n-2j}.$$

Proof. We use this identity in the recursion formula (7.3) and compare the coefficients of ρ^{n+1-2j}. A little rewriting shows that the claim reduces to the Pascal formula (6.9) for the q-binomial coefficients. $\square$

We can write $\rho^k + (-1)^k\rho^{-k}$ as an integral polynomial P_k in $\theta = \rho - \rho^{-1}$. It satisfies the recursion relation

$$\theta P_k = P_{k+1} - P_{k-1}.$$

It is possible to write P_k in terms of Tschebischev- or Jacobi-polynomials. The last proposition thus gives

$$\beta_n(\theta) = \sum_{j=0}^{[n/2]} (-1)^j q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} P_{n-2j}(\theta).$$

The product formula (7.1) was a consequence of representation theory. In view of the applications in section 8 it is desirable to have a proof which uses only the recursive definition of the γ-polynomials. We now give such a proof. By (7.2), it suffices to consider the case $\alpha = \beta = 1$.

Second proof of (7.1). We write

$$C_k^{m,n} = q^{mn-k(k+1)/2}\delta^k[k]!\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix}.$$

and want to show

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} C_k^{m,n}\gamma_{m-k}\gamma_{n-k}.$$

Denote the right hand side by $\gamma(m,n)$. Then $\gamma(m,n)=\gamma(n,m)$. We will use the recursion (6.5) and the Pascal formula (6.9), with q replaced by q^{-1}, to show $\gamma(m+1,n)=\gamma(m,n+1)$. Since $\gamma(m+n,0)=\gamma_{m+n}$, we are then done. We set $\gamma_k = 0$ for $k<0$, then we can sum just over $k \geq 0$. The C-coefficients satisfy the following Pascal type relation

(9.5)
$$C_k^{m+1,n} = q^{n-k}C_k^{m,n} + \delta q^{n-k+1}q^{m-k}[n-k+1]C_{k-1}^{m,n}.$$

The verification uses the Pascal formula for $\begin{bmatrix} m+1 \\ k \end{bmatrix}$ and a little rewriting. We now apply this relation in the sum $\gamma(m+1,n)$ and obtain (with an index shift $k \to k+1$ in the second summand)

$$\begin{aligned}\gamma(m+1,n) &= \sum_k q^{n-k}C_k^{m,n}\gamma_{m-k+1}\gamma_{n-k} \\ &\quad +\sum_k \left(\delta[n-k]q^{n-k-1}\gamma_{n-k-1}\right) q^{m-k}C_k^{m,n}\gamma_{m-k}.\end{aligned}$$

In the second sum we apply the recursion to the factor in brackets and obtain

$$\gamma(m+1,n) = \sum_k C_k^{m,n}(q^{n-k}\gamma_{m-k+1}\gamma_{n-k} + q^{m-k}\gamma_{m-k}\gamma_{n-k+1} - q^{n+m-2k}\theta\gamma_{m-k}\gamma_{n-k}).$$

Since $\gamma(m,n)=\gamma(n,m)$, we obtain $\gamma(m,n+1)$ by interchanging m and n in the last identity. This permutes the first two summands in the bracket and leaves the third. □

10. The universal cylinder twist

In this section we work with operators on integrable U-modules. These are $\mathfrak{K}$-linear weak endomorphisms of the category U-Int. Left multiplication by $x \in U$ is such an operator; it will be denoted by the same symbol or by l_x. If t is an operator, then $\mu(t)$ is the operator on U-Int $\times$ U-Int which is given by the action of t on tensor products of modules. If τ denotes the twist operator, then we define $\tau(t) = \tau \circ t \circ \tau$. We have the compatibility $\mu(l_x) = l_{\mu(x)}$ and $\tau\mu(l_x) = l_{\tau\mu(x)}$. The operators $\mu(t)$ and $\tau(t)$ are again weak endomorphisms of the categories involved.

Typical such operators which are not elements of U themselves are the universal R-matrix R and its factors κ and Ψ, see (6.1). We also use the operators $L = T'_{i,1}$ and $L^{\#} = T''_{i,1}$ of Lusztig [**??**, p. 42].

Since R acts by U-linear maps each operator t satisfies the standard relation

(10.1) $$R \circ \mu(t) = \tau\mu(t) \circ R$$

of a braiding.

An operator t is called a *universal cylinder twist* on U-Int if it is invertible and satisfies the analogue of (1.4)

(10.2) $$\mu(t) = \tau R(1 \otimes t)R(t \otimes 1)$$

(10.3) $$\tau R(1 \otimes t)R(t \otimes 1) = (t \otimes 1)\tau R(1 \otimes t)R.$$

We denote by t_V the action of t on the module V. Then (1.3) holds, if we use R to define the braiding. Recall the operator $t(\alpha, \beta)$, defined at the end of section 6. Here is the main result:

(10.4) Theorem. *Suppose $\alpha\beta = -q$. Then $t(\alpha, \beta)$ is a universal cylinder twist.*

We treat the case $(\alpha, \beta) = (-q, 1)$ in detail and reduce the general case formally to this one. We skip the notation α, β and work with $t = LT$. Note that L is Lusztig's operator referred to above. We collect a few properties of L in the next lemma.

(10.5) Lemma. *The operator L satisfies the following identities:*

(1) $LEL^{-1} = -KF$, $LFL^{-1} = -EK^{-1}$, $LKL^{-1} = K^{-1}$.

(2) $\mu(L) = (L \otimes L)\Psi = \tau R(L \otimes L)\kappa^{-1}$.

(3) $\kappa(L \otimes 1) = (L \otimes 1)\kappa^{-1}$, $\kappa(1 \otimes L) = (1 \otimes L)\kappa^{-1}$.

(4) $(L \otimes L)\Psi(L \otimes L)^{-1} = \kappa \circ \tau\Psi \circ \kappa^{-1}$.

PROOF. For (1), in the case $L^{\#}$, see [**??**, Proposition 5.2.4.]. A simple computation from the definitions yields (3) and (4). For the first equality in (2) see [**??**, Proposition 5.3.4]; the second one follows by using (3) and (4). □

In the universal case one of the axioms for a cylinder twist is redundant:

(10.6) Proposition. *If the operator t satisfies* (8.2) *then also* (8.3)*.*

PROOF. Apply τ to (8.2) and use (8.1). □

Proof of theorem (8.4). The operator L is invertible. The operator T is invertible since its constant term is 1. Thus it remains to verify (8.2). We show that this identity is equivalent to

(10.7) $$\mu(T) = \kappa(1 \otimes T)\kappa^{-1} \circ (L^{-1} \otimes 1)\Psi(L \otimes 1) \circ (T \otimes 1),$$

given the relations of Lemma (8.5). Given (8.2), we have

$$\mu(T) = \mu(L^{-1})\tau(R)(1 \otimes LT)\kappa\Psi(LT \otimes 1).$$

We use (8.5.2) for $\mu(L^{-1})$, cancel $\tau(R)$ and its inverse, and then use (8.5.3); (8.7) drops out. Similarly backwards.

In order to prove (8.7), one verifies the following identities from the definitions

$$\kappa(1 \otimes T)\kappa^{-1} = \sum_{k=0}^{\infty} \frac{\gamma_k}{[k]!}(K^k \otimes E^k)$$

$$(L^{-1} \otimes 1)\Psi(L \otimes 1) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k-1)/2} \frac{\delta^k}{[k]!} K^k E^k \otimes E^k.$$

Using this information, we compute the coefficient of $K^r E^s \otimes E^r$ on the right hand side of (8.7) to be

$$\sum_{n=0}^{\min(r,s)} (-1)^n q^{-n(n-1)/2} \frac{\delta^n}{[n]![s-n]![r-n]!} \gamma_{s-n}\gamma_{r-n}.$$

The coefficient of the same element in $\mu(T)$ is, by the q-binomial formula, equal to

$$q^{-rs} \frac{1}{[s]![r]!} \gamma_{r+s}.$$

Equality of these coefficients is exactly the product formula (7.1) in the case $(\alpha, \beta) = (-q, 1)$. This finishes the proof of the theorem in this special case.

A similar proof works in the general case. A formal reduction to the special case uses the following observation. Write $\alpha = q^{\zeta}$. Then, formally, $L(\alpha, \beta) = K^{\zeta} L$, in case $\alpha\beta = -q$. This is used to deduce from lemma (8.5) similar properties for $L_{\#} = L(\alpha, \beta)$, in particular

$$L_{\#}^{-1} F L_{\#} = \alpha^{-1} \beta q K E.$$

The final identity leads to (7.1) in the general case. □

We point out that the main identity in the construction of the universal twist involves only the Borel subalgebra of U generated by E, K. Of course, there is a similar theory based on F, K and another braiding. The constructions of section 6 show that the universal twist is determined by its action on the 2-dimensional module V_1. Hence our main theorem gives all possible universal cylinder twists associated to the given *braided* category U-Int.

11. Binomial coefficients

Suppose x, y, v are elements of a ring R which satisfy

$$yx = vxy, \quad vx = xv, \quad vy = yv.$$

One shows by induction that

(11.1) $$(x+y)^n = \sum_{k=0}^{n} (n, k; v) x^k y^{n-k}$$

and that the coefficients $(n, k; v)$ satisfy the *Pascal formula*

$$(n,k;v) = (n-1,k-1;v) + v^k(n-1,k;v). \tag{11.2}$$

This inductive formula also shows that $(n,k;v)$ can be defined as a polynomial in $\mathbb{Z}[v]$ and then specialized to any ring. The polynomials $(n,k;v)$ are called *Gauss polynomials* or *v-binomial coefficients*. They are computed as follows. Set

$$(n;v) = \frac{v^n-1}{v-1} = 1 + v + v^2 + \cdots + v^{n-1} \tag{11.3}$$

$$(n;v)! = (1;v)(2;v)\cdots(n;v). \tag{11.4}$$

Then

$$(n,k;v) = \frac{(n;v)!}{(k;v)!(n-k;v)!}. \tag{11.5}$$

For the proof of (A.5) one shows that the right hand side of (A.5) satisfies (A.2). For $v=1$ the polynomial $(n,k;v)$ reduces to the ordinary binomial coefficient $\binom{n}{k}$. Therefore one also finds notations like

$$(n,k;v) = \binom{n}{k}_v.$$

We also need other versions of binomial coefficients. Let $\mathfrak{A} = \mathbb{Z}[v,v^{-1}]$ and set

$$[n] = [n;v] = \begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{v^n - v^{-n}}{v - v^{-1}}, \qquad n \in \mathbb{Z} \tag{11.6}$$

$$[n]! = [n;v]! = \prod_{s=1}^{n} [s], \qquad n \in \mathbb{N}_0 \tag{11.7}$$

$$\begin{bmatrix} n \\ t \end{bmatrix} = [n,t;v] = \frac{[n]!}{[t]![n-t]!}, \qquad 0 \le t \le n. \tag{11.8}$$

The following Pascal formula holds

$$\begin{bmatrix} k+1 \\ i \end{bmatrix} = v^i \begin{bmatrix} k \\ i \end{bmatrix} + v^{-k+i-1} \begin{bmatrix} k \\ i-1 \end{bmatrix} \tag{11.9}$$

for $1 \le i \le k$. With an indeterminate z we have the identity

$$\prod_{j=0}^{k-1} (1 + v^{2j} z) = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix} v^{i(k-1)} z^i. \tag{11.10}$$

This is proved by induction (multiply by $1 + v^{2k}z$ and use (A.9)). This can be rewritten as follows

$$\prod_{j=0}^{k-1} (1 + v^j z) = \sum_{i=0}^{k} v^{i(i-1)/2} \binom{k}{i}_v z^i. \tag{11.11}$$

With indeterminates u and z this yields

(11.12) $$\prod_{j=0}^{k-1}(u - v^j z) = \sum_{i=0}^{k}(-1)^i v^{i(i-1)/2}\binom{k}{i}_v u^{k-i}z^i.$$

From (A.10) we see $[k,t] \in \mathfrak{A}$. Note that $[k,t]$ is invariant under $v \mapsto v^{-1}$ and therefore contained in $\mathbb{Z}[v+v^{-1}]$. From (A.10) one obtains for $z=-1$ and $k \geq 1$ the identity

(11.13) $$\sum_{t=0}^{k}(-1)^t v^{t(k-1)}\begin{bmatrix} k \\ t \end{bmatrix} = 0.$$

If $yx = v^2xy$, then

(11.14) $$(x+y)^n = \sum_{t=0}^{n} v^{t(n-t)}\begin{bmatrix} n \\ t \end{bmatrix} x^t y^{n-t}.$$

One verifies

(11.15) $$(n,k;v^2) = v^{k(n-k)}[n,k;v].$$

We set for $k \in \mathbb{Z}$ and $t \in \mathbb{N}$ in general

(11.16) $$\begin{bmatrix} k \\ t \end{bmatrix} = \prod_{s=1}^{t}\frac{v^{k-s+1} - v^{-k+s-1}}{v^s - v^{-s}}.$$

Then we have

(11.17) $$\begin{bmatrix} k \\ t \end{bmatrix} = (-1)^t\begin{bmatrix} -k+t-1 \\ t \end{bmatrix}$$

(11.18) $$\begin{bmatrix} k \\ t \end{bmatrix} = 0, \quad 0 \leq k < t; \qquad \begin{bmatrix} -1 \\ t \end{bmatrix} = (-1)^t, \quad t \geq 0.$$

For $k \in \mathbb{N}$, this coincides with the previously defined polynomials.

(11.19) *For $a,b \in \mathbb{Z}$ and $u \in \mathbb{N}$ we have*

$$\begin{bmatrix} a+b \\ u \end{bmatrix} = \sum_{s+t=u} v^{at-bs}\begin{bmatrix} a \\ s \end{bmatrix}\begin{bmatrix} b \\ t \end{bmatrix}.$$

In this sum $s \geq 0$ and $t \geq 0$.

Proof. Let $a,b \geq 0$. Then we obtain with (A.10)

$$\begin{aligned}\sum_{u=0}^{a+b} v^{u(a+b-1)}\begin{bmatrix} a+b \\ u \end{bmatrix} z^u &= \prod_{j=0}^{a+b-1}(1+v^{2j}z) \\ &= \prod_{j=0}^{a-1}(1+v^{2j}z)\prod_{h=0}^{b-1}(1+v^{2h}(v^{2a}z))\end{aligned}$$

$$= \sum_{s=0}^{a} v^{s(a-1)} \begin{bmatrix} a \\ s \end{bmatrix} z^s \cdot \sum_{t=0}^{b} v^{t(b-1)} \begin{bmatrix} b \\ t \end{bmatrix} v^{2at} z^t.$$

This can be viewed as formal identity in indeterminates v, v^a, v^b. Since it holds for the special values v^a, v^b for $a, b \geq 0$, it is formally true, and so holds also for $a, b \in \mathbb{Z}$. $\square$

(11.20) *For $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$ the following identities hold*

$$\sum_{r=0}^{m} v^{r(r-1)/2}(v-1)^r (r)_v^! \binom{m}{r}_v \binom{n}{r}_v = v^{nm}$$

$$\sum_{r=0}^{m} v^{-r(r+1)/2+r(m+n)}(v-v^{-1})^r [r]! \begin{bmatrix} m \\ r \end{bmatrix} \begin{bmatrix} n \\ r \end{bmatrix} = v^{2mn}.$$

Proof. The second formula is a rewriting of the first one. $\square$

(11.21) Exercises and supplements.

1. Let F be a field with q elements. The number of k-dimensional subspaces of an n-dimensional F-vector space equals $(n, k; q)$.

2. The Pascal formula (A.9) is not a relation in $\mathbb{Z}[v + v^{-1}]$. Therefore deduce from it the equality

$$\begin{bmatrix} n+1 \\ s \end{bmatrix} = [s+1] \begin{bmatrix} n \\ s \end{bmatrix} + [-n+s] \begin{bmatrix} n \\ s-1 \end{bmatrix}.$$

3. Let $RSU(2)$ denote the complex representation ring of the group $SU(2)$. It is isomorphic to $\mathbb{Z}[w]$. The restriction to the maximal torus $T \cong S^1$ of diagonal matrices yields the embedding

$$r\colon \mathbb{Z}[w] = RSU(2) \to RS^1 = \mathbb{Z}[v, v^{-1}], \quad w \mapsto v + v^{-1}.$$

Let W_k denote the k-dimensional irreduzible representation of $SU(2)$. Then $r(W_k) = [k; v]$. The rings $RSU(2)$ and RS^1 are special λ-rings, and r is a λ-homomorphism. Let

$$\lambda_z(x) = 1 + \lambda^1(x)z + \lambda^2(x)z^2 + \cdots,$$

with an indeterminate z. If $x = V$ is a representation, then $\lambda^k(x) = \Lambda^k(x)$ is the k-th exterior power of x; also $\Lambda^k(V) = 0$ for $k > \dim V$. In particular

$$\lambda(v^s) = 1 + v^s z, \qquad s \in \mathbb{Z},$$

since v^s is one-dimensional. The property $\lambda_z(x+y) = \lambda_z(x)\lambda_z(y)$ yields

$$\lambda_z(rW_k) = \prod_{j=0}^{k-1} \lambda_z(v^{k-1-2j}) = \prod_{j=0}^{k-1}(1 + v^{k-1-2j}z).$$

This gives

$$\text{(11.22)} \qquad r(\Lambda^i W_k) = \begin{bmatrix} k \\ i \end{bmatrix}.$$

Symmetric powers yield a second λ-ring structure. Let $s^k(V) = S^k(V)$ for a representation V and

$$s_z(x) = 1 + s^1(x)z + s^2(x)z^2 + \cdots.$$

Then

$$\text{(11.23)} \qquad s_z(x+y) = s_z(x)s_z(y), \qquad s_{-z}(x) \cdot \lambda_z(x) = 1.$$

This yields

$$\text{(11.24)} \qquad S^j(W_k) = \Lambda^{k-1}(W_{j+k-1})$$

and the character formula

$$\text{(11.25)} \qquad r(S^j W_k) = (-1)^j \begin{bmatrix} -k \\ j \end{bmatrix}.$$

Deduce the identity

$$\text{(11.26)} \qquad \prod_{j=0}^{k-1} \frac{1}{1+v^{2j}z} = \sum_{t=0}^{\infty} \begin{bmatrix} -k \\ t \end{bmatrix} v^{t(k-1)} z^k$$

for all $k \geq 1$.

4 Knot algebra

1. Braid groups, Coxeter groups, and related algebras

These notes are based in one way or another on the geometric idea of braids, in particular braids in the cylinder. These will be the objects in the next two sections. As a preparation, this section collects some algebraic material of a general nature which will be used in subsequent sections. Standard references are [**??**] and [**??**].

(1.1) Coxeter matrix. A *Coxeter matrix* (S, m) consists of a finite set S and a symmetric mapping m: $S \times S \to \mathbb{N} \cup \{\infty\}$ with $m(s, s) = 1$ and $m(s, t) \geq 2$ for $s \neq t$. A Coxeter matrix (S, m) is often specified by its *weighted Coxeter graph* $\Gamma(S, m)$. It has S as its set of vertices and an edge with weight $m(s, t)$ connecting s and t whenever $m(s, t) \geq 3$. Usually, the weight $m(s, t) = 3$ is omitted in the notation. ♡

Associated to a Coxeter matrix are various interrelated algebraic objects: Braid groups, Coxeter groups, Hecke algebras, Temperley-Lieb algebras, Brauer algebras, Birman-Wenzl algebras. We are going to define some of them.

(1.2) Braid groups. The *braid group* $Z(S, m)$ associated to the Coxeter matrix (S, m) has generators $(g_s \mid s \in S)$ and relations

(1.3) $$g_s g_t g_s \ldots = g_t g_s g_t \ldots,$$

with $m(s, t)$ factors alternating on each side whenever $2 \leq (s, t) < \infty$. We call (1.1) the *braid relations.* ♡

(1.4) Coxeter groups. The *Coxeter group* $C(S, m)$ has generators $(s \mid s \in S)$, relations (1.1) and in addition the relations $s^2 = 1$ for each $s \in S$. An important combinatorial invariant of a Coxeter group is the *length fuctions* l: $C(s, m) \to \mathbb{N}_0$. A *reduced expression* for $x \in C(S, m)$ is a family $(s_1, s_2, \ldots, s_r)$ of elements $s_j \in S$ of minimal length $r = l(x)$ such that $x = s_1 \cdot s_2 \cdot \ldots \cdot s_r$. ♡

Define an equivalence relation on S by $s \sim t$ if and only if there exists a sequence $s = s_1, \ldots, s_p = t$ such that $m(s_j, s_{j+1})$ is odd.

(1.5) Proposition. *Two elements $s, t \in S$ are conjugate in $C = C(S, m)$ if and only if $s \sim t$. The commutator quotient C/C' is elementary abelian of order 2^n, where n is the cardinality of $S/\sim$. Similarly, the commutator quotient Z/Z' for $Z = Z(S, m)$ is isomorphic to $\mathbb{Z}^n$, with basis a representative set of $S/\sim$.* □

Consider families (words) $(s_1, s_2, \ldots, s_r)$ of elements $s_j \in S$. The defining relations of $C(S, m)$ yield two *elementary moves* for such words:

(1) Remove a subword (s, s).
(2) If $s \neq t$ and $m(s, t) < \infty$, replace $(s, t, s, \ldots)$ by $(t, s, t, \ldots)$; each sequence is supposed to be of length $m(s, t)$.

The following theorem of Tits gives good information about the word problem in Coxeter groups ([**??**], [**??**]).

(1.6) Theorem. *A word is a reduced expression of an element if and only if it cannot be shortened by elementary moves. Two reduced words define the same element in the group if and only if one can be transformed into the other by moves of type* 2. □

(1.7) Corollary. *Let* $(s_1, \dots, s_r)$ *be a reduced expression of* $w \in C(S,m)$*. Then the number* $l(c;w)$ *of indices* k *such that* s_k *is contained in the conjugacy class* $c \in S/\sim$ *is independent of the reduced expression.* □

We call $(l(c;w) \mid c \in S/\sim)$ the *graded length* of w in $C(S,m)$. The graded length function gives a *Poincaré series* of the Coxeter group C in indeterminates $(x_c \mid c \in S/\sim)$

$$P(x_c) = \sum_{w \in C} \left(\prod_{c \in S/\sim} x_c^{l(c;w)} \right). \tag{1.8}$$

(1.9) Corollary. *Let* M *be a monoid and* $\varphi\colon S \to M$ *be a map such that for all* $s \neq t$ *the two products of length* $m(s,t) < \infty$

$$\varphi(s)\varphi(t)\varphi(s)\cdots = \varphi(t)\varphi(s)\varphi(t)\cdots.$$

Then for any reduced word $(s_1, \dots, s_r)$ *the value* $\varphi(s_1)\cdots\varphi(s_r)$ *only depends on the product* $w = s_1 \cdots s_r$. □

As a special case we obtain:

(1.10) Theorem. *Suppose* $(s_1, \dots, s_r)$ *is a reduced expression of* $x \in C(S,m)$*. Then the product* $g_x := g_{s_1} g_{s_2} \cdots g_{s_r} \in Z(S,m)$ *is independent of the reduced expression and only depends on* x. □

(1.11) Hecke algebras. Let $\mathfrak{K}$ be a commutative ring. Choose parameters $P = ((a_s, b_s) \in \mathfrak{K}^2 \mid s \in S)$, constant along conjugacy classes of S. The *Hecke algebra* $H(S,m;P)$ associated to this parameter set is the associative algebra with 1 over $\mathfrak{K}$ with generators $(x_s \mid s \in S)$, braid relations (1.1) and quadratic relations

$$x_s^2 = a_s x + b_s. \tag{1.12}$$

The *standard Hecke algebra* $H_q(S,m)$ with parameter $q \in \mathfrak{K}$ is obtained as the special case $a_s = q - 1, b_s = q$. ♡

(1.13) Proposition. *Additively, the Hecke algebra is a free* $\mathfrak{K}$*-module with basis*

$$g_w = g_{s_1} \cdots g_{s_r} \quad w = (s_1, \dots, s_r) \quad \text{reduced.} \tag{1.14}$$

The action of the generators g_s *on the basis elements is given by*

(1.15)
$$g_s g_w = \begin{cases} g_{sw} & l(w) < l(sw) \\ a_s + b_s & l(w) > l(sw) \end{cases}.$$

There are similar statements for right multiplication. □

(1.16) Temperley-Lieb algebras. If $m(S \times S) \subset \{1,2,3\}$, we define the associated *Temperley-Lieb algebra* $T_d(S,m)$ as follows. It is the associative algebra with 1 over $\mathfrak{K}$ with generators $(e_s \mid s \in S)$ and relations

(1.17)
$$\begin{array}{rcll} e_s^2 & = & de_s & \\ e_s e_t & = & e_t e_s & m(s,t) = 2 \\ e_s e_t e_s & = & e_s & m(s,t) = 3. \end{array}$$

(1.18) Proposition. *Suppose* $p \in \mathfrak{K}^*$, $q = p^2$, $d = p+p^{-1}$. *Then the assignment* $x_s \mapsto pe_s - 1$ *yields a surjection* φ: $H_q(S,m) \to T_d(S,m)$. *The kernel of* φ *is the twosided ideal generated by the elements* $x(s,t) = x_s x_t x_s + x_s x_t + x_t x_s + x_s + x_t + 1$; *here* (s,t) *runs over the pairs* (s,t) *with* $m(s,t) = 3$.

Proof. (Compare [**??**, 2.11].) One verifies easily that φ respects the defining relations of the Hecke algebra. Certainly, φ is surjective. Let $I \subset H_q(S,m)$ denote the ideal generated by the $x(s,t)$ for (s,t) with $m(s,t) = 3$. We define a homomorphism ψ: $T_d(S,m) \to H_q(S,m)/I$ by $\psi(e_s) = p^{-1}(x_s + 1)$. One verifies that this is compatible with (1.2) and that $x(s,t)$ is contained in the kernel of φ. Hence φ induces φ: $H_q/I \to T_d$. By construction, φ and ψ are inverse homomorphisms. □

The preceding construction can, in particular, be applied to Coxeter matrices of ADE-type. The resulting algebras are then finite dimensional. The structure of TA_{n-1} associated to the linear graph A_{n-1} with n vertices is well known, see [**??**]; this is the classical Temperley-Lieb algebra.

By way of example we show that $T_d(S,m)$ is non-zero. This is done by constructing a standard module which arises from the reflection representation of the Hecke algebra.

We work with a field $\mathfrak{K}$. Let V denote the free $\mathfrak{K}$-module with basis $\{v_s \mid s \in S\}$. We define a symmetric bilinear form B on V by

$$\begin{array}{ll} B(v_s, v_s) = q + 1 & \\ B(v_s, v_t) = p & m(s,t) = 3 \\ B(v_s, v_t) = 0 & m(s,t) = 2. \end{array}$$

We define a linear map X_s: $V \to V$ by $X_s(v) = qv_s - B(v_s, v)v$. Then $X_s(v_s) = -v_s$, and $X_s(v) = v$ for v in the orthogonal complement of v_s. We assume $q+1 \in \mathfrak{K}^*$. Then V is the orthogonal direct sum of $\mathfrak{K}v_s$ and $(\mathfrak{K}v_s)^\perp$. On the latter, X_s acts as multiplication by q. Hence X_s satisfies the quadratic equation $X_s^2 = (q-1)X_s + q$ of the Hecke algebra.

The determinant $d_{s,t}$ of B on the submodule $\langle\, v_s, v_t\,\rangle$ generated by v_s and v_t equals

$$d_{s,t} = \begin{cases} (q+1)^2 & m(s,t) = 2 \\ q^2+q+1 & m(s,t) = 3. \end{cases}$$

We therefore also assume $q^2+q+1 \in \mathfrak{K}^*$. Then V is the orthogonal direct sum of $\langle\, v_s, v_t\,\rangle$ and $\langle\, v_s, v_t\,\rangle^\perp$. On the latter subspace, X_s and X_t act as multiplication by q. The action of X_s and X_t on $\langle\, v_s, v_t\,\rangle$ in the basis v_s, v_t is given by

$$X_s = \begin{pmatrix} -1 & p \\ 0 & q \end{pmatrix}, \quad X_t = \begin{pmatrix} q & 0 \\ p & -1 \end{pmatrix},$$

in the case $m(s,t) = 3$. A simple computation shows

$$X_s X_t X_s = X_t X_s X_t = \begin{pmatrix} 0 & -pq \\ -pq & 0 \end{pmatrix}.$$

Thus we have constructed the reflection representation V of $H_q(S,m)$.

The assignment ω: $H_q(S,m) \to H_q(S,m)$, $\quad x_s \mapsto -qx_s^{-1}$ is an involutive automorphism of the Hecke algebra. It transforms V into a new module $W = V^\omega$.

(1.19) Proposition. *The module W factors over the homomorphism φ of* (1.4).

Proof. We set $Y_s = -qX_s^{-1}$. We have to show that the operator

$$Y_{s,t} = Y_s Y_t Y_s + Y_s Y_t + Y_t Y_s + Y_s + Y_t + 1$$

acts on V as the zero map. We compute that Y_s+1 and Y_t+1 act on $\langle\, v_s, v_t\,\rangle$ in the basis v_s, v_t through the matrices

$$Z_s = \begin{pmatrix} q+1 & -p \\ 0 & 0 \end{pmatrix}, \quad Z_t = \begin{pmatrix} 0 & 0 \\ -p & q+1 \end{pmatrix}.$$

This is used to verify on $\langle\, v_s, v_t\,\rangle$ the relation $Z_s Z_t Z_s = qZ_s$. A formal calculation, using the quadratic equation for Y_s, yields

$$(Y_s+1)(Y_t+1)(Y_s+1) - q(Y_s+1) = Y_{s,t}.$$

Therefore $Y_{s,t}$ acts as zero on $\langle\, v_s, v_t\,\rangle$. Since X_s is multiplication by q on $\langle\, v_s, v_t\,\rangle^\perp$, we see that $-qX_s^{-1}+1$ is the zero map. □

We give a more direct construction of a $T_d(S,m)$-module which does not use the reflection representation of the Hecke algebra. Let $A = (a_{st})$ denote a symmetric $S \times S$-matrix over $\mathfrak{K}$. We consider the associative algebra $T(A)$ over $\mathfrak{K}$ with generators $(Z_s \mid s \in S$ and relations

$$\begin{aligned} Z_s^2 &= a_{ss} Z_s \\ Z_s Z_t Z_s &= a_{st} a_{ts} Z_s. \end{aligned}$$

Then a simple verification from the definitions gives:

(1.20) Proposition. *Let V be the $\mathfrak{K}$-module with basis $(v_s \mid s \in S$. The operators $Z_s(v_t) = a_{st}v_s$ make V into a $T(A)$-module. (Hence each Z_s has rank at most one on V.)* □

The matrix $A = (a_{st})$ is called *indecomposable*, if there is no partition $S = S_1 \coprod S_2$ with $a_{uv} = 0$ for $u \in S_1$, $v \in S_2$.

(1.21) Proposition. *Let $\mathfrak{K}$ be a field. Suppose A is indecomposable and* $\det(A) \neq 0$. *The the module V of the previous proposition is a simple $T(A)$-module.*

Proof. We have $Z_s(\sum_j a_j v_j = (\sum_j a_j a_{sj})v_s$. Suppose $v = \sum_j a_j v_j \neq 0$. Since $\det(A) \neq 0$, not all $Z_s v$ are zero. If $0 \neq M \subset V$ is a $T(A)$-submodule, then there exists $s \in S$ with $v_s \in M$. Suppose $v_t \notin M$. Since $Z_t v_s = a_{ts}v_t \in M$, we must have $a_{ts} = 0$. This contradicts the indecomposability of A. Hence all v_t are containes in M. □

In the case of a Coxeter graph, we set $a_{ss} = d$, $a_{st} = 1$ for $m(s,t) = 3$, and $a_{st} = 0$ for $m(s,t) = 2$. Then V becomes a module over $T_d(S,m)$. Also, $\det(A)$ is a non-trivial monic polynomial in d, hence in general not zero.

The oriented Temperley-Lieb algebra T^oB_n is the associative $\mathfrak{K}$-algebra with 1 generated by $e_0, e_1, \dots, e_{n-1}$ with relations

$$
\begin{array}{rcll}
e_0^2 & = & De_0 & \\
e_1e_0e_1 & = & Fe_1 & \\
e_j^2 & = & d_kd_j, & j \geq 1,\ j \equiv k(2) \\
e_ie_je_i & = & e_i & |i-j| = 1;\ i,j \geq 1 \\
e_ie_j & = & e_je_i & |i-j| \geq 2.
\end{array}
\tag{1.22}
$$

Here $D, F, d_1 d_2 \in \mathfrak{K}$ are given parameters. This algebra is a quotient of the Hecke algebra $B_n(q,Q)$ for suitable parameters q, Q. We use still more parameters $u_1, u_2, U, V \in \mathfrak{K}$.

(1.23) Proposition. *Let $n \geq 3$. The assignment $g_j \mapsto u_k e_j - 1$ for $j \geq 1$, $j \equiv k(2)$ and $t \mapsto Ue_0 + V$ yields a homomorphism*

$$\varphi\colon HB_n(q,Q) \to TB_n(d_1, d_2, D, F)$$

if and only if the following relations between the parameters hold:

(1) $u_j d_j = q + 1$

(2) $u_1 u_2 = q$

Proof. We first show the following lemma.

(1.24) Lemma. *The elements $g_0 = x + ye_0$ and $g_1 = a + a^{-1}e_1$ in $TB_2(d,D,F)$ with $d = -a^2 - a^{-2}$ satisfy the four braid relation $g_0g_1g_0g_1 = g_1g_0g_1g_0$ if and onlx if*

$$y(D + a^{-2}F) = x(a^{-4} - 1).$$

Proof. We set $A = a^{-1}$. The four braid relation is equivalent to

$$e_1(g_0^2 + A^2 g_0 e_1 g_0) = (g_0^2 + A^2 g_0 e_1 g_0)e_1.$$

An element of the form

$$z = \alpha + \beta e_0 + \gamma e_1 + \delta e_0 e_1 + \varepsilon e_1 e_0 + \eta e_0 e_1 e_0$$

commutes with e_1 if and only if $\varepsilon = \delta$ and $\beta + \delta d + \eta F = 0$. For $z = g_0^2 + A^2 g_0 e_1 g_0$ we have

$$\alpha = x^2, \beta = 2xy + y^2 D, \gamma = A^2 x^2, \delta = A^2 xy = \varepsilon, \eta A^2 y^2.$$

This leads to the claim of the lemma. We have used in the proof that TB_2 is a 6-dimensional algebra with basis $1, e_0, e_1, e_0e_1, e_1e_0, e_0e_1e_0$. □

Remarks. We remark that under the conclusion of the lemma the element $g_0g_0g_0g_1$ lies in the center of TB_2. The element $x_1 = -a^{-1}g_1$ satisfies $x_1^2 = (q-1)x_1 + q$ with $q = A^4$. The element g_0 satisfies $g_0^2 = (2x + yD)g_0 - x^2 - xyD$. The flexibility relation $g_0 e_0 = e_0$ holds if and only if $x + yD = 1$. In the case $D = F$, the condition of the lemma reduces to $yD = x(a^{-2} - 1)$. Together with the flexibility this implies

$$x = a^2, \quad yD = 1 - a^2.$$

Under these conditions also the relation $g_0g_1g_0g_1e_1 = e_1$ holds. All these relations are relevant for the Kauffman functor into the category TB, see ??. ♡

We continue with the proof of the proposition. ?? □

2. Braid groups of type B

The braid group ZB_n associated to the Coxeter graph B_n is, by definition, the group generated by $t, g_1, \dots, g_{n-1}$ with relations

(2.1)
$$\begin{array}{llcll} (1) & g_i g_j g_i & = & g_j g_i g_j, & |i-j| = 1 \\ (2) & g_i g_j & = & g_j g_i, & |i-j| \geq 2 \\ (3) & t g_i & = & g_i t, & i \geq 2 \\ (4) & t g_1 t g_1 & = & g_1 t g_1 t. & \end{array}$$

We also need another presentations of this group.

Let $Z'B_n$ be the group with generators $c, g_1, \dots, g_{n-1}$ and relations

(2.2)
$$\begin{array}{llcll} (1) & g_i g_j g_i & = & g_j g_i g_j, & |i-j| = 1 \\ (2) & g_i g_j & = & g_j g_i, & |i-j| \geq 2 \\ (3) & c g_i & = & g_{i-1} c, & i \geq 2 \\ (4) & c^2 g_1 & = & g_{n-1} c^2. & \end{array}$$

We abbreviate $g = g_{n-1}g_{n-2}\cdots g_1$.

(2.3) Proposition. *The assignment* $\varphi(g_i) = g_i$, $1 \leq j \leq n-1$, *and* $\varphi(t) = g^{-1}c$ *defines an isomorphism* $\varphi\colon ZB_n \to Z'B_n$.

Proof. The relations (1) and (2) yield in both groups

(2.4) $$g_{i-1}g = gg_i, \qquad i > 1.$$

We define in ZB_n (resp. $Z'B_n$) an element c (resp. t) by $gt = c$. From (1), (2) and (8.4) we see that the relations $cg_i = g_{i-1}c$ and $g_it = tg_i$ are equivalent for $i > 1$.

We set $h = g_{n-1}\cdots g_2$, $k = g_{n-2}\cdots g_1$ and infer from (8.4)

(2.5) $$gh = kg.$$

We use this to show that the relations $c^2g_1 = g_{n-1}c^2$ and $tg_1tg_1 = g_1tg_1t$ are equivalent, provided (1), (2), and (3) hold. We compute

$$g_{n-1}^{-1}c^2g_1 = g_{n-1}^{-1}g_{n-1}kthg_1tg_1 = khtg_1tg_1$$

$$c^2 = gthg_1t = ghtg_1t = kgtg_1t = khg_1tg_1t$$

and see the equivalence. □

The braid group $Z\tilde{A}_{n-1}$ of the Coxeter graph with n vertices $\tilde{A}_{n-1}$ has, by definition, generators $g_1, \ldots, g_n$ and relations

(2.6) $$\begin{aligned} g_ig_jg_i &= g_jg_ig_j, & m(i,j) = 3 \\ g_ig_j &= g_jg_i, & m(i,j) = 2. \end{aligned}$$

Indices will be considered $\bmod n$ in this case. We have $m(i,j) = 3$ if and only if $i \equiv j \pm 1 \bmod n$. All this holds for $n \geq 3$. For $n = 2$, the group is the free group generated by g_1 and g_2.

The graph $\tilde{A}_{n-1}$ has an automorphism which permutes the vertices cyclically. We have an induced automorphism s of $Z\tilde{A}_{n-1}$ given by

$$s(g_i) = g_{i-1}, \qquad i \bmod n.$$

The n-th power of s is the identity.

We use s to form the semi-direct product

(2.7) $$Z\tilde{A}_{n-1} \to G_n \to \mathbb{Z};$$

the generator $1 \in \mathbb{Z}$ acts through s on $Z\tilde{A}_{n-1}$. There is a similar semi-direct product where $\mathbb{Z}$ is replaced by $\mathbb{Z}/nk$. The semi-direct product is the group structure on the set $Z\tilde{A}_{n-1} \times \mathbb{Z}$ defined by $(x,m)\cdot(y,n) = (x \cdot s^m(y), m+n)$. The group G_n has the following description by generators and relations. Let G'_n denote the group with generators $s, g_1, \ldots, g_n$ and relations (8.6) for the g_j together with

(2.8) $$sg_i = g_{i-1}s, \qquad i \bmod n.$$

(2.9) Proposition. *The assignment $\psi(g_i) = (g_i, 0)$ and $\psi(s) = (e, 1)$ yields an isomorphism $\psi\colon G'_n \to G_n$ (neutral element e).*

Proof. One verifies that ψ is compatible with relations (8.6) and (8.8). This is obvious for (8.6). The relation $(e,1)(x,0)(e,1)^{-1} = (s(x), 0)$ is used to show compatibility with (8.8).

An element $x \in Z\tilde{A}_{n-1}$ has an image $x' \in G'_n$, induced by $g_i \mapsto g_i$. This assignment has the property $(s(x))' = sx's^{-1}$. We have the homomorphism $G_n \to G'_n$, $(x, m) \mapsto x's^m$, by (8.4). It is inverse to ψ. □

(2.10) Proposition. *The assignment $\alpha(g_i) = g_i$, $1 \le i \le n-1$, and $\alpha(c) = s$ defines an isomorphism $\alpha\colon Z'B_n \to G'_n$.*

Proof. The assignment is compatible with the relations of $Z'B_n$, since

$$\alpha(c^2 g_1 c^{-2}) = s^2 g_1 s^{-2} = s g_n s^{-1} = g_{n-1}.$$

An inverse to α is induced by the assignment $\beta(g_i) = g_i$, $\beta(g_n) = cg_1g^{-1}$, and $\beta(s) = c$. In order to see that β is well defined, one has to check, in particular, the relations

$$g_{n-1}g_ng_{n-1} = g_ng_{n-1}g_n, \qquad g_1g_ng_1 = g_ng_1g_n.$$

In the first case, this amounts to the equalitiy of

$$g_{n-1}cg_1c^{-1}g_{n-1} = c^2g_1c^{-1}g_1cg_1c^{-2}$$

and

$$cg_1c^{-1}g_{n-1}cg_1c^{-1} = cg_1cg_1c^{-1}g_1c^{-1}.$$

We compute

$$cg_1g_2g_1c^{-1} = cg_2g_1g_2c^{-1} = cg_2c^{-1}cg_1c^{-1}cg_2c^{-1} = g_1cg_1c^{-1}g_1$$

and hence

$$c(g_1cg_1c^{-1}g_1)c^{-1} = c^2g_1g_2g_1c^{-2}.$$

On the other hand, $g_1c^{-1}g_1cg_1 = g_1g_2g_1$. This yields the desired equality.

The second relation above leads to the same situation. □

If we combine the foregoing, we obtain a semi-direct product

(2.11) $$Z\tilde{A}_{n-1} \to ZB_n \to \mathbb{Z}.$$

In terms of the original generators, the inclusion $Z\tilde{A}_{n-1} \subset ZB_n$ is given by

(2.12) $$g_n \mapsto gtg_1t^{-1}g^{-1}; \quad g_i \mapsto g_i, \quad 1 \le i \le n-1.$$

The homomorphism $ZB_n \to \mathbb{Z}$ in (8.11) is given by $g_i \mapsto 0$ and $t \mapsto 1$.

Different types of Weyl groups (= Coxeter groups) are related to these braid groups. We have the Coxeter groups $W\tilde{A}_{n-1}$ and WB_n associated to the graphs $\tilde{A}_{n-1}$ and B_n. In addition, we will also use a group $W^\infty B_n$. It is obtained from ZB_n by adding the relations $g_j^2 = 1$, but no relation for t. The reason for introducing this group is a semi-direct product in analogy to (8.11). The arguments which lead to (8.11) also give a semi-direct product

$$W\tilde{A}_{n-1} \to W^\infty B_n \to \mathbb{Z}.$$

We give another interpretation and describe these groups as groups of permutations.

Let $t_n \colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x+n$ be the translation by n. Let P_n denote the group of t_n-equivariant permutations $\sigma \colon \mathbb{Z} \to \mathbb{Z}$. Equivariance means $\sigma(i+n) = \sigma(i)+n$. Hence σ induces $\overline{\sigma} \colon \mathbb{Z}/n \to \mathbb{Z}/n$, and $\sigma \mapsto \overline{\sigma}$ is a homomorphism $\pi \colon P_n \to S_n$ onto the symmetric group S_n.

(2.13) Proposition. *The kernel of π is isomorphic to $\mathbb{Z}^n$. The group P_n is isomorphic to the semi-direct product $\mathbb{Z}^n \to P'_n \to S_n$ in which S_n acts on $\mathbb{Z}^n$ by permutations.*

PROOF. Let $\sigma_1 \in P_n$. Then there exists a permutation α of $\{1, \dots, n\}$ and an n-tuple $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that $\sigma(i+tn) = \alpha(i)+(k_i+t)n$. We denote this map by $\sigma_1 = \sigma(\alpha; k_1, \dots, k_n)$. Suppose $\sigma_2 = \sigma(\beta; l_1, \dots, l_n)$ is another permutation written in this form. Then

$$\sigma_2 \circ \sigma_1 = \sigma(\beta\alpha; l_{\alpha(1)} + k_1, \dots, l_{\alpha(n)} + k_n).$$

If we think of $P'_n = S_n \times \mathbb{Z}^n$ as sets, then the desired isomorphism is given by $(\alpha; k_1, \dots, k_n) \mapsto \sigma(\alpha; k_1, \dots, k_n)$. □

The semi-direct product P'_n has a normal subgroup Q'_n which is given as a semi-direct product

$$N \to Q'_n \to S_n \tag{2.14}$$

with $N = \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \subset \mathbb{Z}^n$. The homomorphism

$$\varepsilon \colon P'_n \to \mathbb{Z}, \quad (\alpha; k_1, \dots, k_n) \mapsto \sum k_i$$

is a surjection with kernel Q'_n. The canonical sequence

$$Q'_n \to P'_n \to \mathbb{Z} \tag{2.15}$$

is itself a semi-direct product; the assignment $1 \mapsto (\mathrm{id}; 1, 0, \dots, 0)$ gives a splitting of ε. Under the isomorphism (8.13) the subgroup Q'_n corresponds to the subgroup

$$Q_n = \{\sigma \in P_n \mid 1+2+\cdots+n = \sigma(1)+\cdots+\sigma(n)\}.$$

(2.16) Proposition. *The groups $W^\infty B_n$ and P_n are isomorphic. The groups $W\tilde{A}_{n-1}$ and Q_n are isomorphic. The element g_i is mapped to the transposition*

$(i, i+1)$, $i \in n\mathbb{Z}$. The element t is mapped to $\sigma(i) = i + n$ for $i \equiv 1 \bmod n$ and $\sigma(j) = j$ otherwise.

The proof is given after the proof of (8.21). In the proof of (8.16) we use the following:

(2.17) Lemma. *The elements*

$$t_0 = t, \quad t_1 = g_1 t g_1, \quad \dots, \quad t_{n-1} = g_{n-1} \dots g_2 g_1 t g_1 g_2 \dots g_{n-1}$$

of the braid group ZB_n pairwise commute.

Proof. We set

$$g(i,j) = g_i g_{i+1} \dots g_j, \qquad i \le j$$
$$g(i,j) = g_i g_{i-1} \dots g_j, \qquad i \ge j.$$

The braid relations imply immediately

$$g(1,j) g_{j+1} g(j,1) = g(j+1,2) g_1 g(2,j+1)$$

and (8.5) yields

$$g(2,j+1) g(1,j+1) = g(1,j+1) g(1,j).$$

By commutation of g_j-elements, it suffices to show $t_i t_{i+1} = t_{i+1} t_i$. We compute

$$\begin{aligned} t_j t_{j+1} &= g(j,1) t g(1,j) g_{j+1} g(j,1) t g(1,j+1) \\ &= g(j,1) t g(j+1,2) g_1 g(2,j+1) t g(1,j+1) \\ &= g(j,1) g(j+1,2) t g_1 t g(2,j+1) g(1,j+1) \\ &= g(j,1) g(j+1,2) [t g_1 t g_1] g(2,j+1) g(1,j). \end{aligned}$$

A similar computation works for $t_{j+1} t_j$. □

The semi-direct product relation (8.13), (8.16) between $W^\infty B_n$ and WA_{n-1} has a counterpart for the braid groups. The homomorphism

$$\lambda\colon ZB_n \to ZA_{n-1}, \quad g_j \mapsto g_j, \quad t \mapsto 1$$

splits by $g_j \mapsto g_j$. Therefore we have a semi-direct product

(2.18) $$K_n \to ZB_n \to ZA_{n-1}.$$

The elements

$$y_0 = t, \quad , y_1 = g_1 t g_1^{-1}, \quad \dots, \quad y_{n-1} = g_{n-1} \dots g_1 t g_1^{-1} \dots g_{n-1}^{-1}$$

are contained in the kernel K_n of λ.

(2.19) Lemma. *The elements y_j have the following conjugation properties with respect to ZA_{n-1}:*

(1) $g_k^{-1} y_j g_k = y_j, \qquad k > j,\ k < j-1$
(2) $g_k^{-1} y_k g_k = y_{k-1},$
(3) $g_k^{-1} y_{k-1} g_k = y_{k-1} y_k y_{k-1}^{-1}.$

Proof. (2) follows directly from the definitions.
(1) If $k > j$, then g_k commutes with every generator in the definition of y_j. In the case $k < j-1$ one uses the commutation relation between generators and $g_{k+1} g_k g_{k+1}^{-1} = g_k^{-1} g_{k+1} g_k$ (and the inverse) to cancel g_k^{-1} and g_k.
(3) is proved by induction on k. The verification for $k=0$ is easy. We calculate with (1) and (2)

$$g_k^{-1} y_k y_{k+1} y_k^{-1} g_k = y_{k-1} y_{k+1} y_{k-1}^{-1} = g_{k+1} y_{k-1} y_k y_{k-1}^{-1} g_{k+1}^{-1}.$$

On the other hand, by (1) and (2)

$$\begin{aligned} g_{k+1}^{-1} g_k^{-1} g_{k+1}^{-1} y_k g_{k+1} g_k g_{k+1} &= g_k^{-1} g_{k+1}^{-1} g_k^{-1} y_k g_k g_{k+1} g_k \\ &= g_k^{-1} g_{k+1}^{-1} y_{k-1} g_{k+1} g_k \\ &= g_k^{-1} y_{k-1} g_k. \end{aligned}$$

This yields the induction step. □

(2.20) Proposition. *The group K_n is the free group generated by $y_0, \ldots, y_{n-1}$.*

Proof. By the previous Lemma, the group K_n^0 generated by the $y_0, \ldots, y_{n-1}$ is invariant under conjugation by elements of ZA_{n-1}. Since $t \in K_n^0$ and t together with ZA_{n-1} generates ZB_n, we must have equality $K_n^0 = K_n$.

Let F_n denote the free group generated by $y_0, \ldots, y_{n-1}$. We define homomorphisms $\gamma_1, \ldots, \gamma_{n-1}\colon F_n \to F_n$ by imitating (8.20):
(1) $\gamma_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\gamma_k(y_k) = y_{k-1},$
(3) $\gamma_k(y_{k-1}) = y_{k-1} y_k y_{k-1}^{-1}.$
We claim:

(2.21) Lemma. *The γ_j are automorphisms and satisfy the braid relations*

$$\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j, \qquad |i-j| = 1, \quad \textit{and} \quad \gamma_i \gamma_j = \gamma_j \gamma_i, \qquad |i-j| \geq 2.$$

Proof. First we check that the homomorphism $\delta_k\colon F_n \to F_n$
(1) $\delta_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\delta_k(y_{k-1}) = y_k,$
(3) $\delta_k(y_k) = y_k^{-1} y_{k-1} y_k$
is inverse to γ_k. Hence γ_k is an isomorphism. Since γ_k fixes y_j for $j \notin \{k-1, k\}$, the second braid relation is obviously satisfied. For the first relation, the reader may check the following values of $\gamma_1\gamma_2\gamma_1$ and $\gamma_2\gamma_1\gamma_2$ on y_0, y_1, y_2:

$$y_0 \mapsto y_0 y_1 y_2 y_1^{-1} y_0^{-1}, \quad y_1 \mapsto y_0 y_1 y_1^{-1}, \quad y_2 \mapsto y_0.$$

We use this Lemma to define a semi-direct product

$$\textbf{(2.22)} \qquad F_n \to \Gamma_n \to ZA_{n-1},$$

in which $g_j \in ZA_{n-1}$ acts on F_n through δ_j. By (8.19) and $K_n^0 = K_n$, we have a canonical epimorphism $\mu\colon \Gamma_n \to ZB_n$. We show that μ is an isomorphism. As a set, $\Gamma_n = F_n \times ZA_{n-1}$. An inverse to μ has to send $g_j \mapsto (1, g_j)$ and $t \mapsto (y_0, 1)$. We have to check that this assignment is compatible with the relations of ZB_n. This is obvious for the g_j. Moreover:

$$\begin{array}{rcl} tg_1tg_1 & \mapsto & (y_0,1)(1,g_1)(y_0,1)(1,g_1) \\ & = & (y_0,g_1)(y_0,g_1) \\ & = & (y_0\delta_1(y_0), g_1^2) \\ & = & (y_0y_1, g_1^2) \end{array}$$

$$\begin{array}{rcl} g_1tg_1t & \mapsto & (1,g_1)(y_0,1)(1,g_1)(y_0,1) \\ & = & (y_1,g_1)(y_1,g_1) \\ & = & (y_1\delta(y_1), g_1^2) \\ & = & (y_0y_1, g_1^2). \end{array}$$

This finishes the proof of Proposition (8.20). □

Proof of (8.16). The elements t_j of (8.17) and the elements y_j coincide in $W^\infty B_n$, since $g_j = g_j^{-1}$ in this group. Lemma (8.19) shows that conjugation $y \mapsto g_k^{-1} y g_k$ acts on the set $(y_0, \dots, y_{n-1})$ by interchanging y_{k-1} and y_k. The proof of (8.20) is now easily adapted to show the isomorphism $W^\infty B_n \cong P_n'$. This isomorphism restricts to an isomorphism $W\tilde{A}_{n-1} \cong Q_n'$. □

We now apply the previous results to Hecke algebras. We have the Hecke algebras HA_{n-1}, $H\tilde{A}_{n-1}$, and HB_n associated to the corresponding Coxeter graphs. We consider algebras over the ground ring $\mathfrak{K}$. The first one is given by generators $g_1, \dots, g_{n-1}$, the braid relations between them and the quadratic relations $g_j^2 = (q-1)g_j + q$ with a parameter $q \in \mathfrak{K}$. The second one has generators $g_1, \dots, g_n$, the braid relations (8.6) and the same quadratic relations. The algebra HB_n has generators $t, g_1, \dots, g_{n-1}$, the braid relations (8.1), the quadratic relations above for the g_j and $t^2 = (Q-1)t + Q$ with another parameter $Q \in \mathfrak{K}$. If we omit the quadratic relation for t, then we obtain the definition of $H^\infty B_n$. This is not a Hecke algebra in the formal sense, i. e. associated to a Coxeter graph. It is a deformation of the group algebra of $W^\infty B_n$.

We know from Hecke algebra theory that an additive basis of the Hecke algebra is in bijective correspondence with the elements of the Coxeter group. There is a similar relation between $W^\infty B_n$ and $H^\infty B_n$. In order to derive it, we relate $H\tilde{A}_{n-1}$ and $H^\infty B_n$.

The algebra $H\tilde{A}_{n-1}$ has an automorphism τ given by $\tau(g_i) = g_{i-1}$ (indices $\bmod n$). We define the twisted tensor product over the ground ring $\mathfrak{K}$

(2.23) $$H\tilde{A}_{n-1} \otimes \mathfrak{K}[\tau, \tau^{-1}] =: H_n^\infty$$

by the multiplication rule $(x \otimes \tau^k) \cdot (y \otimes \tau^l) = (x \cdot \tau^k(y), \tau^{k+l})$ for $k, l \in \mathbb{Z}$ and $x, y \in H\tilde{A}_{n-1}$.

(2.24) Proposition. *The algebra* (8.23) *is canonically isomorphic to* $H^\infty B_n$.

PROOF. We use the isomorphism (8.3) to redefine the algebra $H^\infty B_n$ by generators $c, g_1, \dots, g_{n-1}$ relations (8.2) and the quadratic relations for the g_j. The assignment $g_j \mapsto g_j \otimes 1$, $c \mapsto 1 \otimes \tau$ induces a homomorphism $H^\infty B_n \to H\tilde{A}_{n-1} \otimes H_n^\infty$. We have a homomorphism $H\tilde{A}_{n-1} \to H^\infty B_n$, $x \mapsto x'$ induced by $g_j \mapsto g_j$ with $g_n = gtg_1t^{-1}g^{-1}$ in $H^\infty B_n$ (see (8.12)). This extends to a homomorphism $H_n^\infty \to H^\infty B_n$ by $x \otimes \tau^k \mapsto x' \cdot c^k$, since $\tau(y)' = cy'c^{-1}$. These homomorphisms are inverse to each other. □

(2.25) Corollary. *Suppose* $(b_j \mid j \in J)$ *is a* $\mathfrak{K}$*-basis of* $H\tilde{A}_{n-1}$. *Then* $(b_j' c^k \mid j \in J, k \in \mathbb{Z})$ *is a* $\mathfrak{K}$*-basis of* $H^\infty B_n$. □

3. Braids of type B

We use a theorem of Brieskorn [**??**] to derive some geometric interpretations of the braid group ZB_n. The starting point is the reflection representation of the Weyl group WB_n. This group is a semi-direct product

(3.1) $$(\mathbb{Z}/2)^n \to WB_n \to S_n.$$

It acts on complex n-space $\mathbb{C}^n$ as follows:

(1) S_n acts by permuting the coordinates.

(2) $(\mathbb{Z}/2)^n$ act by sign changes $(z_1, \dots, z_n) \mapsto (\varepsilon_1 z_1, \dots, \varepsilon_n z_n)$, $\varepsilon_i \in \{\pm 1\}$.

This group contains the reflections in the hyperplanes

$$z_i = \pm z_j, \quad i \neq j; \quad \text{and} \quad z_j = 0.$$

Let X denote the complement of these hyperplanes. From the theory of finite reflection groups it is known, that $W = WB_n$ acts freely on X. Brieskorn [**??**] shows:

(3.2) Theorem. *The braid group* ZB_n *is isomorphic to the fundamental group* $\pi_1(X/W)$ *of the orbit space* X/W. □

If we think of WB_n as the Coxeter group with generators $t, g_1, \dots, g_{n-1}$, then g_j acts as the transposition $(j, j+1)$ and t as $z_1 \mapsto -z_1$.

We use (9.2) to give several interpretations of ZB_n by braids.

We remove the hyperplanes $z_j = 0$ from $\mathbb{C}^n$. It remains the n-fold product $\mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{C}^{*n}$. Removal of the remaining reflection hyperplanes yields the space X of n-tuples $(z_j) \in \mathbb{C}^{*n}$ with pairwise different squares z_j^2.

The *configuration space* $C^n(\mathbb{C}^*)$ is the space of subsets of $\mathbb{C}^*$ with cardinality n. As topological space it is defined as Y/S_n where $Y \subset \mathbb{C}^{*n}$ is the set of n-tuples (y_j) with pairwise different components.

(3.3) Proposition. *X/W is homeomorphic to $C^n(\mathbb{C}^*)$.*

Proof. We arrive at X/W in two steps: First we form $Y' = X/(\mathbb{Z}/2)^n$ and then we divide out the S_n-action. The map $(z_j) \mapsto (z_j^2)$ yields an S_n-equivariant homeomorphism $Y' \to Y$. □

By (9.2) and (9.3), $ZB_n \cong \pi_1(C^n(\mathbb{C}^*))$. The elements of $\pi_1(C^n(\mathbb{C}^*))$ will be interpreted as braids in the cylinder (cylindrical braids). We take $(1, \omega, \cdots, \omega^{n-1})$, $\omega = \exp(2\pi i/n)$, as base point in $C^n(\mathbb{C}^*)$. A loop in $C^n(\mathbb{C}^*)$ lifts to a path

$$w\colon I \to Y, \quad t \mapsto (w_1(t), \dots, w_n(t))$$

with this initial point. Thus we have

(1) $w(0) = (1, \omega, \dots, \omega^{n-1})$.
(2) $w(1) = (\sigma(1), \dots, \sigma(\omega^{n-1}))$, with a permutation σ of the set $\mathbb{Z}/n = \{1, \omega, \dots, \omega^{n-1}\}$.
(3) For $j \neq k$ we have $w_j(t) \neq w_k(t)$.

These data yield a braid z_w with n strings in $\mathbb{C}^* \times [0,1]$ from $\mathbb{Z}/n \times 0$ to $\mathbb{Z}/n \times 1$

$$z_w(t) = \{w_1(t), \dots, w_n(t)\} \times t.$$

Homotopy classes of loops correspond to isotopy classes of such braids. Multiplication of loops corresponds to concatenation of braids, as usual. Thus we have:

(3.4) Theorem. *The braid group ZB_n is the group of n-string braids in the cylinder $\mathbb{C}^* \times [0,1]$.* □

A second interpretation is by symmetric braids in $\mathbb{C} \times [0,1]$. This was already used in [**??**]. We take the base point $(1, 2, \dots, n) \in X$. We lift a loop in X/W to a path

$$w\colon I \to X, \quad t \mapsto (w_1(t), \dots, w_n(t)).$$

Then we have:

(1) $w(0) = (1, 2, \dots, n)$.
(2) $w(1) = (\pm\sigma(1), \dots, \pm\sigma(n))$ with a permutation σ of $\{1, \dots, n\}$.
(3) For $j \neq k$ we have $w_j(t) \neq \pm w_k(t)$.
(4) $w_j(t) \neq 0$.

Let $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$. The data yield a braid with $2n$ strings in $\mathbb{C} \times [0,1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$, namely

$$t \mapsto \{-w_n(t), \dots, -w_1(t), w_1(t), \dots, w_n(t)\} \times t.$$

These braids are $\mathbb{Z}/2$-equivariant with respect to $(z,t) \mapsto (-z,t)$ and are therefore called *symmetric*. The theorem of Brieskorn thus gives:

(3.5) Theorem. *The group ZB_n is isomorphic to the group of symmetric braids with $2n$ strings.* □

Symmmetric braids are drawn as ordinary braids but with additional symmetry with respect to the axis $0 \times [0,1]$.

The symmetry is not the reflection in the axis, but corresponds to a spacial rotation about this axis. The relation $tg_1tg_1 = g_1tg_1t$ appears in this context as a generalized Reidemeister move.

Braids in the cylinder with n strings can be visualized as ordinary braids with $n+1$ strings — the axis of the cylinder is the additional string. This method has been used by Lambropoulou [**??**]. It allows the reduction of B_n-type braids to ordinary Artin braids, also with respect to proofs. The theorem of Brieskorn is then not used.

The twofold covering, ramified along the axis, of the cylinder produces a symmetric braid from a cylindrical one — and vice versa.

The cylinder $\mathbb{C}^* \times [0,1]$ has the universal covering $\mathbb{C} \times [0,1]$. Lifting cylindrical braids with n strings produces n-periodic infinite braids in $\mathbb{C} \times [0,1]$ from $\mathbb{Z} \times 0$ to $\mathbb{Z} \times 1$. They are invariant with respect to the translation $(z,t) \mapsto (z+n,t)$. This gives yet another interpretation of ZB_n by n-periodic braid pictures.

The relation between ZB_n and $Z\tilde{A}_{n-1}$ has the following geometric origin or counterpart. The map

$$\mathbb{C}^{*n} \to \mathbb{C}^*, \quad (z_1, \ldots, z_n) \mapsto z_1 \cdot \ldots \cdot z_n$$

is S_n-equivariant and induces therefore a map from the configuration space

$$\alpha\colon C^n(\mathbb{C}^*) \to \mathbb{C}^*.$$

(3.6) Lemma. *The map α is a fibre bundle.*

Proof. Let

$$H = \{(z_1, \ldots, z_n) \in \mathbb{C}^{*n} \mid \textstyle\prod z_j = 1\}.$$

This is an S_n-invariant subset. The map

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} H \to \mathbb{C}^{*n}, \quad (z, z_1, \ldots, z_n) \mapsto (zz_1, \ldots, zz_n)$$

is an S_n-equivariant homeomorphism. Thus γ is the fibre bundle with fibre H assoziated to the $\mathbb{Z}/n$-principal bundle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. In $\mathbb{C}^{*n}$ we have to remove the subset

$$C = \{(z_1, \ldots, z_n) \mid \text{there exists } i \neq j \text{ such that } z_i = z_j\}.$$

Let $D = H \cap C$. Then γ induces an S_n-equivariant homeomorphism

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus D) \to \mathbb{C}^{*n} \setminus S.$$

This yields the fibre bundle description

$$\mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus T)/S_n \to \mathbb{C}^*$$

for the configuration space. □

We apply the fundamental group to this fibration and obtain the exact sequence

$$1 \to \operatorname{kernel} \alpha_* \to ZB_n \to \mathbb{Z} \to 0.$$

It can be shown that this is the sequence (8.11), i. e. $Z\tilde{A}_{n-1}$ is the fundamental group of the fibre of α.

Our next aim is to describe an additive basis of the Hecke algebra $H^\infty B_n$ by geometric means, i. e. by specifying a certain canonical set of basic braids.

A cylindrical braid with n strings is called *descending*, if for $i < j$ the i-th string is always overcrossing the j-th string. The i-th string is the one starting at ω^i, $0 \le i \le n-1$. Overcrossing means the following: We look radially and orthogonally from infinity onto the axis. The braid is in general position if we only see transverse double points. The first string we meet, coming from infinity, is the overcrossing one.

(3.7) Theorem. *The descending braids form a $\mathfrak{K}$-basis of the algebra $H^\infty B_n$. The descending braids with winding number zero form a $\mathfrak{K}$-basis of the algebra $H\tilde{A}_{n-1}$.*

We use (8.11) to reduce the first statement to the second. For the latter Hecke algebra we have the canonical basis related to the elements of reduced form in the Weyl group, and elements of the Weyl group will be shown to correspond to descending braids. We use the description of the Weyl group elements as n-periodic permutations of $\mathbb{Z}$. We represent such a permutation by n straight lines $c_1, \dots, c_n$ in the strip $\mathbb{R} \times [0,1]$ starting at $\{1, \dots, n\} \times 0$ such that c_i and c_j have at most one crossing, and then repeat with period n. By slightly moving the endpoints of the c_j we can assume that the curves are in general position. The resulting crossings are used to write the permutation as a product of reflections. This product is reduced, in the sense of Coxeter group theory. It is geometrically obvious that the same configuration of crossings can be realized by a descending braid.

(3.8) Proposition. *The set*

$$\mathfrak{C} = \{y_{n-1}^k g_{n-1} g_{n-2} \dots g_j \mid k \in \mathbb{Z}, 1 \le j \le n\}$$

is a system of representatives for the left cosets of the inclusion $W^\infty B_{n-1} \subset W^\infty B_n$.

Proof. This is an immediate consequence of the semi-direct product description. The powers of y_{n-1} are representatives for cosets of $V_{n-1} \subset V_n$, and the products $g_{n-1} \dots g_j$ are representatives for the cosets of $S_{n-1} \subset S_n$. □

We use this Proposition to derive the following result of Lambropoulou and Przytycki which was proved by them in a purely algebraic manner.

(3.9) Theorem. *Let $\mathfrak{B}$ be the canonical basis of $H^\infty B_{n-1}$. Then $\{bc \mid b \in \mathfrak{B}, c \in \mathfrak{C}\}$ is a basis of $H^\infty B_n$.*

Proof. Represent a basis element of $H^\infty B_n$ by a descending braid. □

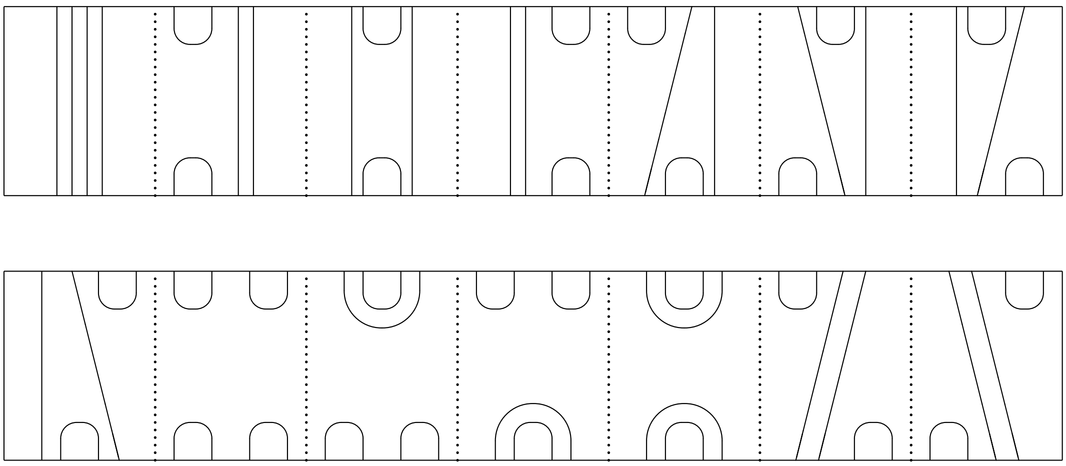
4. Categories of bridges

Let $k \in \mathbb{N}_0$ be a natural number. We set $[+k] = \{1, \ldots, k\}$. For $k = 0$ this is the empty set. Let $k + l = 2n$. A (k,l)-*bridge* is an isotopy class of n smooth disjoint arcs (= embedded intervals) in the strip $\mathbb{R} \times [0,1]$ with boundary set $P(k,l) := [+k] \times 0 \cup [+l] \times 1$, meeting $\mathbb{R} \times \{0,1\}$ transversely. Isotopy is ambient isotopy relative $\mathbb{R} \times \{0,1\}$. A bridge is a purely combinatorial object: It is uniquely determined by specifying which pairs of $P(k,l)$ are connected by an arc of the bridge. In this sense, a bridge is a free involution of $P(k,l)$ with the additional condition which expresses the disjointness of the arcs. An involution s without fixed points belongs to a bridge if and only if for all pairs $(i, s(i))$ and $(j, s(j))$ the inequality

$$(j - i)(s(j) - i)(j - s(i))(s(j) - s(i)) > 0$$

holds.

We illustrate the 14 $(4,4)$-bridges in the next figure.



We define categories TA and T^oA based on bridges. Let $\mathfrak{K}$ be a commutative ring and $d, d_+ d_- \in \mathfrak{K}$ given parameters. The category TA depends on the choice of d, the categoriy T^oA on the choice of d_+, d_-.

The objcets of TA are the symbols $[+k]$, $k \in \mathbb{N}_0$. The category is a $\mathfrak{K}$-category, i. e. morphism sets are $\mathfrak{K}$-modules and composition is $\mathfrak{K}$-bilinear. The morphism module from $[+k]$ to $[+l]$ is the free $\mathfrak{K}$-module on the set of (k,l)-bridges. This is the zero module for $k + l$ odd. In the case $k = l = 0$ we identify the empty bridge with $1 \in \mathfrak{K}$ and thus have $\mathfrak{K}$ as the morphism module.

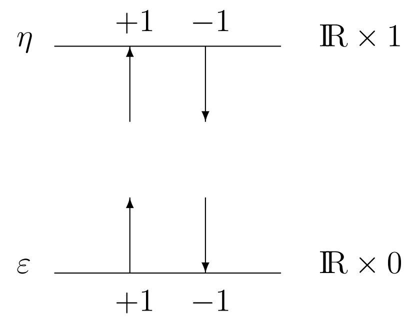
Since composition is assumed to be bilinear, we only have to define the composition of bridges. Let V be a (k,l)-bridge and U be a $l,m)$-bridge. We place U on top of V and shrink the resulting figure in $\mathbb{R} \times [0,2]$ to $UV \subset \mathbb{R} \times [0,1]$. The figure UV may contain loops in the interior of $\mathbb{R} \times [0,1]$, say $l(U,V)$ in number. Let $U \wedge V$ be the figure which remains after the loops have been removed. The composition in TA is now defined by

$$U \circ V = d^{l(U,V)} U \wedge V.$$

We make TA into a strict tensor $\mathfrak{K}$-category. We set $[+k] \otimes [+l] := [+(k+l)]$ and place the corresponding bridges next to each other.

The endomorphism algebra $\mathrm{Hom}([+n],[+n])$ of $[+n]$ in TA will be denoted by $T[+n]$ or TA_{n-1}. It is called a *Temperley-Lieb algebra.*

The categorie T^oA is an oriented version of TA. The objects of T^oA are the functions ε: $[+n] \to \{\pm 1\}$, $n \in \mathbb{N}_0$. We denote such a function as a sequence $(\varepsilon(1), \varepsilon(2), \ldots)$. Suppose ε: $[+k] \to \{\pm 1\}$ and η: $[+l] \to \{\pm 1\}$ are given. A (ε, η)-bridge is a (k,l)-bridge with an orientation of each arc such that the orientations match with the signs in the following manner.

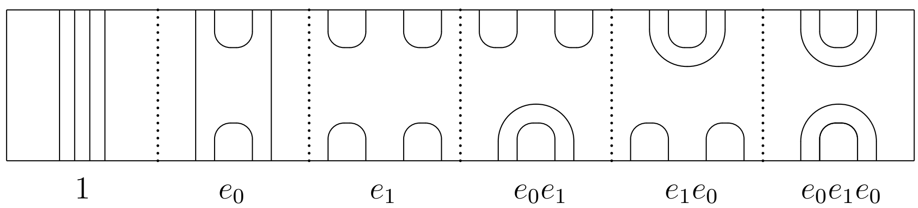


The morphism set $\mathrm{Hom}(\varepsilon, \eta)$ is the free $\mathfrak{K}$-module on the set of (ε, η)-bridges. Composion and tensor product are defined as for TA, but we take orinetations of loops into account. Suppose UV contains $l(U,V,+)$ loops with positive orientation and $l(U,V,-)$ with negative orientation. Then we set

$$U \circ V = d_+^{l(U,V,+)} d_-^{l(U,V,-)} U \wedge V.$$

Note that orientations match in UV.

We now define a new type of bridges, called *symmetric bridges*. We set $[\pm k] = \{-k, \ldots, -1, 1, \ldots, k\}$. A symmetric (k,l)-bridge is represented by a system of $k+l$ disjoint smooth arcs in the strip $\mathbb{R} \times [0,1]$ with boundary set $Q(k,l) = [\pm k] \times 0 \cup [\pm l] \times 1$ meeting $\mathbb{R} \times \{0,1\}$ transversely, and which has the following equivariance property: If an arc connects $(x, \varepsilon) \in Q(k,l)$ with $(y, \eta) \in Q(k,l)$, then there exists another arc which connects $(-x, \varepsilon)$ with $(-y, \eta)$. Also in the present situation, two figures which connect the same points define the same bridge, i. e. a symmetric bridge is a free involution of $Q(k,l)$. As an illustration we show the symmetric $(2,2)$-bridges; the symbols underneath will be explained in ??.



The $\mathbb{Z}/2$-equivariance property of symmetric bridges leads to another graphical presentation: Just consider $\mathbb{Z}/2$-orbits. Thus we consider a system of $k+l$ disjoint smooth arcs in the half-strip $[0, \infty[\times[0,1]$ with boundary set in $P(k,l)$ together with a certain set in $0\times]0,1[$; each arc must have at least one boundary point in $P(k,l)$. The arcs with only one point in $P(k,l)$ are called half-arcs. The location of the boundary points on the axis $0 \times [0,1]$ does not belong to the structure of the bridge.

We use this presentation of symmetric bridges when we now define the category TB. The objects of this category are again the symbols $[+k]$, $k \in \mathbb{N}_0$. The category TB is a $\mathfrak{K}$-category. It depends on the choice of two parameters $c, d \in \mathfrak{K}$. The morphisms set from $[+k]$ to $[+l]$ is the free $\mathfrak{K}$-module on symmetric (k,l)-bridges. The composition of two such bridges U, V as in the TA-case: Place U above V and get UV. Suppose in UV there are $l(U,V)$ loops and $k(U,V)$ half-loops, the latter being arcs with both boundary points in $0 \times [0,1]$. Let $U \wedge V$ denote the bridge which remains after loops and half-loops have been removed. We define

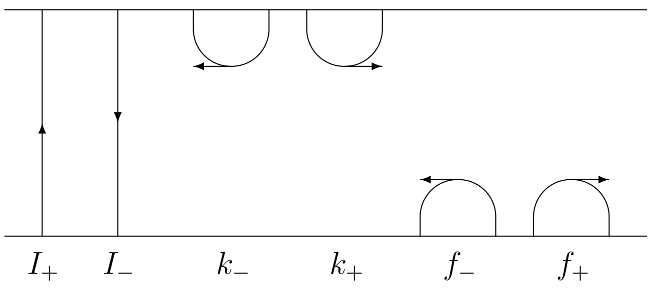
$$U \circ V = c^{k(U,V)} d^{l(U,V)} U \wedge V.$$

The endomorphism algebra of $[+n]$ in this category is denoted TB_n.

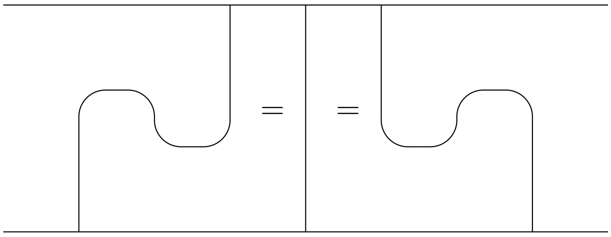
There is again an oriented version T^oB which depends on parameters $c_\pm, d_\pm$. Objcets are functions ε: $[+k] \to \{\pm 1\}$. The morphism set from ε to η is the free $\mathfrak{K}$-modules on the oriented symmetric (ε, η)-bridges. In order to define the composition we count the positive and negative loops and half-loops and use the parameters d_+, d_-, c_+, c_-, respectively.

The category TB is a strict tensor module $\mathfrak{K}$-category. The action $*$: $TB \times TA \to TB$ is defined on objects by $[+k] * [+l] = [+(k+l)]$ and on morphisms by placing an A-bridge left to a B-bridge. We think of TA as a subsactegory of TB. It is clear that $*$ restricts to $\otimes$. In a similar manner T^oB is a strict tensor module $\mathfrak{K}$-category over T^oA. These structures defines action pairs (TB, TA) and (T^oB, T^oA) in the sense of (2.1).

The categories defined so far have a simple description by generators and relations. The generating process uses category ruloes and tensor product rules. The generators of T^oA are the following elemntary bridges $k_\pm, f_\pm$, together with the identities $I_\pm$.



Similarly, TA has generators I, k, f without orientation. The following figure demonstrates a typical relation between these generators.



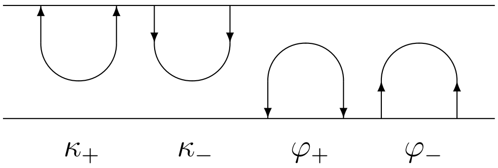
The relations in the case of T^oA are the geometric relations

(4.1) $$(f_\pm \otimes I_\pm)(I_\pm \otimes k_\pm) = I_\pm, \quad (I_\pm \otimes f_\pm)(k_\pm \otimes I_\pm) = I_\pm$$

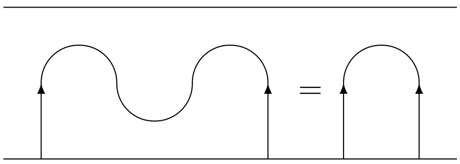
and the algebraic relations

(4.2) $$I_\pm = \mathrm{id} = 1, \quad f_- k_+ = d_+, \quad f_+ k_- = d_-.$$

The category T^oB has additional generators as follows(drawn as symmetric bridges with dotted symmety axis)



A typical geometric realtion is shown in the next figure.



The relations are **(4.3)** $\kappa_\pm = (\varphi_\pm \otimes I_\pm) \circ k_\pm, \quad \varphi_\pm = f_\pm \circ (\kappa_\pm \otimes I_\mp).$

(4.4) $$\varphi_- \kappa_+ = c_+, \quad \varphi_+ \kappa_- = c_-.$$

Similar realations without $\pm$-signs hold for TA and TB The geometric definition of the categories contains a positivity: The categories TA, TB, $T^\circ A$, $T^\circ B$ can be defines over $K = \mathbb{Z}[d]$, $\mathbb{Z}[c,d]$, $\mathbb{Z}[d_+,d_-]$, $\mathbb{Z}[d_+,d_-,c_+,c_-]$.

We now describe dualities in these categories. Suppose ε: $[+k] \to \{\pm 1\}$ is given. The dual object is ε^*: $[+k] \to \{\pm 1\}$, $\varepsilon^*(j) = -\varepsilon(j)$. Since dualities are compatible with tensor products (see ??), it suffices to define the dualities for the generating objects $1_\pm$: $[+1] \to \pm 1$. A left duality in T^oA is given by

$$k_- = b \colon \emptyset \to 1_+ \otimes 1_+^*$$

$$f_- = d \colon 1_+^* \otimes 1_+ \to \emptyset$$

$$k_+ = b\colon \emptyset \to 1_- \otimes 1_-^*$$

$$f_+ = d\colon 1_-^* \otimes 1_- \to \emptyset$$

A right duality is given by reversing the orientations

$$k_+ = a\colon \emptyset \to 1_+^* \otimes 1_+$$

$$f_+ = c\colon 1_+ \otimes 1_+^* \to \emptyset$$

$$k_- = a\colon \emptyset \to 1_-^* \otimes 1_-$$

$$f_- = c\colon 1_- \otimes 1_-^* \to \emptyset$$

In the case of TA we set $[+k]^* = [+k]$. Left and right duality coincide. They are defined by

$$k\colon \emptyset \to [+1] \otimes [+1], \quad f\colon [+1] \otimes [+1] \to \emptyset$$

on the generating object.

The dualities above can be extended to dualities in the sense of (3.1) of the actions pairs (T^oB, T^oA) and (TB, TA). By (3.4) and (3.5) it suffices again to consider the generating objects. We define

$$\beta = \kappa_-\colon \emptyset \to 1_+^*$$

$$\delta = \varphi_-\colon 1_+ \to \emptyset$$

$$\beta = \kappa_+\colon \emptyset \to 1_-^*$$

$$\delta = \varphi_+\colon 1_- \to \emptyset$$

$$\alpha = \kappa_+\colon \emptyset \to 1_+$$

$$\gamma = \varphi_+\colon 1_+^* \to \emptyset$$

$$\alpha = \kappa_-\colon \emptyset \to 1_-$$

$$\gamma = \varphi_-\colon 1_-^* \to \emptyset.$$

5. Representations of bridge categories

We study tensor $\mathfrak{K}$-functors from the categories TA and TB an their oriented versions into the category of $\mathfrak{K}$-modules.

We begin with TA and TB. The object $[+1]$ is mapped to a $\mathfrak{K}$-module V. Compatibility with the tensor product means that $[+k]$ is mapped to the k-fold tensor producr $V^{\otimes k}$; for $k = 0$ this is $\mathfrak{K}$. We only study the case when V is a free $\mathfrak{K}$-module with basis $v_1, \ldots, v_n$. We write tensor product also just by juxta-position, i. e. $v \otimes w = vw \in V \otimes W = VW$.

The tensor functors from TA correspond bijectivle to pairs of linear maps associated to the generators k and f and satisfying the relations (??). We denote them with the same symbol and write them in matrix form (k^t is the transpose of k)

$$k\colon \mathfrak{K} \to VV,\ 1 \mapsto \sum_{ij} k_{ij} v_i v_j, \qquad f\colon VV \to \mathfrak{K},\ v_i v_j \mapsto f_{ij}.$$

(5.1) Proposition. *The relations* (??) *and* (??) *are satisfied if and only if the matrices $k = (k_{ij})$ and $f = (f_{ij})$ are inverse to each other and $d = \mathrm{Tr}(fk^t)$.* □

Representations of TB need two more data

$$\kappa\colon \mathfrak{K} \to V,\ 1 \mapsto \sum_j \kappa_j v_j, \qquad \varphi\colon V \to \mathfrak{K},\ v_j \mapsto \varphi_j.$$

(5.2) Proposition. *The relations* (??) *and* (??) *are satisfied if and only if $(\kappa_1, \ldots, \kappa_n)(f_{ij}) = (\varphi_1, \ldots, \varphi_n)$ and $c = \sum_j \varphi_j \kappa_j$.* □

We are mainly interested in isomorphism classes of representations. From (??) and (??) we see that a representaion of TA is determinded by (V, k) and a representation of TB by (V, k, κ). Suppose (V, k, κ) and (V', k', κ') define two such functors. A linear isomorphism

$$B\colon V \to V', \quad v_k \mapsto \sum_l b_{lk} v'_l$$

is an isomorphism of functors if and only if

(5.3) $\quad f = f' \circ (B \otimes B), \quad (B \otimes B) \circ k = k', \quad \varphi' \circ B = \varphi, \quad B \circ \kappa = \kappa'.$

We use the notation $B = (b_{ij})$ also for the matrix. The relations (??) are equivalent to the relations

(5.4) $\qquad BkB^t = k', \quad B^t f' B = f, \quad \kappa B^t = \kappa', \quad \varphi' B = \varphi.$

(5.5) Proposition. *Isomorphism classes of representations of TA on a free module of rank n correspond bijectively to equivlanece classes of $k \in GL(n, \mathfrak{K})$ under the relation $k \sim BkB^t$ with $d = \mathrm{Tr}(k^{-1}k^t)$. Isomorphism classes of representations of TB correspond bijctively to equivalence classes of pairs $(k, \kappa) \in GL(n, \mathfrak{K}) \times \mathfrak{K}^n$ under the relation $(k, \kappa) \sim (BkB^t, \kappa B^t)$ with $d = \mathrm{Tr}(k^{-1}k^t)$ and $c = \langle\, \kappa, \kappa k^{-1} \,\rangle$.* □

(5.6) Example. We consider 2×2-matrices over the complex numbers. For $C \in GL(2, \mathbb{C})$ we set $d(C) = \mathrm{Tr}(C(C^t)^{-1})$. Then $C' = BCB^t$ implies $d(C) = d(C')$. We distinguish two cases:

(1) There exists a basis v_1, v_2 of $\mathbb{C}^2$ such that $v_j C v_j^t = 0$ for $j = 1, 2$.

(2) There does not exist such a basis.

In the first case, C is equivalent to a matrix of the form

$$\begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}$$

and two matrices C and C' of this type are equivalent if $d(C) = d(C')$. in the second case C is equivalent to

$$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}.$$

We compute for

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad d(C) = 2 - \frac{(b-c)^2}{\det(C)}.$$

The first normal form leads to $d(C) = 2 - (\lambda + \lambda^{-1})^2 = -\lambda^2 - \lambda^{-2}$. We can still change λ to $\pm\lambda^{\pm 1}$. If $\kappa = (\kappa_1, \kappa_2)$, then $c = \kappa_1\kappa_2(\lambda - \lambda^{-1})$. If $\lambda^4 \neq 1$ and the normal form of C is fixed, we can change (κ_1, κ_2) to $(\alpha\kappa_1, \alpha^{-1}\kappa_2)$. ♡

We now describe the variants for T^oA and T^oB. The objects $+1, -1$ are mapped to V, W^*. We think of W^* as the dual of W with dual basis w^j of w_j. The functors from T^oB are specified by the following data:

(1) $f_+\colon VW^* \to K, \quad v_i w^j \mapsto f_i^j$
(2) $f_-\colon W^*V \to K, \quad w^i v_j \mapsto f_j^i(-)$
(3) $k_+\colon K \mapsto W^*V, \quad 1 \mapsto \sum k_i^j w^i v_j$
(4) $k_-\colon K \to VW^*, \quad 1 \mapsto \sum k_i^j(-) v_j w^i$
(5) $\kappa_+\colon K \to V, \quad 1 \mapsto \sum \kappa^i v_i$
(6) $\kappa_-\colon K \to W^*, \quad 1 \mapsto \sum \kappa_i(-) w^i$
(7) $\varphi_+\colon W^* \to K, \quad w^i \mapsto \varphi^i$
(8) $\varphi_-\colon V \to K, \quad v_i \mapsto \varphi_i(-)$.

(5.7) Proposition. *The data above define a tensor $\mathfrak{K}$-functor if and only if the following relations hold:*

(1) *The matrices $k_+ = (k_i^j)$, $f_+ = (f_i^j)$ are inverse to each other.*
(2) *The matrices $k_- = (k_i^j(-))$, $f_- = (f_i^j(-))$ are inverse to each other.*
(3) $\mathrm{Tr}(f_-k_+) = d_+$, $\mathrm{Tr}(f_+k_-) = d_-$.
(4) *Define column vectors φ_+, κ_+ with components (φ^i), (κ^i). Then $k_+\varphi_+ = \kappa_+$.*
(5) *Define row vectors φ_-, κ_- with components $(\varphi_i(-))$, $(\kappa_i(-))$. Then $\varphi_-k_- = \kappa_-$.*
(6) $\varphi_- \cdot \kappa_+ = c_+$, $\kappa_- \cdot \varphi_+ = c_-$.

□

Suppose we have another set of data, denoted with a bar. Let

$$B\colon V \to \bar{V},\ v_k \mapsto \sum_l b_k^l \bar{v}_l, \qquad C\colon W^* \to \bar{W}^*,\ w^k \mapsto \sum_l c_l^k \bar{w}^l$$

be isomorphisms. We use also the notation $B = (b_k^l)$, $C = (c_j^i)$. Then B, C define an isomorphism of functors if and only if the following relations hold (in terms of matrices and vectors)

$$\begin{array}{rclcrcl} C\bar{f}_+B &=& f_+ & \qquad & B\kappa_+ &=& \bar{\kappa}_+ \\ B\bar{f}_-C &=& f_- & & \kappa_-C &=& \bar{\kappa}_- \\ Bk_+C &=& \bar{k}_+ & & C\bar{\varphi}_+ &=& \varphi_+ \\ Ck_-B &=& \bar{k}_- & & \bar{\varphi}_-B &=& \varphi_- \end{array}$$

(5.8) Theorem. *Isomorphism classes of representations of TA on a module of rank n correspond bijectively to conjugacy classes of $v \in GL(n,\mathfrak{K})$ such that $\mathrm{Tr}(v^{\pm 1}) = d_\pm$. Isomorphism classes of representations of TB on a module of rank n correspond bijectively to equivalence classes of triples $(v,\kappa_+,\kappa_-) \in GL(n,\mathfrak{K}) \times \mathfrak{K}^n \times \mathfrak{K}^n$ under the relation $(k_+,\kappa_+,\kappa_-) \sim (Bk_+B^{-1}, B\kappa_+, \kappa_-B^{-1})$ for $B \in GL(n,\mathfrak{K})$ with $\mathrm{Tr}(k_+^{\pm 1}) = d_\pm$ and $\varphi_\mp\kappa_\pm = c_\pm$.*

Proof. Each functor is isomorphic to a functor with $V = W$ and $k_- = f_- = \mathrm{id}$. An isomorphism between such functors is given by $B \in GL(n,\mathfrak{K})$ with $bk_+B^{-1} = \bar{k}_+$, $B\kappa_+ = \bar{\kappa}_+$, $\kappa_-B^{-1} = \bar{\kappa}_-$. The other data are then given by $f_+ = k_+^{-1}$, $\varphi_- = \kappa_-$, $\varphi_+ = k_+^{-1}\kappa_+$. □

(5.9) Example. The following is a generic two-dimensional example with $d_+ = d_-$. We set $V = W^*$ and define

$$\begin{array}{rl} k_+\colon \mathfrak{K} \to VV, & 1 \mapsto Av_1v_2 - A^{-1}v_2v_1 \\ f_-\colon VV \to \mathfrak{K}, & v_1v_2 \mapsto A^{-1}, v_2v_1 \mapsto -A \\ \kappa_+\colon \mathfrak{K} \to V, & v_j \mapsto \kappa^j \\ \varphi_-\colon V \to \mathfrak{K}, & 1 \mapsto \varphi^1v_1 + \varphi^2v_2. \end{array}$$

Then $d = d_+ = d_- = -A^2 - A^{-2}$, $c_+ = \kappa^1\varphi^1 + \kappa^2\varphi^2$, $c_- = -A^{-2}\kappa^1\varphi^1 - A^2\kappa^2\varphi^2$.♡

6. General categories of bridges

This section introduces some general terminology for certain graphical categories and algebras.

A free involution $\sigma\colon P \to P$ of a set P is calles a *P-bridge*. A free involution of P is a partition of P into 2-element subsets $\{i, \sigma(i)\}$, called the *arcs* or *strings* of the bridge. A bridge is called *oriented* if its arcs are ordered sets $\{a_1, a_2\}$.

We study bridges with a geometric terminology. Suppose $\sigma\colon P \to P$ is a bridge. The *geometric realization* $|\sigma|$ of σ is the one-dimensional simplicial complex with P as set of 0-simplices and a 1-simplex for each arc $\{i, \sigma(i)\}$ with i and $\sigma(i)$ as boundary points. We say that the arc *connects* its boundary points. The arcs are the components of $|\sigma|$.

A (P,Q)-bridge is a bridge on the disjoint union $P \coprod Q$. An arc of a (P,Q)-bridge σ is called *horizontal* if its boundary points are either contained in P or in Q. The other arcs are called *vertical*.

We use a graphical notation for (P,Q)-bridges σ. We think of $P \subset \mathbb{R} \times 0$, $Q \subset \mathbb{R} \times 1$ and we draw an arc in $\mathbb{R} \times [0,1]$ from i to $\sigma(i)$. The notation horizontal and vertical is evident in this context. The horizontal arcs with endpoints in P are called the *lower* part of the bridge, the horizontal arcs with endpoints in Q the *upper* part.

(6.1) Remark. *Suppose P has $2n$ elements. The number of P-bridges is*

$$(2n-1)\cdot(2n-3)\cdots 3\cdot 1.$$

PROOF. There are $2n-1$ possibilities to connect a fixed element of P. Having fixed this connection, a set with $2n-2$ elements remains. Now use induction. □

We will use bridges with further properties.

Let G be a group and suppose P and Q are G-sets. A G-equivariant (P,Q)-bridge is a G-equivariant free involution σ of $P \amalg Q$. Equivariant means: $\sigma(gi) = g\sigma(i)$ for $g \in G$ and $i \in P \amalg Q$.

Suppose the bridge $\sigma\colon P \to P$ is G-equivariant. We have an induced G-action on $|\sigma|$. The action on the 0-simplices is given. If $\{i, \sigma(i)\}$ is a 1-simplex, then, by equivariance, $\{gi, g\sigma(i)\}$ is a 1-simplex. It can happen that these simplices coincide. This is the case if g is in the isotropy group G_i of i. If $g \in G$ acts non-trivially on $\{i, \sigma(i)\}$, then

$$gi = \sigma(i), \quad g\sigma(i) = i, \quad g^2 i = i$$

and hence $g^2 \in G_i$. Geometrically, g acts as reflection in the barycentre of the 1-simplex $\{i, \sigma(i)\}$ in this case.

In the sequel we only consider G-sets P with the following additional properties:

(1) The isotropy groups are finite.
(2) The orbit set is finite.
(3) G acts effectively on each orbit.

Under these hypotheses we have:

(6.2) Proposition. *Let σ be a G-equivariant (P,Q)-bridge. Then the following holds:*

(1) *The G-action respects lower, upper and horizontal arcs.*
(2) *The G-action on $|\sigma|$ is proper.*
(3) *The orbit space $|\sigma|/G$ is a compact one-dimensional* CW*-complex.*

PROOF. (1) is clear from the definitions.
(2) The G-action on the barycentric subdivision $|\sigma|'$ of $|\sigma|$ is a cellular action with finite isotropy groups.
(3) This follows since G acts cellularly on $|\sigma|'$ and the orbit space has a finite number of cells.

Suppose σ is a G-equivariant (P,Q)-bridge and τ a G-equivariant (Q,R)-bridge. Consider the G-space $|\tau| \cup_Q |\sigma|$. The components of this space can be of different type. Consider the G-orbit $B = Gx$ of a component x. Let H denote the isotropy group of the component x. Then B/G is homeomorphic to x/H. The orbit space of $|\tau| \cup |\sigma$ is compact. Hence x/H is compact. We use:

(6.3) Lemma. *There is no proper action of a discrete group on* $[0,1[$ *with compact orbit space.* □

This Lemma tells us that the components of $|\tau| \cup |\sigma|$ are not homeomorphic to $[0,1[$. Since the components are one-dimensional manifolds (with or without boundary), there are three cases:

(6.4) *A componente of* $|\tau| \cup |\sigma|$ *is homeomorphic to* $[0,1]$, S^1, *or* $]0,1[$. □

(6.5) Proposition. *The components of* $|\tau| \cup |\sigma|$ *which are homeomorphic to* $[0,1]$ *define a* G*-equivariant* (P,R)*-bridge.*

Proof. If the component is homeomorphic to $[0,1]$, then the boundary points are contained in $P \coprod R$.

For each point in $P \coprod R$ there exists a component of $|\tau| \cup |\sigma|$ with this point as boundary point. Since components of type $[0,1[$ do not exist, the component has a second boundary point in $P \coprod R$. □

We denote the bridge in (10.5) by $\tau \wedge \sigma$. The components of $|\tau| \cup |\sigma|$ which are homeomorphic to S^1 are called *cycles*, the components which are homeomorphic to $]0,1[$ are called *snakes*.

(6.6) Remark. Let $H = G_x$ be the subgroup of elements which map the component x into itself. Then H acts effectively and properly on the one-dimensional manifold x. Therefore we have, up to H-homeomorphism, the following possibilities:

(1) Suppose $x \cong S^1$. Then $H \cong \mathbb{Z}/m$ or $H \cong D_{2m}, m \geq 1$, and the action is by the usual action of a subgroup of $O(2)$.

(2) Suppose $x \cong \mathbb{R}$. Then $H \cong \mathbb{Z}$ or $H \cong D_\infty$, and the action is by the usual action as a subgroup of the group of affine transformations. ♡

Let $Z(\tau,\sigma)$ denote the orbit set of the components of $|\tau| \cup |\sigma|$ which are cycles or snakes. The G-orbits of components in $Z(\tau,\sigma)$ are counted according to types. The *type* of a component x consists of the conjugacy class of G_x together with the G_x-homeomorphism type of the G_x-action. The group $\mathbb{Z}/2$ has two different actions on S^1, by rotation or by reflection. (In the latter case it is the group D_2.) It is an observation of H. Reich [**??**] that these two actions should be distinguished.

Let C denote the set of possible types. We denote by $k(c,\tau,\sigma)$ the number of elements in $Z(\tau,\sigma)$ of type c.

After these preparations we define the category $F(G)$ of G-bridges. The *objects* of $F(G)$ are the G-sets as above, i. e. with finite isotropy groups, finite orbit set and effictive action on orbits.

We fix a ground ring $\mathfrak{K}$. The *morphism set* $\mathrm{Mor}(P,Q)$ is the free $\mathfrak{K}$-module on the set of G-equivariant (P,Q)-bridges.

In order to define the *composition* of morphisms we fix a map $d\colon C \to \mathfrak{K}$, called the *parameter function*. The composition of morphisms $\mathrm{Mor}(Q,R) \times \mathrm{Mor}(P,Q) \to \mathrm{Mor}(P,R)$ is assumed to be $\mathfrak{K}$-bilinear. The composition of bridges is defined to be

$$\tau \circ \sigma \prod_{c \in C} d(c)^{k(c,\tau,\sigma}\tau \wedge \sigma.$$

The *identity* $P \to P$ is represented by the bridge $\iota\colon P \amalg P \to P \amalg P$ which connects $i \in P$ vertically with $i \in P$. We have $|\sigma| \cup |\iota| \cong |\sigma|$ and $|\iota| \cup |\sigma| \cong |\sigma|$, if defined.

Associativity of composition follows from a geometrical consideration: The cycles and snakes of $|\tau| \cup |\sigma| \cup |\rho|$ are those of $|\tau| \cup |\sigma|$, plus those of $|\sigma| \cup |\rho|$, plus those of $|\tau \wedge \sigma| \cup |\rho|$ (equal to those of $|\tau| \cup |\sigma \wedge \rho|$).

We shall mostly work with suitable subcategories of $F(G)$. For instance, we could use only free G-sets. Or we restrict the morphisms; this will be the case in the Temperley-Lieb categories.

The composition of bridges with only vertical strings is again a bridge of this form. No cycles or snakes appear. The vertical (P,P)-bridges under composition can be identified with the group of G-equivariant permutations of P.

7. Representations of Hecke algebras

The Hecke algebra $H_k(q,Q)$ of type B_k is the associative algebra with 1 over $\mathfrak{K}$ with generators $t, g_1, \dots, g_{k-1}$ and relations (1.3) together with the quadratic relations $t^2 = (Q-1)t + Q$ and $g_j^2 = (q-1)g_j + q$. From (1.5) we obtain the R-matrices $X = pX_n(p)$ or $X = -pX_n(p^{-1})$. They satisfy the quadratic equation $X^2 = (q-1)X + q$ with $q = p^2$. Let F be the matrix (1.6) with $w = Q-1$ and $z = Q$. Then F satisfies $F^2 = (Q-1)F + Q$. The assignment (1.4) therefore yields a tensor representation of $H_k(q,Q)$ on $V^{\otimes k}$. See [??] for representations of this algebra in general.

Let $u\colon V \to V$ be the diagonal matrix $\mathrm{Dia}(p^{n-1}, p^{n-3}, \dots, p^{-n+3}, p^{-n+1})$. Denote by $l_x\colon V^{\otimes k} \to V^{\otimes k}$ left multiplication with $x \in H_k(q,Q)$. We define the *quantum trace*

$$\mathrm{Tr}\colon H_k(q,Q) \to \mathfrak{K}, \quad x \mapsto \mathrm{Sp}(l_x \circ u^{\otimes k}),$$

where Sp denotes the ordinary trace of linear algebra. Restricted to the Hecke algebra H_k generated by $g_1, \dots, g_{k-1}$ this is the Markov trace which leads to the two-variable HOMFLY-PT generalization of the Jones-polynomial, see [??]. Also in our case Tr has the trace property $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and is a Markov trace. We do not prove this here since a thorough study of such traces can be found in [??,??], [??].

8. Temperley-Lieb algebras

The Temperley-Lieb algebra $T_k(d,D,F)$ of type B_k is the associative algebra with 1 over $\mathfrak{K}$ generated by $e_0, e_1, \dots, e_{k-1}$ and relations

$$\begin{array}{rcll} e_0^2 &=& De_0 & \\ e_1e_0e_1 &=& Fe_1 & \\ e_j^2 &=& de_j, & j \geq 1 \\ e_ie_je_i &=& e_i, & |i-j|=1;\ i,j\geq 1 \\ e_ie_j &=& e_je_i, & |i-j|\geq 2. \end{array}$$

Here $d, D, F \in \mathfrak{K}$ are given parameters. The algebras $T_k(d,D,F)$ were studied in [**??**] from a geometrical point of view (with slightly restricted parameters). We consider the representation of the previous section with a two-dimensional V. The matrices

$$E_0 = \begin{pmatrix} a & b \\ x & y \end{pmatrix}, \quad E = \begin{pmatrix} 0 & & & \\ & p & -1 & \\ & -1 & p^{-1} & \\ & & & 0 \end{pmatrix}.$$

satisfy

$$E_0^2 = (a+y)E_0, \quad E^2 = (p+p^{-1})E, \quad E(E_0\otimes 1)E = (pa+p^{-1}y)E$$

$$(E\otimes 1)(1\otimes E)(E\otimes 1) = E\otimes 1, \quad (1\otimes E)(E\otimes 1)(1\otimes E) = 1\otimes E.$$

Therefore we obtain a representation of $T_k(d,D,F)$ with parameters $d = p+p^{-1}$, $D = a+y$, $F = pa+p^{-1}y$ on $V^{\otimes k}$ by the assignment $e_0 \mapsto E_0\otimes 1\otimes\cdots\otimes 1$ and $e_i \mapsto 1\otimes\cdots\otimes E\otimes\cdots\otimes 1$ where E acts on the factors $i, i+1$. □

We now relate this representation to the one for the Hecke algebra $H_k(q,Q)$ obtained in the previous section. The following proposition is easily proved by verifying the definig relations of the Hecke algebra. We fix the following parameters:

$$q = p^2, \quad d = p+p^{-1}, \quad D = a(1+Q), \quad F = a(p+p^{-1}Q).$$

Here $a \in \mathcal{K}$ is an invertible parameter. The special case $Q = P^2$ and $a = P^{-1}$ is worth attention.

(8.1) Proposition. *The assignment $t \mapsto a^{-1}e_0 - 1$, $g_j \mapsto pe_j - 1$ defines a surjective homomorphism $\varphi\colon H_k(q,Q) \to T_k(d,D,F)$.* □

The representations of $T_k(d,D,F)$ and $H_k(q,Q)$ are related via φ, if we use E as above and

$$E_0 = a\begin{pmatrix} 1 & \beta \\ \alpha & Q \end{pmatrix}.$$

The tensor representation of $T_k = T_k(d,D,F)$ is the fundamental representation in the sense of the following Theorem. We assume that we are working with a semi-simple algebra $H_k(q,Q)$ (compare [**??**, p. 477]) and a surjective homomorphism φ. The algebra $T_k = T_k(d,D,F)$ has irreducible modules $M_j(k)$, $0 \le j \le k$, of dimension $\binom{k}{j}$; by restriction from T_k to T_{k-1} the modules decompose as follows:

$$\mathrm{res} M_j(k) \cong M_{j-1}(k-1) \oplus M_j(k-1),$$

where $M_j(k) = 0$ if $j \notin \{0,\dots,k\}$ (compare [**??**]).

(8.2) Theorem. *The module $V^{\otimes k}$ is the direct sum of the irreducible T_k-modules.*

Proof. The proof uses the fact that

$$\omega = (\beta v_0 - v_1) \otimes (\beta v_0 + \beta^{-1} v_1) \otimes (P v_0 - p^{-2} v_1) \otimes \cdots$$

generates the subspace of $V^{\otimes k}$ on which t and all g_j act as multiplication by -1. It is easily verified that ω has this property. That there are no other vectors with this property, up to scalar multiples, is shown by induction over i by considering vectors of the form

$$(\beta v_0 - v_1) \otimes \cdots \otimes (\beta v_0 + (-p)^{-i} v_1) \otimes z_{i+1}.$$

The theorem is proved by induction on n. By construction, $\mathrm{res} V^{\otimes k} \cong 2V^{\otimes(k-1)}$. Suppose we have a decomposition

$$V^{\otimes k} = \bigoplus_{j=0}^{k} r_j M_j(k)$$

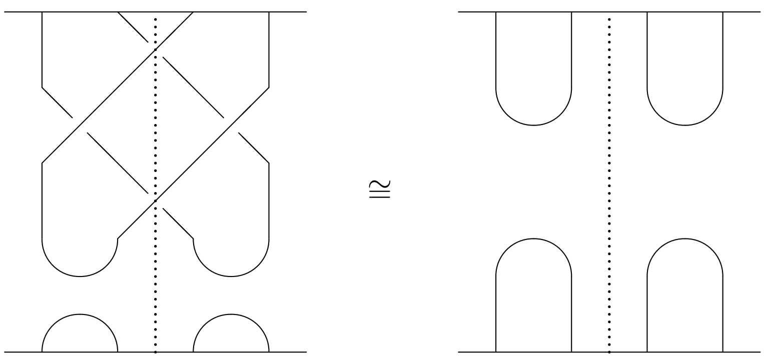
into irreducible summands. By induction and the preceding remarks, $r_j + r_{j+1} = 2$ for $j = 0,\dots,k-1$. We have already seen that (with suitable indexing) $r_0 = 1$. Hence all $r_j = 1$. □

9. Birman–Wenzl–Murakami algebras of type B

We define algebras $BW(B_k, 2)$ which are B-type analogues of the Birman-Wenzl [**??**], [**??**] and Murakami [**??**] algebras. They are generated by $t, g_1, \dots, g_{k-1}$ with relations ($\lambda, q, \alpha \in \mathfrak{K}$ suitable parameters)

$$\begin{array}{lrcll}
(1) & g_i g_j g_i & = & g_j g_i g_j & |i-j| = 1 \\
(2) & g_i g_j & = & g_j g_i & |i-j| \ge 2 \\
(3) & (g_i - \lambda)(g_i - q)(g_i + q^{-1}) & = & 0 & \\
(4) & e_i g_j^{\pm 1} e_i & = & \lambda^{\mp 1} e_i & |i-j| = 1 \\
(5) & t g_1 t g_1 & = & g_1 t g_1 t & \\
(6) & t g_i & = & g_i t & i > 1 \\
(7) & t^2 & = & \alpha t + q^{-1} & \\
(8) & t g_1 t e_1 & = & e_1, &
\end{array}$$

where e_j is determined by $(q-q^{-1})(1-e_j) = g_j - g_j^{-1}$. The generators $g_1, \dots, g_{k-1}$ with relations (1) – (4) define the ordinary Birman–Wenzl–Murakami algebra $BW(A_{k-1})$ of type A. The 2 in our notation refers to the quadratic relation (7) for t. There are analogous algebras $BW(B_k, l)$, where t satisfies a polynomial equation of degree l (possibly with additional relations involving t for geometric reasons). We write $BW(B_k, \infty)$, if t satisfies no polynomial equation. For the geometric relevance of $BW(A_k)$ we refer to [**??**]. We use [**??**] for algebraic information. A geometric representation of elements in $BW(B_k, l)$ is by symmetric tangles in the sense of [**??**]. The geometric meaning of (8) is given in the following figure of symmetric tangle pictures with dotted symmetry axis.



Since relation (8) is inhomogeous, one cannot work with arbitrary quadratic relations (7). The algebraic analysis (in the sense of [**??**]) of $BW(B_k, 2)$ and other algebras has been carried out in [**??**].

The results of section 3 yield tensor representations of $BW(B_k, 2)$ for suitable parameters λ, α. The matrix $X = X_n(B)$ satisfies $(X - q^{-2n})(X-q)(X+q^{-1}) = 0$ and the matrix $X = X_n(C)$ satisfies $(X + q^{-2n-1})(X-q)(X+q^{-1}) = 0$. We write

$$E_n(B) = 1 - \frac{X_n(B) - X_n(B)^{-1}}{q - q^{-1}}$$

and similarly for C_n. Then these matrices satisfy (see [**??**])

$$(E_n(B) \otimes 1)(1 \otimes X_n(B)^{\pm 1})(E_n(B) \otimes 1) \quad = \quad q^{\pm 2n}(E_n(B) \otimes 1)$$

and similarly for C_n with $-q^{\pm(2n+1)}$ in place of $q^{\pm 2n}$. We let $F_n(B) = p^{-1}F$ with F as in (1.8) and $F_n(C)$ as in (1.9) with $\beta = p^{-1}$ and write $Y = F \otimes 1$ with our new F's. Then $F_n(B)$ satisfies $(F - q^{-1})(F+1) = 0$ and $F_n(C)$ satisfies $F^2 = wF + q^{-1}$. Finally, with these normalizations we have:

(9.1) Proposition. *In the cases B_n and C_n the relation $YXYE = E$ is satisfied.*

Proof. As in previous proofs we decompose $V \otimes V$ into invariant subspaces. On the D_8-orbits the matrices E are zero. For the remaining part we use the notations of section 3. The identity in question is then equivalent to the two equations

$$(AZA + qBC)E = E, \qquad (AZB + qBD)E = 0.$$

Inspection shows that, with our new normalization of Y, the matrix $AZA+qBC$ is actually the unit matrix, thus the first equation holds. The eigenvalue relation (2.2, II) yields by the very definition of E the second equation. □

We remark that the non-zero part of E is a symmetric matrix of rank one and therefore determined by its first row. This row is in the case B_n

$$(q^{-2n+1}, q^{-2n+2}, \dots, q^{-n}, p^{-2n+1}, q^{-n+1}, \dots, 1)$$

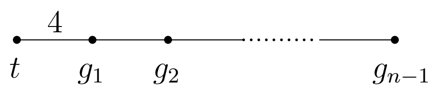
and in the case C_n

$$(-q^{-2n}, \dots, -q^{-n-1}, q^{-n+1}, \dots, 1).$$

The tensor representation is defined as in (1.4).

10. Tensor representations of braid groups

The braid group ZB_n associated to the Coxeter graph B_n



B_n

with n vertices has generators $t, g_1, \dots, g_{n-1}$ and relations:

(10.1)
$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & i>1 \\ g_ig_j & = & g_jg_i & |i-j|\geq 2 \\ g_ig_jg_i & = & g_jg_ig_j & |i-j|=1. \end{array}$$

We recall: The group ZB_n is the group of braids with n strings in the cylinder $(\mathbb{C}\setminus 0)\times[0,1]$ from $\{1,\dots,n\}\times 0$ to $\{1,\dots,n\}\times 1$. This topological interpretation is the reason for using the cylinder terminology. For the relation between the root system B_n and ZB_n see [**??**].

Let V be a $\mathfrak{K}$-module. Suppose $X\colon V\otimes V \to V\otimes V$ and $F\colon V\to V$ are $\mathfrak{K}$-linear automorphisms with the following properties:

(1) X is a Yang-Baxter operator, i. e. satisfies the equation

$$(X\otimes 1)(1\otimes X)(X\otimes 1) = (1\otimes X)(X\otimes 1)(1\otimes X)$$

on $V\otimes V\otimes V$.

(2) With $Y = F\otimes 1_V$, the *four braid relation* $YXYX = XYXY$ is satisfied.

If (1) and (2) hold, we call (X, F) a *four braid pair.* For the construction of four braid pairs associated to standard R-matrices see [**??**]. For a geometric interpretation of (2) in terms of symmetric braids with 4 strings see [**??**].

Given a four braid pair (X, F), we obtain a tensor representation of ZB_n on the n-fold tensor power $V^{\otimes n}$ of V by setting:

(10.2)
$$\begin{array}{rcl} t & \mapsto & F \otimes 1 \otimes \cdots \otimes 1 \\ g_i & \mapsto & X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \end{array}$$

The X in X_i acts on the factors i and $i+1$.

These representations give raise to further operators, if we apply them to special elements in the braid groups. We set

$$t(1) = t, \quad t(j) = g_{j-1}g_{j-2}\cdots g_1 t g_1 g_2 \cdots g_{j-1}, \quad t_n = t(1)t(2)\cdots t(n)$$

$$g(j) = g_j g_{j+1} \cdots g_{j+n-1}, \quad x_{m,n} = g(m)g(m-1)\cdots g(1).$$

The elements $t(j)$ pairwise commute. We denote by $T_n\colon V^{\otimes n} \to V^{\otimes n}$ and $X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ the operators induced by t_n and $x_{m,n}$, respectively.

(10.3) Proposition. *The following identities hold*

$$T_{m+n} = X_{n,m}(T_n \otimes 1)X_{m,n}(T_m \otimes 1) = (T_m \otimes 1)X_{n,m}(T_n \otimes 1)X_{m,n}.$$

Proof. We use some fact about Coxeter groups [**??**, CH. IV, §1]. If we adjoin the relations $t^2 = 1$ and $g_j^2 = 1$ to (2.1) we obtain the Coxeter group CB_n. The element t_n is given as a product of n^2 generators t, g_j. The uniquely determined element of CB_n has length n^2 and is equal to t_n. The element $x_{n,m}t_n x_{m,n} t_m$ of CB_{m+n} has length $(m+n)^2$ and therefore equals t_{m+n} in CB_{m+n}. By a fundamental fact about braid groups [**??**, CH. IV, §1.5, Prop. 5], the corresponding elements in the braid group are equal. We now apply the tensor representation and obtain the first equality in (2.3). □

For later use we record:

(10.4) Proposition. *The element t_n is contained in the center of ZB_n.* □

5 The universal twist

1. Cylinder forms

Let $A = (A, m, e, \mu, \varepsilon)$ be a bialgebra (over the commutative ring $\mathfrak{K}$) with multiplication m, unit e, comultiplication μ, and counit ε. Let $r\colon A \otimes A \to \mathfrak{K}$ be a linear form. We associate to left A-comodules M, N a $\mathfrak{K}$-linear map

$$z_{M,N}\colon M \otimes N \to N \otimes M, \quad x \otimes y \mapsto \sum r(y^1 \otimes x^1) y^2 \otimes x^2,$$

where we have used the formal notation $x \mapsto \sum x^1 \otimes x^2$ for a left A-comodule structure $\mu_M\colon M \to A \otimes M$ on M. (See [**??**, p. 186] formula (5.9) for our map $z_{M,N}$ and also formula (5.8) for a categorical definition.) We call r a *braid form* on A, if the $z_{M,N}$ yield a braiding on the tensor category A-COM of left A-comodules. We refer to [**??**, Def. VIII.5.1 on p. 184] for the properties of r which make it into a braid form and (A, r) into a cobraided bialgebra. (What we call braid form is called universal R-form in [**??**].)

Let (C, μ, ε) be a coalgebra. Examples of our μ-convention for coalgebras are $\mu(a) = \sum a_1 \otimes a_2$ and $(\mu \otimes 1)\mu(a) = \sum a_{11} \otimes a_{12} \otimes a_2$; if we set $\mu_2(a) = (\mu \otimes 1)\mu$, then we write $\mu_2(a) = \sum a_1 \otimes a_2 \otimes a_3$. The counit axiom reads in this notation $\sum \varepsilon(a_1)a_2 = a = \sum \varepsilon(a_2)a_1$. The multiplication in the dual algebra C^* is denoted by $*$ and called convolution: If $f, g \in C^*$ are $\mathfrak{K}$-linear forms on C, then the convolution product $f * g$ is the element of C^* defined by $a \mapsto \sum f(a_1)g(a_2)$. The unit element of the algebra C^* is ε. Therefore g is a (convolution) inverse of f, if $f * g = g * f = \varepsilon$. We apply this formalism to the coalgebras A and $A \otimes A$. If f and g are linear forms on A, we denote their exterior tensor product by $f \hat{\otimes} g$; it is the linear form on $A \otimes A$ defined by $a \otimes b \mapsto f(a)g(b)$. The twist on $A \otimes A$ is $\tau(a \otimes b) = b \otimes a$.

Let now (A, r) be a cobraided bialgebra with braid form r. A linear form $f\colon A \to \mathfrak{K}$ is called a *cylinder form* for (A, r), if it is convolution invertible and satisfies

(1.1) $$f \circ m = (f \hat{\otimes} \varepsilon) * r\tau * (\varepsilon \hat{\otimes} f) * r = r\tau * (\varepsilon \hat{\otimes} f) * r * (f \hat{\otimes} \varepsilon).$$

In terms of elements and the μ-convention, (1.1) assumes the following form:

(1.2) Proposition. *For any two elements $a, b \in A$ the identities*

$$f(ab) = \sum f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3) = \sum r(b_1 \otimes a_1) f(b_2) r(a_2 \otimes b_3) f(a_3)$$

hold.

Proof. Note that a four-fold convolution product is computed by the formula

$$(f_1 * f_2 * f_3 * f_4)(x) = \sum f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4).$$

We apply this to the second term in (1.1). The value on $a \otimes b$ is then

$$(f(a_1) \cdot \varepsilon(b_1)) \cdot r(b_2 \otimes a_2) \cdot (\varepsilon(a_3) \cdot f(b_3)) \cdot r(a_4 \otimes b_4).$$

By the counit axiom, we can replace $\sum \varepsilon(b_1) \cdot b_2 \otimes b_3 \otimes b_4$ by $\sum b_1 \otimes b_2 \otimes b_3$ (an exercise in the μ-convention), and $\sum a_1 \otimes a_2 \otimes \varepsilon(a_3) \cdot a_4$ can be replaced by $\sum a_1 \otimes a_2 \otimes a_3$. This replacement yields the second expression in (1.2). The third expression is verified in a similar manner. The first value is obtained from the definition of $f \circ m$. □

A cylinder form f (in fact any linear form) yields for each left A-comodule M a $\mathfrak{K}$-linear endomorphism

$$t_M \colon M \to M, \quad x \mapsto \sum f(x^1)x^2.$$

If $\varphi \colon M \to N$ is a morphism of comodules, then $\varphi \circ t_M = t_N \circ \varphi$. Since t_M is in general not a morphism of comodules we express this fact by saying: The t_M constitute a *weak endomorphism of the identity functor* of A-COM. We call t_M the *cylinder twist* on M. The axiom (1.1) for a cylinder form has the following consequence.

(1.3) Proposition. *The linear map t_M is invertible. For any two comodules M, N the identities*

$$t_{M\otimes N} = z_{N,M}(t_N \otimes 1_M)z_{M,N}(t_M \otimes 1_N) = (t_M \otimes 1_N)z_{N,M}(t_N \otimes 1_M)z_{M,N}$$

hold.

Proof. Let g be a convolution inverse of f. Define the endomorphism $s_M \colon M \to M$ via $x \mapsto \sum g(x^1)x^2$. Then

$$s_M t_M(x) = \sum f(x^1)g(x^{21})x^{22} = \sum \varepsilon(x^1)x^2 = x,$$

by the definition of a convolution inverse and the counit axiom. Hence s_M is inverse to t_M.

In order to verify the second equality, we insert the definitions and see that the second map is

$$x \otimes y \mapsto \sum f(x^1)r(y^1 \otimes x^{21})f(y^{21})r(y^{221} \otimes x^{221})y^{222} \otimes x^{222}$$

while the third map is

$$x \otimes y \mapsto \sum r(y^1 \otimes x^1)f(y^{21})r(y^{21} \otimes x^{221})f(x^{221})y^{222} \otimes x^{222}.$$

The coassociativity of the comodule structure yields a rewriting of the form

$$\sum y^1 \otimes y^{21} \otimes y^{221} \otimes y^{222} = \sum (y^1)_1 \otimes (y^1)_2 \otimes (y^1)_3 \otimes y^2$$

and one has a similar formula for x. We now apply (1.2) in the case where $(a,b) = (x^1, y^1)$.

By definition of the comodule structure of $M \otimes N$, the map $t_{M\otimes N}$ has the form $x \otimes y \mapsto \sum f(x^1y^1)x^2 \otimes y^2$. Again we use (1.2) in the case where $(a,b) = (x^1, y^1)$ and obtain the first equality of (1.3). □

2. Cylinder twist

We also mention dual notions. Let A be a bialgebra with a universal R-matrix $R \in A \otimes A$. An element $v \in A$ is called a (universal) *cylinder twist* for (A, R), if it is invertible and satisfies

(2.1) $$\mu(v) = (v \otimes 1) \cdot \tau R \cdot (1 \otimes v) \cdot R = \tau R \cdot (1 \otimes v) \cdot R \cdot (v \otimes 1).$$

The R-matrix $R = \sum a_r \otimes b_r$ induces the braiding

$$z_{M,N} \colon M \otimes N \to N \otimes M,\ x \otimes y \mapsto \sum b_r y \otimes a_r x.$$

Let $t_M \colon M \to M$ be the induced cylinder twist defined by $x \mapsto vx$. Again the t_M form a weak endomorphism of the identity functor. If v is not central in A, then the t_M are not in general A-module morphisms. The relations (1.3) also holds in this context.

In practice one has to consider variants of this definition. The universal R-matrix for the classical quantum groups A is not contained in the algebra $A \otimes A$, but rather is an operator on suitable modules. The same phenomenon will occur for the cylinder twist. Will see an example of this situation in Section 8.

If a ribbon algebra is defined as in [**??**, p. 361], then the element θ^{-1}, *loc. cit.*, is a cylinder twist in the sense above.

3. Cylinder forms from four braid pairs

Let V be a free $\mathfrak{K}$-module with basis $\{v_1, \dots, v_n\}$. Associated to a Yang-Baxter operator $X \colon V \otimes V \to V \otimes V$ is a bialgebra $A = A(V, X)$ with braid form r, obtained via the FRT-construction (see [**??**, VIII.6 for the construction of A and r). We show that a four braid pair (X, F) induces a canonical cylinder form on (A, r).

Recall that A is a quotient of a free algebra $\tilde{A}$. We use the model

$$\tilde{A} := \bigoplus_{n=0}^{\infty} \operatorname{Hom}(V^{\otimes n}, V^{\otimes n}).$$

The multiplication of $\tilde{A}$ is given by the canonical identification $E_k \otimes E_l \cong E_{k+l}, furnished by f \otimes g \mapsto f \otimes g$ where $E_k = \operatorname{Hom}(V^{\otimes k}, V^{\otimes k})$. The canonical basis of E, given by $T_i^j \colon v_k \mapsto \delta_{i,k} v_j$ of E_1, induces the basis

$$T_i^j = T_{i_1}^{j_1} \otimes \cdots \otimes T_{i_k}^{j_k}$$

of E_k, where, in multi-index notation, $i = (i_1, \dots, i_k)$ and $j = (j_1, \dots, j_k)$. The comultiplication of $\tilde{A}$ is given by $\mu(T_i^j) = \sum_k T_i^k \otimes T_k^j$ while the counit of $\tilde{A}$ is given by $\varepsilon(T_i^j) = \delta_i^j$.

In Section 2 we defined an operator $T_k \in E_k$ from a given four braid pair (X, F). We express T_k in terms of our basis

$$T_k(v_i) = \sum_j F_i^j v_j$$

again using the multi-index notation $v_i = v_{i_1} \otimes \cdots \otimes v_{i_k}$ when $i = (i_1, \dots, i_k)$. We use these data in order to define a linear form

$$\tilde{f}\colon \tilde{A} \to \mathfrak{K}, \quad T_i^j \mapsto F_i^j.$$

(3.1) Theorem. *The linear form $\tilde{f}$ factors through the quotient map $\tilde{A} \to A$ and induces a cylinder form f for (A, r).*

Proof. Suppose the operator $X = X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ has the form $X(v_i \otimes v_j) = \sum_{ab} X_{ij}^{ab} v_a \otimes v_b$. We define a form $\tilde{r}\colon \tilde{A} \otimes \tilde{A} \to \mathfrak{K}$ by defining

$$\tilde{r}\colon E_k \otimes E_l \to \mathfrak{K}, \quad T_i^a \otimes T_j^b \mapsto X_{ji}^{ab}.$$

The form $\tilde{r}$ factors through the quotient $A \otimes A$ and induces r.

Claim: The forms $\tilde{r}$ and $\tilde{f}$ satisfy (1.1) and (1.2). *Proof of the Claim:* In the proof we use the following summation convention: Summation occurs over an upper-lower index. We can then write $\mu_2(T_i^c) = T_i^k \otimes T_k^a \otimes T_a^c$ and $\mu_2(T_j^d) = T_j^l \otimes T_l^b \otimes T_b^d$. The equality (1.2) amounts to

$$F_{ij}^{cd} = F_i^k X_{kj}^{la} F_l^b X_{ba}^{cd} = X_{ij}^{lk} F_l^b X_{bk}^{ad} F_a^c.$$

These equations are also a translation of (2.3) into matrix form. This completes the proof of the claim.

We have to show that $\tilde{f}$ maps the kernel I of the projection $\tilde{A} \to A$ to zero. But this is a consequence of (1.2), applied in the case $b = 1$, since one of the terms a_1, a_2, a_3 is contained in I and $\tilde{r}$ is the zero map on $I \otimes \tilde{A}$ and $\tilde{A} \otimes I$.

It remains to show that f is convolution invertible. The pair (X^{-1}, F^{-1}) is a four braid pair. Let $\bar{r}$ and $\bar{f}$ be the induced operators on $\tilde{A}$. Then $\tilde{f} * \bar{f} = \varepsilon = \bar{f} * \tilde{f}$ on $\tilde{A}$, and (1.2) holds for $(\bar{f}, \bar{r})$ in place of (f, r). The Yang-Baxter operator X^{-1} defines the same quotient A of $\tilde{A}$ as X. Hence the kernel ideal obtained from X^{-1} equals I; therefore $\bar{f}(I) = 0$. □

We have the comodule structure map $V \to A \otimes V$ defined via $v_i \mapsto \sum_j T_i^j \otimes v_j$. One has a similar formula for $V^{\otimes k}$ using multi-index notation. By construction we have:

(3.2) Proposition. *The cylinder form f induces on $V^{\otimes k}$ the cylinder twist $t_{V^{\otimes k}} = T_k$.* □

4. The example sl_2

We illustrate the theory with the quantum group $\mathcal{O}(sl_2)$. For simplicity we work over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$.

Let V be a two-dimensional $\mathfrak{K}$-module with basis $\{v_1, v_2\}$. In terms of the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$ the matrix

(4.1)
$$X = q^{-1/2}\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}$$

defines a Yang-Baxter operator. The FRT-construction associates to X the algebra $A = \mathcal{O}(sl_2)$ The matrix

(4.2)
$$F = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix}$$

yields a four braid pair (X, F) for arbitrary parameters with invertible $\alpha\beta$. (See [**??**], also for an n-dimensional generalization.) Recall the quantum plane $P = \mathfrak{K}\{x,y\}/(xy - qyx)$ as a left A-comodule. The operator $T_2 = (F \otimes 1)X(F \otimes 1)X$ on $V \otimes V$ has the matrix (with $\delta = q - q^{-1}$)

$$\begin{pmatrix} 0 & 0 & 0 & \beta^2 \\ 0 & \alpha\beta\delta & \alpha\beta & q\beta\theta \\ 0 & \alpha\beta & 0 & \beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix} = \begin{pmatrix} F^{11}_{11} & F^{11}_{12} & F^{11}_{21} & F^{11}_{22} \\ F^{12}_{11} & F^{12}_{12} & F^{12}_{21} & F^{12}_{22} \\ F^{21}_{11} & F^{21}_{12} & F^{21}_{21} & F^{21}_{22} \\ F^{22}_{11} & F^{22}_{12} & F^{22}_{21} & F^{22}_{22} \end{pmatrix}$$

with respect to the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$. This is also the matrix of values of the cylinder form f

$$f\begin{pmatrix} aa & ac & ca & cc \\ ab & ad & cb & cd \\ ba & bc & da & dc \\ bb & bd & db & dd \end{pmatrix}.$$

(The notation means: If we apply f to entries of the matrix we obtain the matrix for T_2 diplayed above.) Let $\det_q = ad - qbc$ be the quantum determinant. It is a group-like central element of A. The quotient of A by the ideal generated by $\det_q$ is the Hopf algebra $SL_q(2)$.

(4.3) Proposition. *The form f has the value $-q^{-1}\alpha\beta$ on $\det_q$. If $-q^{-1}\alpha\beta = 1$, then f factors over $SL_q(2)$.*

Proof. The stated value of $f(\det_q)$ is computed from the data above. We use the fact that
$$r(x \otimes \det_q) = r(\det_q \otimes x) = \varepsilon(x).$$
(See [**??**, p. 195].) From (1.2) we obtain, for $a \in A$ and $b = \det_q$, that

$$\begin{aligned} f(ab) &= \sum f(a_1)r(b_1 \otimes a_2)f(b_2)r(a_3 \otimes b_3) \\ &= \sum f(a_1)\varepsilon(a_2)f(\det_q)\varepsilon(a_3) \\ &= f(a), \end{aligned}$$

by using the assumption that $f(\det_q) = 1$ together with the counit axiom. $\square$

We consider the subspace $W = V_2$ of the quantum plane generated by x^2, xy, and y^2. We have

$$\begin{aligned} \mu_P(x^2) &= b^2 \otimes y^2 + (1+q^{-2})ab \otimes xy + a^2 \otimes x^2 \\ \mu_P(xy) &= bd \otimes y^2 + (ad + q^{-1}bc) \otimes xy + ac \otimes x^2 \\ \mu_P(y^2) &= d^2 \otimes y^2 + (1+q^{-2})cd \otimes xy + c^2 \otimes x^2. \end{aligned}$$

This yields the following matrix for t_W with respect to the basis $\{x^2, xy, y^2\}$:

$$\begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & (q+q^{-1})\beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In the Clebsch-Gordan decomposition $V \otimes V = V_2 \oplus V_0$ the subspace V_0 (the trivial irreducible module) is spanned by $u = v_2 \otimes v_1 - q^{-1} v_1 \otimes v_2$. This is the eigenvector of X with eigenvalue $-q^{-3/2}$. It is mapped by T_2 to $-q^{-1}\alpha\beta u$. If we require this to be the identity we must have $\alpha\beta = -q$. We already obtained this condition by considering the quantum determinant.

The matrix of t_W with respect to the basis $\{w_1 = x^2, w_2 = \sqrt{1+q^{-2}}xy, w_3 = y^2\}$ is

(4.4) $$F_2 = \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & \sqrt{1+q^2}\beta\theta \\ \alpha^2 & \sqrt{1+q^2}\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In case $\alpha = \beta$ this matrix is symmetric.

The R-matrix X on $W \otimes W$ with respect to the *lexicographic basis* consisting of elements $w_i \otimes w_j$ with $w_1 = x^2, w_2 = \sqrt{1+q^{-2}}xy$, and $w_3 = y^2$ has the form

(4.5) $$X_2 = \left|\begin{array}{ccc|ccc|ccc} \hline q^2 & & & & & & & & \\ & \delta^* & & 1 & & & & & \\ & & \mu & & \lambda & & q^{-2} & & \\ \hline & 1 & & 0 & & & & & \\ & & \lambda & & 1 & & & & \\ & & & & & \delta^* & & 1 & \\ \hline & & q^{-2} & & & & 0 & & \\ & & & & & 1 & & 0 & \\ & & & & & & & & q^2 \\ \hline \end{array}\right|$$

It makes use of the identities $\delta^* = q^2 - q^{-2}$, $\mu = \delta^*(1-q^{-2})$, and $\lambda = q^{-1}\delta^*$. By construction, (X_2, F_2) is a four braid pair.

One has the the problem of computing t_W on irreducible comodules W. We treat instead the more familiar dual situation of modules over the quantized universal enveloping algebra.

5. The cylinder braiding for U-modules

The construction of the cylinder form is the simplest method to produce a universal operator for the cylinder twist. In order to compute the cylinder twist explicitly, we pass to the dual situation of the quantized universal enveloping algebra U. One can formally dualize comodules to modules and thus obtain a cylinder braiding for suitable classes of U-modules from the results of the previous sections. But we rather start from scratch.

We work with the Hopf algebra $U = U_q(sl_2)$ as in [**??**]. As an algebra, it is the the associative algebra over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$ generated[4] by K, K^{-1}, E, and F subject to the relations $KK^{-1} = K^{-1}K = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, and $EF - FE = (K - K^{-1})/(q - q^{-1})$. Its coalgebra structure is defined by setting $\mu(K) = K \otimes K$, $\mu(E) = E \otimes 1 + K \otimes E$, $\mu(F) = F \otimes K^{-1} + 1 \otimes F$, $\varepsilon(K) = 1$, and $\varepsilon(E) = \varepsilon(F) = 0$. A left U-module M is called *integrable* if the following condition holds:

(1) $M = \bigoplus M^n$ is the direct sum of weight spaces M^n on which K acts as multiplication by q^n for $n \in \mathbb{Z}$.

(2) E and F are locally nilpotent on M.

Let U-INT denote the *category of integrable U-modules* and U-linear maps. (It would be sufficient to consider only finite dimensional such modules.) An integrable U-module M is semi-simple: It has a unique isotypic decomposition $M = \bigoplus_{n \geq 0} M(n)$ with $M(n)$ isomorphic to a direct sum of copies of the irreducible module V_n. The module V_n has a $\mathfrak{K}$-basis $x_0, x_1, \dots, x_n$ with $F(x_i) = [i+1]x_{i+1}$, $E(x_i) = [n-i+1]x_{i-1}$, $x_{-1} = 0$, $x_{n+1} = 0$; moreover, $x_i \in V_n^{n-2i}$. The category of integrable U-modules is *braided.* The braiding is induced by the universal R-matrix $R = \kappa \circ \Psi$ with

$$\textbf{(5.1)} \qquad \Psi = \sum_{n \geq 0} q^{n(n-1)/2} \frac{(q - q^{-1})^n}{[n]!} F^n \otimes E^n$$

and $\kappa = q^{H \otimes H/2}$. Note that Ψ is a well-defined operator on integrable U-modules. (This operator is called $\bar{\Theta}$ in [**??**, section 4.1] and L_i' in [**??**, p. 46].) The operator κ acts on $M^m \otimes N^n$ as multiplication by $q^{mn/2}$. If we view H as the operator $H\colon M^m \to M^m$ given by $x \mapsto mx$, then $q^{H \otimes H/2}$ is a suggestive notation for κ. The braiding $z_{M,N}\colon M \otimes N \to N \otimes M$ is $\tau \circ R$, i. e., the action of R followed by the interchange operator $\tau\colon x \otimes y \mapsto y \otimes x$.

A four braid pair (X, F) on the vector space V yields a tensor representation of ZB_n on $V^{\otimes n}$. We start with the standard four braid pair determined by (5.1) and (5.2) on the two-dimensional U-module $V = V_1$. Let $T_n\colon V^{\otimes n} \to V^{\otimes n}$ be the associated cylinder twist as defined in Section 2. By the Clebsch-Gordan decomposition, V_n is contained in $V^{\otimes n}$ with multiplicity 1. Similarly, $V_{m+n} \subset V_m \otimes V_n$ with multiplicity one [**??**, VII.7].

[4]There is another use of the letter F. It has nothing to do with the 2×2-matrix F in (4.2).

(5.2) Lemma. *There exists a projection operator $e_n\colon V^{\otimes n} \to V^{\otimes n}$ whose image, V_n, commutes with T_n.*

Proof. Let H_n be the Hecke algebra over $\mathfrak{K}$ generated by $x_1, \dots, x_{n-1}$ with braid relations $x_i x_j x_i = x_j x_i x_j$ for $|i-j| = 1$ and $x_j x_i = x_i x_j$ for $|i-j| > 1$ and quadratic relations $(x_i+1)(x_i-q^2) = 0$. Since X satisfies $(X-q^{1/2})(X+q^{-3/2}) = 0$, we obtain an action of H_n from the action of $ZA_{n-1} \subset ZB_n$ on $V^{\otimes n}$ if we let x_i act as $q^{3/2}g_i$. Since T_n comes from a central element of ZB_n as noted in (2.4), the H_n-action commutes with T_n. It is well known that there exists an idempotent $e_n \in H_n$ for which $e_n V^{\otimes n} = V_n$. (This is quantized Schur-Weyl duality.) This fact implies the assertion of the Lemma. □

(5.3) Corollary. *The subspace $V_n \subset V^{\otimes n}$ is T_n-stable.* □

A similar proof shows that all summands in the isotypic decomposition of $V^{\otimes n}$ are T_n-stable.

We denote by τ_n the restriction of T_n to V_n; and we denote by $\tau_{m,n} = z_{n,m}(\tau_n \otimes 1)z_{m,n}(\tau_m \otimes 1)$ the induced operator on $V_m \otimes V_n$ where $z_{m,n}$ denotes the braiding on $V_m \otimes V_n$.

(5.4) Lemma. *The subspace $V_{m+n} \subset V_m \otimes V_n$ is $\tau_{m,n}$-stable. The induced morphism equals τ_{m+n}.*

Proof. Consider $V_m \otimes V_n \subset V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes(m+n)}$. The projection operator $e_m \otimes e_n$ is again obtained from the action of a certain element of the Hecke algebra H_{m+n}. Hence $V_m \otimes V_n$ is T_{m+n}-stable and the action on the subspace V_{m+n} is τ_{m+n}. We now use the equality (2.3)

$$T_{m+n} = X_{n,m}(T_n \otimes 1)X_{m,n}(T_m \otimes 1).$$

The essential fact is that $X_{m,n}$ is the braiding on $V^{\otimes m} \otimes V^{\otimes n}$. It induces, by naturality of the braiding, the braiding $z_{m,n}$ on $V_m \otimes V_n$. □

Let $A(n) = (\alpha_i^j(n))$ be the matrix of τ_n with respect to $x_0, \dots, x_n$. In the next theorem we derive a recursive description of $A(n)$. We need more notation to state it. Define inductively polynomials γ_k by $\gamma_{-1} = 0$, $\gamma_0 = 1$ and, for $k > 0$,

$$\alpha\gamma_{k+1} = q^k\theta\gamma_k + \beta q^{k-1}\delta[k]\gamma_{k-1}. \tag{5.5}$$

Here $\delta = q - q^{-1}$, and $\gamma_k = \gamma_k(\theta, q, \alpha, \beta)$ is a polynomial in θ with coefficients in $\mathbb{Z}[q, q^{-1}, \alpha^{-1}, \beta]$. Let $D(n)$ denote the codiagonal matrix with $\alpha^k\beta^{n-k}q^{k(n-k)}$ in the k-th row and $(n-k)$-th column and zeros otherwise. (We enumerate rows and columns from 0 to n.) Let $B(n)$ be the upper triangular matrix

$$B(n) = \begin{pmatrix} \gamma_0 & \begin{bmatrix} n \\ 1 \end{bmatrix}\gamma_1 & \begin{bmatrix} n \\ 2 \end{bmatrix}\gamma_2 & \cdots & \gamma_n \\ & \gamma_0 & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}\gamma_1 & \cdots & \gamma_{n-1} \\ & & \cdots & \cdots & \cdots \\ & & & \gamma_0 & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}. \tag{5.6}$$

Thus the $(n-k)$-th row of $B(n)$ is

$$0,\dots,0,\begin{bmatrix}k\\0\end{bmatrix}\gamma_0,\begin{bmatrix}k\\1\end{bmatrix}\gamma_1,\begin{bmatrix}k\\2\end{bmatrix}\gamma_2,\dots,\begin{bmatrix}k\\k-1\end{bmatrix}\gamma_{k-1},\begin{bmatrix}k\\k\end{bmatrix}\gamma_k.$$

(5.7) Theorem. *The matrix $A(n)$ is equal to the product $D(n)B(n)$.*

Proof. The proof is by induction on n. We first compute the matrix of $\tau_{n,1}$ on $V_n \otimes V_1$ and then restrict to V_{n+1}. In order to display the matrix of $\tau_{n,1}$ we use the basis

$$x_0 \otimes x_0, \dots, x_n \otimes x_0, x_0 \otimes x_1, \dots, x_n \otimes x_1.$$

The matrix of $\tau_{n,1}$ has the block form

$$\begin{pmatrix} 0 & \beta A(n) \\ \alpha A(n) & A'(n) \end{pmatrix}.$$

The matrix $A'(n)$ is obtained from $A(n)$ in the following manner: Let $\alpha_0,\dots,\alpha_n$ denote the columns of $A(n)$ and $\beta_0,\dots,\beta_n$ the columns of $A'(n)$. We claim that

$$\beta_i = \alpha q^{2i-n}\theta\alpha_i + \beta q^{2i-n-1}\delta[n-i+1]\alpha_{i-1} + \alpha\delta[i+1]\alpha_{i+1},$$

with $\alpha_{-1} = \alpha_{n+1} = 0$.

Recall that $\tau_{n,1} = (\tau_n \otimes 1)z_{1,n}(\tau_1 \otimes 1)z_{n,1}$. In our case the universal R-matrix has the simple form

$$R = \kappa \circ (1 + (q-q^{-1})F \otimes E).$$

For the convenience of the reader we display the four steps in the calculation of $\tau_{n,1}$, separately for $x_i \otimes x_0$ and $x_i \otimes x_1$.

$$\begin{aligned} x_i \otimes x_0 &\mapsto q^{(n-2i)/2} x_0 \otimes x_i \\ &\mapsto \alpha q^{(n-2i)/2} x_1 \otimes x_i \\ &\mapsto \alpha x_i \otimes x_1 \\ &\mapsto \sum_j \alpha\alpha_i^j x_j \otimes x_0. \end{aligned}$$

$$\begin{aligned} x_i \otimes x_1 &\mapsto q^{-(n-2i)/2} x_1 \otimes x_i + \delta[i+1]q^{(n-2i-2)/2} x_0 \otimes x_{i+1} \\ &\mapsto q^{-(n-2i)/2}(\beta x_0 + \theta x_1) \otimes x_i + \alpha\delta[i+1]q^{(n-2i-2)/2} x_1 \otimes x_{i+1} \\ &\mapsto \beta x_i \otimes x_0 + \beta q^{-n+2i-1}\delta[n-i+1] x_{i-1} \otimes x_1 \\ &\quad + q^{2i-n}\theta x_i \otimes x_1 + \alpha\delta[i+1] x_{i+1} \otimes x_1 \\ &\mapsto \sum_j \alpha_i^j x_i \otimes x_0 + \sum_j \beta q^{2i-n+1}\delta[n-i+1]\alpha_{i-1}^j x_j \otimes x_1 \\ &\quad + \sum_j q^{2i-n}\theta\alpha_i^j x_j \otimes x_1 + \sum_j \alpha\delta[i+1]\alpha_{i+1}^j x_j \otimes x_1. \end{aligned}$$

This proves the claim about the matrix for $\tau_{n,1}$.

We now use the following fact about the Clebsch-Gordan decomposition (it is easily verified in our case, but see e. g. [**??**, VII.7] for more general results): In the Clebsch-Gordan decomposition $V_n \otimes V_1 = V_{n+1} \oplus V_{n-1}$ a basis of V_{n+1} is given by

$$y_j = \frac{F^j}{[j]!}(x_0 \otimes x_0) = q^{-j} x_j \otimes x_0 + x_{j-1} \otimes x_1.$$

We apply $\tau_{n,1}$ to the y_j. Since there are no overlaps between the coordinates of the y_j, we can directly write $\tau_{n,1}(y_j)$ as a linear combination of the y_k.

We assume inductively that $A(n)$ has *bottom-right triangular form*, i. e., zero entries above the codiagonal, with codiagonal as specified by $D(n)$. Then $A'(n)$ has a nonzero line one step above the codiagonal and is bottom-right triangular otherwise. From the results so far we see that the columns of $A(n+1)$, enumerated from 0 to $n+1$, are obtained inductively as follows: The 0-th row is $(0, \dots, 0, \beta^{n+1})$. Below this 0-th row the j-th column, for $0 \le j \le n+1$, has the form

(5.8)
$$\alpha q^j \alpha_j + q^{2j-n-2} \theta \alpha_{j-1} + \beta q^{2j-n-3} \delta[n-j+2] \alpha_{j-2}.$$

From this recursive formula one derives immediately that the codiagonal of $A(n)$ is given by $D(n)$.

Finally, we prove by induction that $A(n)$ is as claimed. The element in row k and column $n-k+j$ equals

$$\alpha^k \beta^{n-k} q^{k(n-k)} \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j.$$

For $n = 1$, we have defined τ_1 as $A(1)$. For the inductive step we use (6.8) in order to determine the element of $A(n)$ in column $n-k+j$ and row $k+1$. The assertion is then equivalent to the following identity:

$$\begin{aligned} & \alpha^k \beta^{n-k} q^{k(n-k)} \left(\alpha \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j + q^{n-2k+2j-2} \theta \begin{bmatrix} k \\ j-1 \end{bmatrix} \gamma_{j-1} \right. \\ & \qquad \left. + \beta q^{n-2k+2j-3} \delta[k-j+2] \begin{bmatrix} k \\ j-2 \end{bmatrix} \gamma_{j-2} \right) \\ = \; & \alpha^{k+1} \beta^{n-k} q^{(n-k)(k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \gamma_j. \end{aligned}$$

We cancel α-, β-, and q-factors, use the Pascal formula

(5.9)
$$\begin{bmatrix} a+1 \\ b \end{bmatrix} = q^b \begin{bmatrix} a \\ b \end{bmatrix} + q^{-a+b-1} \begin{bmatrix} a \\ b-1 \end{bmatrix}$$

and the identity

$$\delta[k-j+2] \begin{bmatrix} k \\ j-2 \end{bmatrix} = \begin{bmatrix} k \\ j-1 \end{bmatrix} [j-1]$$

and see that the identity in question is equivalent to the recursion formula (6.5) defining the γ-polynomials. This completes the proof. □

We now formulate the main result of this section in a different way. First, we note that it was not essential to work with the function field $\mathfrak{K}$. In fact, $\mathfrak{K}$ could have been any commutative ring and q, α, and β could have been any suitable parameters in it. We think of θ as being an indeterminate.

Let $L(\alpha, \beta)$ be the operator on integrable U-modules which acts on V_n via

$$x_j \mapsto \alpha^{n-j}\beta^j q^{j(n-j)} x_{n-j}.$$

Let

$$T(\alpha,\beta) = \sum_{k=0}^{\infty} \gamma_k \frac{E^k}{[k]!}; \tag{5.10}$$

$T(\alpha, \beta)$ is well-defined as an operator on integrable U-modules. Then (6.7) can be expressed as follows:

(5.11) Theorem. *The operator $t(\alpha,\beta)$ defined by setting $t(\alpha,\beta) = L(\alpha,\beta) \circ T(\alpha,\beta)$ acts on V_n as τ_n.* □

In Section 8 we give another derivation of this operator from the universal point of view.

One can develop a parallel theory by starting with the four braid pair (X^{-1}, F^{-1}). This leads to matrices which are *top-left triangular*, i. e., zero entries below the codiagonal. By computing the inverse of (5.1) and of (5.2) we see that, in the case $(\alpha, \beta) = (1, 1)$, we have to replace (q, θ) by $(q^{-1}, -\theta)$.

The following proposition may occasionally be useful. Introduce a new basis $u_0, \dots, u_n$ in V_n by

$$x_i = q^{-i(n-i)/2} \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}}\, u_i.$$

Then a little computation shows:

(5.12) Proposition. *Suppose $\alpha = \beta$. With respect to the basis (u_i) the R-matrix and the matrix for τ_n are symmetric.* □

6. The γ-polynomials

For later use we derive some identities for the γ-polynomials of the previous section. A basic one, (7.1), comes from the compatibility of the cylinder twist with tensor products. Again we use $\delta = q - q^{-1}$. We give two proofs of (7.1).

(6.1) Theorem. *The γ-polynomials satisfy the product formula*

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} \alpha^{-k}\beta^k q^{mn-k(k+1)/2}\delta^k [k]! \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \gamma_{m-k}\gamma_{n-k}.$$

First proof of (7.1). The first proof is via representation theory. We have a unique U-submodule of $V_m \otimes V_n$ which is isomorphic to V_{m+n} (Clebsch-Gordan decomposition). We use the symbol V_{m+n} also for this module. The vector $x_m \otimes x_n$ is contained in this module and satisfies $F(m_m \otimes x_n) = 0$. The latter property characterizes $x_m \otimes x_n$ inside V_{m+n} up to a scalar (lowest weight vector, F-primitive vector).

We consider

$$\tau_{m,n} = (\tau_m \otimes 1) z_{n,m} (\tau_n \otimes 1) z_{m,n}$$

on $V_{m+n} \subset V_m \otimes V_n$ where it equals τ_{m+n}. We first express this equality formally in terms of matrices and then evaluate the formal equation by a small computation. We already have intruced the matrices for τ_m in Section 6

$$\tau_m(x_j) = \sum_k \alpha_j^k(m) x_k.$$

We write

$$z_{n,m}(x_j \otimes x_m) = \sum_{u,v} r_{jm}^{uv} x_u \otimes x_v.$$

From the form of the action of E and F on the modules V_t and from the form of the universal R-matrix we see that the sum is over (u,v) with $u+v = j+m$. Since $Fx_m = 0$ for $x_m \in V_m$, we also observe directly $z_{m,n}(x_m \otimes x_n) = q^{mn/2} x_n \otimes x_m$. The two expressions for τ_{m+n}, applied to $x_m \otimes x_n \in V_{m+n} \subset V_m \otimes V_n$, now yield the formal identity

$$\alpha_{m+n}^{m+n}(m+n) = \sum_{k \ge 0} q^{mn/2} \alpha_n^{n-k}(n) r_{n-k,m}^{m-k,n} \alpha_{m-n}^m(n).$$

By (6.7) we have

$$\begin{aligned} \alpha_{m+n}^{m+n}(m+n) &= \alpha^{m+n} \gamma_{m+n}, \\ \alpha_n^{n-k}(n) &= \alpha^{n-k} \beta^k q^{(n-k)k} \gamma_{n-k}, \quad \text{and} \\ \alpha_{m-k}^m(m) &= \alpha^m \begin{bmatrix} m \\ k \end{bmatrix} \gamma_{m-k}. \end{aligned}$$

This already yields a relation of type (7.1). It remains to compute the coefficient $r_{n-k,m}^{m-k,n}$. For this purpose we use the definition $z_{n,m} = \tau \circ \kappa \circ \Psi$ of the braiding, the action of E and F on vectors x_j, and the explicit form (6.1) of the operator Ψ. Put together, this yields

$$z_{n,m}(x_j \otimes x_m) = \sum_{k \ge 0} v^{\bullet(k)} \delta^k [j+1] \cdots [j+k] x_{m-k} \otimes x_{j+k}$$

with

$$\bullet(k) = k(k-1)/2 + (n-2j-2k)(2k-m)/2.$$

We now have enough data to rewrite the formal identity above and give it the form (7.1). □

The dependence of γ_k on the parameters α and β is not essential. Define, inductively, polynomials γ'_k in θ over $\mathbb{Z}[q,q^{-1}]$ by setting $\gamma'_{-1}=0, \gamma'_0=1$ and, for $k\geq 0$,

$$\gamma'_{k+1}=q^k\theta\gamma'_k+q^{k-1}\delta[k]\gamma'_{k-1},$$

i. e., by setting $\gamma'_k(\theta,q)=\gamma_k(\theta,q,1,1)$. A simple rewriting of the recursion formula then yields the identity

(6.2)
$$\gamma_k(\theta,q,\alpha,\beta)=\gamma'_k\left(\frac{\theta}{\sqrt{\alpha\beta}},q\right)\left(\frac{\beta}{\alpha}\right)^{k/2}.$$

Note that γ'_k contains only powers θ^l with $l\equiv k \bmod 2$.

Normalize the γ' to obtain monic polynomials $\beta_k(\theta)=q^{-k(k-1)/2}\gamma'_k(\theta)$. The new polynomials are determined by the recursion relation

(6.3) $\beta_{-1}=0,\quad \beta_0=1,\quad$ and $\beta_{k+1}=\theta\beta_k+(1-q^{-2k})\beta_{k-1}\quad$ for $k\geq 0$.

In order to find an explicit expression for the β_k, we introduce a new variable ρ via the quadratic relation $\theta=\rho-\rho^{-1}$. We then consider the recursion formally over the ring $\mathbb{Z}[q,q^{-1},\rho,\rho^{-1}]$. Let us set

$$B_n(\rho)=\sum_{j=0}^{n}(-1)^jq^{-j(n-j)}\begin{bmatrix}n\\ j\end{bmatrix}\rho^{n-2j}.$$

(6.4) Proposition. *The polynomials β satisfy the identity*

$$\beta_k(\rho-\rho^{-1})=B_k(\rho).$$

Proof. We verify the recursion (7.3) with θ replaced by $\rho-\rho^{-1}$ and β_k replaced by B_k. We use the definition of the B_k in the right hand side of (7.3). Then the coefficient of ρ^{k+1-2j}, for $1\leq j\leq k$, turns out to be

$$(-1)^jq^{-j(k-j}\left(\begin{bmatrix}k\\ j\end{bmatrix}+q^{k-2j+1}\begin{bmatrix}k\\ j-1\end{bmatrix}-q^{-j}\delta[k]\begin{bmatrix}k-1\\ j-1\end{bmatrix}\right).$$

We use the identity

$$[k]\begin{bmatrix}k-1\\ j-1\end{bmatrix}=[k-j+1]\begin{bmatrix}k\\ j-1\end{bmatrix}$$

and arrive at

$$(-1)^jq^{-j(k-j}\left(\begin{bmatrix}k\\ j\end{bmatrix}+q^{-k-1}\begin{bmatrix}k\\ j-1\end{bmatrix}\right).$$

The Pascal formula (6.9) now shows that this is the coefficient of ρ^{k+1-2j} in B_{k+1}. It is easy to check that the coefficients of $\rho^{\pm(k+1)}$ on both sides coincide. □

We can write $\rho^k+(-1)^k\rho^{-k}$ as an integral polynomial P_k in θ where $\theta=\rho-\rho^{-1}$. That polynomial satisfies the recursion relation

$$\theta P_k = P_{k+1} - P_{k-1}.$$

It is possible to write P_k in terms of Tschebischev- or Jacobi-polynomials. The last proposition says that

$$\beta_n(\theta) = \sum_{j=0}^{[n/2]} (-1)^j q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} P_{n-2j}(\theta).$$

The product formula (7.1) was a consequence of representation theory. In view of the applications to be made in Section 8 it is desirable to have a proof which uses only the recursive definition of the γ-polynomials. We now give such a proof. By (7.2), it suffices to consider the case $\alpha=\beta=1$.

Second proof of (7.1). We write

$$C_k^{m,n} = q^{mn-k(k+1)/2}\delta^k[k]!\begin{bmatrix} m \\ k \end{bmatrix}\begin{bmatrix} n \\ k \end{bmatrix},$$

and want to show that

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} C_k^{m,n}\gamma_{m-k}\gamma_{n-k}.$$

Denote the right hand side by $\gamma(m,n)$. Then $\gamma(m,n)=\gamma(n,m)$. We will use the recursion (6.5) and the Pascal formula (6.9), with q replaced by q^{-1}, to show $\gamma(m+1,n)=\gamma(m,n+1)$. Since $\gamma(m+n,0)=\gamma_{m+n}$ the proof will then be complete. Set $\gamma_k=0$ for $k<0$. We can then sum just over $k\geq 0$. The C-coefficients satisfy the following Pascal type relation

(6.5) $$C_k^{m+1,n} = q^{n-k}C_k^{m,n} + \delta q^{n-k+1}q^{m-k}[n-k+1]C_{k-1}^{m,n}.$$

The verification that this is so uses the Pascal formula for $\begin{bmatrix} m+1 \\ k \end{bmatrix}$ and a little rewriting. Next we apply this relation in the sum $\gamma(m+1,n)$ and obtain (with an index shift $k\to k+1$ in the second summand) the identity

$$\begin{aligned}\gamma(m+1,n) &= \sum_k q^{n-k}C_k^{m,n}\gamma_{m-k+1}\gamma_{n-k} \\ &\quad + \sum_k \left(\delta[n-k]q^{n-k-1}\gamma_{n-k-1}\right) q^{m-k}C_k^{m,n}\gamma_{m-k}.\end{aligned}$$

In the second sum apply the recursion to the factor in parentheses to obtain the identity

$$\gamma(m+1,n) = \sum_k C_k^{m,n}(q^{n-k}\gamma_{m-k+1}\gamma_{n-k} + q^{m-k}\gamma_{m-k}\gamma_{n-k+1} - q^{n+m-2k}\theta\gamma_{m-k}\gamma_{n-k}).$$

Since $\gamma(m,n)=\gamma(n,m)$, we obtain $\gamma(m,n+1)$ upon interchanging m and n in the foregoing identity: That interchanges the first two summands in the parentheses and leaves the third fixed. □

7. The universal cylinder twist

In this section we work with operators on integrable U-modules. These are $\mathfrak{K}$-linear weak endomorphisms of the category U-INT. Left multiplication by $x \in U$ is such an operator; it will be denoted by x or by l_x. If t is an operator, then $\mu(t)$ is the operator on U-INT $\times$ U-INT which is given by the action of t on tensor products of modules. If τ denotes the twist operator, then we define $\tau(t) = \tau \circ t \circ \tau$. We have the compatibilities $\mu(l_x) = l_{\mu(x)}$ and $\tau\mu(l_x) = l_{\tau\mu(x)}$. The operators $\mu(t)$ and $\tau(t)$ are again weak endomorphisms of the categories involved.

Typical examples of such operators which are not themselves elements of U are the universal R-matrix R and its factors κ and Ψ, (See (6.1).) as are the operators $L = T'_{i,1}$ and $L^{\#} = T''_{i,1}$ of Lusztig [**??**, p. 42].

Since R acts by U-linear maps each operator t satisfies the standard relation

(7.1) $$R \circ \mu(t) = \tau\mu(t) \circ R$$

of a braiding.

An operator t is called a *universal cylinder twist* on U-INT if it is invertible and satisfies the analogue of (1.4), namely,

(7.2) $$\mu(t) = \tau R(1 \otimes t)R(t \otimes 1) \qquad \text{and}$$

(7.3) $$\tau R(1 \otimes t)R(t \otimes 1) = (t \otimes 1)\tau R(1 \otimes t)R.$$

We denote by t_V the action of t on the module V. Then (1.3) holds if we use R to define the braiding. Recall the operator $t(\alpha, \beta)$ defined at the end of Section 6. Here is the main result, proved following (8.6).

(7.4) Theorem. *Suppose $\alpha\beta = -q$. Then $t(\alpha, \beta)$ is a universal cylinder twist.*

We treat the case $(\alpha, \beta) = (-q, 1)$ in detail and reduce the general case formally to this one. We skip the notation α, β and work with $t = LT$. Note that L is Lusztig's operator referred to above. We collect a few properties of L in the next lemma.

(7.5) Lemma. *The operator L satisfies the following identities:*
(1) $LEL^{-1} = -KF$, $LFL^{-1} = -EK^{-1}$, $LKL^{-1} = K^{-1}$.
(2) $\mu(L) = (L \otimes L)\Psi = \tau R(L \otimes L)\kappa^{-1}$.
(3) $\kappa(L \otimes 1) = (L \otimes 1)\kappa^{-1}$, $\kappa(1 \otimes L) = (1 \otimes L)\kappa^{-1}$.
(4) $(L \otimes L)\Psi(L \otimes L)^{-1} = \kappa \circ \tau\Psi \circ \kappa^{-1}$.

Proof. For (1), in the case $L^{\#}$, see [**??**, Proposition 5.2.4.]. A simple computation from the definitions yields (3) and (4). For the first equality in (2) see [**??**, Proposition 5.3.4]; the second one follows by using (3) and (4). □

In the universal case one of the axioms for a cylinder twist is redundant, namely:

(7.6) Proposition. *If the operator t satisfies* (8.2)*, then it also satisfies* (8.3)*.*

Proof. Apply τ to (8.2) and use (8.1). $\square$

Proof of theorem (8.4). The operator L is invertible. The operator T is invertible since its constant term is 1. Thus it remains to verify (8.2). We show that (8.2) is equivalent to

(7.7) $$\mu(T) = \kappa(1 \otimes T)\kappa^{-1} \circ (L^{-1} \otimes 1)\Psi(L \otimes 1) \circ (T \otimes 1),$$

given the relations of Lemma (8.5). Given (8.2), we have

$$\mu(T) = \mu(L^{-1})\tau(R)(1 \otimes LT)\kappa\Psi(LT \otimes 1).$$

We use (8.5.2) for $\mu(L^{-1})$, cancel $\tau(R)$ and its inverse, and then use (8.5.3); (8.7) drops out. In like manner, (8.2) follows from (8.1).

In order to prove (8.7), one verifies the following identities from the definitions:

$$\kappa(1 \otimes T)\kappa^{-1} = \sum_{k=0}^{\infty} \frac{\gamma_k}{[k]!}(K^k \otimes E^k) \qquad \text{and}$$

$$(L^{-1} \otimes 1)\Psi(L \otimes 1) = \sum_{k=0}^{\infty}(-1)^k q^{-k(k-1)/2}\frac{\delta^k}{[k]!}K^k E^k \otimes E^k.$$

Using this information, we compute the coefficient of $K^r E^s \otimes E^r$ on the right hand side of (8.7) to be

$$\sum_{n=0}^{\min(r,s)} (-1)^n q^{-n(n-1)/2}\frac{\delta^n}{[n]![s-n]![r-n]!}\gamma_{s-n}\gamma_{r-n}.$$

The coefficient of the same element in $\mu(T)$ is, by the q-binomial formula, equal to

$$q^{-rs}\frac{1}{[s]![r]!}\gamma_{r+s}.$$

Equality of these coefficients is exactly the product formula (7.1) in the case where $(\alpha, \beta) = (-q, 1)$. This finishes the proof of the theorem in this special case.

A similar proof works in the general case. Specifically, a formal reduction to the special case uses the following observation. Write $\alpha = q^{\zeta}$. Then, formally, $L(\alpha, \beta) = K^{\zeta}L$ in case $\alpha\beta = -q$. This fact is used to deduce similar properties for $L_{\#} = L(\alpha, \beta)$ from lemma (8.5), in particular

$$L_{\#}^{-1}FL_{\#} = \alpha^{-1}\beta q K E.$$

The final identity leads to (7.1) in the general case. $\square$

We point out that the main identity in the construction of the universal twist involves only the Borel subalgebra of U generated by E and K. Of course, there is a similar theory based on F and K and another braiding. The constructions of section 6 show that the universal twist is determined by its action on the 2-dimensional module V_1. Hence our main theorem gives all possible universal cylinder twists associated to the given *braided* category U-Int.

8. The structure of the cylinder twist

In this section we study the internal structure of element t_n of the braid group ZB_n, called the cylinder twist. The main result of this section is of a technical nature and gives the eigenspace structure of t_n. This result will be used in the next section in order to derive an algebraic model for the Temperley-Lieb category of type B.

In this section V denotes the U-module V_1. The cylinder twist $t_n \in ZB_n$ and the elements $t(j)$ were already defined. The elements $t(j)$ pairwise commute.

We study the eigenspace structure of the cylinder twist t_n on $V^{\otimes n}$ based on the tensor representation with the matrices $g(p) = g$ and $t(1,1,\theta) = t$.

$$t = t(\rho) = \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}, \qquad g = g(p) = \begin{pmatrix} p & & & \\ & 0 & p^{-1} & \\ & p^{-1} & p - p^3 & \\ & & & p \end{pmatrix}$$

with $\theta = \rho - \rho^{-1}$, $p^2 = q$, $\delta = q - q^{-1}$. We only consider the generic case that the $\rho^a q^b$, $(a,b) \in \mathbb{Z}^2$ are pairwise different. We use the basis $v_{-1} = x_0$, $v_1 = x_1$ of V and have written g in the antilexicographical basis. We will construct in the generic case 2^n eigenvectors of t_n and compute its eigenvalues.

We need some notation in order to state the result. Let $P(n)$ be the set of all functions $\{1, 2, \dots, n\} \to \{\pm 1\}$. We associate to $e \in P(n)$ another function $e^* \in P(n)$ defined by

$$e^*(j) = \prod_{k=1}^{j} e(k).$$

The assignment $e \mapsto e^*$ is a bijection of $P(n)$. For $e \in P(n)$ we denote by $e' \in P(n-1)$ the restriction of e to $\{1, \dots, n-1\}$.

Given $e \in P(n)$, we define inductively

$$\lambda(e) = \alpha(e) q^{\beta(e)} \rho^{\gamma(e)}, \quad \alpha(e) \in \{\pm 1\}, \quad \beta(e) \in \mathbb{Z}, \quad \gamma(e) \in \mathbb{Z}$$

as follows: For $e \in P(1)$ we set

$$\alpha(e) = e(1), \quad \beta(e) = 0, \quad \gamma(e) = e(1),$$

i. e. $\lambda(e) = \rho$ in case $e(1) = 1$ and $\lambda(e) = -\rho^{-1}$ in case $e(1) = -1$. For $e \in P(n)$, $n > 1$, we set

$$\lambda(e) = e(n)(q\lambda(e'))^{e(n)}$$

hence

$$\begin{aligned} \alpha(e) &= e(n)\alpha(e') \\ \beta(e) &= e(n)(\beta(e') + 1) \\ \gamma(e) &= e(n)\gamma(e'). \end{aligned}$$

This recursive definition yields

$$\alpha(e) = e^*(n), \quad \gamma(e) = e^*(n), \quad \beta(e) = e^*(n) \cdot \sum_{j=1}^{n-1} e^*(j) \quad n > 1.$$

We define inductively $x(e) \in V^{\otimes n}$ by

$$x(e) = x(e') \otimes (v_{-1} + \lambda(e)v_1), \qquad n \geq 1$$

(in case $n = 1$ the term $x(e')$ does not appear).

(8.1) Theorem. *The $\mathbb{Z}^n$-module $V^{\otimes n}$ decomposes into 2^n pairwise different one-dimensional modules. The vectors $x(e)$ are eigenvectors of $t(n)$ with eigenvalue $\lambda(e)$ and simultaneous eigenvectors for the $\mathbb{Z}^n$-action.*

We set

$$k_e = |\{j \mid e^*(j) = 1\}|, \qquad \ell_e = 2k_e - n.$$

Then we have:

(8.2) Theorem. *The vector $x(e)$ is an eigenvector of t_n with eigenvalue*

$$\mu(e) = (-1)^{n-k_e} \rho^{\ell_e} p^{\ell_e^2 - n}.$$

From (5.2) be see that t_n has $n+1$ different eigenvalues (generic case), namely according to the value of k_e. The module $V^{\otimes n}$ decomposes into $n+1$ irreducible ZB_n-modules $M_j(n)$. The element t_n is contained in the center of ZB_n. Thus t_n acts as a scalar on $M_j(n)$. The eigenspaces of t_n are the modules $M_j(n)$. The dimension of $M_j(n)$ is $\binom{n}{j}$. There are $\binom{n}{j}$ functions $e \in P(n)$ with $k_e = j$. We choose the indexing such that $M_j(n)$ belongs to $k_e = j$. See [**??**] for more information.

Proof of (5.1). Induct over n. A simple computation shows that $v_{-1} + \rho v_1$ and $v_{-1} - \rho^{-1} v_1$ are eigenvectors of $t(\rho)$ with eigenvalues ρ and $-\rho^{-1}$, respectively. We also need an explicit computation in the case $n = 2$. The operator $t(2)$ has in the basis $v_{-1} \otimes v_{-1}$, $v_1 \otimes v_{-1}$, $v_{-1} \otimes v_1$, $v_1 \otimes v_1$ the matrix

$$t(2, \rho) = t(2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \theta q^{-1} & \delta \\ 0 & 1 & \delta & \theta q \end{pmatrix}.$$

A direct computation gives the following eigenvectors and eigenvalues in accordance with (5.1).

eigenvector			eigenvalue
$(v_{-1} + \rho v_1)$	$\otimes$	$(v_{-1} + q\rho v_1)$	$q\rho$
$(v_{-1} + \rho v_1)$	$\otimes$	$(v_{-1} - q^{-1}\rho^{-1} v_1)$	$-q^{-1}\rho^{-1}$
$(v_{-1} - \rho^{-1} v_1)$	$\otimes$	$(v_{-1} - q\rho^{-1} v_1)$	$-q\rho^{-1}$
$(v_{-1} - \rho^{-1} v_1)$	$\otimes$	$(v_{-1} + q^{-1}\rho v_1)$	$q^{-1}\rho$

For the induction step we decompose $V^{\otimes n} = V^{\otimes(n-2)} \otimes V^2$ and use the defining relation

$$t(n) = (1_{n-2} \otimes g)(t(n-1) \otimes 1_1)(1_{n-2} \otimes g).$$

The morphisms $t(n-j)\otimes 1_j$, $0\le j\le n-1$ pairwise commute. Thus, for each simultaneous eigenvector $u\in V^{\otimes(n-1)}$ of the $t(j)$, $1\le j\le n-1$, the subspace $u\otimes V$ is $t(n)$-stable. The reason is that in the generic case the simultaneous eigenspaces have multiplicity one, as follows easily by induction. By induction, we assume that u has the form $w\otimes z$, $w\in V^{\otimes(n-2)}$, $z\in V$. By induction again, the map $t(n-1)$ acts on $w\otimes V$ with eigenvectors of the form

$$w\otimes(v_{-1}+\lambda v_1),\quad w\otimes(v_{-1}-\lambda^{-1}v_1),$$

i. e. as $t(\lambda)$ in the basis $w\otimes v_{-1}, w\otimes v_1$. Therefore $t(n)$ acts on $w\otimes V\otimes V$ as $t(2,\lambda)$ with the following eigenvectors and eigenvalues.

eigenvector	eigenvalue
$w\otimes(v_{-1}+\lambda v_1)\otimes(v_{-1}+q\lambda v_1)$	$q\lambda$
$w\otimes(v_{-1}+\lambda v_1)\otimes(v_{-1}-q^{-1}\lambda^{-1}v_1)$	$-q^{-1}\lambda^{-1}$
$w\otimes(v_{-1}-\lambda^{-1}v_1)\otimes(v_{-1}-q\lambda^{-1}v_1)$	$-q\lambda^{-1}$
$w\otimes(v_{-1}-\lambda^{-1}v_1)\otimes(v_{-1}+q^{-1}\lambda v_1)$	$q^{-1}\lambda$

This gives the induction step. □

Proof of (5.2). By definition $t_n=t(1)t(2)\cdots t(n)$, where $t(j)$ also denotes the action $t(j)\otimes 1_{n-j}$ on $V^{\otimes j}\otimes V^{\otimes(n-j)}$. Inductively we see that the $x(e)$ are eigenvectors of each $t(j)$. The eigenvalue belongs to $e|\{1,\dots,j\}=:e^{(j)}$. We have to multiply the eigenvalues in order to obtain the eigenvalue $\mu(e)$ of t_n

$$\begin{aligned}\mu(e) &= q^{b(e)}\cdot\rho^{c(e)}\cdot\prod_{j=1}^n e^*(j)\\ b(e) &= \sum_{j=1}^n \beta(e^{(j)})\\ c(e) &= \sum_{j=1}^n \gamma(e^{(j)}).\end{aligned}$$

With our definition of $k_e=k$ we have $c(e)=2k-n$. We note that $b(e)$ is the second elementary symmetric function $\sum_{i<j}e^*(i)e^*(j)$. In order to compute it, we determine the coefficient of x^{n-2} in

$$\prod_{j=1}^n(x-e^*(j))=(x-1)^k(x+1)^{n-k}.$$

We obtain

$$-k(n-k)+\binom{k}{2}+\binom{n-k}{2}=\frac{1}{2}(\ell^2-n).$$

Similarly, by considering the constant term,

$$\prod_{j=1}^n e^*(j)=(-1)^{n-k}.$$

This proves (5.2). □

The preceding results will also be used to obtain information about the eigenspace structure of the cylinder twist on the irreducible U-modules V_n.

The proof of (5.1) also yields the following result.

(8.3) Theorem. *Let $x(e) \in V^{\otimes m}$ be as above and $v \in V^{\otimes n}$. Then*

$$t_{m+n}(\rho)(x(e) \otimes v) = \mu(e)x(e) \otimes t_n(q\lambda(e))(v).$$

There are reasons [**??**] to consider instead of $t(\rho)$ matrices and the cylinder twist based on $t(\alpha, \beta, \theta)$. We can reduce formally to the previously considered case $\alpha = \beta = 1$ as follows. Let D be the diagonal matrix $\mathrm{Dia}(\lambda_1, \lambda_2)$. Then $D \otimes D$ commutes with g. We can therefore make the basis change with D. This leads to $t(\lambda_1\lambda_1^{-1}\alpha, \lambda_1\lambda_2^{-1}\beta, \theta)$. Thus set $\mu = \lambda_2\lambda_1^{-1}$ and determine μ by $\mu^2 = \beta\alpha^{-1}$. Then we are reduced to $t(\gamma, \gamma, \theta), \ \ \gamma = \mu\alpha = \mu^{-1}\beta$. Finally, consider $\gamma^{-1}t(\gamma, \gamma, \theta)$.

We write $t(n, \rho)$ for the map $t(n)$ in order to show its dependend on ρ. The maps $t(j)$ commute. Let $W \subset V^{\otimes m}$ denote an eigenspace of $t(m)$. Then the subspace $W \otimes V^{\otimes n} \subset V^{\otimes(m+n)}$ is $t(m+n)$-stable.

The following special case is used in the next section. Suppose $\alpha\beta = -q$ and $\theta = ip(\rho - \rho^{-1})$. In that case the eigenvalues of t_n in (5.2) have to be multiplied by $(ip)^n$. If we further specialize to the setting of section 4, then $\rho = -ip = -ia$ and the eigenvalues become

(8.4) $$p^{l(l+1)}, \qquad l = 2k - n, \, 0 \le k \le n.$$

These eigenvalues are still pairwise different.

The vectors $x(e)$ are formally definable with suitable parameters q, ρ in an integral domain $\mathfrak{K}$. We use on $V^{\otimes n}$ a symmetric bilinear form which makes the $v_{e(1)} \otimes \cdots \otimes v_{e(n)} =: v_e$, $e \in P(n)$ into an orthonormal basis. Then we have:

(8.5) Theorem. *The vectors $x(e)$ are pairwise orthogonal. They are a basis of $V^{\otimes n}$ provided*

$$(1 + \rho^2) \prod_{j=1}^{n-1} (1 + q^{2j}\rho^2)(1 + q^{2j}\rho^{-2})$$

is invertible in $\mathfrak{K}$.

Proof of (5.3). Induct over n. The vectors $(1, \rho)$ and $(1, -\rho^{-1})$ are orthogonal. Set

$$x(e) = \sum_{f \in P(n)} \lambda(e, f)v_f$$

with $v_f = v_{f(1)} \otimes \cdots \otimes v_{f(n)}$. In the induction step we have to consider two vectors of the form

$$x(e_1) \otimes (v_{-1} + \lambda_1 v_1), \qquad x(e_2) \otimes (v_{-1} + \lambda_2 v_1).$$

We have

$$\sum_f \lambda(e_1, f)\lambda(e_2, f) + \lambda_1\lambda_2 \sum_f \lambda(e_1, f)\lambda(e_2, f) = 0,$$

since, by induction, the first sum ist zero. For the second assertion we have to study the transition matrix from the v_e to the $x(e)$. For $n = 1$ we have

$$\begin{vmatrix} 1 & \rho \\ 1 & -\rho^{-1} \end{vmatrix} = -\rho^{-1}(1+\rho^2).$$

We assume inductively that the $x(e)$, $e \in P(n-1)$ are a basis of $V^{\otimes(n-1)}$. Then the $x(e) \otimes v_{-1}$ and $x(e) \otimes v_1$ are a basis of $V^{\otimes n}$. If $x(e)$ has eigenvalue $\lambda(e)$, then the transition matrix to the vectors

$$x(e) \otimes (v_{-1} + q\lambda(e)v_1), \qquad x(e) \otimes (v_{-1} - q^{-1}\lambda(e)^{-1}v_1)$$

consists of 2×2-blocks with determinant

$$\begin{vmatrix} 1 & q\lambda(e) \\ 1 & -q^{-1}\lambda(e) \end{vmatrix} = -q^{-1}\lambda(e)^{-1}(1+q^2\lambda(e)^2).$$

Thus we require that the $1 + q^2\lambda(e)^2$ be invertible. We have $\lambda(e)^2$ of the form $q^{2\alpha}\rho^{\pm 2}$. Without essential restriction we can assume $\alpha \geq 0$. Thus the invertibility of the product in (5.3) suffices. □

In the last proof we have assumed that we have a basis for $V^{\otimes 1}, \dots, V^{\otimes n}$. But the transition determinant for $V^{\otimes j}$ is a factor of the determinant for $V^{\otimes(j+1)}$.

9. The cylinder twist on irreducible modules

We consider the representation of the braid groups ZB_n on $V^{\otimes n}$ given by the four braid pair (X, F) with the standard R-matrix g from section 5 and

$$t = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix}, \quad \alpha\beta = -q.$$

The cylinder twist t_n is compatible with the Clebsch-Gordan decomposition and induces on the unique irreducible component $V_n \subset V^{\otimes n}$ a morphism τ_n. A matrix $(F_{k,\ell})$ for τ_n in the standard basis $x_0, \dots, x_n$ of the U-module V_n was computed in [**??**]. The result is

$$F_{k,\ell} = \alpha^k \beta^{n-k} q^{k(n-k)} \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j$$

with $k + \ell = n + j$. These entries are zero for $j < 0$. The γ_j are polynomials determined by the recursion relation $\gamma_{-1} = 0$, $\gamma_0 = 1$ and

$$\alpha\gamma_{k+1} = q^k\theta\gamma_k + \beta q^{k-1}(q^k - q^{-k})\gamma_{k-1}$$

for $k > 0$.

We want to work with symmetric matrices. For this purpose we make the following assumptions about the ground field $\mathfrak{K}$. It has characteristic zero and q is transcendental over $\mathbb{Q}$. We assume given square roots

$$p^2 = q, \quad \gamma^2 = \alpha\beta = -q, \quad \sigma^2 = \alpha\beta^{-1}.$$

Then there exists $\varepsilon = \pm 1$ such that $\sigma\beta = \varepsilon\gamma$. We set $\alpha(\ell) = \sigma^{-\ell}$. Then $\alpha(\ell)\alpha(k)^{-1}\alpha^\ell\beta^{n-\ell} = \varepsilon^n\sigma^j\gamma^n$. We assume given square roots $\sqrt{[n]}$ of the quantum numbers and use these to define the square roots of the quantum binomial coefficients

$$[n]!^{1/2} = [1]^{1/2}[2]^{1/2}\cdots[n]^{1/2}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}^{1/2} = \frac{[n]!^{1/2}}{[k]!^{1/2}[n-k]!^{1/2}}.$$

We choose the basis $z_0, \dots, z_n$ defined by

$$x_k = \sigma^{-k}p^{-k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}^{1/2} z_k.$$

In this basis, the operator τ_n has the symmetric matrix

(9.1) $$F_{k,\ell} = \varepsilon^n\sigma^j\gamma^n p^{k(n-k)+\ell(n-\ell)}\begin{bmatrix} k \\ j \end{bmatrix}^{1/2}\begin{bmatrix} \ell \\ j \end{bmatrix}^{1/2}\gamma_j$$

with $k+\ell = n+j$. In this basis also the R-matrix (= braiding) on $V_n \otimes V_n$, obtained from the universal R-matrix, has a symmetric matrix. It is independent of σ.

Guided by the Kauffman calculus of section 4, we specialize to the case $\theta = q+1 = 1-\gamma^2$. In that case, we use the renormalized polynomials β_k defined by $\gamma_k = \sigma^{-k}p^{k(k-1)}\beta_k$. They satisfy the recursion relation $\beta_{k+1} = (\gamma^{-1}-\gamma)\beta_k + (1-q^{-2k})\beta_{k-1}$.

(9.2) Proposition. *The β-polynomials have the following product decomposition*

$$\beta_k = (-\gamma)^k\prod_{j=1}^{k}(1+q^{-j}).$$

Proof. By definition, $\beta_{-1} = 0$ and $\beta_0 = 1$. We verify that the right hand side satisfies the recursion formula for the β_k

$$\begin{aligned}
& (\gamma^{-1}-\gamma)(-\gamma)^k\prod_{j=1}^{n}(1+q^{-k}) + (1-q^{-2k})(-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= {} & \left[(\gamma^{-1}-\gamma)(-\gamma)(1+q^{-k}) + (1-q^{-2k})\right](-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= {} & (1+q^{-k})\left[-1-q+1-q^{-k}\right](-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= {} & (-\gamma)^2(1+q^{-k})(1+q^{-k-1})(-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= {} & \beta_{k+1}.
\end{aligned}$$

This finishes the proof by induction. □

The quantum binomial coefficients are Laurent polynomials in q. In the sequel we have to use the same polynomials with q replaced by p. We use the notation $\begin{bmatrix} n \\ k \end{bmatrix}_p$ for these binomial coefficients. The following Theorem will be used in the next section.

(9.3) Theorem. *The vector*

$$\varepsilon^n \sum_{k=0}^{n} \gamma^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2} z_k$$

is an eigenvector of the matrix (7.1) *for the eigenvalue* 1.

Proof. By matrix multiplication, the claim is equivalent to the identities

$$\gamma^n \sum_{j=0}^{k} F_{k,n-k+j} \gamma^{k-j-n} \begin{bmatrix} n \\ k-j \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2} = \gamma^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2}.$$

We insert the value (7.1) and see that these identities are equivalent to

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} n-k+j \\ j \end{bmatrix} \begin{bmatrix} n \\ n-k+j \end{bmatrix}_p p^{\bullet} \prod_{\nu=1}^{j} (1+q^{-\nu}) = (-1)^k q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_p$$

with $\bullet = k(n-k) + \ell(n-\ell) + j(j-1)$ and $\ell = n-k+j$. We set

$$\prod_{\nu=1}^{j} (1+q^{-\nu}) = p^{-j(j+1)/2} \pi_j \quad \text{with} \quad \pi_j = \prod_{\nu=1}^{j} (p^j + p^{-j}).$$

We compute

$$\begin{bmatrix} n-k+j \\ j \end{bmatrix} \begin{bmatrix} n \\ n-k+j \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_p^{-1} = \frac{\pi_{n-k+j}}{\pi_j \pi_{n-k}} \begin{bmatrix} k \\ j \end{bmatrix}_p$$

and use this to put the claimed identities into the form ($t = n-k$)

$$\sum_{j=0}^{k} (-1)^j p^{\#} \begin{bmatrix} k \\ j \end{bmatrix}_p \pi_{t+j} = (-1)^k q^{-k(t+1)} \pi_t$$

with $\# = j(k-t-1) - j(j+1)/2$.

It is now possible to verify these identities by induction over k. For $k=0$ it reduces to $\pi_t = \pi_t$ and thus holds for all t.

We rewrite the left hand side of the identity in question for $k+1$ by using the Pascal formula

$$\begin{bmatrix} k+1 \\ j \end{bmatrix}_p = p^{-j} \begin{bmatrix} k \\ j \end{bmatrix}_p + p^{j-k+1} \begin{bmatrix} k \\ j-1 \end{bmatrix}_p.$$

We obtain

$$\sum_{j=0}^{k}(-1)^j p^{j(k-t)-j(j+3)/2}\left[\begin{array}{c} k \\ j \end{array}\right]\pi_{t+j}+\sum_{j=1}^{k+1}(-1)^j p^{j(k-t)-j(j-1)/2-k+1}\left[\begin{array}{c} k \\ j-1 \end{array}\right]\pi_{t+j}.$$

By induction, the first sum equals $(-1)^k q^{-k(t+1)}\pi_t$. In the second sum we replace j by $j-1$. Then we see, again by induction, that it equals

$$-q^{-k(t+1)+k}p^{-t-1}\pi_{t+1} = -q^{-k(t+1)}(1+q^{-t-1})(-1)^k\pi_t.$$

Altogether we obtain the correct result. □

We symmetrize the vector (7.3). Suppose $\bar{\gamma}^2=\gamma$. We use

$$\kappa_n = \sum_{k=0}^{n}\bar{\gamma}^{n-2k}\left[\begin{array}{c} n \\ k \end{array}\right]_p \left[\begin{array}{c} n \\ k \end{array}\right]^{-1/2} z_k.$$

It is sensible to consider (z_k) as an orthogonal basis of V_n. In that case the norm-square of κ_k is

$$A(n)=\sum_{k=0}^{n}\gamma^{n-2k}\left[\begin{array}{c} n \\ k \end{array}\right]_p^2\left[\begin{array}{c} n \\ k \end{array}\right]^{-1} = \pi_n^{-1}\sum_{k=0}^{n}\gamma^{n-2k}\pi_j\pi_{n-j}\left[\begin{array}{c} n \\ k \end{array}\right]_p.$$

It turns out that $A(n)$ also has the following product decomposition.

(9.4) Theorem. *The following identity holds for $n \geq 1$*

$$[2]_p\sum_{k=0}^{n}\gamma^{n-2k}\pi_j\pi_{n-j}\left[\begin{array}{c} n \\ k \end{array}\right]_p = (\gamma+\gamma^{-1})^n[n+1]_p^!.$$

A direct verification of this identity does not seem easy. We shall obtain it at the end of the next section from the representation theory of the Temperley-Lieb category and the structure of the Jones-Wenzl idempotent. For later use we mention already at this point a q-analogue of a well known formula

(9.5) $$\sum_{j=1}^{n}[j] = \left[\begin{array}{c} n+1 \\ 2 \end{array}\right]_p.$$

There is no difficulty to prove this by induction.

By (7.4), we can also write

$$A(n)=\frac{(\gamma+\gamma^{-1})^n}{[n]!}\prod_{j=1}^{n}\left[\begin{array}{c} j+1 \\ 2 \end{array}\right]_p.$$

6 Categories of ribbons

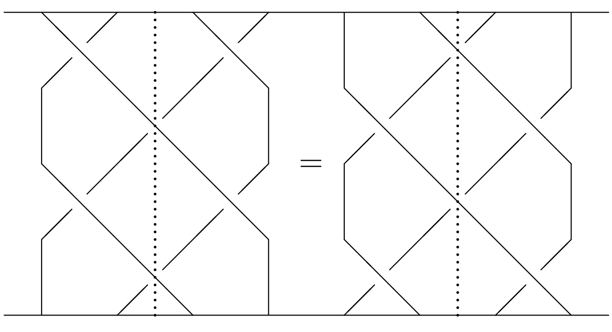
1. Categories of ribbons

We recall some standard notions about tangles, ribbons, and the associated tensor categories. By analogy, we transport these notions to objects in the cylinder. We introduce two versions of a graphical calculus for cylinder ribbons. New phenomena are related to the axis of the cylinder. As new tools we introduce the so called "rooted ribbons", ribbons which may end on the axis. Using the notions of tensor categories and tensor module categories, we describe (without proof) some of the ribbon categories by generators and relations (in the sense of [**??**] or [**??**]).

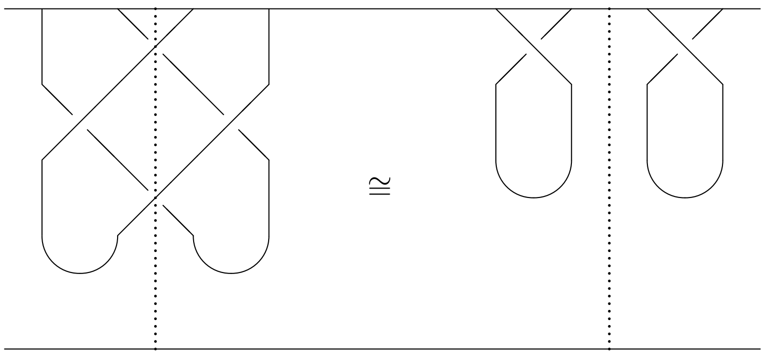
A (k,l)-tangle is (a smooth isotopy class relative to the boundary of) a compact one-dimensional submanifold of $\mathbb{C}\times[0,1]$ such that the set of its boundary points is $\{1,\ldots,k\}\times 0\cup\{1,\ldots,l\}\times 1$. A ribbon is a tangle with a normal framing; the framing vector at the boundary always points to $-\infty$. In the graphical calculus, a tangle is represented by a generic immersion of a one-manifold into the strip $]0,\infty[\,\times[0,1]$ together with overcrossing information at the double points. (For our applications it is convenient not to use immersions into $\mathbb{R}\times[0,1]$.) The tangles or ribbons and their graphical analogues form a tensor category. Objects are the natural numbers and the morphisms from k to l are the (k,l)-tangles or ribbons. There are oriented versions. For simplicity we mainly work with unoriented objects in this paper. We refer to Turaev [**??**] and [**??**, Ch.I] for detailed back ground information about these tensor categories and their presentation by generators and relations. The graphical category of ribbons will be denoted by RA.

In this paper, we are concerned with tangles and ribbons in the cylinder $\mathbb{C}^*\times[0,1]$. The definition of tangles and ribbons is completely analogous to the ordinary case recalled above. The only difference is that now everything takes place in the cylinder $\mathbb{C}^*\times[0,1]$.

We use two versions of a graphical calculus for cylinder ribbons. The first one is based on generic immersions into $\mathbb{R}\times[0,1]$ which are symmetric with respect to the axis $0\times[0,1]$. This setting was already used in [**??**]. There are two additional Reidemeister type moves. They are represented graphically as follows (the axis is dotted).

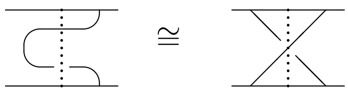


This first relation is called the *four braid relation*. In order to understand the twist in the right part of the next figure observe that the untwisting of the left part is by a rotation about 180°.



The second relation also has an upside-down version. We call these the *untwist relation*.

Because of the $\mathbb{Z}/2$-symmetry of the figures it suffices to consider essentially the part in $[0,\infty[\times[0,1]$. This leads to the second version of the graphical calculus (compare [**??**]). A symmetric crossing of the axis will then be represented by the left part of the next figure.



The first version is obtained from the second one by taking the two-fold covering ramified along the axis. One could also pass to the universal covering of the cylinder; this would yield infinite but periodic tangles.

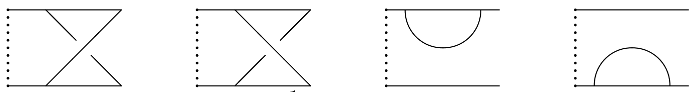
The trefoil has a symmetric picture with three crossings on the axis. In the second version this becomes an unknotted circle which winds two times about the axis. The symmetric Hopf link corresponds to an unknotted circle which winds about the axis just once. The figure eight knot has a symmetric representative with the axis passing the knot twice. This is not allowed at present but later when we consider rooted tangles.

The category of these cylinder ribbons will be denoted by RB. The letters A and B in RA, RB refer to Coxeter graphs of type A, B. The reason is that the corresponding braid groups are part of these ribbon categories.

We can place an ordinary graph to the right of a cylinder ribbon graph without producing new double points. This process makes RB into a tensor module category over RA; see the formal definitions in the next section. Actually, by placing one cylinder into a second one we can make RB into a tensor category. It turns out that this is not suitable for our purposes. Again there are oriented versions. The natural framing of the strand in version 2 above (intendend in the drawing) is the one where the normal vector always points to the axis. If we intend to draw this with the black board framing, then we have to add a twist. But the natural framing of the components which do not touch the axis is the black board framing. This convention has to be kept in mind when the figures in this text are interpreted.

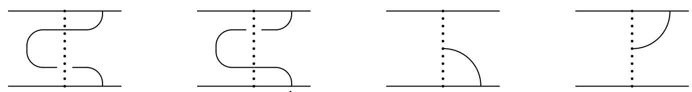
If one wants to develop skein invariants for cylinder ribbons (as we will do), one is led to consider more general ribbons. They will be called *rooted cylinder ribbons*. These are framed tangles represented by embeddings of compact one-manifolds in $\mathbb{C} \times [0,1]$ where the circle components are contained in $\mathbb{C}^* \times]0,1[$ and the interval components have their boundary points in $]0,\infty[\times\{0,1\} \cup \{0\}\times]0,1[$. Thus, some components may have one or two boundary points on the interior of the axis. An isotopy is allowed to move the points on the axis, the isotopy respects the axis setwise but not pointwise. The graphical calculus uses immersions into $[0,\infty[\times]0,1[$, except that there may be some crossings of the axis as for RB. Let RRB denote the graphical category of rooted cylinder ribbons. Again this is a tensor module category over RA. We point out that the objects of the categories under consideration are the natural numbers $n \in \mathbb{N}_0$ and a morphism from k to l is a ribbon graph Γ with $\Gamma \cap (\mathbb{R} \times [0,1]) = \{1,\dots,k\} \times 0 \cup \{1,\dots,l\} \times 1$. The symbol 1_n denotes the identity of the object n of RRB.

The basis of this paper is the description of RRB as a tensor module category over RA by generators and relations. The generators of RA are



X $\qquad X^{-1}$ $\qquad k$ $\qquad f$

and the additional generators of RRB are



F $\qquad F^{-1}$ $\qquad \kappa$ $\qquad \varphi$

For RB one only needs the additional generators F, F^{-1}. The relations for the generators of RA are known [**??**, **??**]. The following version suffices for the unoriented category.

(1.1) Relations for RA.

(1) $XX^{-1} = 1_2 = X^{-1}X$

(2) $(X \otimes 1_1)(1_1 \otimes X)(X \otimes 1_1) = (1_1 \otimes X)(X \otimes 1_1)(1_1 \otimes X)$

(3) $(1_1 \otimes f)(k \otimes 1_1) = 1_1 = (f \otimes 1_1)(1_1 \otimes k)$

(4) $(f \otimes 1_1) = (1_1 \otimes f)(X^{\pm 1} \otimes 1_1)(1_1 \otimes X^{\pm 1})$
$(k \otimes 1_1) = (X^{\pm 1} \otimes 1_1)(1_1 \otimes X^{\pm 1})(1_1 \otimes k)$

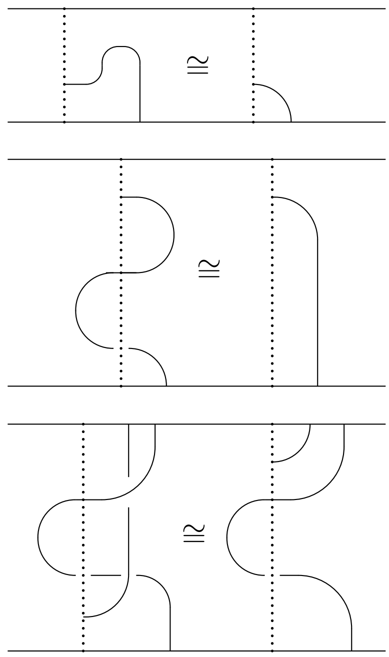
The additional relations involving $F, F^{-1}, \kappa, \varphi$ are as follows. For RB one only needs (1), (2), and (3).

(1.2) Additional relations for RRB.

(1) $FF^{-1} = 1_1 = F^{-1}F$
(2) $X_2 := X(F \otimes 1_1)X(F \otimes 1_1) = (F \otimes 1_1)X(F \otimes 1_1)X$
(3) $X_2 k = k, \quad fX_2 = f$
(4) $(\varphi \otimes 1_1)k = \kappa, \quad f(\kappa \otimes 1_1) = \varphi$
(5) $F\kappa = \kappa, \quad \varphi F = \varphi$
(6) $X(F \otimes 1_1)X(\kappa \otimes 1_1) = (\kappa \otimes 1_1)F, \quad (\varphi \otimes 1_1)X(F \otimes 1_1)X = F(\varphi \otimes 1_1)$

We do not prove in this paper that (1.2) contains a complete set of additional relations since our interest is in algebraic realizations of the relations. We have already illustrated (2) and (3) as the four braid relation and the untwist relation. Here are figures for (4), (5), and (6). There are also upside-down versions.

(4) (5) (6)



We can linearize the categories above. Let $\mathfrak{K}$ be a commutative ring. The morphisms sets are replaced by the free $\mathfrak{K}$-module on the set of morphisms and the composition of morphisms is extended $\mathfrak{K}$-bilinearly.

2. Skein relations

We go on to discuss quotients of the linearized categories by skein relations. New are: Additional relations related to the axis of the cylinder.

A skein invariant for $(0,0)$-ribbons in RRB with values in $\mathfrak{K}$, in the spirit of the Kauffman polynomial [**??**], introduces additional local relations (written in terms of generators) as follows. The symbols $C_1, \dots, C_9$ are suitable parameters in $\mathfrak{K}$; but they cannot be chosen arbitrarily.

(2.1) Skein relations.
(1) $X - X^{-1} = C_1(1_2 - kf)$
(2) $fX = C_2 f, \quad Xk = C_2 k$

(3) $fk = C_3 1_0$

(4) $C_4^{-1}F + C_4 F^{-1} = C_5(\kappa\varphi - 1_1)$

(5) $\varphi\kappa = C_6 1_0$

(6) $C_7((\varphi \otimes 1_1)X(\kappa \otimes 1_1) - \kappa\varphi) = C_8 1_1 + C_9 F$

A version of relation (1.3.6) is due to Häring-Oldenburg [**??**]. The local modifications of the Kauffman polynomial are (1.3.1), (1.3.2), and (1.3.3). One can, of course, contemplate other skein relations; compare sections 4 for an example. We do not consider the general case in this paper. The reader should draw figures to understand the geometric meaning of (1.3). The next Proposition is a justification for (1.3).

(2.2) Proposition. *Relations of the type above suffice to compute the value of a rooted cylinder* $(0,0)$*-ribbon. This value is called a skein invariant.*

PROOF. By the ordinary theory and relation (1.3.6) we can remove all crossings which do not lie on the axis. What remains are circles which wind around the axis, say n times. The relation (1.3.6) is now used to evaluate such ribbons by induction on n, in the presence of the other axioms. The value of $f(F \otimes 1)k$ is $C_9^{-1}C_3((C_2^{-1} - 1)C_7 - C_8)$. □

Coherence of the axioms and geometry tell that the following is a reasonable set of relations between the parameters $C_2 - C_2^{-1} = C_1(1 - C_3), C_4 + C_4^{-1} = C_5(C_6 - 1), C_2C_4^2 = 1, C_5 = C_7, C_8 = C_1C_4, C_9 = -C_1C_4^{-1}$. In section 3 we construct a representation of RRB which yields a skein invariant of RRB with parameters $C_1 = q^2 - q^{-2}$, $C_2 = q^{-4}$, $C_4 = q^2$, $C_5 = \rho^2 + \rho^{-2}$. The other values are given by the relations above. (Here q and ρ are suitable elements in $\mathfrak{K}$.)

Motivated by the geometric examples of the previous section we develop the new notion of a tensor category with cylinder braiding. This is based on an additional structure in a braided tensor category, the cylinder twist. We extend the notion of duality in tensor categories by introducing „rootings“ and „corootings“. In the graphical calculus for tensor categories these use rooted tangles. We mention that the graphical calculus for tensor categories can be extended to these new structures. We also need what we call tensor module categories: Categories with an action of a tensor category. The abstract viewpoint is also helpful in that the geometry does not quite determine what the correct notions should be. (Remark: From a topological viewpoint one would like to replace the cylinder by „surface times interval“. In this more general context, tensor module categories are inadequate or, at least, too special.) As a justification we mention that suitable categories of representations of Lie type quantum groups carry the additional structure of a cylinder braiding (a structure which cannot be seen in the classical limit); see [**??**] [**??**]. It is more or less clear that [**??**, Chapter I] can be extended to our setting.

3. A representation of rooted cylinder ribbons

We construct a tensor module functor from the category of unoriented rooted cylinder ribbons (= a representation of the category). This functor is based on the presentation of the category by generators and relations (section 1) and amounts to finding suitable matrix identities. One basic identity is the four braid relation. New are additional identities required by the categorical viewpoint. The identities can be checked by mere computation. A full conceptual understanding is still missing. At moment it seems like a miracle that the matrices forced on us by the four braid relation satisfy all the other identities.

Let $U = U_q(sl_2)$ be the quantum enveloping algebra over the field $\mathfrak{K}$ of characteristic zero.

Let V_n be the left U-module with basis $x_0, x_1, \dots, x_n$ and action $F(x_i) = [i+1]x_{i+1}$, $E(x_i) = [n-i+1]x_{i-1}$, $Kx_i = q^{n-2i}x_i$. Here $[k] = (q^k - q^{-k})/(q-q^{-1})$. The universal R-matrix is the operator

$$\mathcal{R} = q^{H\otimes H/2} \sum_{n\geq 0} q^{n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!} F^n \otimes E^n$$

on finite-dimensional U-modules. It makes the category of these modules into a braided tensor category. The operator $q^{H\otimes H/2}$ acts on the tensor product of weight spaces $M^m \otimes N^n$ as multiplication by $q^{mn/2}$. (The weight space M^m of M is the K-eigenspace of M for the eigenvalue q^m.)

The representation of the category RRB is based on the module $V = V_2$. We use the basis $w_0 = x_0$, $w_1 = (1+q^{-2})^{1/2}x_1$, $w_2 = x_2$. In general, we use on $V \otimes W$ the antilexicographical basis in order to display matrices. The universal R-matrix then gives the R-matrix $X = X_3$ of the introduction on $V\otimes V$ in the antilexicographical basis. From the properties of the universal R-matrix we see that X is a Yang-Baxter operator, i. e. it satisfies (1.1.2). It consists of blocks of size 1, 2, and 3 and satisfies the characteristic equation $(X-q^{-4})(X-q^2)(X+q^{-2}) = 0$. An eigenvector for the eigenvalue q^{-4} is $(0,0,-q,0,1,0,-q^{-1},0,0)$. We therefore define a linear map $f\colon V\otimes V \to \mathfrak{K}$ with this matrix and a linear map $k\colon \mathfrak{K} \to V\otimes V$ with the transposed matrix ($1 \in \mathfrak{K}$ basis). We have a decomposition $V\otimes V = V_4 \oplus V_2 \oplus V_0$ of U-modules [**??**, VII.7]. Here V_4, V_2, V_0 are the eigenspaces of X for the eigenvalues $q^2, -q^{-2}, q^{-4}$. In particular, f and k are morphisms of U-modules. They satisfy the duality relations (1.1.3). Since f and k are U-linear, the naturality of the braiding yields the relations (1.1.4). Therefore we see:

(3.1) Proposition. *We obtain a tensor representation of the category* RA *of unoriented ribbon tangles if we map the generators X, f, k to the linear maps above with the same names. This representation leads to the Kauffman polynomial with local modifications* (1.3) *and parameters $C_1 = q^2 - q^{-2}$, $C_2 = q^{-4}$, $C_3 = [3]$.* □

Our aim is to extend this representation to a representation of RRB. Let $F = F_3$ be the 3×3-matrix of the introduction. The inverse of F is obtained from F by reflection in the codiagonal and $p \mapsto p^{-1}$, $\rho \mapsto \rho^{-1}$. The matrix F

satisfies the equation $(F-1)(F+q^2\rho^2)(F+q^2\rho^{-2})=0$. We mentioned already that (X,F) is a four braid pair, i. e. (1.2.2) holds. We point out that F is not an endomorphism of the U-module $V=V_2$.

An eigenvector of F for the eigenvalue 1 is $(p\omega,\theta,-p^{-1}\omega)$. We therefore define linear maps $\kappa\colon \mathfrak{K}\to V$ and $\varphi\colon V\to\mathfrak{K}$ with this matrix divided by $\sqrt{\rho^2+\rho^{-2}}$ and its transpose. These normalizations yield the identities (1.2.5).

One verifies the duality relations (1.2.3), (1.2.4), and (1.2.6). As an aid for computations we display the matrix of XYX. For its general structure see [**??**]. We use the abbreviation $\gamma=1-q^2-q^2\theta^2$. The four braid relation is equivalent to the fact that each block in the following matrix commutes with F.

$$XYX=\begin{pmatrix} 0 & 0 & -qI\\ 0 & -q^2I & M\\ -qI & M & N\end{pmatrix}$$

with

$$M=\begin{pmatrix} -p^{-1}\omega\theta & -q\delta^* & 0\\ -q\delta^* & -p^3\omega\theta & -q^2\delta^*\\ 0 & -q^2\delta^* & -p^7\omega\theta\end{pmatrix}$$

$$N=\begin{pmatrix} q^{-4}\gamma-q^2\lambda^2 & -p^3\lambda\omega\theta & -q^3\mu\\ -p^3\lambda\omega\theta & \gamma-q^2\delta^{*2} & -p^7\omega\theta\delta^*\\ -q^3\mu & -p^7\omega\theta\delta^* & q^4\gamma\end{pmatrix}.$$

Using this matrix the reader can verify (1.2.6). One can also use this matrix to verify (1.2.3) in the form $XYXk=Y^{-1}k$. These relations are quite unlikely from the computational point of view. Therefore we explain the structure of this result in a moment. If we collect the results obtained so far we see:

(3.2) Proposition. *If we map the generators F, κ, and φ to the linear maps with the same name we obtain a tensor representation Φ of* RRB *into the category of $\mathfrak{K}$-vector spaces which extends the representation* (3.1). □

Finally, we explain the skein relations. We define a matrix $\mathcal{E}$ by $(\rho^2+\rho^{-2})(\mathcal{E}-I)=q^{-2}F+q^2F^{-1}$, see (1.3.4). Then

$$\textbf{(3.3)}\qquad (\rho^2+\rho^{-2})\mathcal{E}=\begin{pmatrix} q[2] & p\omega\theta & -[2]\\ p\omega\theta & \theta^2 & -p^{-1}\omega\theta\\ -[2] & -p^{-1}\omega\theta & q^{-1}[2]\end{pmatrix}.$$

We have the following identities, in particular (1.3.5),

$$\textbf{(3.4)}\qquad \mathcal{E}=\kappa\varphi,\quad \varphi\kappa=\left(\frac{q^2+q^{-2}}{\rho^2+\rho^{-2}}+1\right)1_0,\quad \mathcal{E}F=F\mathcal{E}=\mathcal{E}.$$

One computes that the operator $(\rho^2+\rho^{-2})(\varphi\otimes 1)X(\kappa\otimes 1)$ has the following matrix

$$Z=\begin{pmatrix} q^3[2] & p\omega\theta & -q^{-2}[2]\\ p\omega\theta & q[2]\delta^*+\theta^2 & p\omega\lambda\theta-p^{-1}w\theta\\ -q^{-2}[2] & p\omega\lambda\theta-p^{-1}\omega\theta & q[2]\mu+\theta^2\delta^*+q[2]\end{pmatrix}.$$

Using this, the following skein relation of type (1.3.6) is easily verified

(3.5) $$(\rho^2 + \rho^{-2})(Z - \mathcal{E}) = (q^4 - 1)I + (q^{-4} - 1)F = \delta^*(q^2 I - q^{-2}F).$$

We mention the following values of the representation

$$\Phi(f(F^{\pm 1} \otimes 1)k) = q^{\pm 4}(1 - \rho^2 - \rho^{-2}).$$

We now give some information about the validity of the relations (1.2.3) and (1.2.6).

The cylinder twist $t_{V \otimes V} = YXYX$ on $V \otimes V$ commutes with X and $Y = F \otimes 1$. Therefore the eigenspaces of X and Y are stable under $t_{V \otimes V}$. Thus, if we consider the eigenspace for the eigenvalue 1 of Y, we see that there exists a linear map $\tilde{F}$ which satisfies

$$t_{V \otimes V}(\kappa \otimes 1) = XYX(\kappa \otimes 1) = (\kappa \otimes 1)\tilde{F}.$$

It is therefore not too surprising that $F = \tilde{F}$ does the job. For completeness we communicate the eigenspace structure. In section 5 we consider eigenspace structures in general.

The first table gives the information for $t_{V \otimes V}$.

Eigenvalue	$q^8\rho^4$	$q^8\rho^{-4}$	1	$-q^2\rho^2$	$-q^2\rho^{-2}$
Multiplicity	1	1	3	2	2
Module	V_1^+	V_1^-	V_3	V_2^+	V_2^-

The third row gives names to the eigenspaces. The decomposition

$$V \otimes V = V_1^+ \otimes V_1^- \oplus V_2^+ \oplus V_2^- \oplus V_3$$

is the decomposition into irreducible representations of the braid group

$$ZB_2 = \langle X, Y \mid XYXY = YXYX \rangle.$$

Although the eigenspaces have multiplicities there is a canonical decomposition into one-dimensional eigenspaces. This comes from the action of $F \otimes 1$. We assume here that F has three different eigenvalues (generic case). The next table gives the eigenvalue of X and Y on the modules above.

	V_1^+	V_1^-	V_3	V_2^+	V_2^-
X	q^2	q^2	$q^{-4}, q^2, -q^{-2}$	$q^2, -q^{-2}$	$q^2, -q^{-2}$
Y	$-q^2\rho^2$	$-q^2\rho^{-2}$	$1, -q^2\rho^2, -q^2\rho^{-2}$	$1, -q^2\rho^2$	$1, -q^2\rho^{-2}$

The spaces V_4, V_2, V_0 of the Clebsch-Gordan decomposition are invariant under $t_{V \otimes V}$. They split into different eigenspaces with eigenvalues as in the following table.

V_4	$q^8\rho^4, q^8\rho^{-4}$, 1, $-q^2\rho^2, -q^2\rho^{-2}$
V_2	1, $-q^2\rho^2, -q^2\rho^{-2}$
V_0	1

We mention the following values of the representation

$$\Phi(f(F^{\pm 1}\otimes 1)k) = q^{\pm 4}(1-\rho^2-\rho^{-2}).$$

Since Φ is a tensor representation it maps the object n to $M = V^{\otimes n}$. The iteration of k and f yield the duality maps $k(n)\colon \mathfrak{K} \to M\otimes M$ and $f(n)\colon M\otimes M \to \mathfrak{K}$. The quantum trace Trq of a $\mathfrak{K}$-linear map $l\colon M \to M$ is defined by the composition

$$f(n)\circ(l\otimes 1)\circ k(n)\colon 1 \mapsto \operatorname{Trq}(l).$$

This trace can be computed as a linear algebra trace Sp. Let $u\colon V \to V$ denote the morphism with diagonal $\mathrm{Dia}(q^{-2},1,q^2)$. Then $\operatorname{Trq}(l) = \operatorname{Sp}(l\circ u^{\otimes n})$. The proof is by linear algebra. Suppose M has basis (v_i) and let in general

$$k(1) = \sum k_{ij} v_i \otimes v_j, \quad f(v_i\otimes v_j) = f_{ij}.$$

Let $u\colon V \to V$ have matrix $(u_{rs}) = (\sum_j f_{rj}k_{sj})$. Then

$$\operatorname{Sp}(f(l\otimes 1)k) = \operatorname{Sp}((l\otimes 1)\circ kf) = \operatorname{Sp}(l\circ u).$$

4. The Kauffman functor

In this section we extend the Kauffman functor (Kauffman bracket) [**??**] to cylinder ribbons. This will be a tensor module functor $\mathcal{K}\colon$ RRB $\to$ TB into the Temperley-Lieb category TB of symmetric bridges.

In the graphical calculus, the Kauffman functor $\mathcal{K}$ is defined by the following local modifications. The parameter a is the usual one for the the Kauffman bracket; the parameters x, y, D have to be determined yet.

(4.1) Local modifications.

(1) $\mathcal{K}(X) = a1_2 + a^{-1}kf$
(2) $\mathcal{K}(fk) = (-a^2-a^{-2})1_0$
(3) $\mathcal{K}(F) = x1_1 + y\kappa\varphi$
(4) $\mathcal{K}(\varphi\kappa) = D1_0$.

The first two are the standard moves in the definition of the Kauffman bracket. The parameters x, y, and D have to be chosen correctly so as to be compatible with the relations of the category RRB.

A computation as in [**??**] shows that the parameters are compatible with the four braid relation (1.1.2) if and only if $x(a^{-2}-1) = yD$. The relation (1.2.5) is satisfied if and only if $x + yD = 1$. These two conditions give

(4.2) $$x = a^2, \quad y = D^{-1}(1-a^2).$$

We assume (4.2) from now on.

One verifies with these parameters that also the relation (1.1.6) holds. Therefore we obtain a well-defined functor RRB $\to$ TB which extends the classical Kauffman functor RA $\to$ TA and is compatible with tensor products.

We now construct a tensor representation of RRB which factors over the Kauffman functor $\mathcal{K}$: RRB $\to$ TB and induces a module-theoretic description of TB (see section 6).

The tensor representation is now based on the fundamental two-dimensional module $V = V_1$ of $U = U_q(sl_2)$. Here $q = a^2$ is again not a root of unity.

We map X to the R-matrix (in the antilexicographical basis) $X_2 = g(a)$ of the introduction. We assume that a square root i of -1 is contained in $\mathfrak{K}$ in order to produce the most symmetric representation. The morphisms

$$f\colon V \otimes V \to \mathfrak{K}, \qquad k\colon \mathfrak{K} \to V \otimes V$$

are defined by the matrix $(0, ia, (ia)^{-1}, 0)$ and its transpose. The following proposition is known. The proof is analogous to the proof for RRB given below in section 6.

(4.3) Proposition. *The values for X, k, f above yield a tensor functor from* RA *into the category of U-modules which factors over the Kauffman functor* $\mathcal{K}$: RA $\to$ TA *and induces a bijection*

$$\mathrm{Hom}_{\mathrm{TA}}([m],[n]) \cong \mathrm{Hom}_U(V^{\otimes m}, V^{\otimes n})$$

for $m, n \in \mathbb{N}_0$. □

We extend this functor as follows. The morphism F is sent to

$$F = \begin{pmatrix} 0 & v \\ u & a^2+1 \end{pmatrix} \quad \text{with} \quad uv = -a^2.$$

We make the choice $u = v = ia$. We define

$$\varphi\colon V \to \mathfrak{K}, \quad \kappa\colon \mathfrak{K} \to V$$

by $\sqrt{\frac{D}{1-a^2}}(ia, 1)$ and its transpose.

(4.4) Proposition. *With the data above the relations* (1.2) *and* (4.1) *hold. Thus we obtain a representation of* RRB *which factors over the Kauffman functor for* RRB*.* □

The Kauffman functor assumes the following values on "the symmetric unknots" (unknotted circles which wind about the axis once):

$$\mathcal{K}(f(F \otimes 1)k) = -a^3(a + a^{-1}), \qquad \mathcal{K}(f(F^{-1} \otimes 1)k) = -a^{-3}(a + a^{-1}).$$

These values differ slightly from those in [**??**], since the category theory dictates a different choice of parameters. By closing ribbons one obtains, as usual, invariants of framed links in the cylinder. The Kauffman calculus uses the parameter D. But as long as we consider invariants of links in RB we don't need D since we can work with the representation of RB obtained from X and F.

Using the representation above, these invariants are obtained as quantum traces in the sense of the theory of tensor categories. The quantum trace can be computed as an ordinary trace in the following manner.

(4.5) Proposition. *Let* u: $V \to V$ *be the morphism with diagonal matrix* $\mathrm{Dia}(-a^{-2}, -a^2)$. *Suppose a framed link* L *is obtained as a closure of an* (n,n)*-ribbon with value* α_L: $V^{\otimes n} \to V^{\otimes n}$ *of the representation. Then the invariant* $\mathcal{K}(L) \in \mathfrak{K}$ *is the ordinary linear algebra trace of the linear map* $\alpha_L \circ u^{\otimes n}$. □

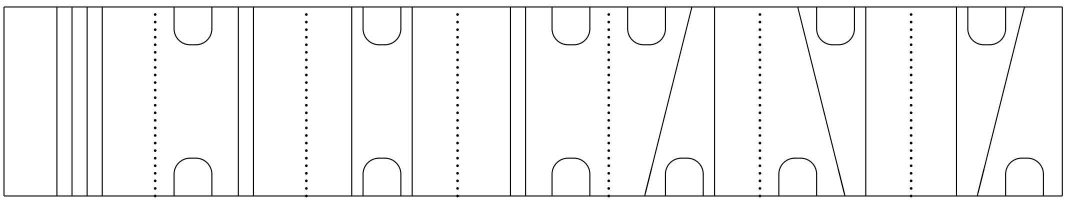
5. Categories of bridges

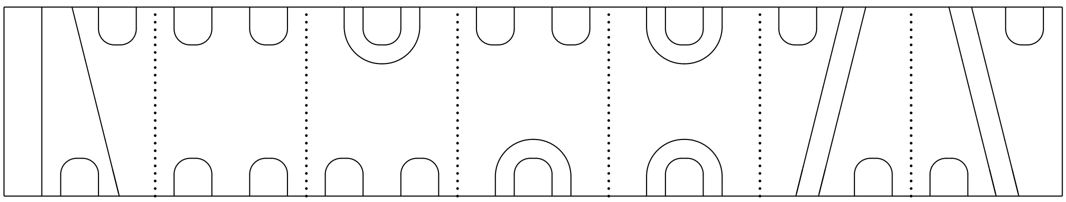
Let $k \in \mathbb{N}_0$ be a natural number. We set $[+k] = \{1, \dots, k\}$. For $k = 0$ this is the empty set. Let $k + l = 2n$. A (k,l)-*bridge* is an isotopy class of n smooth disjoint arcs (= embedded intervals) in the strip $\mathbb{R} \times [0,1]$ with boundary set $P(k,l) := [+k] \times 0 \cup [+l] \times 1$, meeting $\mathbb{R} \times \{0,1\}$ transversely. Isotopy is ambient isotopy relative $\mathbb{R} \times \{0,1\}$. A bridge is a purely combinatorial object: It is uniquely determined by specifying which pairs of $P(k,l)$ are connected by an arc of the bridge. In this sense, a bridge is a free involution of $P(k,l)$ with the additional condition which expresses the disjointness of the arcs. An involution s without fixed points belongs to a bridge if and only if for all pairs $(i, s(i))$ and $(j, s(j))$ the inequality

$$(j - i)(s(j) - i)(j - s(i))(s(j) - s(i)) > 0$$

holds.

We illustrate the 14 $(4,4)$-bridges in the next figure.





We define categories TA and T^oA based on bridges. Let $\mathfrak{K}$ be a commutative ring and $d, d_+ d_- \in \mathfrak{K}$ given parameters. The category TA depends on the choice of d, the category T^oA on the choice of d_+, d_-.

The objcets of TA are the symbols $[+k]$, $k \in \mathbb{N}_0$. The category is a $\mathfrak{K}$-category, i. e. morphism sets are $\mathfrak{K}$-modules and composition is $\mathfrak{K}$-bilinear. The morphism module from $[+k]$ to $[+l]$ is the free $\mathfrak{K}$-module on the set of (k,l)-bridges. This is the zero module for $k+l$ odd. In the case $k=l=0$ we identify the empty bridge with $1 \in \mathfrak{K}$ and thus have $\mathfrak{K}$ as the morphism module.

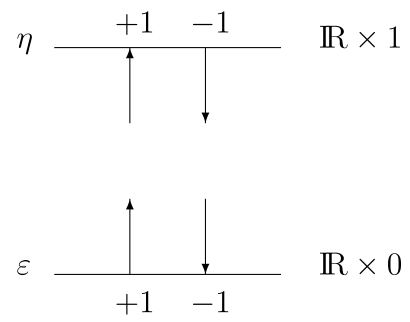
Since composition is assumed to be bilinear, we only have to define the composition of bridges. Let V be a (k,l)-bridge and U be a (l,m)-bridge. We place U on top of V and shrink the resulting figure in $\mathbb{R} \times [0,2]$ to $UV \subset \mathbb{R} \times [0,1]$. The figure UV may contain loops in the interior of $\mathbb{R} \times [0,1]$, say $l(U,V)$ in number. Let $U \wedge V$ be the figure which remains after the loops have been removed. The composition in TA is now defined by

$$U \circ V = d^{l(U,V)} U \wedge V.$$

We make TA into a strict tensor $\mathfrak{K}$-category. We set $[+k] \otimes [+l] := [+(k+l)]$ and place the corresponding bridges next to each other.

The endomorphism algebra $\mathrm{Hom}([+n],[+n])$ of $[+n]$ in TA will be denoted by $T[+n]$ or TA_{n-1}. It is called a *Temperley-Lieb algebra*.

The category $T^o A$ is an oriented version of TA. The objects of $T^o A$ are the functions $\varepsilon\colon [+n] \to \{\pm 1\}$, $n \in \mathbb{N}_0$. We denote such a function also as a sequence $(\varepsilon(1), \varepsilon(2), \ldots)$. Suppose $\varepsilon\colon [+k] \to \{\pm 1\}$ and $\eta\colon [+l] \to \{\pm 1\}$ are given. An (ε,η)-bridge is a (k,l)-bridge with an orientation of each arc such that the orientations match with the signs as indicated in the following figure.



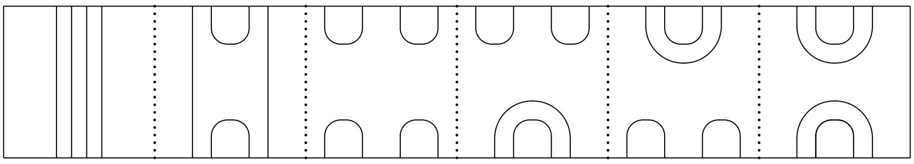
The morphism set $\mathrm{Hom}(\varepsilon,\eta)$ is the free $\mathfrak{K}$-module on the set of (ε,η)-bridges. Composition and tensor product are defined as for TA, but we take orientations of loops into account. Suppose UV contains $l(U,V,+)$ loops with positive orientation and $l(U,V,-)$ with negative orientation. Then we set

$$U \circ V = d_+^{l(U,V,+)} d_-^{l(U,V,-)} U \wedge V.$$

Note that orientations match in UV.

6. Symmetric bridges

We now define a new type of bridges, called *symmetric bridges*. We set $[\pm k] = \{-k, \ldots, -1, 1, \ldots, k\}$. A symmetric (k,l)-bridge is represented by a system of $k+l$ disjoint smooth arcs in the strip $\mathbb{R} \times [0,1]$ with boundary set $Q(k,l) = [\pm k] \times 0 \cup [\pm l] \times 1$ meeting $\mathbb{R} \times \{0,1\}$ transversely, and which has the following equivariance property: If an arc of the bridge connects $(x,\varepsilon) \in Q(k,l)$ with $(y,\eta) \in Q(k,l)$, then there exists another arc of the bridge which connects $(-x,\varepsilon)$ with $(-y,\eta)$. Also in the present situation, two figures which connect the same points define the same bridge, i. e. a symmetric bridge is a free involution of $Q(k,l)$. As an illustration we show the symmetric $(2,2)$-bridges; the symbols underneath will be explained in ??.



1 e_0 e_1 e_0e_1 e_1e_0 $e_0e_1e_0$

The $\mathbb{Z}/2$-equivariance property of symmetric bridges leads to another graphical presentation: Just consider $\mathbb{Z}/2$-orbits. Thus we consider a system of $k+l$ disjoint smooth arcs in the half-strip $[0,\infty[\,\times[0,1]$ with boundary set in $P(k,l)$ together with a certain set in $0\times\,]0,1[$; each arc must have at least one boundary point in $P(k,l)$. The arcs with only one point in $P(k,l)$ are called half-arcs. The location of the boundary points on the axis $0\times[0,1]$ does not belong to the structure of the bridge.

We use this presentation of symmetric bridges when we now define the category TB. The objects of this category are again the symbols $[+k]$, $k \in \mathbb{N}_0$. The category TB is a $\mathfrak{K}$-category. It depends on the choice of two parameters $c, d \in \mathfrak{K}$. The morphism set from $[+k]$ to $[+l]$ is the free $\mathfrak{K}$-module on symmetric (k,l)-bridges. The composition of two such bridges U, V is defined as in the TA-case: Place U above V and get UV. Suppose in UV there are $l(U,V)$ loops and $k(U,V)$ half-loops, the latter being arcs with both boundary points in $0\times[0,1]$. Let $U \wedge V$ denote the bridge which remains after loops and half-loops have been removed. We define

$$U \circ V = c^{k(U,V)} d^{l(U,V)} U \wedge V.$$

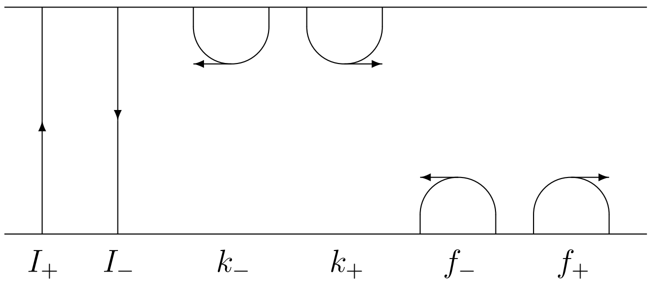
The endomorphism algebra of $[+n]$ in this category is denoted TB_n.

There is again an oriented version T^oB which depends on parameters $c_\pm, d_\pm$. Objects are functions $\varepsilon\colon [+k] \to \{\pm1\}$. The morphism set from ε to η is the free $\mathfrak{K}$-modules on the oriented symmetric (ε,η)-bridges. In order to define the composition we count the positive and negative loops and half-loops and use the parameters d_+, d_-, c_+, c_-, respectively.

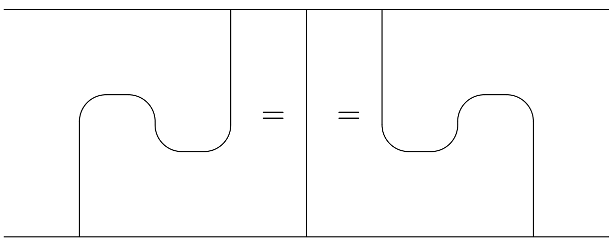
The category TB is a strict tensor module $\mathfrak{K}$-category over TA. The action $*\colon TB \times TA \to TB$ is defined on objects by $[+k] * [+l] = [+(k+l)]$ and on morphisms by placing an A-bridge left to a B-bridge. We think of TA as a subcategory of TB. It is clear that $*$ restricts to $\otimes$. In a similar manner T^oB is a strict tensor module $\mathfrak{K}$-category over T^oA.

7. Presentation of the categories

The categories defined so far have a simple description by generators and relations. The generating process uses category rules and tensor product rules. The generators of T^oA are the following elementary bridges $k_\pm, f_\pm$, together with the identities $I_\pm$.



Similarly, TA has generators I, k, f without orientation. The following figure demonstrates a typical relation between these generators.



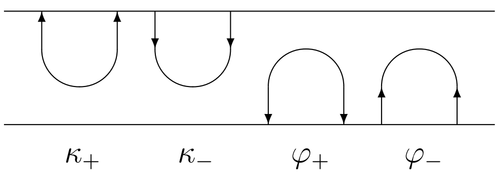
The relations in the case of T^oA are the geometric relations

(7.1) $$(f_\pm \otimes I_\pm)(I_\pm \otimes k_\pm) = I_\pm, \quad (I_\pm \otimes f_\pm)(k_\pm \otimes I_\pm) = I_\pm$$

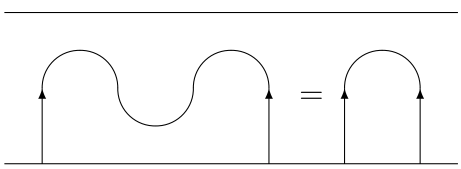
and the algebraic relations

(7.2) $$I_\pm = \mathrm{id} = 1, \quad f_-k_+ = d_+, \quad f_+k_- = d_-.$$

The category T^oB has additional generators as follows (drawn as symmetric bridges with dotted symmety axis)



A typical geometric relation is shown in the next figure.



The relations are

(7.3) $$\kappa_\pm = (\varphi_\pm \otimes I_\pm) \circ k_\pm, \quad \varphi_\pm = f_\pm \circ (\kappa_\pm \otimes I_\mp).$$

(7.4) $$\varphi_- \kappa_+ = c_+, \quad \varphi_+ \kappa_- = c_-.$$

Similar relations without $\pm$-signs hold for TA and TB The geometric definition of the categories contains a positivity: The categories TA, TB, $T^\circ A$, $T^\circ B$ can be defined over $\mathfrak{K} = \mathbb{Z}[d]$, $\mathbb{Z}[c,d]$, $\mathbb{Z}[d_+, d_-]$, $\mathbb{Z}[d_+, d_-, c_+, c_-]$.

We now describe dualities in these categories. Suppose ε: $[+k] \to \{\pm 1\}$ is given. The dual object is ε^*: $[+k] \to \{\pm 1\}$, $\varepsilon^*(j) = -\varepsilon(j)$. Since dualities are compatible with tensor products, it suffices to define the dualities for the generating objects $1_\pm$: $[+1] \to \pm 1$. A left duality in $T^o A$ is given by

$$k_- = b\colon \emptyset \to 1_+ \otimes 1_+^*$$
$$f_- = d\colon 1_+^* \otimes 1_+ \to \emptyset$$
$$k_+ = b\colon \emptyset \to 1_- \otimes 1_-^*$$
$$f_+ = d\colon 1_-^* \otimes 1_- \to \emptyset$$

A right duality is given by reversing the orientations

$$k_+ = a\colon \emptyset \to 1_+^* \otimes 1_+$$
$$f_+ = c\colon 1_+ \otimes 1_+^* \to \emptyset$$
$$k_- = a\colon \emptyset \to 1_-^* \otimes 1_-$$
$$f_- = c\colon 1_- \otimes 1_-^* \to \emptyset$$

In the case of TA we set $[+k]^* = [+k]$. Left and right duality coincide. They are defined by

$$k\colon \emptyset \to [+1] \otimes [+1], \quad f\colon [+1] \otimes [+1] \to \emptyset$$

on the generating object.

The dualities above can be extended to dualities in the sense of (3.1) of the actions pairs $(T^o B, T^o A)$ and (TB, TA). By (3.4) and (3.5) it suffices again to consider the generating objects. We define

$$\beta = \kappa_-\colon \emptyset \to 1_+^*$$
$$\delta = \varphi_-\colon 1_+ \to \emptyset$$
$$\beta = \kappa_+\colon \emptyset \to 1_-^*$$
$$\delta = \varphi_+\colon 1_- \to \emptyset$$
$$\alpha = \kappa_+\colon \emptyset \to 1_+$$
$$\gamma = \varphi_+\colon 1_+^* \to \emptyset$$
$$\alpha = \kappa_-\colon \emptyset \to 1_-$$
$$\gamma = \varphi_-\colon 1_-^* \to \emptyset.$$

8. An algebraic model for TB

In ?? we have constructed a representation of RRB which induces a representation $\mathcal{L}$ of TB. It yields on morphisms

$$\mathcal{L}\colon \mathrm{Hom}_{\mathrm{TB}}(r,s) \to \mathrm{Hom}(V^{\otimes r}, V^{\otimes s}).$$

By construction, the morphisms in the image of $\mathcal{L}$ commute with the twists $\mathcal{L}(\alpha)\circ t_r = t_s \circ \mathcal{L}(\alpha)$. Let $\mathrm{Hom}_t(V^{\otimes r}, V^{\otimes s})$ be the subspace of $\mathcal{L}$-linear maps $h\colon V^{\otimes r} \to V^{\otimes s}$ with $h\circ t_r = t_s \circ h$. The algebraic model for TB is given by

(8.1) Theorem. *The linear map*

$$\mathcal{L}\colon \mathrm{Hom}_{\mathrm{TB}}(r,s) \to \mathrm{Hom}_t(V^{\otimes r}, V^{\otimes s})$$

is an isomorphism.

Proof. Let $\{k\}$ denote the largest integer below k. It was shown in [**??**] that $\mathrm{Hom}_{\mathrm{TB}}(r,s)$ has dimension $\binom{r+s}{\{(r+s)/2\}}$. We first verify that $\mathrm{Hom}_t(r,s)$ has the same dimension. The eigenvector $x(e)$ in (5.2) has multiplicity $\binom{n}{k}$. Since the eigenvalues (5.5) are pairwise different, the dimension in question equals

$$\sum_k \binom{r}{k}\binom{s}{\tilde{k}}$$

with $2k-r = 2\tilde{k}-s$ in case $r+s$ is even. A well-known formula for binomial coefficients shows the claim to be correct. A similar argument works if $s+r$ is odd. (Remark: For $r+s$ even a similar proof works for general parameters ρ.)

By the dimension count above it suffices to show injectivity. By dualization, it suffices to consider the case $s=0$ (or $r=0$). By the results of [**??**] about the Markov trace, the composition of morphisms

$$\mathrm{Hom}(0,r)\times \mathrm{Hom}(r,0) \to \mathrm{Hom}(0,0) = \mathfrak{K}$$

is a perfect pairing. Since the Markov trace is a quantum trace this pairing can be computed from the corresponding bilinear form via $\mathcal{L}$. Therefore $\mathcal{L}$ has to be injective. □

9. The category of coloured cylinder ribbons

By way of example, we construct the category of unoriented rooted cylinder ribbons coloured by representations of $U_q(sl_2)$. We describe (= define) this category by generators and relations (in the context of tensor module categories). We construct a representation of this category which extends the Kauffman functor of section 4. It is seen that most of our techniques and results obtained so far have to be used. It would be interesting to extend to approach of [**??**, Chapter XII] to our setting. In any case, this reference contains relevant additional information for this and the next section.

We consider unoriented rooted cylinder ribbons with components coloured by the irreducible U-modules V_n, $n \in \mathbb{N}$. The meaning of a colouring is as in [**??**, Ch. I]. There is an associated category. The objects of this category are sequences $(j_1, \dots, j_r)$ with $j_k \in \mathbb{N}$ (the empty sequence for $r = 0$). The morphisms from $(j_1, \dots, j_r)$ to $(k_1, \dots, k_k)$ are the coloured rooted (r, s)-ribbons; a component which ends in $(a, 0)$ carries the colour V_a; a component which ends in $(b, 1)$ carries a colour V_b. We call this category RRB($\mathbb{N}$). It is a tensor module category over the tensor category RA($\mathbb{N}$) of coloured ordinary ribbons. The category RRB($\mathbb{N}$) has the following presentation by generators and relations. The relations are coloured versions of (1.1) and (1.2).

(9.1) Generators.

(1) $X_{m,n}\colon (m,n) \to (n,m)$, $\quad X^{-1}_{m,n}\colon (n,m) \to (m,n)$
(2) $k_m\colon \emptyset \to (m,n)$, $\quad f_m\colon (m,n) \to \emptyset$
(3) $t_m\colon (m) \to (m)$, $\quad t^{-1}_m\colon (m) \to (m)$
(4) $\kappa_m\colon \emptyset \to (m)$, $\quad \varphi_m\colon (m) \to \emptyset$

(9.2) Relations.

(1) $X_{m,n}X^{-1}_{m,n} = 1_{(m,n)} = X^{-1}_{n,m}X_{n,m}$
(2) $(X_{n,p} \otimes 1_{(m)})(1_{(n)} \otimes X_{m,p})(X_{m,n} \otimes 1_{(p)}) = (1_{(p)} \otimes X_{m,n})(X_{m,p} \otimes 1_{(n)})(1_{(m)} \otimes X_{n,p})$
(3) $(1_{(m)} \otimes f_m)(k_m \otimes 1_{(m)}) = 1_{(m)} = (f_m \otimes 1_{(m)})(1_{(m)} \otimes k_m)$
(4) $(f_m \otimes 1_{(p)}) = (1_{(p)} \otimes f_m)(X_{m,p} \otimes 1_{(p)})(1_{(m)} \otimes X_{m,p})$
$(k_p \otimes 1_{(m)}) = (1_p \otimes X_{m,p})(X_{m,p} \otimes 1_p)(1_{(m)} \otimes k_p)$
and similar relations with $X_{m,p}$ replaced by $X^{-1}_{p,m}$
(5) $t_m t^{-1}_m = 1_{(m)} = t^{-1}_m t_m$
(6) $t_{(m,n)} := X_{n,m}(t_n \otimes 1_{(m)})X_{m,n}(t_m \otimes 1_{(n)}) = (t_m \otimes 1_{(n)})X_{n,m}(t_n \otimes 1_{(m)})X_{m,n}$
(7) $t_{(m,m)}k_m = k_m$, $\quad f_m t_{(m,n)} = f_m$
(8) $(\varphi_m \otimes 1_{(m)})k_m = \kappa_m$, $\quad f_m(\kappa_m \otimes 1_{(m)}) = \varphi_m$
(9) $t_m\kappa_m = \kappa_m$, $\quad \varphi_m t_m = \varphi_m$
(10) $t_{(m,n)}(\kappa_m \otimes 1_{(n)}) = (\kappa_m \otimes 1_{(n)})t_n$, $\quad (\varphi_m \otimes 1_{(n)})t_{(m,n)} = t_n(\varphi_m \otimes 1_{(n)})$

Our aim is to construct a representation of this category. On object level it maps $(j_1, \dots, j_r)$ to $V_{j_1} \otimes \dots \otimes V_{j_r}$. We now specify the values of the generators and verify the relations. The values of the generators are denoted with the same symbol.

We take for $X_{m,n}\colon V_m \otimes V_n \to V_n \otimes V_m$ the braiding (R-matrix) which is induced from the universal R-matrix of section 3. The operator $t_m\colon V_m \to V_m$ is the cylinder twist of section 7 (there denoted τ_m, for distiction). These data satisfy the relations (1), (2), (5), (6).

We have a Clebsch-Gordan decomposition of U-modules

$$V_m \otimes V_m = V_{2m} \otimes V_{2m-2} \oplus V_{2m-2} \oplus \dots \oplus V_0.$$

The morphism $k_m\colon \mathfrak{K} \to V_m \otimes V_m$ corresponds to the inclusion of V_0 and the morphisms $f_m\colon V_m \otimes V_m \to \mathfrak{K}$ to the projection onto V_0. These conditions determine them up to a scalar multiple. The naturality of the braiding yields relation (4). We have to normalize k_m and f_m. A normalization of k_m yields, by (3), a normalization of f_m. In order to specify the normalization, we use the basis (z_k) of section 7. We set

$$k_m(1) = \sum_{j=0}^{m} \gamma^{m-2j} z_j \otimes z_{m-j},$$

and define f_m by the transposed matrix, i. e.

$$f_m(z_j \otimes z_{m-j}) = \gamma^{m-2j} \quad \text{and} \quad f_m(z_k \otimes z_\ell) = 0$$

otherwise. Then (3) holds. We note $f_m k_m(1) = [m+1]_{\gamma^2}$. In order to satisfy (9), we have to take for κ_m and φ_m eigenvalues of t_m for the eigenvalue 1. Here we use (7.3). We set

$$\kappa_m(1) = \varepsilon^m \sum_{k=0}^{m} \bar{\gamma}^{m-2k} \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}^{-1/2} z_k.$$

Since the matrix for t_m is symmetric, we define φ_m by the transposed matrix

$$\varphi_m(z_k) = \varepsilon^m \bar{\gamma}^{m-2k} \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}^{-1/2}.$$

These choices yield the relations (9).

If $\kappa_m(1) = \sum_{k=0}^{m} a_k z_k$, then the second relation in (8) gives

$$f_m(\kappa_m \otimes 1_{(m)})(z_\ell) = \gamma^{2\ell-m} a_{m-\ell}.$$

With our choices this equals $\varphi_m(z_\ell)$. It is here that we need the precise structure of $\kappa_m(1)$ provided by (7.3). Similarly for the first relation in (8).

The relations (7) hold, since the cylinder twist commutes with U-linear maps, in particular $t_{(m,m)}$ is compatible with the Clebsch-Gordan decomposition.

It remains to verify (10). These relations do not depend on the normalization of κ_m and φ_m. The second one is the transposition of the first one. The first relation is a consequence of (5.3). There we have shown that a similar result holds if the V_k are replaced by $V^{\otimes k}$ throughout. Hence we have realized all relations.

The connection between $V^{\otimes m}$ and V_m is as follows. Let j_m be the Jones-Wenzl idempotent in the Temperley-Lieb algebra $\mathrm{Hom}_{\mathrm{RA}}(m,m) = T_m$[5]. Via the representation of RA the element j_m yields a projection operator on $V^{\otimes m}$ with image isomorphic to V_m. From section 5 we know that the m-fold tensor product

$$z := (\bar{\gamma} v_{-1} + \bar{\gamma}^{-1} v_1) \otimes \cdots \otimes (\bar{\gamma} v_{-1} + \bar{\gamma}^{-1} v_1)$$

[5]This is often denoted by f_m in the literature; unfortunately we already used this symbol.

is an eigenvector with eigenvalue 1 for the cylinder twist. The model $V_m \subset V^{\otimes m}$ has the standard basis $z_0, \dots, z_m$ with $z_0 = v_{-1} \otimes \dots \otimes v_{-1}$. We assume the value $\varepsilon = 1$.

(9.3) Proposition. *The projection $j_m(z)$ is the vector* (8.5) *with* $\varepsilon = 1$.

Proof. It certainly is an eigenvector for the eigenvalue 1, by naturality of the cylinder twist. Thus $j_m(z) = \lambda\kappa_m(1)$ for some scalar λ. In order to determine the scalar we consider the coefficient of z_0 in $j_m(z)$. By the structure of the Jones-Wenzl idempotent, this coefficient is $\bar{\gamma}^n$. In order to see this, express j_m as a linear combination of the standard graphical basis of the Temperley-Lieb algebra T_m and observe that all basis elements except 1 map $z_0 \in V^{\otimes m}$ to zero.□

The following recursion formula for j_m is due to Hermisson [**??**]. Let $e_1, \dots, e_{m-1}$ be the standard generators of the Temperley Lieb algebra. Write

$$e(m, n) = e_{m-1}e_{m-2} \dots e_n.$$

Then

(9.4)
$$j_m = j_{m-1} \cdot \frac{1}{[m]} \sum_{j=1}^{m} [j]e(m, j).$$

Suppose b is a bridge in T_m and also the corresponding morphism in $V^{\otimes m} \to V^{\otimes m}$. Then

(9.5)
$$[b\rangle := \varphi_{V^{\otimes m}} b \kappa_{V^{\otimes m}} = (\gamma + \gamma^{-1})^m.$$

Thus, by (8.7),

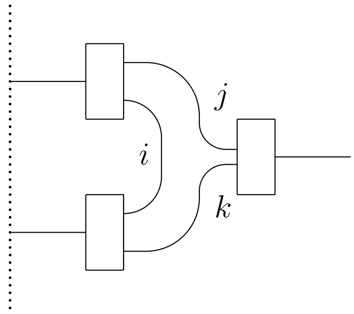
(9.6)
$$[j_m\rangle = (\gamma + \gamma^{-1})^m \prod_{k=1}^{m} \frac{1}{[k]} \left(\sum_{j=1}^{k} [j] \right).$$

On the other hand this value equals $\varphi_m \kappa_m(1)$. If we use (7.5) we see that the equality of these values yields the identity (7.6).

10. Trivalent graphs

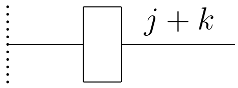
We apply the results of the previous section to extend the results of [**??**] to our setting . The presentation will be brief and assumes detailed knowledge of [**??**]. We consider a category of trivalent weighted graphs where some edges end on the axis. The result (9.1) gives a recursion formula for the evaluation of such graphs in the sense of [**??**].

The $\mathbb{N}$-coloured Temperley-Lieb category TA($\mathbb{N}$) has a natural extension to a category ot trivalent graphs [**??**], [**??**], [**??**], [**??**]. There is a corresponding extension of TB. A trivalent vertex in TA($\mathbb{N}$) is defined in graphical notation as Figure 8.1 in [**??**, p. 552]. See the figure below for $\omega_{i,j,k}$; the boxes in that figure represent Jones-Wenzl idempotents. We here consider trivalent graphs where some edges end on the axis. The evaluation of such graphs by the method of [**??**], say, reduces to the determination of



$$= \omega_{i,j,k}$$

in terms of



$$= \omega_{j+k}.$$

(10.1) Theorem. *There exists a scalar* $[i,j,k\,\rangle$ *such that*

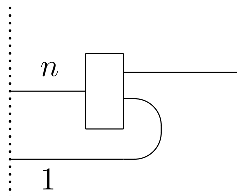
$$\omega_{i,j,k} = [i,j,k\,\rangle \omega_{j+k}.$$

Let $\alpha_n = (\gamma + \gamma^{-1})[n]^{-1}([1] + [2] + \cdots + [n])$. *We have the recursion relations*

$$\omega_{i,j,k} = \alpha_{i+k}\omega_{i-1,j,k} + \frac{[j]}{[i+j]}\alpha_{i+j-1}\omega_{i-1,j-1,k+1}$$

and $[0,j,k\,\rangle = 1$, $[i,0,k\,\rangle = \alpha_{i+k}[i-1,0,k\,\rangle$.

Proof. We use the method of [**??**] and our earlier results. A first step evaluates



We apply the second recursion formula (8.7) for the Jones-Wenzl idempotent j_n and obtain the value $\alpha_n\omega_{n-1}$. At the same time we obtain

$$(*) \qquad \omega_{i,0,k} = \alpha_{i+k}\omega_{i-1,0,k}$$

and

$$\omega_{i,0,k} = [i,0,k\,\rangle \omega_{i+k}.$$

The value $[0,j,k\,\rangle = 1$ comes from the definitions; similarly $\omega_{0,j,k} = \omega_{j+k}$.

We now consider the case $i > 0, j > 0$ and apply the standard recursion formula [**??**, (4.2.a) on p. 531] for j_{i+j}. Then $(*)$ above in conjunction with [**??**, Lemma 2] yield the recursion formula for $\omega_{i,j,k}$ as stated in (9.1). □