Traces for braid groups and Hecke algebras of type B

Knot theories and root systems. Part III

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1. Traces on groups

Let G be a group and $\mathfrak{K}$ a commutative ring. A ($\mathfrak{K}$-valued) *trace* on G is a function $T: G \to \mathfrak{K}$ such that for all $g, h \in G$

(1.1)
$$T(gh) = T(hg).$$

Equivalently, a trace is a function constant on conjugacy classes. A trace extends to a $\mathfrak{K}$-linear map $T: \mathfrak{K}G \to \mathfrak{K}$ from the group algebra $\mathfrak{K}G$ such that (1.1) holds for any two elements g, h in the group algebra.

Suppose $\tau: G \to G$ is an automorphism and T a trace. We call T (strongly) *τ-invariant* if for all $g, h \in G$ the relation

(1.2)
$$T(g \cdot \tau(h)) = T(g \cdot h)$$

holds. If we set $g = 1$, we have the ordinary τ-invariance $T(\tau(h)) = T(h)$. If T is τ_1- and τ_2-invariant, then also τ_1^{-1}- and $\tau_1\tau_2$-invariant. If Γ is a group of automorphisms of G, then a trace is called Γ-invariant, if T is τ-invariant for each $\tau \in \Gamma$. It suffices to check Γ-invariance for a generating set of Γ.

Suppose T_i is a trace on G_i $(i = 1, 2)$. Then $(g_1, g_2) \mapsto T_1(g_1)T_2(g_2)$ is a trace on $G_1 \times G_2$. We want to generalize this to semi-direct products.

Let $\alpha: \Gamma \to \operatorname{Aut} G$ be a group of G-automorphisms. The semi-direct product $G \times_\alpha \Gamma$ is a group structure on the set $G \times \Gamma$ defined by

$$(g, \sigma)(h, \tau) := (g \cdot \sigma(h), \sigma\tau).$$

In this group structure we have

(1.3)
$$(1, \sigma)(g, 1)(1, \sigma)^{-1} = (\sigma(g), 1).$$

We will use the following fact several times.

(1.4) Lemma. *A pair of group homomorphisms $\lambda: G \to H$ and $\mu: \Gamma \to H$ defines via $(g, \sigma) \mapsto \lambda(g)\mu(\sigma)$ a homomorphism $\varphi: G \times_\alpha \Gamma \to H$ if and only if for all $g \in G$ and $\sigma \in \Gamma$ the relation $\lambda(\sigma(g)) = \mu(\sigma)\lambda(g)\mu(\sigma)^{-1}$ holds. Each homomorphsm φ has this form for a unique pair (λ, μ).* □

Is $\alpha\colon \Gamma \to \operatorname{Aut}(G)$ is an antihomorphism, we define the semi-direct product $\Gamma{}_\alpha\!\times G$ with multiplication $(\sigma, g)(\tau, h) = (\sigma\tau, \tau(g)h)$.

The following is immediately verified from the definitions.

(1.5) Proposition. *Let S be a Γ-invariant trace on G and U a trace on Γ. Then*

$$T\colon G \times_\alpha \Gamma \to \mathfrak{K}, \quad (g, \sigma) \mapsto S(g)U(\sigma)$$

is a trace on $G \times_\alpha \Gamma$. □

If $\varphi\colon G \to H$ is a group homomorphism and T a trace on H, then $T \circ \varphi$ is a trace on G. Any function $T\colon G \to \mathfrak{K}$ on an abelian group G is a trace. Characters of finite dimensional representations are traces.

2. Braid groups of type B

The braid group ZB_n associated to the Coxeter graph B_n is, by definition, the group generated by $t, g_1, \dots, g_{n-1}$ with relations

(2.1)
$$\begin{array}{llcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j|=1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j|\geq 2 \\ (3) & tg_i & = & g_it, & i \geq 2 \\ (4) & tg_1tg_1 & = & g_1tg_1t. & \end{array}$$

For certain applications we need other presentations of this group.

Let $Z'B_n$ be the group with generators $c, g_1, \dots, g_{n-1}$ and relations

(2.2)
$$\begin{array}{llcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j|=1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j|\geq 2 \\ (3) & cg_i & = & g_{i-1}c, & i \geq 2, \\ (4) & c^2g_1 & = & g_{n-1}c^2. & \end{array}$$

We abbreviate $g = g_{n-1}g_{n-2}\cdots g_1$.

(2.3) Proposition. *The assignment $\varphi(g_i) = g_i$, $1 \leq j \leq n-1$, and $\varphi(t) = g^{-1}c$ defines an isomorphism $\varphi\colon ZB_n \to Z'B_n$.*

Proof. The relations (1) and (2) yield in both groups

(2.4)
$$g_{i-1}g = gg_i, \qquad i > 1.$$

We define in ZB_n (resp. $Z'B_n$) an element c (resp. t) by $gt = c$. From (1), (2) and (2.4) we see that the relations $cg_i = g_{i-1}c$ and $g_it = tg_i$ are equivalent for $i > 1$.

We set $h = g_{n-1}\cdots g_2$, $k = g_{n-2}\cdots g_1$ and infer from (2.4)

(2.5)
$$gh = kg.$$

We use this to show that $c^2g_1 = g_{n-1}c^2$ and $tg_1tg_1 = g_1tg_1t$ are equivalent, provided (1), (2), and (3) hold. We compute

$$g_{n-1}^{-1}c^2g_1 = g_{n-1}^{-1}g_{n-1}kthg_1tg_1 = khtg_1tg_1$$

$$c^2 = gthg_1t = ghtg_1t = kgtg_1t = khg_1tg_1t$$

and see the equivalence. □

The braid group $Z\tilde{A}_{n-1}$ of the Coxeter graph with n vertices $\tilde{A}_{n-1}$ has, by definition, generators $g_1, \dots, g_n$ and relations

(2.6)
$$\begin{array}{rcll} g_ig_jg_i & = & g_jg_ig_j, & m(i,j)=3 \\ g_ig_j & = & g_jg_i, & m(i,j)=2. \end{array}$$

Indices will be considered $\bmod n$ in this case. We have $m(i,j) = 3$ if and only if

$i \equiv j \pm 1 \bmod n$. All this holds for $n \geq 3$. For $n = 2$, the group is the free group generated by g_1 and g_2.

The graph $\tilde{A}_{n-1}$ has an automorphism which permutes the vertices cyclically. We have an induced automorphism s of $Z\tilde{A}_{n-1}$ given by

$$s(g_i) = g_{i-1}, \qquad i \bmod n.$$

The n-th power of s is the identity.

We use s to form the semi-direct product

(2.7)
$$Z\tilde{A}_{n-1} \to G_n \to \mathbb{Z};$$

the generator $1 \in \mathbb{Z}$ acts through s on $Z\tilde{A}_{n-1}$. The semi-direct product is the group structure on the set $Z\tilde{A}_{n-1} \times \mathbb{Z}$ defined by $(x, m) \cdot (y, n) = (x \cdot s^m(y), m+n)$. The group G_n has the following description by generators and relations. Let G'_n denote the group with generators $s, g_1, \ldots, g_n$ and relations (2.6) together with

(2.8)
$$sg_i = g_{i-1}s, \qquad i \bmod n.$$

(2.9) Proposition. *The assignment $\psi(g_i) = (g_i, 0)$ and $\psi(s) = (e, 1)$ yields an isomorphism $\psi\colon G'_n \to G_n$ (neutral element e).*

Proof. One verifies that ψ is compatible with relations (2.6) and (2.8). This is obvious for (2.6). The relation $(e, 1)(x, 0)(e, 1)^{-1} = (s(x), 0)$ is used to show compatibility with (2.8).

An element $x \in Z\tilde{A}_{n-1}$ has an image $x' \in G'_n$, induced by $g_i \mapsto g_i$. This assignment has the property $(s(x))' = sx's^{-1}$. We have the Homomorphism $G_n \to G'_n$, $(x, m) \mapsto x's^m$ by (1.4). It is inverse to *psi*. □

(2.10) Proposition. *The assignment $\alpha(g_i) = g_i$, $1 \leq i \leq n-1$, and $\alpha(c) = s$ defines an isomorphism $\alpha\colon Z'B_n \to G'_n$.*

Proof. The assignment is compatible with the relations of $Z'B_n$, since

$$\alpha(c^2 g_1 c^{-2}) = s^2 g_1 s^{-2} = sg_n s^{-1} = g_{n-1}.$$

An inverse to α is induced by the assignment $\beta(g_i) = g_i$, $\beta(g_n) = cg_1 g^{-1}$, and $\beta(s) = c$. In order to see that β is well defined, one has to check, in particular, the relations

$$g_{n-1}g_n g_{n-1} = g_n g_{n-1} g_n, \qquad g_1 g_n g_1 = g_n g_1 g_n.$$

In the first case, this amounts to the equalitiy of

$$g_{n-1} c g_1 c^{-1} g_{n-1} = c^2 g_1 c^{-1} g_1 c g_1 c^{-2}$$

and

$$c g_1 c^{-1} g_{n-1} c g_1 c^{-1} = c g_1 c g_1 c^{-1} g_1 c^{-1}.$$

We compute

$$c g_1 g_2 g_1 c^{-1} = c g_2 g_1 g_2 c^{-1} = c g_2 c^{-1} c g_1 c^{-1} c g_2 c^{-1} = g_1 c g_1 c^{-1} g_1$$

and hence

$$c(g_1 c g_1 c^{-1} g_1) c^{-1} = c^2 g_1 g_2 g_1 c^{-2}.$$

On the other hand, $g_1 c^{-1} g_1 c g_1 = g_1 g_2 g_1$. This yields the desired equality.

The second relation above leads to the same situation. □

If we combine the foregoing, we obtain a semi-direct product

(2.11)
$$Z\tilde{A}_{n-1} \to ZB_n \to \mathbb{Z}.$$

In terms of the original generators, the inclusion $Z\tilde{A}_{n-1} \subset ZB_n$ is given by

(2.12)
$$g_n \mapsto gtg_1t^{-1}g^{-1}; \quad g_i \mapsto g_i, \quad 1 \le i \le n-1.$$

The homomorphism $ZB_n \to \mathbb{Z}$ in (2.14) is given by $g_i \mapsto 0$ and $t \mapsto 1$.

Different types of Weyl groups (= Coxeter groups) are related to these braid groups. We have the Coxeter groups $W\tilde{A}_{n-1}$ and WB_n associated to the graphs $\tilde{A}_{n-1}$ and B_n. In addition, we will also use a group $W^\infty B_n$. It is obtained from ZB_n by adding the relations $g_j^2 = 1$, but no relation for t. The reason for introducing this group is a semi-direct product in analogy to (2.14). The arguments which lead to (2.14) also give a semi-direct product

$$W\tilde{A}_{n-1} \to W^\infty B_n \to \mathbb{Z}.$$

We give another interpretation and describe these groups as groups of permutations.

Let $t_n\colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x+n$ be the translation by n. Let P_n denote the group of t_n-equivariant permutations $\sigma\colon \mathbb{Z} \to \mathbb{Z}$. Equivariance means $\sigma(i+n) = \sigma(i)+n$. Hence σ induces $\overline{\sigma}\colon \mathbb{Z}/n \to \mathbb{Z}/n$, and $\sigma \mapsto \overline{\sigma}$ is a homomorphism $\pi\colon P_n \to S_n$ onto the symmetric group S_n.

(2.13) Proposition. *The kernel of π is isomorphic to $\mathbb{Z}^n$. The group P_n is isomorphic to the semi-direct product $\mathbb{Z}^n \to P_n' \to S_n$ in which S_n acts on $\mathbb{Z}^n$ by permutations.*

Proof. Let $\sigma_1 \in P_n$. Then there exists a permutation α of $\{1,\dots,n\}$ and an n-tuple $(k_1,\dots,k_n) \in \mathbb{Z}^n$ such that $\sigma(i+tn) = \alpha(i)+(k_i+t)n$. We denote this map by $\sigma_1 = \sigma(\alpha; k_1,\dots,k_n)$. Suppose $\sigma_2 = \sigma(\beta; l_1,\dots,l_n)$ is another permutation written in this form. Then

$$\sigma_2 \circ \sigma_1 = \sigma(\beta\alpha; l_{\alpha(1)} + k_1, \dots, l_{\alpha(n)} + k_n).$$

If we think of $P_n' = S_n \times \mathbb{Z}^n$ as sets, then the desired isomorphism is given by $(\alpha; k_1,\dots,k_n) \mapsto \sigma(\alpha; k_1,\dots,k_n)$. □

The semi-direct product P_n' has a normal subgroup Q_n' which is given as a semi-direct product

(2.14)
$$N \to Q_n' \to S_n$$

with $N = \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \subset \mathbb{Z}^n$. The homomorphism

$$\varepsilon\colon P'_n \to \mathbb{Z}, \quad (\alpha; k_1, \dots, k_n) \mapsto \sum k_i$$

is a surjection with kernel Q'_n. The canonical sequence

(2.15) $$Q'_n \to P'_n \to \mathbb{Z}$$

is itself a semi-direct product; the assignment $1 \mapsto (\mathrm{id}; 1, 0, \dots, 0)$ gives a splitting of ε. Under the isomorphism (2.13) the subgroup Q'_n corresponds to the subgroup

$$Q_n = \{\sigma \in P_n \mid 1 + 2 + \cdots + n = \sigma(1) + \cdots + \sigma(n)\}.$$

(2.16) Proposition. *The groups $W^\infty B_n$ and P_n are isomorphic. The groups $W\tilde{A}_{n-1}$ and Q_n are isomorphic. The element g_i is mapped to the transposition $(i, i+1)$, $i \in n\mathbb{Z}$. The element t is mapped to $\sigma(i) = i + n$ for $i \equiv 1 \bmod n$ and $\sigma(j)j$ otherwise.*

The proof is given after the proof of (2.21). In the proof of (2.17) we use the following:

(2.17) Lemma. *The elements*

$$t_0 = t, \quad t_1 = g_1 t g_1, \quad \dots, \quad t_{n-1} = g_{n-1} \dots g_2 g_1 t g_1 g_2 \dots g_{n-1}$$

of the braid group ZB_n pairwise commute.

Proof. We set

$$g(i,j) = g_i g_{i+1} \dots g_j, \qquad i \le j$$
$$g(i,j) = g_i g_{i-1} \dots g_j, \qquad i \ge j.$$

The braid relations imply immediately

$$g(1,j) g_{j+1} g(j,1) = g(j+1,2) g_1 g(2, j+1)$$

and (2.5)

$$g(2, j+1) g(1, j+1) = g(1, j+1) g(1, j).$$

By commutativity of g_j-elements, it suffices to show $t_i t_{i+1} = t_{i+1} t_i$. We compute

$$\begin{aligned} t_j t_{j+1} &= g(j,1) t g(1,j) g_{j+1} g(j,1) t g(1, j+1) \\ &= g(j,1) t g(j+1, 2) g_1 g(2, j+1) t g(1, j+1) \\ &= g(j,1) g(j+1,2) t g_1 t g(2, j+1) g(1, j+1) \\ &= g(j,1) g(j+1,2) [t g_1 t g_1] g(2, j+1) g(1, j). \end{aligned}$$

A similar computation works for $t_{j+1} t_j$. □

The semi-direct product relation (2.13, (2.17) between $W^\infty B_n$ and WA_{n-1} has a counterpart for the braid groups. The homomorphism

$$\lambda\colon K_n \to ZA_{n-1}, \quad g_j \mapsto g_j, \quad t \mapsto 1$$

splits by $g_j \mapsto g_j$. Therefore we have a semi-direct product

(2.18)
$$ZP_n \to ZB_n \to ZA_{n-1}.$$

The elements

$$y_0 = t, \quad , y_1 = g_1 t g_1^{-1}, \quad \dots, \quad y_{n-1} = g_{n-1} \dots g_1 t g_1^{-1} \dots g_{n-1}^{-1}$$

are contained in the kernel K_n of λ.

(2.19) Lemma. *The elements y_j have the following conjugation properties with respect to ZA_{n-1}:*
(1) $g_k^{-1} y_j g_k = y_j, \qquad k > j,\ k < j-1$
(2) $g_k^{-1} y_k g_k = y_{k-1},$
(3) $g_k^{-1} y_{k-1} g_k = y_{k-1} y_k y_{k-1}^{-1}.$

Proof. (2) follows directly from the definitions.
(1) If $k > j$, then g_k commutes with every generator in the definition of y_j. In the case $k < j-1$ one uses the commutation relation between generators and $g_{k+1} g_k g_{k+1}^{-1} = g_k^{-1} g_{k+1} g_k$ (and the inverse) to cancel g_k^{-1} and g_k.
(3) is proved by induction on k. The verification for $k = 0$ is easy. We calculate with (1) and (2)

$$g_k^{-1} y_k y_{k+1} y_k^{-1} g_k = y_{k-1} y_{k+1} y_{k-1}^{-1} = g_{k+1} y_{k-1} y_k y_{k-1}^{-1} g_{k+1}^{-1}.$$

On the other hand, by (1) and (2)

$$\begin{aligned} g_{k+1}^{-1} g_k^{-1} g_{k+1}^{-1} y_k g_{k+1} g_k g_{k+1} &= g_k^{-1} g_{k+1}^{-1} g_k^{-1} y_k g_k g_{k+1} g_k \\ &= g_k^{-1} g_{k+1}^{-1} y_{k-1} g_{k+1} g_k \\ &= g_k^{-1} y_{k-1} g_k. \end{aligned}$$

This yields the induction step. □

(2.20) Proposition. *The group K_n is the free group generated by $y_0, \dots, y_{n-1}$.*

Proof. By the previous Lemma, the group K_n^0 generated by the $y_0, \dots, y_{n-1}$ is invariant under conjugation by elements of ZA_{n-1}. Since $t \in K_n^0$ and t together with ZA_{n-1} generates ZB_n, we must have equality $K_n^0 = K_n$.

Let F_n denote the free group generated by $y_0, \dots, y_{n-1}$. We define homomorphisms $\gamma_1, \dots, \gamma_{n-1} \colon F_n \to F_n$ by immitating (2.20):
(1) $\gamma_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\gamma_k(y_k) = y_{k-1},$
(3) $\gamma_k(y_{k-1}) = y_{k-1} y_k y_{k-1}^{-1}.$
We claim:

(2.21) Lemma. *The γ_j are automorphisms and satisfy the braid relations*

$$\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j, \qquad |i-j| = 1, \quad \textit{and} \quad \gamma_i \gamma_j = \gamma_j \gamma_i, \qquad |i-j| \geq 2.$$

Proof. First we check that the homomorphism $\delta_k \colon F_n \to F_n$

(1) $\delta_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\delta_k(y_{k-1}) = y_k,$
(3) $\delta_k(y_k) = y_k^{-1} y_{k-1} y_k$
is inverse to γ_k. Hence γ_k is an isomorphism. Since γ_k fixes y_j for $j \notin \{k-1, k\}$, the second braid relation is obviously satisfied. For the first relation, the reader may check the following values of $\gamma_1\gamma_2\gamma_1$ and $\gamma_2\gamma_1\gamma_2$ on y_0, y_1, y_2:

$$y_0 \mapsto y_0 y_1 y_2 y_1^{-1} y_0^{-1}, \quad y_1 \mapsto y_0 y_1 y_1^{-1}, \quad y_2 \mapsto y_0.$$

We use this Lemma to define a semi-direct product

(2.22)
$$F_n \to \Gamma_n \to ZA_{n-1},$$

in which $g_j \in ZA_{n-1}$ acts on F_n through δ_j. By (2.19) and $K_n^0 = K_n$, we have a canonical epimorphism $\mu\colon \Gamma_n \to ZB_n$. We show that μ is an isomorphism. As a set, $\Gamma_n = F_n \times ZA_{n-1}$. An inverse to μ has to send $g_j \mapsto (1, g_j)$ and $t \mapsto (y_0, 1)$. We have to check that this assignment is compatible with the relations of ZB_n. This is obvious for the g_j. Moreover:

$$\begin{aligned} tg_1tg_1 &\mapsto (y_0,1)(1,g_1)(y_0,1)(1,g_1) \\ &= (y_0,g_1)(y_0,g_1) \\ &= (y_0\delta_1(y_0), g_1^2) \\ &= (y_0y_1, g_1^2) \end{aligned}$$

$$\begin{aligned} g_1tg_1t &\mapsto (1,g_1)(y_0,1)(1,g_1)(y_0,1) \\ &= (y_1,g_1)(y_1,g_1) \\ &= (y_1\delta(y_1), g_1^2) \\ &= (y_0y_1, g_1^2). \end{aligned}$$

This finishes the proof of Proposition (2.21). □

Proof of (2.17). The elements t_j of (2.18) and the elements y_j coincide in $W^\infty B_n$, since $g_j = g_j^{-1}$ in this group. Lemma (2.20) shows that conjugation $y \mapsto g_k^{-1} y g_k$ acts on the set $(y_0, \dots, y_{n-1})$ by interchanging y_{k-1} and y_k. The proof of (2.21) is now easily adapted to show the isomorphism $W^\infty B_n \cong P_n'$. This isomorphism restricts to an isomorphism $W\tilde{A}_{n-1} \cong Q_n'$. □

We now apply the previous results to Hecke algebras. We have the Hecke algebras HA_{n-1}, $H\tilde{A}_{n-1}$, and HB_n associated to the corresponding Coxeter graphs. We consider algebras over the ground ring $\mathfrak{K}$. The first one is given by generators $g_1, \dots, g_{n-1}$, the braid relations between them and the quadratic relations $g_j^2 = (q-1)g_j + q$ with a parameter $q \in \mathfrak{K}$. The second one has generators $g_1, \dots, g_n$, the braid relations (2.8) and the same quadratic relations. The algebra HB_n has generators $t, g_1, \dots, g_{n-1}$, the braid relations (2.1), the quadratic relations above for the g_j and $t^2 = (Q-1)t + Q$ with another parameter $Q \in \mathfrak{K}$.

If we omit the quadratic relation for Q, then we obtain the definition of $H^\infty B_n$. This is not a Hecke algebra in the formal sense, i. e. associated to a Coxeter graph. It is a deformation of the group algebra of $W^\infty B_n$.

We know from Hecke algebra theory that an additive basis of the Hecke algebra is in bijective correspondence with the elements of the Coxeter group. There is a similar relation between $W^\infty B_n$ and $H^\infty B_n$. In order to derive it, we relate $H\tilde{A}_{n-1}$ and $H^\infty B_n$.

The algebra $H\tilde{A}_{n-1}$ has an automorphism τ given by $\tau(g_i) = g_{i-1}$ (indices $\bmod n$). We define the twisted tensor product over the ground ring $\mathfrak{K}$

(2.23)
$$H\tilde{A}_{n-1} \otimes \mathfrak{K}[\tau, \tau^{-1}] =: H_n^\infty$$

by the multiplication rule $(x \otimes \tau^k) \cdot (y \otimes \tau^l) = (x \cdot \tau^k(y), \tau^{k+l})$ for $k, l \in \mathbb{Z}$ and $x, y \in H\tilde{A}_{n-1}$.

(2.24) Proposition. *The algebra (2.24) is canonically isomorphic to $H^\infty B_n$.*

Proof. We use the isomorphism (2.3) to redefine the algebra $H^\infty B_n$ by generators $c, g_1, \dots, g_{n-1}$ relations (2.2) and the quadratic relations for the g_j. The assignment $g_j \mapsto g_j \otimes 1$, $c \mapsto 1 \otimes \tau$ induces a homomorphism $H^\infty B_n \to H\tilde{A}_{n-1} \otimes H_n^\infty$. We have a homomorphism $H\tilde{A}_{n-1} \to H^\infty B_n$, $x \mapsto x'$ induced by $g_j \mapsto g_j$ with $g_n = gtg_1t^{-1}g^{-1}$ in $H^\infty B_n$ (see (2.12)). This extends to a homomorphism $H_n^\infty \to H^\infty B_n$ by $x \otimes \tau^k \mapsto x' \cdot c^k$, since $\tau(y)' = cy'c^{-1}$. These homomorphisms are inverse to each other. □

(2.25) Corollary. *Suppose $(b_j \mid j \in J)$ is a $\mathfrak{K}$-basis of $H\tilde{A}_{n-1}$. Then $(b_j' c^k \mid j \in J, k \in \mathbb{Z})$ is a $\mathfrak{K}$-basis of $H^\infty B_n$.* □

3. Markov traces on braid groups

We use the semi-direct product (2.19), (2.21)

$$F_n \to ZB_n \to ZA_{n-1}$$

in order to construct traces on ZB_n by (1.5). This requires a ZA_{n-1}-invariant trace on F_n.

Let $s\colon \mathbb{Z} \to \mathfrak{K}$ be any function with $s(0) = 1$, called *parameter function*. We define a trace $T_s\colon V_n \to \mathfrak{K}$ on the free abelian multiplicative group V_n with basis $y_0, \dots, y_{n-1}$ by

(3.1)
$$T_s(y_0^{k(0)} \cdots y_{n-1}^{k(n-1)}) = \prod_{j=0}^{n-1} s(k(j)).$$

Let $F_n \to V_n$ be the abelianization. The trace T_s on V_n lifts to a trace T_s on F_n.

(3.2) Proposition. *The trace T_s on F_n is ZA_{n-1}-invariant.*

PROOF. It suffices to check invariance for the generators g_i. This is obvious from (2.20). □

If U is any trace on ZA_{n-1} and T_s the trace (3.2), we call the induced trace (1.5) on ZB_n the *s-extension* U_s of U.

Most important for applications to knot theory are Markov traces on the groups ZA_n. We recall: A sequence $U = (U^n)$ of traces U^n on ZA_{n-1} is called a *Markov trace* on $ZA = (ZA_n)$, provided

(3.3)
$$\begin{aligned} U^{n+1}|ZA_{n-1} &= U^n \\ U^{n+1}(xg_n^{\pm 1}) &= \alpha^{\pm 1}\beta^{-1}U^n(x) \end{aligned}$$

for $x \in ZA_{n-1}$ with units $\alpha, \beta \in \mathfrak{K}^*$.

Markov traces on $ZB = (ZB_n)$ have been constructed by Lambropoulou and Przytycki [10]. The next Proposition states that the s-extension of a Markov trace (U^n) is a Markov trace in their sense.

(3.4) Proposition. *Let $(T^n) = (U_s^n)$ denote the family of s-extensions of a Markov trace (U^n) on ZA. Then the following holds:*

(1) $T^{n+1}|ZB_n = T^n$.

(2) $T^{n+1}(xg_n^{\pm 1}) = z_\pm T^n(x)$ *for* $x \in ZB_n$, $z_\pm = \alpha^{\pm 1}\beta_{-1}$.

(3) $T^{n+1}(xy_n^k) = s(k)T^n(x)$ *for* $x \in ZB_n$.

PROOF. In terms of the semi-direct product $ZB_n = F_n \cdot ZA_{n-1}$, the inclusion $ZB_n \to ZB_{n+1}$ is given by $g_i \mapsto g_i$ and $y_i \mapsto y_i$ for $0 \le i \le n-1$. The s-traces T_s on F_n are compatible with $F_n \to F_{n+1}$, $y_i \mapsto y_i$. This yields (1).

Suppose $x = (h, \sigma) \in F_n \cdot ZA_{n-1}$. Then

$$xy_n^k = (h, \sigma)(y_n^k, 1) = (h \cdot \sigma(y_n^k), \sigma).$$

But $\sigma \in ZA_{n-1}$ acts trivially on y_n^k. Therefore

$$T(h \cdot \sigma(y_n^k)) = T(hy_n^k) = T_(h)s(k).$$

This shows (3); and (2) is equally simple. □

The generalized Hecke algebra $H^\infty B_n$ is the quotient of the group algebra $\mathfrak{K}ZB_n$ by the ideal generated by the $\zeta_i := g_i^2 - (q-1)g_i - q$, provided $q \in \mathfrak{K}^*$.

(3.5) Proposition. *Suppose the trace U: $ZA_{n-1} to \mathfrak{K}$ factors over the quotient maps $\mathfrak{K}ZA_{n-1} \to HA_{n-1}$. Then $T = U_s$: $\mathfrak{K}ZB_n \to \mathfrak{K}$ factors over the quotient map $\mathfrak{K}ZB_n \to H^\infty B_n$.*

Proof. We have to show $T(x\zeta_i y) = T(yx\zeta_i) = 0$ for all $y, x \in \mathfrak{K}ZB_n$. Write $yx = z$ in the form $\sum_j \lambda_j u_j v_j$ with $\lambda_j \in \mathfrak{K}$, $u_j \in F_n$, and $v_j \in ZA_{n-1}$. Then

$$T(z\zeta_i) = \sum \lambda_j T(u_j v_j \zeta_i) = \sum \lambda_j T_s(u_j) U(v_j \zeta_i).$$

By hypothesis, $U(v_j\zeta_i) = 0$. □

Lambropoulou and Przytycki show that there is a unique trace on the family $H^\infty B_n$ with the properties of the previous Proposition (3.4), normalized by $T(1) = 1$. Their proof requires the construction of an inductive basis for the family $H^\infty B_n$. We give a geometric interpretation of their result in section 4.

4. Braids of type B

We use a theorem of Brieskorn [**??**] to derive some geometric interpretations of the braid group ZB_n. The starting point is the reflection representation of the Weyl group WB_n. This group is a semi-direct product

$$(\mathbb{Z}/2)^n \to WB_n \to S_n. \tag{4.1}$$

It acts on complex n-space $\mathbb{C}^n$ as follows:

(1) S_n acts by permuting the coordinates.

(2) $(\mathbb{Z}/2)^n$ act by sign changes $(z_1, \dots, z_n) \mapsto (\varepsilon_1 z_1, \dots, \varepsilon_n z_n)$, $\varepsilon_i \in \{\pm 1\}$.

This group contains the reflections in the hyperplanes

$$z_i = \pm z_j, \quad i \neq j; \quad \text{and} \quad z_j = 0.$$

Let X denote the complement of these hyperplanes. From the theory of finite reflection groups it is known, that $W = WB_n$ acts freely on X. Brieskorn [**??**] shows:

(4.2) Theorem. *The braid group ZB_n is isomorphic to the fundamental group $\pi_1(X/W)$ of the orbit space X/W.* □

If we think of WB_n as the Coxeter group with generators $t, g_1, \dots, g_{n-1}$, then g_j acts as the transposition $(j, j+1)$ and t as $z_1 \mapsto -z_1$.

We use (5.2) to give several interpretations of ZB_n by braids.

We remove the hyperplanes $z_j = 0$ from $\mathbb{C}^n$. It remains the n-fold product $\mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{C}^{*n}$. Removal of the remaining reflection hyperplanes yields the space X of n-tuples $(z_j) \in \mathbb{C}^{*n}$ with pairwise different squares z_j^2.

The *configuration space* $C^n(\mathbb{C}^*)$ is the space of subsets of $\mathbb{C}^*$ with cardinality n. As topological space it is defined as Y/S_n where $Y \subset \mathbb{C}^{*n}$ is the set of n-tuples (y_j) with pairwise different components.

(4.3) Proposition. *X/W is homeomorphic to $C^n(\mathbb{C}^*)$.*

Proof. We arrive at X/W in two steps: First we form $Y' = X/(\mathbb{Z}/2)^n$ and then we divide out by the S_n-action. The map $(z_j) \mapsto (z_j^2)$ yields an S_n-equivariant homeomorphism $Y' \to Y$. □

By (5.2) and (5.3), $ZB_n \cong \pi_1(C^n(\mathbb{C}^*))$. The elements of $\pi_1(C^n(\mathbb{C}^*))$ will be interpreted as braids in the cylinder (cylindrical braids). We take $(1, \omega, \cdots, \omega^{n-1})$, $\omega = \exp(2\pi i/n)$, as base point in $C^n(\mathbb{C}^*)$. A loop in $C^n(\mathbb{C}^*)$ lifts to a path

$$w \colon I \to Y, \quad t \mapsto (w_1(t), \dots, w_n(t))$$

with this initial point. Thus we have

(1) $w(0) = (1, \omega, \dots, \omega^{n-1})$.

(2) $w(1) = (\sigma(1), \dots, \sigma(\omega^{n-1}))$, with a permutation σ of the set $\mathbb{Z}/n = \{1, \omega, \dots, \omega^{n-1}\}$.

(3) For $j \neq k$ we have $w_j(t) \neq w_k(t)$.

These data yield a braid z_w with n strings in $\mathbb{C}^* \times [0,1]$ from $\mathbb{Z}/n \times 0$ to $\mathbb{Z}/n \times 1$

$$z_w(t) = \{w_1(t), \ldots, w_n(t)\} \times t.$$

Homotopy classes of loops correspond to isotopy classes of such braids. Multiplication of loops corresponds to concatenation of braids, as usual. Thus we have:

(4.4) Theorem. *The braid group ZB_n is the group of n-string braids in the cylinder $\mathbb{C}^* \times [0,1]$.* □

A second interpretation is by symmetric braids in $\mathbb{C} \times [0,1]$. This was already used in [3]. We take the base point $(1,2,\ldots,n) \in X$. We lift a loop in X/W to a path

$$w \colon I \to X, \quad t \mapsto (w_1(t), \ldots, w_n(t)).$$

Then we have:
(1) $w(0) = (1,2,\ldots,n)$.
(2) $w(1) = (\pm\sigma(1), \ldots, \pm\sigma(n))$ with a permutation σ of $\{1,\ldots,n\}$.
(3) For $j \neq k$ we have $w_j(t) \neq \pm w_k(t)$.
(4) $w_j(t) \neq 0$.

Let $[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$. The data yield a braid with $2n$ strings in $\mathbb{C} \times [0,1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$, namely

$$t \mapsto \{-w_n(t), \ldots, -w_1(t), w_1(t), \ldots, w_n(t)\} \times t.$$

These braids are $\mathbb{Z}/2$-equivariant with respect to $(z,t) \mapsto (-z,t)$ and are therefore called *symmetric*. The theorem of Brieskorn thus gives:

(4.5) Theorem. *The group ZB_n is isomorphic to the group of symmetric braids with $2n$ strings.* □

Symmmetric braids are drawn as ordinary braids but with additional symmetry with respect to the axis $0 \times [0,1]$. Here are figures for the generators t and g_j.

??

The symmetry is not the reflection in the axis, but corresponds to a spacial rotation about this axis. The relation $tg_1tg_1 = g_1tg_1t$ appears in this context as a generalized Reidemeister move.

Braids in the cylinder with n strings can be visualized as ordinary braids with $n+1$ strings — the axis of the cylinder is the additional string. This method has been used by Lambropoulou [**??**]. It allows the reduction of B_n-type braids to ordinary Artin braids, also with respect to proofs. The theorem of Brieskorn is then not used.

The twofold covering, ramified along the axis, of the cylinder produces a symmetric braid from a cylindrical one — and vice versa.

The cylinder $\mathbb{C}^* \times [0,1]$ has the universal covering $\mathbb{C} \times [0,1]$. Lifting cylindrical braids with n strings produces n-periodic infinite braids in $\mathbb{C} \times [0,1]$ from $\mathbb{Z} \times 0$

to $\mathbb{Z} \times 1$. They are invariant with respect to the translation $(z,t) \mapsto (z+n,t)$. This gives yet another interpretation of ZB_n by n-periodic braid pictures.

The relation between ZB_n and $Z\tilde{A}_{n-1}$ has the following geometric origin or counterpart. The map

$$\mathbb{C}^{*n} \to \mathbb{C}^*, \quad (z_1, \dots, z_n) \mapsto z_1 \cdot \ldots \cdot z_n$$

is S_n-equivariant and induces therefore a map from the configuration space

$$\alpha\colon C^n(\mathbb{C}^*) \to \mathbb{C}^*.$$

(4.6) Lemma. *The map α is a fibre bundle.*

PROOF. Let

$$H = \{(z_1, \dots, z_n) \in \mathbb{C}^{*n} \mid \prod z_j = 1\}.$$

This is an S_n-invariant subset. The map

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} H \to \mathbb{C}^{*n}, \quad (z, z_1, \dots, z_n) \mapsto (zz_1, \dots, zz_n)$$

is an S_n-equivariant homeomorphism. Thus γ is the fibre bundle with fibre H assoziated to the $\mathbb{Z}/n$-principal bundle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. In $\mathbb{C}^{*n}$ we have to remove the subset

$$C = \{(z_1, \dots, z_n) \mid \text{there exists } i \neq j \text{ such that } z_i = z_j\}.$$

Let $D = H \cap C$. Then γ induces an S_n-equivariant homeomorphism

$$\gamma\colon \mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus D) \to \mathbb{C}^{*n} \setminus S.$$

This yields the fibre bundle description

$$\mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus T)/S_n \to \mathbb{C}^*$$

for the configuration space. □

We apply the fundamental group to this fibration and obtain the exact sequence

$$1 \to \operatorname{kernel} \alpha_* \to ZB_n \to \mathbb{Z} \to 0.$$

It can be shown that this is the sequence (2.11), i. e. $Z\tilde{A}_{n-1}$ is the fundamental group of the fibre of α.

Our next aim is to describe an additive basis of the Hecke algebra $H^\infty B_n$ by geometric means, i. e. by specifying a certain canonical set of basic braids.

A cylindrical braid with n strings is called *descending*, if for $i < j$ the i-th string is always overcrossing the j-th string. The i-th string is the one starting at ω^i, $0 \leq i \leq n-1$. Overcrossing means the following: We look radially and orthogonally from infinity onto the axis. The braid is in general position if we only see transverse double points. The first string we meet, coming from infinity, is the overcrossing one.

(4.7) Theorem. *The descending braids form a $\mathfrak{K}$-basis of the algebra $H^\infty B_n$. The descending braids with winding number zero form a $\mathfrak{K}$-basis of the algebra $H\tilde{A}_{n-1}$.*

We use (2.11) to reduce the first statement to the second. For the latter Hecke algebra we have the canonical basis related to the elements of reduced form in the Weyl group, and elements of the Weyl group will be shown to correspond to descending braids. We use the description of the Weyl group elements as n-periodic permutations of $\mathbb{Z}$. We represent such a permutation by n straight lines $c_1, \ldots, c_n$ in the strip $\mathbb{R} \times [0,1]$ starting at $\{1, \ldots, n\} \times 0$ such that c_i and c_j have at most one crossing, and then repeat with period n. By slightly moving the endpoints of the c_j we can assume that the curves are in general position. The resulting crossings are used to write the permutation as a product of reflections. This product is reduced in the sense of Coxeter group theory (see (??)). It is geometrically obvious that the same configuration of crossings can be realized by a descending braid.

(4.8) Proposition. *The set*

$$\mathfrak{C} = \{y_{n-1}^k g_{n-1} g_{n-2} \ldots g_j \mid k \in \mathbb{Z}, 1 \le j \le n\}$$

is a system of representatives for the left cosets of the inclusion $W^\infty B_{n-1} \subset W^\infty B_n$.

Proof. This is an immediate consequence of the semi-direct product description. The powers of y_{n-1} are representatives for cosets of $V_{n-1} \subset V_n$, and the products $g_{n-1} \ldots g_j$ are representatives for the cosets of $S_{n-1} \subset S_n$. □

We use this Proposition to derive the following result of Lambropoulou and Przytycki which was proved by them in a purely algebraic manner. The relation to standard Hecke algebra bases and the interpretation by descending braids seems more transparent, though.

(4.9) Theorem. *Let $\mathfrak{B}$ be the canonical basis of $H^\infty B_{n-1}$. Then $\{bc \mid b \in \mathfrak{B}, c \in \mathfrak{C}\}$ is a basis of $H^\infty B_n$.*

Proof. Represent a basis element of $H^\infty B_n$ by a descending braid. □

Recall the construction and definition of a Markov trace in section 2. The last Theorem gives immediately the uniqueness of a Markov trace with given parameters.

(4.10) Corollary. *There exists a unique Markov trace on $H^\infty B_n$ with given parameters $(s(k) \mid k \in \mathbb{Z})$ and z.* □

From a Markov trace (U^n) on ZA one obtains a link invariant. Let $\hat{x}$ denote the Alexander closure of the braid $x \in ZA_{n-1}$. Write x as a product of symbols (crossings) $g_1, g_1^{-1}, \ldots, g_{n-1}, g_{n-1}^{-1}$, and let $w(x)$ denote the resulting sum of exponents (writhe of x). Then a link invariant P is obtained by setting

$$P(\hat{x}) := \alpha^{-w(x)} \beta^n U^n(x)$$

for $x \in ZA_{n-1}$. Related are Markov traces $Tr = (Tr_n)$ on Hecke algebras $HA = (HA_n)$. These are $\mathfrak{K}$-linear maps $Tr_n\colon HA_{n-1} \to \mathfrak{K}$ such that

(1) $Tr_{n+1}|HA_{n-1} = Tr_n$,

(2) $TR_{n+1}(xx_n) = zTr_n(x), \quad x \in HA_{n-1}$

with a parameter $z \in \mathfrak{K}$. Here we use the names $x_1, \dots, x_{n-1}$ for the standard generators of the Hecke algebra because we want to distinguish them from the g_j. The Hecke algebras are defined with a parameter $q \in \mathfrak{K}^*$ which enters the quadratic relation $x_j^2 = (q-1)x_j + q$. The relation between the two notions of Markov traces is the following.

(4.11) Proposition. *Let $q = p^2$ and $\beta(p - p^{-1}) = \alpha - \alpha^{-1}$ with $p^2 \neq 1$. Let $U = (U^n)$ be a Markov trace on ZA with parameters α, β, as defined in section 3. Let $\iota\colon ZA_n \to (HA_n)^*$ be the homomorphism $g_j \mapsto p^{-1}x_j$. Then there exists a unique Markov trace Tr on HA such that $Tr_n \circ \iota = U^n$. It has parameter $z = p^{-1}\alpha\beta^{-1}$. The corresponding link invariant satisfies the skein relation $\alpha P(L+) - \alpha^{-1}P(L_-) = (p - p^{-1})P(L_0)$.* □

Lambropoulou [**??**] has proved a Markov theorem for links of type B (symmetric links). The statement is exactly as in the classical case, here called of type A. A Markov trace $(T^n\colon TB_n \to \mathfrak{K})$ therefore yields an invariant of B-links by setting

$$P(\hat{x}) = \alpha^{-w(x)}\beta^n T^n(x)$$

for $x \in ZB_n$. Here $w(x)$ still counts the exponent sum in terms of the generators g_j.

5. Representations of Hecke algebras and R-matrices

6. References

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