Knot theories and root systems
Part III

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We continue our study of knot theory from the point of view of root systems. The presentation is partly expository, in preparation for later parts of this series.

The first part of this note is concerned with the Coxeter-Dynkin series of type D. It turns out that some relevant algebra can be reduced to the corresponding algebra for the series A and B. We demonstrate this for the Temperley-Lieb algebra TD_n and the Kauffman functor for D-tangles.

1. Hecke algebras and Temperley-Lieb algebras

This section collects some general results. Let S be a finite set. A *Coxeter matrix* is a symmetric mapping

$$m\colon S \times S \to \mathbb{N} \cup \{\infty\}$$

such that $m(s,s) = 1$ and $m(s,t) \geq 2$ for $s \neq t$. A Coxeter matrix (S,m) is often specified by its *Coxeter graph* $\Gamma(S,m)$. It has S as its set of vertices and an edge with weight $m(s,t)$ whenever $m(s,t) \geq 3$. Usually, he weight $m(s,t) = 3$ is omitted in the notation.

The *standard Hecke algebra* $H_q(S,m)$ associated to a Coxeter matrix (S,m) is the associative algebra with 1 over the commutative ring $\mathfrak{K}$ with generators $(x_s \mid s \in S)$ and relations

(1.1)
$$\begin{array}{rcll} x_s^2 & = & (q-1)x_s + q, & q \in K^* \\ x_s x_t x_s \ldots & = & x_t x_s x_t \ldots, & m(s,t) \geq 2 \end{array}$$

($m(s,t)$ factors on each side, alternating). Here $\mathfrak{K}^*$ denotes the unit groups of $\mathfrak{K}$.

Suppose m takes values in $\{1,2,3\}$ — the simply laced case. Then the *Temperley-Lieb algebra* $T_d(S,m)$ is the associative algebra with 1 over $\mathfrak{K}$ with generators $(e_s \mid s \in S)$ and relations

(1.2)
$$\begin{array}{rcll} e_s^2 & = & de_s & d \in \mathfrak{K}^* \\ e_s e_t & = & e_t e_s & m(s,t) = 2 \\ e_s e_t e_s & = & e_s & m(s,t) = 3. \end{array}$$

We shall obtain the Temperley-Lieb algebra as a quotient of a Hecke algebra. For this purpose we assume

(1.3) $$p \in \mathfrak{K}^*, \quad q = p^2, \quad d = p + p^{-1}.$$

(1.4) Proposition. *Under the hypothesis* (1.3) *the assignment* $x_s \mapsto pe_s - 1$ *defines a surjective homomorphism*

$$\varphi\colon H_q(S,m) \to T_d(S,m).$$

The kernel of φ *is the twosided ideal generated by the elements*

$$x(s,t) = x_s x_t x_s + x_s x_t + x_t x_s + x_s + x_t + 1;$$

here (s,t) *runs over the pairs* (s,t) *with* $m(s,t) = 3$*.*

Proof. We verify that φ respects the defining relations of the Hecke algebra. Certainly $\varphi(x_s x_t) = \varphi(x_t x_s)$, whenever $m(s,t) = 2$. Moreover,

$$\begin{aligned} & (pe_s - 1)(pe_s - 1) \\ = \ & p^2 e_s^2 - 2pe_s + 1 \\ = \ & p^2(p + p^{-1})e_s - 2pe_s + 1 \\ = \ & (p^2 - 1)(pe_s - 1) + p^2, \end{aligned}$$

hence compatibility with the quadratic relations of the Hecke algebra. Finally,

$$\begin{aligned} & (pe_s - 1)(pe_t - 1)(pe_s - 1) \\ = \ & p^3 e_s e_t e_s - p^2(e_s e_t + e_t e_s + e_s^2) + p(2e_s + e_t) - 1 \\ = \ & -p^2(e_s e_t + e_t e_s) + pe_s + pe_t - 1, \end{aligned}$$

if $m(s,t) = 3$. The result is symmetric in s,t; hence compatibility with the remaining relations of the Hecke algebra.

Certainly, φ is surjective. Let $I \subset H_q(S,m)$ denote the ideal generated by the $x(s,t)$ for (s,t) with $m(s,t) = 3$. We define a homomorphism

$$\psi\colon T_d(S,m) \to H_q(S,m)/I$$

by

$$\psi(e_s) = p^{-1}(x_s + 1).$$

We have to verify compatibility with (1.2). The first relation follows from the quadratic relation in the Hecke algebra, the second one is obviously satisfied. For the third one we compute modulo I

$$\begin{aligned} & p^{-1}(x_s + 1)p^{-1}(x_t + 1)p^{-1}(x_s + 1) \\ = \ & p^{-3}(x(s,t) + x_s^2 + x_s) \\ = \ & p^{-3}((p^2 - 1)x_s + p^2 + x_s) \\ = \ & p^{-1}(x_s + 1). \end{aligned}$$

A similar computation shows that $x(s,t)$ is contained in the kernel of φ. Hence φ induces $\varphi\colon H_q/I \to T_d$. By construction, φ and ψ are inverse homomorphisms.□

The preceding construction can, in particular, be applied to Coxeter matrices of ADE-type. The resulting algebras are then finite dimensional. The structure of TA_{n-1} associated to the linear graph A_{n-1} with n vertices is well known, see [**??**]; this is the classical Temperley-Lieb algebra. In the following sections 2 and 3 we study the algebras related to the graph D_n with n vertices ($n \geq 4$). In section 4 we briefly discuss D-tangles and the associated Kauffman functor.

But first we present one general result: By way of example we show that $T_d(S,m)$ is non-zero. This is done by constructing a standard module which arises from the reflection representation of the Hecke algebra [**??**].

We work with a field $\mathfrak{K}$. Let V denote the free $\mathfrak{K}$-module with basis $\{v_s \mid s \in S\}$. We define a symmetric bilinear form B on V by

(1.5)
$$\begin{aligned} B(v_s,v_s) &= q+1 & & \\ B(v_s,v_t) &= p & & m(s,t)=3 \\ B(v_s,v_t) &= 0 & & m(s,t)=2. \end{aligned}$$

We define a linear map $X_s\colon V \to V$ by

(1.6)
$$X_s(v) = qv_s - B(v_s,v)v.$$

Then $X_s(v_s) = -v_s$, and $X_s(v) = v$ for v in the orthogonal complement of v_s. We assume $q+1 \in \mathfrak{K}^*$. Then V is the orthogonal direct sum of $\mathfrak{K}v_s$ and $(\mathfrak{K}v_s)^\perp$. On the latter, X_s acts as multiplication by q. Hence X_s satisfies the quadratic equation $X_s^2 = (q-1)X_s + q$ of the Hecke algebra.

The determinant $d_{s,t}$ of B on the submodule $\langle\, v_s, v_t \,\rangle$ generated by v_s and v_t equals

(1.7)
$$d_{s,t} = \begin{cases} (q+1)^2 & m(s,t)=2 \\ q^2+q+1 & m(s,t)=3. \end{cases}$$

We therefore also assume $q^2+q+1 \in \mathfrak{K}^*$. Then V is the orthogonal direct sum of $\langle\, v_s, v_t \,\rangle$ and $\langle\, v_s, v_t \,\rangle^\perp$. On the latter subspace, X_s and X_t act as multiplication by q. The action of X_s and X_t on $\langle\, v_s, v_t \,\rangle$ in the basis v_s, v_t is given by

(1.8)
$$X_s = \begin{pmatrix} -1 & p \\ 0 & q \end{pmatrix}, \quad X_t = \begin{pmatrix} q & 0 \\ p & -1 \end{pmatrix},$$

in the case $m(s,t)=3$. A simple computation shows

(1.9)
$$X_sX_tX_s = X_tX_sX_t = \begin{pmatrix} 0 & -pq \\ -pq & 0 \end{pmatrix}.$$

Thus we have constructed the reflection representation V of $H_q(S,m)$.

The assignment

(1.10)
$$\omega\colon H_q(S,m) \to H_q(S,m), \quad x_s \mapsto -qx_s^{-1}$$

is an involutive automorphism of the Hecke algebra. It transforms V into a new module $W = V^\omega$.

(1.11) Proposition. *The module W factors over the homomorphism φ of* (1.4).

PROOF. We set $Y_s = -qX_s^{-1}$. We have to show that the operator

$$Y_{s,t} = Y_sY_tY_s + Y_sY_t + Y_tY_s + Y_s + Y_t + 1$$

acts on V as the zero map. With (1.8) we compute that $Y_s + 1$ and $Y_t + 1$ act on $\langle v_s, v_t \rangle$ in the basis v_s, v_t through the matrices

$$Z_s = \begin{pmatrix} q+1 & -p \\ 0 & 0 \end{pmatrix}, \quad Z_t = \begin{pmatrix} 0 & 0 \\ -p & q+1 \end{pmatrix}.$$

This is used to verify on $\langle v_s, v_t \rangle$

$$Z_sZ_tZ_s = qZ_s.$$

A formal calculation, using the quadratic equation for Y_s, yields

$$(Y_s + 1)(Y_t + 1)(Y_s + 1) - q(Y_s + 1) = Y_{s,t}.$$

Therefore $Y_{s,t}$ acts as zero on $\langle v_s, v_t \rangle$. Since X_s is multiplication by q on $\langle v_s, v_t \rangle^\perp$, we see that $-qX_s^{-1} + 1$ is the zero map. □

(1.12) Remark. *The determinant of the bilinear form B for the graph D_n in the basis $(v_s \mid s \in S)$ equals $(q+1)(q^{n-1}+1)$. Therefore the form is regular if q is not a root of unity.* ♡

We give a more direct construction of a $T_d(S, m)$-module which does not use the reflection representation of the Hecke algebra. Let $A = (a_{st})$ denote a symmetric $S \times S$-matrix over $\mathfrak{K}$. We consider the associative algebra $T(A)$ over $\mathfrak{K}$ with generators $(Z_s \mid s \in S$ and relations

(1.13)
$$\begin{aligned} Z_s^2 &= a_{ss}Z_s \\ Z_sZ_tZ_s &= a_{st}a_{ts}Z_s. \end{aligned}$$

Then a simple verification from the definitions give:

(1.14) Proposition. *Let V be the $\mathfrak{K}$-module with basis $(v_s \mid s \in S$. The operators $Z_s(v_t) = a_{st}v_s$ make V into a $Z(A)$-module. (Hence each Z_s has rank at most one on V.)* □

The matrix $A = (a_{st})$ is called *indecomposable*, if there is no partition $S = S_1 \coprod S_2$ with $a_{uv} = 0$ for $u \in S_1$, $v \in S_2$.

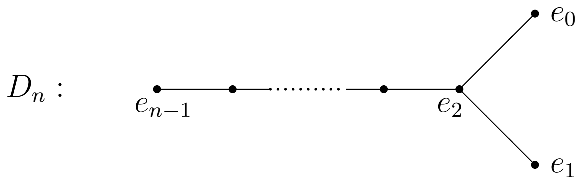
(1.15) Proposition. *Let $\mathfrak{K}$ be a field. Suppose A is indecomposable and* $\det(A) \neq 0$. *The the module V of the previous proposition is a simple $T(A)$-module.*

PROOF. We have $Z_s(\sum_j a_j v_j = (\sum_j a_j a_{sj}) v_s$. Suppose $v = \sum_j a_j v_j \neq 0$. Since $\det(A) \neq 0$, not all $Z_s v$ are zero. If $0 \neq M \subset V$ is a $T(A)$-submodule, then there exists $s \in S$ with $v_s \in M$. Suppose $v_t \notin M$. Since $Z_t v_s = a_{ts} v_t \in M$, we must have $a_{ts} = 0$. This contradicts the indecomposability of A. Hence all v_t are containes in M. □

In the case of a Coxeter graph, we set $a_{ss} = d$, $a_{st} = 1$ for $m(s,t) = 3$, and $a_{st} = 0$ for $m(s,t) = 2$. Then V becomes a module over $T_d(S,m)$. Also, $\det(A)$ is a non-trivial monic polynomial in d, hence in general not zero.

2. The structure of TD_n

The algebra TD_n with parameter $d \in \mathfrak{K}$ is associated to the graph D_n. In the following figure we have specified the names of the generators.



The algebra TD_n will be decomposed into an algebra which belongs to the linear graph A_{n-1} and another algebra which is related to the graph B_n. Here A_{n-1} is the linear Coxeter graph with $n-1$ vertices $e_1, \dots, e_{n-1}$ and $m(e_j, e_{j+1}) = 3$. We use the same notation for the generators of TD_n and TA_{n-1}. The following is easily verified.

(2.1) Proposition. *The assignment $e_0 \mapsto e_1$ and $e_j \mapsto e_j$ $(j \geq 1)$ defines a surjective homomorphism $\alpha\colon TD_n \to TA_{n-1}$.* □

We remark that the automorphism of the graph D_n which interchanges e_0 and e_1 and fixes e_j for $j \geq 2$ induces an involution $\tau\colon TD_n \to TD_n$. We have $\alpha\tau = \alpha$.

(2.2) Proposition. *The kernel of α is the twosided ideal I generated by the difference $e_0 - e_1$. The homomorphism $\alpha_1\colon TA_{n-1} \to TD_n$, $e_1 \mapsto e_j$ is right inverse to α. We therefore have a splitting of modules*

$$TD_n = I \oplus TA_{n-1}.$$

Proof. The inclusion $I \subset \operatorname{kernel}\alpha$ follows from the definitions. The relation $\alpha\alpha_1 = \mathrm{id}$ is obvious. The composition

$$TD_n/I \xrightarrow{\ \alpha\ } TA_{n-1} \xrightarrow{\ \alpha_1\ } TD_n/I$$

is easily seen to be be the identity on generators. Hence $I = \operatorname{kernel}\alpha$. □

We now use the following algebra $T'D_n$ of Temperley-Lieb type: It has generators $\varepsilon_0, \dots, \varepsilon_{n-1}$ and relations

$$\textbf{(2.3)}\qquad \begin{aligned} \varepsilon_j^2 &= d\varepsilon_j & & j \geq 1 \\ \varepsilon_0^2 &= 2\varepsilon_0 & & \\ \varepsilon_i\varepsilon_j &= \varepsilon_j\varepsilon_i & & |i-j| \geq 2 \\ \varepsilon_1\varepsilon_0\varepsilon_1 &= d\varepsilon_1 & & \\ \varepsilon_i\varepsilon_j\varepsilon_i &= \varepsilon_i & & |i-j| = 1; i,j \geq 1. \end{aligned}$$

This is a variant of the algebra of B_n type which has been studied in [**??**],[**??**].

(2.4) Proposition. *The assignment $\beta(e_0) = (\varepsilon_0 - 1)\varepsilon_1(\varepsilon_0 - 1)$ and $\beta(e_j) = \varepsilon_j$ for $j \geq 1$ defines a homomorphism*

$$\beta\colon TD_n \to T'D_n.$$

Proof. For $j \geq 1$, the e_j and ε_j satisfy the same relations; we consider the remaining ones. We use

$$(\varepsilon_0 - 1)^2 = 1, \quad \varepsilon_1(\varepsilon_0 - 1)\varepsilon_1 = 0$$

and verify easily $\beta(e_0^2) = d\beta(e_0)$ and $\beta(e_0e_2e_0) = \beta(e_2e_0e_2)$. Moreover $\beta(e_0e_1) = 0 = \beta(e_1e_0)$. □

The proof of the previous proposition shows that the twosided ideal J generated by e_0e_1 is contained in the kernel of β. The image of β will turn out to be half of $T'D_n$, and J is equal to the kernel of β. In order to prove these statements we introduce the crossed product of $TD_n/J =: \mathfrak{A}$ with the algebra $\mathfrak{K}[\tau]/(\tau^2-1)$ where τ acts via the previously defined involution τ on $\mathfrak{A}$. Formally, this crossed product $\mathfrak{B}$ is defined as the free $\mathfrak{A}$-module with basis $1, \tau$ and multiplication

$$(a + b\tau)\cdot(c + d\tau) := (ac + bd^\tau) + (bc^\tau + ad)\tau,$$

where x^τ denotes the action of τ on x. Note that the ideal J is τ-stable.

(2.5) Proposition. *The assignment $\varepsilon_j \mapsto \varepsilon_j$ $(j \geq 1)$ and $\varepsilon_0 \mapsto 1 + \tau$ defines an isomorphism $\beta_1\colon T'D_n \to \mathfrak{B}$. The image of β corresponds to the subalgebra $\mathfrak{A}$. The kernel of β is equal to J.*

Proof. The relation $\varepsilon_0^2 = 2\varepsilon_0$ corresponds to $(1+\tau)^2 = 1 + 2\tau + 1 = 2(1+\tau)$. The element $\varepsilon_1\varepsilon_0\varepsilon_1$ is mapped to $e_1 \cdot (1+\tau)\cdot e_1 = (e_1 + e_1\tau)\cdot e_1 = e_1^2 + e_1e_0\tau$ and this equals de_1 modulo J. We see that β_1 is well-defined and surjective. The inverse homomorphism is given by β and $\tau \mapsto \varepsilon_0 - 1$. □

The algebras $T'D_n$ and TA_{n-1} have augmentation homomorphisms to $\mathfrak{K}$ which map the generators ε_j and e_j to zero. Let $T''D_n$ denote the image of β. We have a sequence

$$0 \to TD_n \xrightarrow{(\alpha_1,\beta)} TA_{n-1} \oplus T''D_n \to \mathfrak{K} \to 0; \tag{2.6}$$

the map to $\mathfrak{K}$ is the difference of the augmentations. The structure of TD_n will be obtained from the next result.

(2.7) Theorem. *The sequence* (2.6) *is exact.*

Proof. We show that α maps the kernel J of β isomorphically onto the kernel of the augmentation. For this purpose we recall a basis of TA_{n-1}, see [**??**]. We set

$$e(i,j) = e_ie_{i-1}\dots e_j, \quad i \geq j.$$

Then a basis of TA_{n-1} is given by the products

$$e(i_1,j_1)\cdot \ldots \cdot e(i_p,j_p)$$

$$\begin{matrix} 1 \leq i_1 < \ldots < i_p \leq n-1, & 1 \leq j_1 < \ldots < j_p \leq n-1 \\ j_s \leq i_s, & 0 \leq p \leq n-1. \end{matrix} \tag{2.8}$$

We exhibit a basis of J which is mapped onto this basis. We use the notation

$$\bar{e}(i,j) = e_i e_{i-1} \dots e_1 e_0 e_2 \dots e_j, \quad j \geq 2.$$

This element is mapped under α to $de(i,j)$. Recall that $d \in \mathfrak{K}^*$ is a unit.

We show that J is spanned by the elements of the form (2.8) where $e(i_1, j_1)$ is replaced by $\bar{e}(i_1, j_1)$; this finishes the proof of (2.7).

We consider words in the symbols $e_0, \dots, e_{n-1}$. An *elementary reduction* of a word is one of the following replacements: $e_j e_j$ by de_j, $e_i e_j e_i$ by e_i, $e_i e_j$ by $e_j e_i$. Note that the length of the word is not increased. A coefficient of the form d^r may appear. A word is in *reduced form*, if it cannot be shortened by an elementary reduction. The words (2.8) are reduced. Certainly, J is generated by reduced words.

We claim that J is generated by reduced words of the form ae_0e_1b in which a and b do not involve e_0 and e_1.

We know already that J is the ideal generated by words ce_0e_1d. If d contains e_1, say, then the word contains a string of the form e_1xe_1 in which x involves only e_j, $j \geq 2$. A word of this type is never reduced; this follows easily by using (2.8) for x. This shows the claim.

We next consider normal forms of reduced words in J by induction on n. Suppose a reduced word contains two factors e_{n-1}, say a string $e_{n-1}ye_{n-1}$ with y not involving e_{n-1} and of shortest length. Then, by induction, this string must equal $\bar{e}(n-1, n-1)$. If a word contains $z = \bar{e}(n-1, n-1)$, it is not reduced, unless it is equal to z. Therefore z is the only reduced word in J with two appearances of e_{n-1}. Next, consider reduced words which have the form $w = xe_{n-1}y$. By interchanging elements, if necessary, we assume that y has minimal length. Then y necessarily has the form $e(n-2, j)$ or $\bar{e}(n-2, j)$. Since x does not contain e_{n-1}, we can apply the induction hypothesis to x. Since w is reduced, it is easily seen that w has the form (2.8) with $e(i_1, j_1)$ replaced by $\bar{e}(i_1, j_1)$. □

We assume known the structure of TA_{n-1} in the generic case (q not a root of unity) [**??**]. It remains to study the algebra $T''D_n$. This is the subject of the next section.

We conclude this section with some remarks concerning the algebra of the graph D_n: braid groups and Hecke algebras.

Each Coxeter matrix (S, m) has associated to it a *braid group* $Z(S, m)$ with generators $(x_s \mid s \in S)$ and relations $x_s x_t x_s \dots = x_t x_s x_t \dots$ with $m(s,t)$ factors on each side. For the graph D_n we define another braid group $Z'D_n$ with generators $\kappa_0, \dots, \kappa_{n-1}$ and relations

(2.9)
$$\begin{aligned} \kappa_i \kappa_j \kappa_i &= \kappa_j \kappa_i \kappa_j & |i-j| = 1; i,j \geq 1 \\ \kappa_0 \kappa_1 \kappa_0 \kappa_1 &= \kappa_1 \kappa_0 \kappa_1 \kappa_0 & \\ \kappa_i \kappa_j &= \kappa_j \kappa_i & |i-j| \geq 2 \\ \kappa_0^2 &= 1. & \end{aligned}$$

This is a quotient of the group ZB_n for which the last relation is not present.

(2.10) Proposition. *The group $Z'D_n$ is the semidirect product of ZD_n with $\mathbb{Z}/2$. The generator τ of $\mathbb{Z}/2$ acts on ZD_n by the automorphism induced by the graph automorphism.*

Proof. Let G denote the semi-direct product. We define inverse homomorphisms $f\colon G \to Z'D_n$ and $g\colon Z'D_n \to G$ by

$$f\colon \tau, x_0, x_1, \dots, x_{n-1} \mapsto \kappa_0, \kappa_1\kappa_0\kappa_1, \kappa_1, \dots, \kappa_{n-1}$$

$$g\colon \kappa_0, \kappa_1, \dots, \kappa_{n-1} \mapsto \tau, x_1, \dots, x_{n-1}.$$

□

We remark that conjugation by κ_0 corresponds to τ.

We define the Hecke algebra $H'D_n$ as the associative algebra with 1 generated by $\kappa_0, \dots, \kappa_{n-1}$ with braid relations as above and quadratic relations $\kappa_0^2 = 1$ and $\kappa_j^2 = (q-1)\kappa_j + q$ for $j \geq 1$. This is a Hecke algebra of B_n-type where the parameter Q belonging to κ_0 has been specialized to 1. We have an embedding $\tilde{\alpha}\colon HD_n \to H'D_n$, $x_0 \mapsto \kappa_0\kappa_1\kappa_0$, $x_j \mapsto \kappa_j$ for $j \geq 1$. As in the case of the Temperley-Lieb algebras we see:

(2.11) Proposition. *The algebra $H'D_n$ is the crossed product of HD_n with $\mathfrak{K}[\tau]/(\tau^2 - 1)$.* □

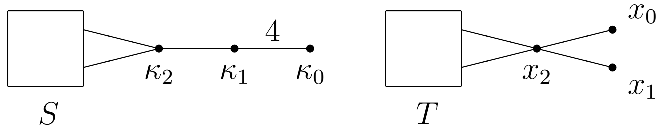
There is a connecting between Hecke algebras and Temperley-Lieb algebras as follows.

(2.12) Proposition. *The algebra $T'D_n$ is a quotient of $H'D_n$ under the homomorphism $\varphi'\colon \kappa_0 \mapsto \varepsilon_0 - 1$, $\kappa_j \mapsto p\varepsilon_j - 1$ $(j \geq 1)$. The diagram*

$$\begin{array}{ccc} HD_n & \xrightarrow{\tilde{\alpha}} & H'D_n \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi'} \\ TD_n & \xrightarrow{\alpha} & T'D_n \end{array}$$

is commutative. □

For use in other situations we formalize the preceding discussion. Suppose we have two Coxeter graphs which differ in a similar way as B_n and D_n do. The following figure illustrates this situation:



The box stands for the same subgraph in both cases and is connected to x_2 (resp. κ_2) by the same weighted edges. The graph T carries the involution τ which interchanges κ_0 and κ_1. Therefore we can form the semi-direct product of the braid group

(2.13) $$ZT \times_\tau \mathbb{Z}/2.$$

We also have the quotient

(2.14) $$Z'T = ZS/(\kappa_0^2 = 1).$$

The same proof as for (.??) shows:

(2.15) Proposition. *The groups $ZT \times_\tau \mathbb{Z}/2$ and $Z'T$ are isomorphic.* □

Also (??) has its analogue in this case.

3. The reduced Temperley-Lieb algebra

This section presents the structure of $T'D_n$ and $T''D_n$ for generic parameters (p not a root of unity). The algebra $T'D_n$ is of the type B_n but not exactly the same. Therefore we have to extend some of results in [**??**] to the present situation.

There exists idempotent elements f_k and g_k in $T'D_n$ with the following properties:

(1) $1 - \frac{1}{2}\varepsilon_0$

(2) $f_k = f_{k-1} + \frac{p^{k-1}+p^{-k+1}}{p^k+p^{-k}} f_{k-1} v e_k f_{k-1}, \quad 1 \le k \le n-1$

(3) $\varepsilon_j f_k = f_k \varepsilon_j = 0, \quad 0 \le k \le k$

(4) $g_0 = \dfrac{1}{2}\varepsilon_0$

(5) $g_k = g_{k-1} + \frac{p^{k-1}+p^{-k+1}}{p^k+p^{-k}} g_{k-1}\varepsilon_k g_{k-1}, \quad 1 \le k \le n-1$

(6) $\varepsilon_j g_k = g_k \varepsilon_j = 0, \quad 1 \le j \le k$

(7) $\varepsilon_0 g_k = g_k \varepsilon_0, \quad 0 \le k \le n-1$

(8) $g_k f_k = f_k g_k = 0, \quad 0 \le k \le n-1$

(9) $\eta(f_k) = 1 - \frac{1}{2}\varepsilon_0$

(10) $\eta(g_k) = \frac{1}{2}\varepsilon_0$

The map η is the augmentation which sends e_j, $j \ge 1$, to zero.

The proof for these assertions is as for [**??**], Satz 5.2, by induction on k. With the help of the central orthogonal idempotents f_{n-1} and g_{n-1} it is shown as in [**??**], Satz (7.1), that the Bratteli diagram of the inclusion $T'D_{n-1} \subset T'D_n$ is the same as for the inclusion $TB_{n-1} \subset TB_n$. In particular, $T'D_n$ has $n+1$ simple modules $M_0(n), M_1(n), \dots, M_n(n)$ with $M_j(n) = N_j$ of dimension $\binom{n}{j}$.

The simple modules of $T''D_n$ are determined via restriction from $T'D_n$.

(3.1) Theorem. *The algebra $T''D_n$ has the following irreducible modules:*

(1) *Suppose $n = 2k+1$. The restrictions* $\mathrm{res}M_j$ *for $j \le k$. Moreover* $\mathrm{res}M_j \cong \mathrm{res}M_{n-j}$.

(2) *Suppose $n = 2k$. The restrictions* $\mathrm{res}M_j$, $j < k$. *In this case* $\mathrm{res}M_j \cong \mathrm{res}M_{n-j}$. *The module* $\mathrm{res}M_k$ *is the direct sum of two simple $T''D_n$-module of the same dimension.*

The proof of (3.1) is by induction on n. One uses the structure of the Bratteli-diagram for $T'D_{n-1} \subset T'D_n$ and the following general fact about the crossed product construction of $T'D_n$ from $T''D_n$.

Let $\mathfrak{A}$ be a semi-simple algebra with an involutive automorphism τ over the field $\mathfrak{K}$ of characteristic zero and let $\mathfrak{B}$ denote the crossed product algebra as described in section 2. If U is an $\mathfrak{A}$-module, let U^τ denote the same vector space with the $\mathfrak{A}$-action twisted by τ. The map $a + b\tau \mapsto a - b\tau$ is an automorphism of $\mathfrak{B}$. If V is a $\mathfrak{B}$-module, then $\bar{V}$ is obtained from V by twisting with this automorphism (conjugate module). A simple $\mathfrak{A}$-module U is called of type I (resp. type II) if $U \cong U^\tau$ (resp. $U \not\cong U^\tau$). A simple $\mathfrak{B}$-module V is called of type I (resp. type II) if $V \not\cong \bar{V}$ (resp. $V \cong \bar{V}$). If U is an $\mathfrak{A}$-module, we call $\mathfrak{B} \otimes_{\mathfrak{A}} U$ the induced $\mathfrak{B}$-module $\mathrm{ind}\, U$. These notations are used in the statement of the following result.

(3.2) Proposition.

(1) *Suppose* V *is a simple* $\mathfrak{B}$*-module of type* I*. Then* $\mathrm{res}V = U$ *is simple of type* I *and* $\mathrm{ind}\, U \cong V \oplus \bar{V}$*.*

(2) *Suppose* V *is a simple* $\mathfrak{B}$*-module of type II. Then* $\mathrm{res}V \cong U \oplus U^\tau$ *and* $V \cong \mathrm{ind}\, U \cong \mathrm{ind}\, U^\tau$ *and* U, U^τ *of type two.*

(3) *Suppose* U *is a simple* $\mathfrak{A}$*-module of type I. Moreover,* $\mathrm{res}V \cong \mathrm{res}\bar{V} \cong U$*.*

(4) *Suppose* U *is a simple* $\mathfrak{A}$*-module of type II. Then* $\mathrm{ind}\, U = V$ *is a simple* $\mathfrak{B}$*-module of type II and* $\mathrm{res}V \cong U \oplus U^\tau$*.*

PROOF. The proof of this proposition is by an adaption of the argument in [**??**], Ch. VI for the proof of Theorem (7.3). □

Proof of (3.1). By (3.2) it suffices to determine the modules $\bar{M}_j(n)$. Let res_{n-1} denote the restriction via $T'D_{n-1} \subset T'D_n$. From the Bratteli diagram we know

$$\begin{array}{l} \mathrm{res}_{n-1} M_j(n) = M_{j-1}(n-1) \oplus M_j(n), \quad 1 \le j \le n-1 \\ \mathrm{res}_{n-1} M_0(n) = M_0(n-1),\ \mathrm{res}_{n-1} M_n(n) = M_{n-1}(n-1). \end{array} \tag{3.3}$$

The isomorphism type of a simple $T'D_n$-module M is therefore determined by $\mathrm{res}_{n-1}M$. We show by induction on n that $\bar{M}_j(n) = M_{n-j}(n)$. Since restriction is compatible with conjugation, the induction step follows from (3.3). The induction starts with the irreducible representations of the group $\mathbb{Z}/2$ generated by τ. □

4. Braids and tangles of type D_n

The permutations σ of

$$[\pm n] = \{-n, \ldots, -1, 1, \ldots, n\}$$

with the property $\sigma(-i) = -\sigma(i)$ form the Weyl group WB_n of the root system B_n. The subgroup of even permutations in WB_n is the Weyl group WD_n of the root system D_n. The group WD_n has order $2^{n-1} \cdot n!$ and is a semi-direct product

$$1 \to (\mathbb{Z}/2)^{n-1} \to WD_n \to S_n \to 1$$

with the symmetric group S_n. The reflection representation of WD_n on $\mathbb{C}^n$ is given as follows: The subgroup S_n acts by permutation of coordinates and $(\mathbb{Z}/2)^{n-1}$ by sign changes $(z_j) \mapsto (\pm z_j)$ with an even number of minus signs. The reflection hyperplanes are given by $z_i = z_j$ and $z_i = -z_j$ for all pairs (i, j) with $i \neq j$. Let X be the complement of the reflection hyperplanes and X/W the orbit space of the free $W = WD_n$ action. Brieskorn [**??**] has shown that the fundamental group $\pi_1(X/W)$ is the braid group ZD_n.

We translate this result and obtain a description of ZD_n by planar braid pictures.

A loop $[w]$ in X/W with base point $(1, \ldots, n)$ can be lifted to X with $(1, \ldots, n)$ as starting point. Let

$$w\colon [0,1] \to X, \quad t \mapsto (w_j(t))$$

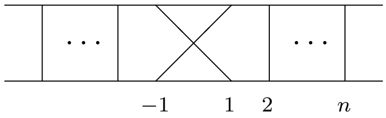
be the resulting path from $(1, \ldots, n)$ to $(\pm\sigma(1), \ldots, \pm\sigma(n))$. Here $\sigma \in S_n$, and the number of minus signs is even. We consider the braid in $\mathbb{C} \times [0, 1]$ with $2n$ strings given by

$$t \mapsto \{-w_n(t), \ldots, -w_1(t), w_1(t), \ldots, w_n(t)\} \times \{t\}.$$

The braid is symmetric with respect to the symmetry $\mathbb{C} \to \mathbb{C}$, $z \mapsto -z$. the strings are $\zeta_{\pm j}\colon t \mapsto (\pm w_j(t), t)$. Since w maps into X, the strings have the following property: The string pairs (ζ_j, ζ_{-j}) and (ζ_k, ζ_{-k}) never meet for $j \neq k$; an intersection would correspond to a point $w_j(t) = \pm w_k(t)$ on a reflection hyperplane. A value $w_j(t)$ is not excluded, though. In this case, the strings ζ_j and ζ_{-j} intersect. Therefore we are not dealing with a braid in the usual sense. Of course, we can always choose representing paths w such that no intersection of ζ_j with ζ_{-j} occurs.

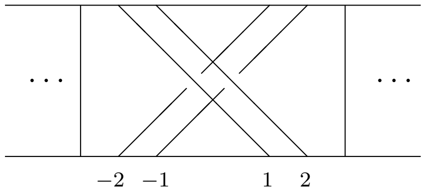
As usual, we consider planar generic projections of braids in the strip $\mathbb{R} \times [0, 1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$ which are symmetric with respect to the axis $0 \times [0, 1]$. The transverse intersection on the axis are ordinary crossings, and the other crossings are over- and undercrossings which appear in symmetric pairs.

In the geometric picture, the extended braid group $Z'D_n$ has generators $\kappa_0, \ldots, \kappa_{n-1}$ with κ_0 given by



κ_0

and κ_j $(j \geq 1)$ given by the symmetrized crossing of the j-th and $1+j$-th string (see the next figure for κ_1). The braid group ZD_n has generators $x_j = \kappa_j$ $(j \geq 1)$ and $x_0 = \kappa_0 \kappa_1 \kappa_0$ represented by



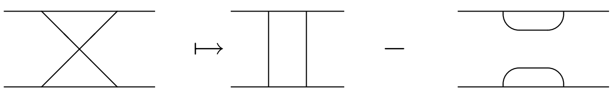
x_0

The relation $\kappa_0^2 = 1$ corresponds to a standard Reidemeister move of type II. It would also be possible to use over- and under-crossing on the axis, but then allow for an interchange of over-crossing and under-crossing on the axis.

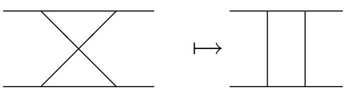
Elements in the subgroup ZD_n have in their geometric picture an even number of crossings on the axis.

The geometric braid groups $Z'D_n$ and ZD_n are included in tangle categories $S'D$ and SD. The category $S'D$ has objects $[\pm n]$, $n \in \mathbb{N}_0$. The morphisms from $[\pm m]$ to $[\pm n]$ are tangle pictures in $\mathbb{R} \times [0,1]$ from $[\pm m] \times 0$ to $[\pm n] \times 1$ which are symmetric with respect to the axis $0 \times [0,1]$. The crossings on the axis are ordinary crossings. Composition is defined by placing one tangle above the other and shrinking of $[0,2]$ to $[0,1]$. The subcategory SD of $S'D$ consists of tangles with an even number of points on the axis. There are similar categories S_0D and $S_0'D$ of oriented tangles. Also, one may consider banded (framed) tangles by not allowing Reidemeister type I moves. The D-tangle categories are analogous to the B-tangle categories [**??**], except for the special treatment of the crossings on the axis. The categories are tensor module categories over the appropriate categories of ordinary tangles. Ordinary tangles are included by symmetrizing.

There is a Kauffman functor from $S'D$ to the category $T'D$ of bridges. In this context one chooses a parameter A with $p = -A^2$. The Kauffman functor resolves a symmetrized ordinary crossing as usual in the definition of the Kauffman bracket [**??**], [**??**]. A crossing on the axis is treated as in the following figure.



There is also a forgetful functor to A-tangles which maps



and takes the $\mathbb{Z}/2$-quotient of the resulting tangle. These two functors correspond to the splitting of the algebra TD_n in section 2.

5. Categories of bridges

This section introduces some general terminology for certain graphical categories.

A free involution $\sigma\colon P \to P$ of a set P is calles a *P-bridge*. A free involution of P is a partition of P into 2-element subsets $\{i, \sigma(i)\}$, called the *arcs* or *strings* of the bridge. A bridge is called *oriented* if its arcs are ordered sets $\{a_1, a_2\}$.

We study bridges with a geometric terminology. Suppose $\sigma\colon P \to P$ is a bridge. The *geometric realization* $|\sigma|$ of σ is the one-dimensional simplicial complex with P as set of 0-simplices and a 1-simplex for each arc $\{i, \sigma(i)\}$ with i and $\sigma(i)$ as boundary points. We say that the arc *connects* its boundary points. The arcs are the components of $|\sigma|$.

A (P, Q)-bridge is a bridge on the disjoint union $P \coprod Q$. An arc of a (P, Q)-bridge σ is called *horizontal* if its boundary points are either contained in P or in Q. The other arcs are called *vertical*.

We use a graphical notation for (P, Q)-bridges σ. We think of $P \subset \mathbb{R} \times 0$, $Q \subset \mathbb{R} \times 1$ and we draw an arc in $\mathbb{R} \times [0, 1]$ from i to $\sigma(i)$. This is illustrated by the next figure.

??

The crossings of the arcs have no significance right now. The notation horizontal and vertical is evident in this context. The horizontal arcs with endpoints in P are called the *lower* part of the bridge, the horizontal arcs with endpoints in Q the *upper* part.

(5.1) Remark. *Suppose P has $2n$ elements. The number of P-bridges is*

$$(2n - 1) \cdot (2n - 3) \cdots 3 \cdot 1.$$

PROOF. There are $2n - 1$ possibilities to connect a fixed element of P. Having fixed this connection, a set with $2n - 2$ elements remains. Now use induction. □

We will use bridges with further properties.

Let G be a group and suppose P and Q are G-sets. A G-equivariant (P, Q)-bridge is a G-equivariant free involution σ of $P \coprod Q$. Equivariant means: $\sigma(gi) = g\sigma(i)$ for $g \in G$ and $i \in P \coprod Q$.

Suppose the bridge $\sigma\colon P \to P$ is G-eqivariant. We have an induced G-action on $|\sigma|$. The action on the 0-simplices is given. If $\{i, \sigma(i)\}$ is a 1-simplex, then, by equivariance, $\{gi, gsigma(i)\}$ is a 1-simplex. It can happen that these simplices coincide. This is the case if g is in the isotropy group G_i of i. If $g \in G$ acts non-trivially on $\{i, \sigma(i)\}$, then

$$gi = \sigma(i), \quad g\sigma(i) = i, \quad g^2 i = i$$

and hence $g^2 \in G_i$. Geometrically, g acts as reflection in the barycentre of the 1-simplex $\{i, \sigma(i)\}$ in this case.

In the sequel we only consider G-sets P with the following additional properties:

(1) The isotropy groups are finite.
(2) The orbit set is finite.
(3) G acts effectively on each orbit.
Under these hypotheses we have:

(5.2) Proposition. *Let σ be a G-equivariant (P,Q)-bridge. Then the following holds:*
(1) *The G-action respects lower, upper and horizontal arcs.*
(2) *The G-action on $|\sigma|$ is proper.*
(3) *The orbit space $|\sigma|/G$ is a compact one-dimensional* CW*-complex.*

PROOF. (1) is clear from the definitions.
(2) The G-action on the barycentric subdivision $|\sigma|'$ of $|\sigma|$ is a cellular action with finite isotropy groups. Now use [**??**,I(3.22)].
(3) This follows since G acts cellularly on $|\sigma|'$ and the orbit space has a finite number of cells [**??**,II(1.15),(1.17)]. □

Suppose σ is a G-equivariant $(P,Q) - bridgeand\tau$ a G-equivariant (Q,R)-bridge. Consider the G-space $|\tau| \cup_Q |\sigma|$. The components of this space can be if different type. Consider the G-orbit $B = Gx$ of a component x. Let H denote the isotropy group of the component x. Then B/G is homeomorphic to x/H. The orbit space of $|\tau| \cup |\sigma$ is compact. Hence x/H is compact. We use:

(5.3) Lemma. *There is no proper action of a discrete group on $[0,1[$ with compact orbit space.* □

This Lemma tells us that the components of $|\tau| \cup |\sigma|$ are not homeomorphic to $[0,1[$. Since the components are one-dimensional manifolds (with or without boundary), there are three cases:

(5.4) *A componente of $|\tau| \cup |\sigma|$ is homeomorphic to $[0,1]$, S^1, or $]0,1[$.* □

(5.5) Proposition. *The components of $|\tau| \cup |\sigma|$ which are homeomorphic to $[0,1]$ define a G-equivariant (P,R)-bridge.*

PROOF. If the component is homeomorphic to $[0,1]$, then the boundary points are contained in $P \coprod R$.

For each point in $P \coprod R$ there exists a component of $|\tau| \cup |\sigma|$ with this point as boundary point. Since components of type $[0,1[$ do not exist, the component has a second boundary point in $P \coprod R$. □

We denote the bridge in (5.5) by $\tau \wedge sigma$. The components of $|\tau| \cup |\sigma|$ which are homeomorphic to S^1 are called *cycles*, the components which are homeomorphic to $]0,1[$ are called *snakes*.

(5.6) Remark. Let $H = G_x$ be the subgroup of elements which map the component x into itself. Then H acts effectively and properly on the one-dimensional manifold x. Therefore we have, up to H-homeomorphism, the following possibilities:

(1) Suppose $x \cong S^1$. Then $H \cong \mathbb{Z}/m$ or $H \cong D_{2m}, m \geq 1$, and the action is by the usual action of a subgroup of $O(2)$.
(2) Suppose $x \cong \mathbb{R}$. Then $H \cong \mathbb{Z}$ or $H \cong D_\infty$ and the action is by the usual action as a subgroup of the group of affine transformations. ♡

Let $Z/\tau, \sigma)$ denote the orbit set of the components of $|\tau| \cup |\sigma|$ which are cycles or snakes. The G-orbits of components in $Z(\tau, \sigma)$ are counted according to types. The *type* of a component x consists of the conjugacy class of G_x together with the G_x-homeomorphism type of the G_x-action. The group $\mathbb{Z}/2$ has to different actions on S^1, by rotation or by reflection. (In the latter case it is the group D_2.) It is an observation of H. Reich [**??**] that these two actions should be distinguished.

Let C denote the set of possible types. We denote by $k(c, \tau, \sigma)$ the number of elements in $Z(\tau, \sigma)$ of type c.

After these preparations we define the category $F(G)$ of G-bridges. The *objects* of $F(G)$ are the G-sets as above, i. e. with finite isotropy groups, finte orbit set and effictive action on orbits.

We fix a ground ring $\mathfrak{K}$. The *morphism set* $\mathrm{Mor}(P, Q)$ is the free $\mathfrak{K}$-module on the set of G-equivariant (P, Q)-bridges.

In order to define the *composition* of morphisms we fix a map d: $C \to \mathfrak{K}$, called the *parameter function*. The composition of morphisms $\mathrm{Mor}(Q, R) \times \mathrm{Mor}(P, Q) \to \mathrm{Mor}(P, R)$ is assumed to be $\mathfrak{K}$-bilinear. The composition of bridges is defined to be

$$\tau \circ \sigma \prod_{c \in C} d(c)^{k(c,\tau,\sigma}\tau \wedge \sigma.$$

The *identity* $P \to P$ is represented by the bridge ι: $P \coprod P \to P \coprod P$ which connects $i \in P$ vertically with $i \in P$. We have $|\sigma| \cup |\iota| \cong |\sigma|$ and $|\iota| \cup |\sigma| \cong |\sigma|$, if defined.

Associativity of composition follows from a geometrical consideration: The cycles and snakes of $|\tau| \cup |\sigma| \cup |\rho|$ are those of $|\tau| \cup |\sigma|$, plus those of $|\sigma| \cup |\rho|$, plus those of $|\tau \wedge \sigma| \cup |\rho|$ (equal to those of $|\tau| \cup |\sigma \wedge \rho|$).

We shall mostly work with suitable subcategories of $F(G)$. For instance, we could use only free G-sets. Or we restrict the morphisms; this will be the case in the Temperley-Lieb categories.

The composition of bridges with only vertical strings is again a bridge of this form. No cycles or snakes appear. The vertical (P, P)-bridges under composition can be identified with the group of G-equivariant permutations of P. We describe this group.

6. Braid groups of type B

The braid group ZB_n associated to the Coxeter graph B_n is, by definition, the group generated by $t, g_1, \ldots, g_{n-1}$ with relations

(6.1)
$$\begin{array}{llcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j|=1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j|\geq 2 \\ (3) & tg_i & = & g_it, & i\geq 2 \\ (4) & tg_1tg_1 & = & g_1tg_1t. & \end{array}$$

For certain applications we need other presentations of this group.

Let $Z'B_n$ be the group with generators $c, g_1, \ldots, g_{n-1}$ and relations

(6.2)
$$\begin{array}{llcll} (1) & g_ig_jg_i & = & g_jg_ig_j, & |i-j|=1 \\ (2) & g_ig_j & = & g_jg_i, & |i-j|\geq 2 \\ (3) & cg_i & = & g_{i-1}c, & i\geq 2, \\ (4) & c^2g_1 & = & g_{n-1}c^2. & \end{array}$$

We abbreviate $g = g_{n-1}g_{n-2}\cdots g_1$.

(6.3) Proposition. *The assignment $\varphi(g_i) = g_i$, $1 \leq j \leq n-1$, and $\varphi(t) = g^{-1}c$ defines an isomorphism φ: $ZB_n \to Z'B_n$.*

Proof. The relations (1) and (2) yield in both groups

(6.4) $$g_{i-1}g = gg_i, \qquad i > 1.$$

We define in ZB_n (resp. $Z'B_n$) an element c (resp. t) by $gt = c$. From (1), (2) and (2.4) we see that the relations $cg_i = g_{i-1}c$ and $g_it = tg_i$ are equivalent for $i > 1$.

We set $h = g_{n-1}\cdots g_2$, $k = g_{n-2}\cdots g_1$ and infer from (2.4)

(6.5) $$gh = kg.$$

We use this to show that $c^2g_1 = g_{n-1}c^2$ and $tg_1tg_1 = g_1tg_1t$ are equivalent, provided (1), (2), and (3) hold. We compute

$$g_{n-1}^{-1}c^2g_1 = g_{n-1}^{-1}g_{n-1}kthg_1tg_1 = khtg_1tg_1$$

$$c^2 = gthg_1t = ghtg_1t = kgtg_1t = khg_1tg_1t$$

and see the equivalence. □

The braid group $Z\tilde{A}_{n-1}$ of the Coxeter graph with n vertices $\tilde{A}_{n-1}$ has, by definition, generators $g_1, \ldots, g_n$ and relations

(6.6)
$$\begin{array}{lcll} g_ig_jg_i & = & g_jg_ig_j, & m(i,j)=3 \\ g_ig_j & = & g_jg_i, & m(i,j)=2. \end{array}$$

Indices will be considered $\bmod n$ in this case. We have $m(i,j) = 3$ if and only if

$i \equiv j \pm 1 \bmod n$. All this holds for $n \geq 3$. For $n = 2$, the group is the free group generated by g_1 and g_2.

The graph $\tilde{A}_{n-1}$ has an automorphism which permutes the vertices cyclically. We have an induced automorphism s of $Z\tilde{A}_{n-1}$ given by

$$s(g_i) = g_{i-1}, \qquad i \bmod n.$$

The n-th power of s is the identity.

We use s to form the semi-direct product

(6.7) $$Z\tilde{A}_{n-1} \to G_n \to \mathbb{Z};$$

the generator $1 \in \mathbb{Z}$ acts through s on $Z\tilde{A}_{n-1}$. The semi-direct product is the group structure on the set $Z\tilde{A}_{n-1} \times \mathbb{Z}$ defined by $(x,m)\cdot(y,n) = (x \cdot s^m(y), m+n)$. The group G_n has the following description by generators and relations. Let G'_n denote the group with generators $s, g_1, \ldots, g_n$ and relations (2.6) together with

(6.8) $$sg_i = g_{i-1}s, \qquad i \bmod n.$$

(6.9) Proposition. *The assignment $\psi(g_i) = (g_i, 0)$ and $\psi(s) = (e, 1)$ yields an isomorphism $\psi\colon G'_n \to G_n$ (neutral element e).*

Proof. One verifies that ψ is compatible with relations (2.6) and (2.8). This is obvious for (2.6). The relation $(e,1)(x,0)(e,1)^{-1} = (s(x),0)$ is used to show compatibility with (2.8).

An element $x \in Z\tilde{A}_{n-1}$ has an image $x' \in G'_n$, induced by $g_i \mapsto g_i$. This assignment has the property $(s(x))' = sx's^{-1}$. We have the Homomorphism $G_n \to G'_n$, $(x,m) \mapsto x's^m$ by (1.4). It is inverse to psi. □

(6.10) Proposition. *The assignment $\alpha(g_i) = g_i$, $1 \leq i \leq n-1$, and $\alpha(c) = s$ defines an isomorphism $\alpha\colon Z'B_n \to G'_n$.*

Proof. The assignment is compatible with the relations of $Z'B_n$, since

$$\alpha(c^2 g_1 c^{-2}) = s^2 g_1 s^{-2} = s g_n s^{-1} = g_{n-1}.$$

An inverse to α is induced by the assignment $\beta(g_i) = g_i$, $\beta(g_n) = cg_1g^{-1}$, and $\beta(s) = c$. In order to see that β is well defined, one has to check, in particular, the relations

$$g_{n-1}g_ng_{n-1} = g_ng_{n-1}g_n, \qquad g_1g_ng_1 = g_ng_1g_n.$$

In the first case, this amounts to the equalitiy of

$$g_{n-1}cg_1c^{-1}g_{n-1} = c^2g_1c^{-1}g_1cg_1c^{-2}$$

and

$$cg_1c^{-1}g_{n-1}cg_1c^{-1} = cg_1cg_1c^{-1}g_1c^{-1}.$$

We compute

$$cg_1g_2g_1c^{-1} = cg_2g_1g_2c^{-1} = cg_2c^{-1}cg_1c^{-1}cg_2c^{-1} = g_1cg_1c^{-1}g_1$$

and hence

$$c(g_1 c g_1 c^{-1} g_1) c^{-1} = c^2 g_1 g_2 g_1 c^{-2}.$$

On the other hand, $g_1 c^{-1} g_1 c g_1 = g_1 g_2 g_1$. This yields the desired equality.

The second relation above leads to the same situation. □

If we combine the foregoing, we obtain a semi-direct product

(6.11)
$$Z\tilde{A}_{n-1} \to ZB_n \to \mathbb{Z}.$$

In terms of the original generators, the inclusion $Z\tilde{A}_{n-1} \subset ZB_n$ is given by

(6.12)
$$g_n \mapsto g t g_1 t^{-1} g^{-1}; \quad g_i \mapsto g_i, \quad 1 \leq i \leq n-1.$$

The homomorphism $ZB_n \to \mathbb{Z}$ in (2.14) is given by $g_i \mapsto 0$ and $t \mapsto 1$.

Different types of Weyl groups (= Coxeter groups) are related to these braid groups. We have the Coxeter groups $W\tilde{A}_{n-1}$ and WB_n associated to the graphs $\tilde{A}_{n-1}$ and B_n. In addition, we will also use a group $W^\infty B_n$. It is obtained from ZB_n by adding the relations $g_j^2 = 1$, but no relation for t. The reason for introducing this group is a semi-direct product in analogy to (2.14). The arguments which lead to (2.14) also give a semi-direct product

$$W\tilde{A}_{n-1} \to W^\infty B_n \to \mathbb{Z}.$$

We give another interpretation and describe these groups as groups of permutations.

Let $t_n \colon \mathbb{Z} \to \mathbb{Z}$, $x \mapsto x+n$ be the translation by n. Let P_n denote the group of t_n-equivariant permutations $\sigma \colon \mathbb{Z} \to \mathbb{Z}$. Equivariance means $\sigma(i+n) = \sigma(i)+n$. Hence σ induces $\overline{\sigma} \colon \mathbb{Z}/n \to \mathbb{Z}/n$, and $\sigma \mapsto \overline{\sigma}$ is a homomorphism $\pi \colon P_n \to S_n$ onto the symmetric group S_n.

(6.13) Proposition. *The kernel of π is isomorphic to $\mathbb{Z}^n$. The group P_n is isomorphic to the semi-direct product $\mathbb{Z}^n \to P'_n \to S_n$ in which S_n acts on $\mathbb{Z}^n$ by permutations.*

Proof. Let $\sigma_1 \in P_n$. Then there exists a permutation α of $\{1, \ldots, n\}$ and an n-tuple $(k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that $\sigma(i+tn) = \alpha(i) + (k_i + t)n$. We denote this map by $\sigma_1 = \sigma(\alpha; k_1, \ldots, k_n)$. Suppose $\sigma_2 = \sigma(\beta; l_1, \ldots, l_n)$ is another permutation written in this form. Then

$$\sigma_2 \circ \sigma_1 = \sigma(\beta\alpha; l_{\alpha(1)} + k_1, \ldots, l_{\alpha(n)} + k_n).$$

If we think of $P'_n = S_n \times \mathbb{Z}^n$ as sets, then the desired isomorphism is given by $(\alpha; k_1, \ldots, k_n) \mapsto \sigma(\alpha; k_1, \ldots, k_n)$. □

The semi-direct product P'_n has a normal subgroup Q'_n which is given as a semi-direct product

(6.14)
$$N \to Q'_n \to S_n$$

with $N = \{(x_1, \dots, x_n) \mid \sum x_i = 0\} \subset \mathbb{Z}^n$. The homomorphism

$$\varepsilon\colon P'_n \to \mathbb{Z}, \quad (\alpha; k_1, \dots, k_n) \mapsto \sum k_i$$

is a surjection with kernel Q'_n. The canonical sequence

(6.15)
$$Q'_n \to P'_n \to \mathbb{Z}$$

is itself a semi-direct product; the assignment $1 \mapsto (\mathrm{id}; 1, 0, \dots, 0)$ gives a splitting of ε. Under the isomorphism (2.13) the subgroup Q'_n corresponds to the subgroup

$$Q_n = \{\sigma \in P_n \mid 1 + 2 + \cdots + n = \sigma(1) + \cdots + \sigma(n)\}.$$

(6.16) Proposition. *The groups $W^\infty B_n$ and P_n are isomorphic. The groups $W\tilde{A}_{n-1}$ and Q_n are isomorphic. The element g_i is mapped to the transposition $(i, i+1)$, $i \in n\mathbb{Z}$. The element t is mapped to $\sigma(i) = i + n$ for $i \equiv 1 \bmod n$ and $\sigma(j)j$ otherwise.*

The proof is given after the proof of (2.21). In the proof of (2.17) we use the following:

(6.17) Lemma. *The elements*

$$t_0 = t, \quad t_1 = g_1 t g_1, \quad \dots, \quad t_{n-1} = g_{n-1} \dots g_2 g_1 t g_1 g_2 \dots g_{n-1}$$

of the braid group ZB_n pairwise commute.

Proof. We set

$$g(i,j) = g_i g_{i+1} \dots g_j, \qquad i \le j$$
$$g(i,j) = g_i g_{i-1} \dots g_j, \qquad i \ge j.$$

The braid relations imply immediately

$$g(1,j) g_{j+1} g(j,1) = g(j+1,2) g_1 g(2, j+1)$$

and (2.5)

$$g(2, j+1) g(1, j+1) = g(1, j+1) g(1, j).$$

By commutativity of g_j-elements, it suffices to show $t_i t_{i+1} = t_{i+1} t_i$. We compute

$$\begin{aligned} t_j t_{j+1} &= g(j,1) t g(1,j) g_{j+1} g(j,1) t g(1, j+1) \\ &= g(j,1) t g(j+1, 2) g_1 g(2, j+1) t g(1, j+1) \\ &= g(j,1) g(j+1, 2) t g_1 t g(2, j+1) g(1, j+1) \\ &= g(j,1) g(j+1, 2) [t g_1 t g_1] g(2, j+1) g(1, j). \end{aligned}$$

A similar computation works for $t_{j+1} t_j$. □

The semi-direct product relation (2.13, (2.17) between $W^\infty B_n$ and $W A_{n-1}$ has a counterpart for the braid groups. The homomorphism

$$\lambda\colon K_n \to Z A_{n-1}, \quad g_j \mapsto g_j, \quad t \mapsto 1$$

splits by $g_j \mapsto g_j$. Therefore we have a semi-direct product

(6.18) $$ZP_n \to ZB_n \to ZA_{n-1}.$$

The elements

$$y_0 = t, \quad , y_1 = g_1 t g_1^{-1}, \quad \dots, \quad y_{n-1} = g_{n-1} \dots g_1 t g_1^{-1} \dots g_{n-1}^{-1}$$

are contained in the kernel K_n of λ.

(6.19) Lemma. *The elements y_j have the following conjugation properties with respect to ZA_{n-1}:*

(1) $g_k^{-1} y_j g_k = y_j, \qquad k > j,\ k < j-1$
(2) $g_k^{-1} y_k g_k = y_{k-1},$
(3) $g_k^{-1} y_{k-1} g_k = y_{k-1} y_k y_{k-1}^{-1}.$

PROOF. (2) follows directly from the definitions.
(1) If $k > j$, then g_k commutes with every generator in the definition of y_j. In the case $k < j-1$ one uses the commutation relation between generators and $g_{k+1} g_k g_{k+1}^{-1} = g_k^{-1} g_{k+1} g_k$ (and the inverse) to cancel g_k^{-1} and g_k.
(3) is proved by induction on k. The verification for $k = 0$ is easy. We calculate with (1) and (2)

$$g_k^{-1} y_k y_{k+1} y_k^{-1} g_k = y_{k-1} y_{k+1} y_{k-1}^{-1} = g_{k+1} y_{k-1} y_k y_{k-1}^{-1} g_{k+1}^{-1}.$$

On the other hand, by (1) and (2)

$$\begin{aligned} g_{k+1}^{-1} g_k^{-1} g_{k+1}^{-1} y_k g_{k+1} g_k g_{k+1} &= g_k^{-1} g_{k+1}^{-1} g_k^{-1} y_k g_k g_{k+1} g_k \\ &= g_k^{-1} g_{k+1}^{-1} y_{k-1} g_{k+1} g_k \\ &= g_k^{-1} y_{k-1} g_k. \end{aligned}$$

This yields the induction step. □

(6.20) Proposition. *The group K_n is the free group generated by $y_0, \dots, y_{n-1}$.*

PROOF. By the previous Lemma, the group K_n^0 generated by the $y_0, \dots, y_{n-1}$ is invariant under conjugation by elements of ZA_{n-1}. Since $t \in K_n^0$ and t together with ZA_{n-1} generates ZB_n, we must have equality $K_n^0 = K_n$.

Let F_n denote the free group generated by $y_0, \dots, y_{n-1}$. We define homomorphisms $\gamma_1, \dots, \gamma_{n-1}\colon F_n \to F_n$ by immitating (2.20):

(1) $\gamma_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\gamma_k(y_k) = y_{k-1},$
(3) $\gamma_k(y_{k-1}) = y_{k-1} y_k y_{k-1}^{-1}.$

We claim:

(6.21) Lemma. *The γ_j are automorphisms and satisfy the braid relations*

$$\gamma_i \gamma_j \gamma_i = \gamma_j \gamma_i \gamma_j, \qquad |i-j| = 1, \quad \textit{and} \quad \gamma_i \gamma_j = \gamma_j \gamma_i, \qquad |i-j| \geq 2.$$

PROOF. First we check that the homomorphism $\delta_k\colon F_n \to F_n$

(1) $\delta_k(y_j) = y_j, \qquad k > j,\ k < j-1$
(2) $\delta_k(y_{k-1}) = y_k,$
(3) $\delta_k(y_k) = y_k^{-1} y_{k-1} y_k$

is inverse to γ_k. Hence γ_k is an isomorphism. Since γ_k fixes y_j for $j \notin \{k-1, k\}$, the second braid relation is obviously satisfied. For the first relation, the reader may check the following values of $\gamma_1\gamma_2\gamma_1$ and $\gamma_2\gamma_1\gamma_2$ on y_0, y_1, y_2:

$$y_0 \mapsto y_0 y_1 y_2 y_1^{-1} y_0^{-1}, \quad y_1 \mapsto y_0 y_1 y_1^{-1}, \quad y_2 \mapsto y_0.$$

We use this Lemma to define a semi-direct product

(6.22) $$F_n \to \Gamma_n \to ZA_{n-1},$$

in which $g_j \in ZA_{n-1}$ acts on F_n through δ_j. By (2.19) and $K_n^0 = K_n$, we have a canonical epimorphism $\mu\colon \Gamma_n \to ZB_n$. We show that μ is an isomorphism. As a set, $\Gamma_n = F_n \times ZA_{n-1}$. An inverse to μ has to send $g_j \mapsto (1, g_j)$ and $t \mapsto (y_0, 1)$. We have to check that this assignment is compatible with the relations of ZB_n. This is obvious for the g_j. Moreover:

$$\begin{aligned} tg_1tg_1 &\mapsto (y_0,1)(1,g_1)(y_0,1)(1,g_1) \\ &= (y_0,g_1)(y_0,g_1) \\ &= (y_0\delta_1(y_0), g_1^2) \\ &= (y_0y_1, g_1^2) \end{aligned}$$

$$\begin{aligned} g_1tg_1t &\mapsto (1,g_1)(y_0,1)(1,g_1)(y_0,1) \\ &= (y_1,g_1)(y_1,g_1) \\ &= (y_1\delta(y_1), g_1^2) \\ &= (y_0y_1, g_1^2). \end{aligned}$$

This finishes the proof of Proposition (2.21). □

Proof of (2.17). The elements t_j of (2.18) and the elements y_j coincide in $W^\infty B_n$, since $g_j = g_j^{-1}$ in this group. Lemma (2.20) shows that conjugation $y \mapsto g_k^{-1} y g_k$ acts on the set $(y_0, \dots, y_{n-1})$ by interchanging y_{k-1} and y_k. The proof of (2.21) is now easily adapted to show the isomorphism $W^\infty B_n \cong P_n'$. This isomorphism restricts to an isomorphism $W\tilde{A}_{n-1} \cong Q_n'$. □

We now apply the previous results to Hecke algebras. We have the Hecke algebras HA_{n-1}, $H\tilde{A}_{n-1}$, and HB_n associated to the corresponding Coxeter graphs. We consider algebras over the ground ring $\mathfrak{K}$. The first one is given by generators $g_1, \dots, g_{n-1}$, the braid relations between them and the quadratic relations $g_j^2 = (q-1)g_j + q$ with a parameter $q \in \mathfrak{K}$. The second one has generators $g_1, \dots, g_n$, the braid relations (2.8) and the same quadratic relations. The algebra HB_n has generators $t, g_1, \dots, g_{n-1}$, the braid relations (2.1), the quadratic relations above for the g_j and $t^2 = (Q-1)t + Q$ with another parameter $Q \in \mathfrak{K}$.

If we omit the quadratic relation for Q, then we obtain the definition of $H^\infty B_n$. This is not a Hecke algebra in the formal sense, i. e. associated to a Coxeter graph. It is a deformation of the group algebra of $W^\infty B_n$.

We know from Hecke algebra theory that an additive basis of the Hecke algebra is in bijective correspondence with the elements of the Coxeter group. There is a similar relation between $W^\infty B_n$ and $H^\infty B_n$. In order to derive it, we relate $H\tilde{A}_{n-1}$ and $H^\infty B_n$.

The algebra $H\tilde{A}_{n-1}$ has an automorphism τ given by $\tau(g_i) = g_{i-1}$ (indices $\operatorname{mod} n$). We define the twisted tensor product over the ground ring $\mathfrak{K}$

(6.23) $$H\tilde{A}_{n-1} \otimes \mathfrak{K}[\tau, \tau^{-1}] =: H_n^\infty$$

by the multiplication rule $(x \otimes \tau^k) \cdot (y \otimes \tau^l) = (x \cdot \tau^k(y), \tau^{k+l})$ for $k, l \in \mathbb{Z}$ and $x, y \in H\tilde{A}_{n-1}$.

(6.24) Proposition. *The algebra (2.24) is canonically isomorphic to $H^\infty B_n$.*

Proof. We use the isomorphism (2.3) to redefine the algebra $H^\infty B_n$ by generators $c, g_1, \ldots, g_{n-1}$ relations (2.2) and the quadratic relations for the g_j. The assignment $g_j \mapsto g_j \otimes 1$, $c \mapsto 1 \otimes \tau$ induces a homomorphism $H^\infty B_n \to H\tilde{A}_{n-1} \otimes H_n^\infty$. We have a homomorphism $H\tilde{A}_{n-1} \to H^\infty B_n$, $x \mapsto x'$ induced by $g_j \mapsto g_j$ with $g_n = g t g_1 t^{-1} g^{-1}$ in $H^\infty B_n$ (see (2.12)). This extends to a homomorphism $H_n^\infty \to H^\infty B_n$ by $x \otimes \tau^k \mapsto x' \cdot c^k$, since $\tau(y)' = c y' c^{-1}$. These homomorphisms are inverse to each other. □

(6.25) Corollary. *Suppose $(b_j \mid j \in J)$ is a $\mathfrak{K}$-basis of $H\tilde{A}_{n-1}$. Then $(b'_j c^k \mid j \in J, k \in \mathbb{Z})$ is a $\mathfrak{K}$-basis of $H^\infty B_n$.* □

7. Braids of type B

We use a theorem of Brieskorn [**??**] to derive some geometric interpretations of the braid group ZB_n. The starting point is the reflection representation of the Weyl group WB_n. This group is a semi-direct product

(7.1)
$$(\mathbb{Z}/2)^n \to WB_n \to S_n.$$

It acts on complex n-space $\mathbb{C}^n$ as follows:
(1) S_n acts by permuting the coordinates.
(2) $(\mathbb{Z}/2)^n$ act by sign changes $(z_1, \dots, z_n) \mapsto (\varepsilon_1 z_1, \dots, \varepsilon_n z_n)$, $\varepsilon_i \in \{\pm 1\}$.
This group contains the reflections in the hyperplanes

$$z_i = \pm z_j, \quad i \neq j; \quad \text{and} \quad z_j = 0.$$

Let X denote the complement of these hyperplanes. From the theory of finite reflection groups it is known, that $W = WB_n$ acts freely on X. Brieskorn [**??**] shows:

(7.2) Theorem. *The braid group ZB_n is isomorphic to the fundamental group $\pi_1(X/W)$ of the orbit space X/W.* □

If we think of WB_n as the Coxeter group with generators $t, g_1, \dots, g_{n-1}$, then g_j acts as the transposition $(j, j+1)$ and t as $z_1 \mapsto -z_1$.

We use (5.2) to give several interpretations of ZB_n by braids.

We remove the hyperplanes $z_j = 0$ from $\mathbb{C}^n$. It remains the n-fold product $\mathbb{C}^* \times \cdots \times \mathbb{C}^* = \mathbb{C}^{*n}$. Removal of the remaining reflection hyperplanes yields the space X of n-tuples $(z_j) \in \mathbb{C}^{*n}$ with pairwise different squares z_j^2.

The *configuration space* $C^n(\mathbb{C}^*)$ is the space of subsets of $\mathbb{C}^*$ with cardinality n. As topological space it is defined as Y/S_n where $Y \subset \mathbb{C}^{*n}$ is the set of n-tuples (y_j) with pairwise different components.

(7.3) Proposition. *X/W is homeomorphic to $C^n(\mathbb{C}^*)$.*

Proof. We arrive at X/W in two steps: First we form $Y' = X/(\mathbb{Z}/2)^n$ and then we divide out by the S_n-action. The map $(z_j) \mapsto (z_j^2)$ yields an S_n-equivariant homeomorphism $Y' \to Y$. □

By (5.2) and (5.3), $ZB_n \cong \pi_1(C^n(\mathbb{C}^*))$. The elements of $\pi_1(C^n(\mathbb{C}^*))$ will be interpreted as braids in the cylinder (cylindrical braids). We take $(1, \omega, \cdots, \omega^{n-1})$, $\omega = \exp(2\pi i/n)$, as base point in $C^n(\mathbb{C}^*)$. A loop in $C^n(\mathbb{C}^*)$ lifts to a path

$$w \colon I \to Y, \quad t \mapsto (w_1(t), \dots, w_n(t))$$

with this initial point. Thus we have
(1) $w(0) = (1, \omega, \dots, \omega^{n-1})$.
(2) $w(1) = (\sigma(1), \dots, \sigma(\omega^{n-1}))$, with a permutation σ of the set $\mathbb{Z}/n = \{1, \omega, \dots, \omega^{n-1}\}$.
(3) For $j \neq k$ we have $w_j(t) \neq w_k(t)$.

These data yield a braid z_w with n strings in $\mathbb{C}^* \times [0,1]$ from $\mathbb{Z}/n \times 0$ to $\mathbb{Z}/n \times 1$

$$z_w(t) = \{w_1(t), \dots, w_n(t)\} \times t.$$

Homotopy classes of loops correspond to isotopy classes of such braids. Multiplication of loops corresponds to concatenation of braids, as usual. Thus we have:

(7.4) Theorem. *The braid group ZB_n is the group of n-string braids in the cylinder $\mathbb{C}^* \times [0,1]$.* □

A second interpretation is by symmetric braids in $\mathbb{C} \times [0,1]$. This was already used in [**??**]. We take the base point $(1,2,\dots,n) \in X$. We lift a loop in X/W to a path

$$w\colon I \to X, \quad t \mapsto (w_1(t), \dots, w_n(t)).$$

Then we have:

(1) $w(0) = (1,2,\dots,n)$.
(2) $w(1) = (\pm\sigma(1), \dots, \pm\sigma(n))$ with a permutation σ of $\{1,\dots,n\}$.
(3) For $j \neq k$ we have $w_j(t) \neq \pm w_k(t)$.
(4) $w_j(t) \neq 0$.

Let $[\pm n] = \{-n, \dots, -1, 1, \dots, n\}$. The data yield a braid with $2n$ strings in $\mathbb{C} \times [0,1]$ from $[\pm n] \times 0$ to $[\pm n] \times 1$, namely

$$t \mapsto \{-w_n(t), \dots, -w_1(t), w_1(t), \dots, w_n(t)\} \times t.$$

These braids are $\mathbb{Z}/2$-equivariant with respect to $(z,t) \mapsto (-z,t)$ and are therefore called *symmetric*. The theorem of Brieskorn thus gives:

(7.5) Theorem. *The group ZB_n is isomorphic to the group of symmetric braids with $2n$ strings.* □

Symmmetric braids are drawn as ordinary braids but with additional symmetry with respect to the axis $0 \times [0,1]$. Here are figures for the generators t and g_j.

??

The symmetry is not the reflection in the axis, but corresponds to a spacial rotation about this axis. The relation $tg_1tg_1 = g_1tg_1t$ appears in this context as a generalized Reidemeister move.

Braids in the cylinder with n strings can be visualized as ordinary braids with $n+1$ strings — the axis of the cylinder is the additional string. This method has been used by Lambropoulou [**??**]. It allows the reduction of B_n-type braids to ordinary Artin braids, also with respect to proofs. The theorem of Brieskorn is then not used.

The twofold covering, ramified along the axis, of the cylinder produces a symmetric braid from a cylindrical one — and vice versa.

The cylinder $\mathbb{C}^* \times [0,1]$ has the universal covering $\mathbb{C} \times [0,1]$. Lifting cylindrical braids with n strings produces n-periodic infinite braids in $\mathbb{C} \times [0,1]$ from $\mathbb{Z} \times 0$

to $\mathbb{Z} \times 1$. They are invariant with respect to the translation $(z,t) \mapsto (z+n,t)$. This gives yet another interpretation of ZB_n by n-periodic braid pictures.

The relation between ZB_n and $Z\tilde{A}_{n-1}$ has the following geometric origin or counterpart. The map

$$\mathbb{C}^{*n} \to \mathbb{C}^*, \quad (z_1,\dots,z_n) \mapsto z_1 \cdot \ldots \cdot z_n$$

is S_n-equivariant and induces therefore a map from the configuration space

$$\alpha \colon C^n(\mathbb{C}^*) \to \mathbb{C}^*.$$

(7.6) Lemma. *The map α is a fibre bundle.*

Proof. Let

$$H = \{(z_1,\dots,z_n) \in \mathbb{C}^{*n} \mid \prod z_j = 1\}.$$

This is an S_n-invariant subset. The map

$$\gamma \colon \mathbb{C}^* \times_{\mathbb{Z}/n} H \to \mathbb{C}^{*n}, \quad (z, z_1,\dots,z_n) \mapsto (zz_1,\dots,zz_n)$$

is an S_n-equivariant homeomorphism. Thus γ is the fibre bundle with fibre H assoziated to the $\mathbb{Z}/n$-principal bundle $\mathbb{C}^* \to \mathbb{C}^*$, $z \mapsto z^n$. In $\mathbb{C}^{*n}$ we have to remove the subset

$$C = \{(z_1,\dots,z_n) \mid \text{there exists } i \neq j \text{ such that } z_i = z_j\}.$$

Let $D = H \cap C$. Then γ induces an S_n-equivariant homeomorphism

$$\gamma \colon \mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus D) \to \mathbb{C}^{*n} \setminus S.$$

This yields the fibre bundle description

$$\mathbb{C}^* \times_{\mathbb{Z}/n} (H \setminus T)/S_n \to \mathbb{C}^*$$

for the configuration space. □

We apply the fundamental group to this fibration and obtain the exact sequence

$$1 \to \operatorname{kernel} \alpha_* \to ZB_n \to \mathbb{Z} \to 0.$$

It can be shown that this is the sequence (2.11), i. e. $Z\tilde{A}_{n-1}$ is the fundamental group of the fibre of α.

Our next aim is to describe an additive basis of the Hecke algebra $H^\infty B_n$ by geometric means, i. e. by specifying a certain canonical set of basic braids.

A cylindrical braid with n strings is called *descending*, if for $i < j$ the i-th string is always overcrossing the j-th string. The i-th string is the one starting at ω^i, $0 \leq i \leq n-1$. Overcrossing means the following: We look radially and orthogonally from infinity onto the axis. The braid is in general position if we only see transverse double points. The first string we meet, coming from infinity, is the overcrossing one.

(7.7) Theorem. *The descending braids form a $\mathfrak{K}$-basis of the algebra $H^\infty B_n$. The descending braids with winding number zero form a $\mathfrak{K}$-basis of the algebra $H\tilde{A}_{n-1}$.*

We use (2.11) to reduce the first statement to the second. For the latter Hecke algebra we have the canonical basis related to the elements of reduced form in the Weyl group, and elements of the Weyl group will be shown to correspond to descending braids. We use the description of the Weyl group elements as n-periodic permutations of $\mathbb{Z}$. We represent such a permutation by n straight lines $c_1, \ldots, c_n$ in the strip $\mathbb{R} \times [0,1]$ starting at $\{1, \ldots, n\} \times 0$ such that c_i and c_j have at most one crossing, and then repeat with period n. By slightly moving the endpoints of the c_j we can assume that the curves are in general position. The resulting crossings are used to write the permutation as a product of reflections. This product is reduced in the sense of Coxeter group theory (see (??)). It is geometrically obvious that the same configuration of crossings can be realized by a descending braid.

(7.8) Proposition. *The set*

$$\mathfrak{C} = \{y_{n-1}^k g_{n-1} g_{n-2} \ldots g_j \mid k \in \mathbb{Z}, 1 \leq j \leq n\}$$

is a system of representatives for the left cosets of the inclusion $W^\infty B_{n-1} \subset W^\infty B_n$.

Proof. This is an immediate consequence of the semi-direct product description. The powers of y_{n-1} are representatives for cosets of $V_{n-1} \subset V_n$, and the products $g_{n-1} \ldots g_j$ are representatives for the cosets of $S_{n-1} \subset S_n$. □

We use this Proposition to derive the following result of Lambropoulou and Przytycki which was proved by them in a purely algebraic manner. The relation to standard Hecke algebra bases and the interpretation by descending braids seems more transparent, though.

(7.9) Theorem. *Let $\mathfrak{B}$ be the canonical basis of $H^\infty B_{n-1}$. Then $\{bc \mid b \in \mathfrak{B}, c \in \mathfrak{C}\}$ is a basis of $H^\infty B_n$.*

Proof. Represent a basis element of $H^\infty B_n$ by a descending braid. □

Recall the construction and definition of a Markov trace in section 2. The last Theorem gives immediately the uniqueness of a Markov trace with given parameters.

(7.10) Corollary. *There exists a unique Markov trace on $H^\infty B_n$ with given parameters $(s(k) \mid k \in \mathbb{Z})$ and z.* □

From a Markov trace (U^n) on ZA one obtains a link invariant. Let $\hat{x}$ denote the Alexander closure of the braid $x \in ZA_{n-1}$. Write x as a product of symbols (crossings) $g_1, g_1^{-1}, \ldots, g_{n-1}, g_{n-1}^{-1}$, and let $w(x)$ denote the resulting sum of exponents (writhe of x). Then a link invariant P is obtained by setting

$$P(\hat{x}) := \alpha^{-w(x)} \beta^n U^n(x)$$

for $x \in ZA_{n-1}$. Related are Markov traces $Tr = (Tr_n)$ on Hecke algebras $HA = (HA_n)$. These are $\mathfrak{K}$-linear maps $Tr_n\colon HA_{n-1} \to \mathfrak{K}$ such that

(1) $Tr_{n+1}|HA_{n-1} = Tr_n$,

(2) $TR_{n+1}(xx_n) = zTr_n(x), \quad x \in HA_{n-1}$

with a parameter $z \in \mathfrak{K}$. Here we use the names $x_1, \dots, x_{n-1}$ for the standard generators of the Hecke algebra because we want to distinguish them from the g_j. The Hecke algebras are defined with a parameter $q \in \mathfrak{K}^*$ which enters the quadratic relation $x_j^2 = (q-1)x_j + q$. The relation between the two notions of Markov traces is the following.

(7.11) Proposition. *Let $q = p^2$ and $\beta(p - p^{-1}) = \alpha - \alpha^{-1}$ with $p^2 \neq 1$. Let $U = (U^n)$ be a Markov trace on ZA with parameters α, β, as defined in section 3. Let $\iota\colon ZA_n \to (HA_n)^*$ be the homomorphism $g_j \mapsto p^{-1}x_j$. Then there exists a unique Markov trace Tr on HA such that $Tr_n \circ \iota = U^n$. It has parameter $z = p^{-1}\alpha\beta^{-1}$. The corresponding link invariant satisfies the skein relation $\alpha P(L+) - \alpha^{-1}P(L_-) = (p - p^{-1})P(L_0)$.* □

Lambropoulou [**??**] has proved a Markov theorem for links of type B (symmetric links). The statement is exactly as in the classical case, here called of type A. A Markov trace ($T^n\colon TB_n \to \mathfrak{K}$) therefore yields an invariant of B-links by setting

$$P(\hat{x}) = \alpha^{-w(x)}\beta^n T^n(x)$$

for $x \in ZB_n$. Here $w(x)$ still counts the exponent sum in terms of the generators g_j.

8. Traces on groups

Let G be a group and $\mathfrak{K}$ a commutative ring. A ($\mathfrak{K}$-valued) *trace* on G is a function $T\colon G \to \mathfrak{K}$ such that for all $g, h \in G$

$$\textbf{(8.1)} \qquad T(gh) = T(hg).$$

Equivalently, a trace is a function constant on conjugacy classes. A trace extends to a $\mathfrak{K}$-linear map $T\colon \mathfrak{K}G \to \mathfrak{K}$ from the group algebra $\mathfrak{K}G$ such that (1.1) holds for any two elements g, h in the group algebra.

Suppose $\tau\colon G \to G$ is an automorphism and T a trace. We call T (strongly) *τ-invariant* if for all $g, h \in G$ the relation

$$\textbf{(8.2)} \qquad T(g \cdot \tau(h)) = T(g \cdot h)$$

holds. If we set $g = 1$, we have the ordinary τ-invariance $T(\tau(h)) = T(h)$. If T is τ_1- and τ_2-invariant, then also τ_1^{-1}- and $\tau_1\tau_2$-invariant. If Γ is a group of automorphisms of G, then a trace is called Γ-invariant, if T is τ-invariant for each $\tau \in \Gamma$. It suffices to check Γ-invariance for a generating set of Γ.

Suppose T_i is a trace on G_i $(i = 1, 2)$. Then $(g_1, g_2) \mapsto T_1(g_1)T_2(g_2)$ is a trace on $G_1 \times G_2$. We want to generalize this to semi-direct products.

Let $\alpha\colon \Gamma \to \operatorname{Aut} G$ be a group of G-automorphisms. The semi-direct product $G \times_\alpha \Gamma$ is a group structure on the set $G \times \Gamma$ defined by

$$(g, \sigma)(h, \tau) := (g \cdot \sigma(h), \sigma\tau).$$

In this group structure we have

$$\textbf{(8.3)} \qquad (1, \sigma)(g, 1)(1, \sigma)^{-1} = (\sigma(g), 1).$$

We will use the following fact several times.

(8.4) Lemma. *A pair of group homomorphisms $\lambda\colon G \to H$ and $\mu\colon \Gamma \to H$ defines via $(g, \sigma) \mapsto \lambda(g)\mu(\sigma)$ a homomorphism $\varphi\colon G \times_\alpha \Gamma \to H$ if and only if for all $g \in G$ and $\sigma \in \Gamma$ the relation $\lambda(\sigma(g)) = \mu(\sigma)\lambda(g)\mu(\sigma)^{-1}$ holds. Each homomorphsm φ has this form for a unique pair (λ, μ).* □

Is $\alpha\colon \Gamma \to \operatorname{Aut}(G)$ is an antihomorphism, we define the semi-direct product $\Gamma\,{}_\alpha{\times}\, G$ with multiplication $(\sigma, g)(\tau, h) = (\sigma\tau, \tau(g)h)$.

The following is immediately verified from the definitions.

(8.5) Proposition. *Let S be a Γ-invariant trace on G and U a trace on Γ. Then*

$$T\colon G \times_\alpha \Gamma \to \mathfrak{K}, \quad (g, \sigma) \mapsto S(g)U(\sigma)$$

is a trace on $G \times_\alpha \Gamma$. □

If $\varphi\colon G \to H$ is a group homomorphism and T a trace on H, then $T \circ \varphi$ is a trace on G. Any function $T\colon G \to \mathfrak{K}$ on an abelian group G is a trace. Characters of finite dimensional representations are traces.