Induction Categories for Compact Lie Groups

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Abstract

We use Euler groups to construct induction categories for Lie groups and suitable families of closed subgroups. Euler groups are universal additive invariants. Induction categories combine ordinary morphisms and transfer morphisms and are the source categories for Mackey functors.

1 Additive Invariants

We define the basic object of our theory. The definition is by a universal property of a type considered in algebraic K-theory. Let G be a locally compact Hausdorff group. Subgroups are assumed to be closed. We use the general concepts and notations of the theory of transformation groups [2].

(1.1) Definition. Let $\mathcal{C}$ be a category of G-spaces and G-maps. An ***additive invariant*** (A, a) for $\mathcal{C}$-spaces Z over X consists of an abelian group A and an assignment $a[f] \in A$ to each G-map $f\colon Z \to X$, $Z \in \mathcal{C}$ such that the following holds:

(1) (Homotopy invariance) Suppose $f_i\colon Z_i \to X$ are $\mathcal{C}$-spaces over X $(i = 0, 1)$. Let $\sigma\colon Z_0 \to Z_1$ be a G-homotopy equivalence such that $f_1\sigma \simeq_G f_0$ ($\simeq_G$ means G-homotopic). Then $a[f_0] = a[f_1]$.

(2) (Additivity) Let

$$\begin{array}{ccc} Z_0 & \longrightarrow & Z_1 \\ \downarrow{\scriptstyle j} & & \downarrow \\ Z_2 & \longrightarrow & Z \end{array}$$

be a pushout in $\mathcal{C}$ and j a closed G-cofibration. Let $f\colon Z \to X$ be a G-map and denote by $f_i\colon Z_i \to X$ the compositions of f with the maps in the diagram. Then $a[f] + a[f_0] = a[f_1] + a[f_2]$.

(3) (Normalization) $a[\emptyset \to X] = 0$. (Assume $\emptyset \in \mathcal{C}$.)

An additive invariant (A, a) is called ***universal***, if every other additive invariant (B, b) is obtained from (A, a) by composing with a unique homomorphism $A \to B$. ◇

The existence of a universal additive invariant is shown by a Grothendieck construction. Let $\mathcal{C}$ be a category of G-spaces and G-maps. Suppose there exists a set $O(\mathcal{C})$ of objects in $\mathcal{C}$ such that any object in $\mathcal{C}$ is G-homotopy equivalent to an object in $O(\mathcal{C})$. Let X be any G-space. Let $F(\mathcal{C}; X)$ be the free abelian group on the set of G-homotopy classes $[f]\colon Z \to X$ of G-maps f with $Z \in O(\mathcal{C})$. Suppose $g\colon Y \to X$ is a G-map from an object in $\mathcal{C}$ and $h\colon Z \to Y$ a G-equivalence to an object in $O(\mathcal{C})$. Then g represents the element $[gh]$ in $F(\mathcal{C}; X)$. Let $R(\mathcal{C}; X)$ be the subgroup generated by elements of the following form:

(1) $[f_1] - [f_2]$, if there exists a G-homotopy equivalence h such that $f_1 h \simeq_G f_2$;

(2) $[f] - [f_1] - [f_2] + [f_0]$, if the maps are related by a pushout diagram in $\mathcal{C}$ of the type (2) in 1.1;

(3) $[\emptyset \to X]$.

Let $U(\mathcal{C}; X) = F(\mathcal{C}; X)/R(\mathcal{C}; X)$ and let $u[f\colon Z \to X]$ denote the element represented by f in $U(\mathcal{C}; X)$. Then (U, u) is a universal additive invariant for G-spaces in $\mathcal{C}$ over X.

If $\mathcal{C}$ is the category of G-spaces which are G-equivalent to finite G-complexes, then we write $U(\mathcal{C}; X) = U(G; X)$. If $\mathcal{F}$ is a family of closed subgroups of G and $\mathcal{C}$ the category of G-spaces which are G-homotopy-equivalent to finite $\mathcal{F}$-complexes, then we write $U(\mathcal{C}; X) = U(G, \mathcal{F}; X)$. If X is a point, we set $U(G, \mathcal{F}; X) = U(G, \mathcal{F})$. A computation of the universal groups uses the Euler characteristic as an essential tool. Therefore we call groups of the type $U(G, \mathcal{F}; X)$ ***Euler groups***.

(1.2) Remarks.

1. Consider the diagrams

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle V} \\ C & \xrightarrow[U]{} & D \end{array} \qquad \begin{array}{ccc} A \times Z & \xrightarrow{u \times \mathrm{id}} & B \times Z \\ \downarrow{\scriptstyle V \times \mathrm{id}} & & \downarrow{\scriptstyle V \times \mathrm{id}} \\ C \times Z & \xrightarrow[U \times \mathrm{id}]{} & D \times Z \end{array} \qquad \begin{array}{ccc} A/G & \xrightarrow{u/G} & B/G \\ \downarrow{\scriptstyle v/G} & & \downarrow{\scriptstyle V/G} \\ C/G & \xrightarrow[U/G]{} & D/G. \end{array}$$

Suppose the first one is a pushout in TOP. Suppose A, B, and C are G-spaces and u, v are G-maps. Then there exists a unique G-action on D such that the diagram is a pushout in G-TOP. If Z is a locally compact G-space, then the second diagram is a pushout in G-TOP. Moreover the third diagram is a pushout in TOP.

2. Let S be a locally compact (K, G)-space. If we apply $S \times_G$ to the first pushout diagram in the previous exercise, then we obtain a pushout in K-Top.

3. Let $i: A \to B$ be a closed G-cofibration. Then the product with a locally compact G-space S is again a closed G-cofibration. If S is a locally compact (K, G)-space, then $S \times_G A \to S \times_G B$ is a closed K-cofibration. $\diamond$

2 Formal Properties

In this section we collect formal properties of the universal groups.

(2.1) Proposition. *A G-map $f: X \to Y$ induces a homomorphism $f_*: U(G, \mathcal{F}; X) \to U(G, \mathcal{F}; Y)$, $[h] \mapsto [fh]$.*

Proof. The assignment in question is certainly additive and compatible with homotopy equivalences. Hence f_* is well-defined by the universal property. $\square$

Let $\mathcal{F}$ be a family of subgroups of H. If H is a closed subgroup of G, we define $\mathcal{F}^G$ as the family of subgroups of G which are conjugate to an element of $\mathcal{F}$. If the H-space Y is an $\mathcal{F}$-space, then the G-space $G \times_H Y$ is an $\mathcal{F}^G$-space.

(2.2) Proposition. *The assignment*

$$[f: Z \to X] \mapsto [G \times_H f: G \times_H Z \to G \times_H X]$$

induces an isomorphism $i_H^G: U(H, \mathcal{F}; X) \to U(G, \mathcal{F}^G; G \times_H X)$.

Proof. We first show that the assignment induces a well-defined homomorphism. This is a consequence of the universal property and the following facts:

(1) If Z is a finite $\mathcal{F}$-complex, then $G \times_H Z$ is a finite $\mathcal{F}^G$-complex.
(2) If we apply the induction construction $G \times_H$ - to an H-pushout of the type 1.1, then we obtain a G-pushout.
(3) An H-cofibration is mapped into a G-cofibration, an H-equivalence into a G-equivalence.

An inverse to i_H^G is constructed as follows. Suppose a G-map $f: Z \to G \times_H X$ of a finite G-complex Z is given. Compose with the projection $p: G \times_H X \to G/H$. Then Z becomes a G-space over G/H by $h = pf$. Let $Z_0 = h^{-1}(eH)$. The assignment $Z \mapsto Z_0$ is an inverse, up to equivariant homeomorphism, to $Z_0 \mapsto G \times_H Z_0$; it maps finite G-complexes to finite H-complexes and is compatible with equivariant pushouts and cofibrations. $\square$

Let $\varphi: H \to G$ be a homomorphism between compact Lie groups. If $\mathcal{F}$ is a G-family, then

$$\varphi^* \mathcal{F} = \{\varphi^{-1}(C) \mid C \in \mathcal{F}\}$$

is an H-family. An $\mathcal{F}$-space yields via φ an $\varphi^*\mathcal{F}$-space $\varphi^* Y$. If Y is a finite G-complex, then $\varphi^* Y$ is not right away a finite H-complex, but certainly H-homotopy equivalent to such a complex (or even H-homeomorphic). It can be shown that a pushout of G-spaces yields under φ^* a pushout of H-spaces and that φ^* maps G-cofibrations to H-cofibrations. These properties, together with the universal properties of the Euler groups, yield:

(2.3) Proposition. *The assignment $[f\colon Z \to X] \mapsto [\varphi^* f\colon \varphi^* Z \to \varphi^* X]$ yields a homomorphism $\varphi^*\colon U(G,\mathcal{F};X) \to U(H,\varphi^*\mathcal{F};\varphi^* X)$. This construction is functorial $(\varphi\psi)^* = \psi^*\varphi^*$.* □

The special case of the previous proposition in which φ is an inclusion of a subgroup is called *restriction* to this subgroup.

Suppose $\mathcal{F}_i$ is a G_i-family ($i = 1, 2$). We have the $G_1 \times G_2$-family

$$\mathcal{F}_1 \times \mathcal{F}_2 = \{H_1 \times H_2 \mid H_i \in \mathcal{F}_i\}.$$

(2.4) Proposition. *The assignment $([f_1],[f_2]) \mapsto [f_1 \times f_2]$ induces a bilinear map $U(G_1,\mathcal{F}_1;X_1) \times U(G_2,\mathcal{F}_2;X_2) \to U(G_1 \times G_2,\mathcal{F}_1 \times \mathcal{F}_2;X_1 \times X_2)$.*

Proof. Fix a G_2-map $f_2\colon Y \to X_2$ from a finite G_2-complex Y into X_2. We show that $[f_1] \mapsto [f_1 \times f_2]$ induces a homomorphism

$$U(G_1,\mathcal{F}_1;X_1) \to U(G_1 \times G_2,\mathcal{F}_1 \times \mathcal{F}_2;X_1 \times X_2).$$

This is a consequence of the fact that the product with Y preserves equivariant pushouts and homotopy equivalences. The fact that pushouts are preserved follows from an adjunction of the type

$$\mathrm{Top}_{G_1 \times G_2}(Z \times Y, P) \cong \mathrm{Top}_{G_1}(Z, \mathrm{Top}_{G_2}(Y, P))$$

where the mapping space Top_G carries the compact-open topology. We now use in a similar manner the universal property of $U(G_2,\mathcal{F}_2;X_2)$ to show that the construction so far induces a homomorphism of $U(G_2,\mathcal{F}_2;X_2)$ into

$$\mathrm{Hom}(U(G_1,\mathcal{F}_1;X_1), U(G_1 \times G_2,\mathcal{F}_1 \times \mathcal{F}_2;X_1 \times X_2)).$$

This homomorphism yields, by adjunction, the desired bilinear map. □

(2.5) Proposition. *Let H and K be compact Lie groups and S a finite (H,K)-complex, i.e., S carries commuting left H- and right K-actions. The assignment*

$$[f\colon Z \to X] \mapsto [S \times_K f\colon S \times_K Z \to S \times_K X]$$

induces a homomorphism $i_H^K(S)\colon U(H;X) \to U(K;S \times_K X)$. □

Let $H \lhd G$. If Z is a finite G-complex, then Z/H can be considered as a finite G/H-complex. If $x \in Z$ has isotropy group K, then $Hx \in Z/H$ has isotropy KH/H. The G-family $\mathcal{F}$ induces the G/H-family $\mathcal{F}/H = \{KH/H \mid K \in \mathcal{F}\}$.

(2.6) Proposition. *Let $H \lhd G$. The assignment*

$$[f\colon Z \to X] \mapsto [f/H\colon Z/H \to X/H]$$

induces a homomorphism $U(G, \mathcal{F}; X) \to U(G/H, \mathcal{F}/H; X/H)$. □

If $d\colon G \to G \times G$ is the diagonal and if $\mathcal{F}_1, \mathcal{F}_2$ are G-families, the relation $d^*(\mathcal{F}_1 \times \mathcal{F}_2) = \mathcal{F}_1 \circ \mathcal{F}_2$ holds. Thus, cartesian product of representatives induces by 2.3 and 2.4 an internal product

$$U(G, \mathcal{F}_1; X) \times U(G, \mathcal{F}_2; X) \to U(G, \mathcal{F}_1 \circ \mathcal{F}_2; X).$$

If $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}$ is multiplicative, we obtain on $U(G, \mathcal{F}; X)$ the structure of a $\mathbb{Z}$-algebra. In particular, if X is a point, we obtain the ***Euler ring*** $U(G, \mathcal{F})$ for a multiplicative family, and the module $U(\mathcal{G}; X)$ over this ring, if $\mathcal{G}$ is $\mathcal{F}$-modular.

(2.7) Proposition. *Let $p\colon E \to B$ be a G-fibration such that the fibre is a finite complex. Then pullback of representatives $X \to B$ along p induces a homomorphism $p^*\colon U(G; B) \to U(G; E)$.*

Proof. □

The groups $U(G; X)$ are additive in X, i.e., it follows immediately from the definitions that we have a canonical isomorphism

$$U(G; X + Y) \cong U(G; X) \oplus U(G; Y)$$

which is induced by the inclusions of the summands.

3 Euler Groups

The ***component category*** $\pi_0(G; X)$ of the G space X has objects the G-homotopy classes $[x]\colon G/H \to X$. A morphism from $[x]\colon G/H \to X$ to $[y]\colon G/K \to X$ is a G-map $\sigma\colon G/H \to G/K$ such that $y\sigma \simeq_G x$. An object $\alpha\colon G/H \to X$ of $\pi_0(G, X)$ can be identified with the path-component X^H_α of $\alpha(eH)$ in $\pi_0(X^H)$. Thus the object set of $\pi_0(G; X)$ is the disjoint union of the sets $\pi_0(X^H)$ for $H \le G$.

The automorphism group $\mathrm{Aut}(\alpha)$ of $\alpha = [x]$ consists of those $\sigma\colon G/H \to G/H$ such that $x\sigma \simeq_G x$. We have $\sigma \in NH/H = WH$ by identifying σ with $\sigma(eH) = nH$, $n \in NH$. We have the canonical action of WH on X^H and $\pi_0(X^H)$; and $W_\alpha H$, the isotropy group of $\alpha \in \pi_0(X^H)$ under this action, is isomorphic to $\mathrm{Aut}(\alpha)$.

Given $f\colon Z \to X$ and $\alpha\colon G/H \to X$, we let

$$Z(f, \alpha) = Z^H \cap f^{-1}(X^H_\alpha).$$

Here $X^H_\alpha \subset X^H$ is the path-component of $\alpha(eH)$. The action of WH on Z^H restricts to an action of $W_\alpha H$ on $Z(f,\alpha)$. In other terms: the space $Z(f,\alpha)$ is an $\mathrm{Aut}(\alpha)$-space.

We denote by $\chi(A)$ the Euler characteristic of a space A. Until further notice we use the homological definition $\chi(A,B) = \sum_{i=0}^\infty (-1)^i \mathrm{rank} H_i(A,B;\mathbb{Z})$ with integral homology. Finite G-complexes Z and their orbit-spaces have such an Euler characteristic.

Finite $\mathcal{F}$-complexes $f_i\colon Z_i \to X$ over X are called ***Euler equivalent*** if and only if for each $\alpha\colon G/H \to X$ with $H \in \mathcal{F}$ the Euler characteristics of the orbit spaces $\chi(Z(f_i,\alpha)/\mathrm{Aut}(\alpha))$ coincide $(i = 1,2)$.

Let $Eu(G,\mathcal{F};X)$ denote the set of equivalence classes under this equivalence relation. Let $[f]$ denote the element in Eu represented by f. We define an associative and commutative composition law on $Eu(G,\mathcal{F};X)$ by disjoint union

$$[f_1\colon Z_1 \to X] + [f_2\colon Z_2 \to X] = [f_1 + f_2\colon Z_1 + Z_2 \to X].$$

This composition law is the structure of an abelian group. We therefore call such groups ***Euler groups***. The neutral element is given by $[\emptyset \to X]$. The existence of inverses is seen as follows. Let A be a finite complex with trivial G-action and Euler characteristic $\chi(A) = -1$. Then one verifies that $f \circ \mathrm{pr}_Z\colon Z \times A \to Z \to X$ represents an inverse of $[f\colon Z \to X]$.

By definition, each α defines a homomorphism

$$\chi_\alpha\colon Eu(G,\mathcal{F};X) \to \mathbb{Z}, \quad [f\colon Z \to X] \mapsto \chi(Z(f,\alpha)/\mathrm{Aut}(\alpha)).$$

If α ranges over the isomorphism classes of objects in $\pi_0(G,\mathcal{F};X)$, we obtain an injective homomorphism

$$eu\colon Eu(G,\mathcal{F};X) \to \prod\nolimits_\alpha \mathbb{Z}, \qquad \alpha \in \mathrm{Iso}\,\pi_0(G,\mathcal{F};X).$$

(3.1) Lemma. *The pair $(Eu(G,\mathcal{F};X), eu)$ is an additive invariant for finite $\mathcal{F}$-complexes over X.*

Proof. Given subcomplexes $Z_0 \subset Z_j$ with $Z = Z_1 \cup Z_2$ and a G-map. Then we have induced inclusions

$$\iota_j\colon Z_0(\alpha,f)/\mathrm{Aut}(\alpha) \subset Z_j(\alpha,f)/\mathrm{Aut}(\alpha)$$

with union $Z(\alpha,f)/\mathrm{Aut}(\alpha)$. We can view the inclusions ι_j as inclusions of finite complexes. The fundamental additivity property $\chi(A\cup B)+\chi(A\cap B) = \chi(A)+\chi(B)$ of the Euler characteristic together with its homotopy invariance yields that $[f\colon Z \to X] \mapsto \chi_\alpha[f]$ is an additive invariant. This is equivalent to the claim. □

(3.2) Theorem. *The group $U(G,\mathcal{F};X)$ is the free abelian group with basis $[\alpha]$ for $\alpha \in \mathrm{Iso}\,\pi_0(G,\mathcal{F};X)$.*

Proof. We verify that the $[\alpha]$, $\alpha \in \operatorname{Iso}\pi_0(G,\mathcal{F};X)$ are linearly independent. For each α we denote a representative by $\alpha\colon G/H_\alpha \to X$. Suppose $x = \sum n_\alpha[\alpha] = 0$ is a linear relation in $U(G,\mathcal{F};X)$ with $n_\alpha \in \mathbb{Z}$. Let $K = H_\gamma$ be a maximal subgroup such that $n_\gamma \neq 0$. Consider

$$\chi_\gamma(\beta\colon G/H_\beta \to X) = \chi(G/H_\beta^K \cap \beta^{-1}(X_\gamma^K)/\operatorname{Aut}(\gamma)).$$

If this is non-zero, then $G/H_\beta^K \neq \emptyset$ and hence $K = H_\gamma$ subconjugate to H_β. By maximality of H_γ, only the summands $[n_\beta][\beta]$ with $H_\gamma \sim H_\beta$ can contribute to $\chi_\gamma(x)$. In this case, the relation

$$gH_\beta \in G/H_\beta^K \cap \beta^{-1}(X_\gamma^K)$$

is equivalent to

$$g^{-1}H_\gamma g = H_\beta, \quad \text{and} \quad \beta(gH_\beta) \simeq \gamma(eH_\gamma) \quad \text{in } X^K.$$

Therefore the morphism $\sigma \in \operatorname{Hom}(G/H_\gamma, G/H_\beta)$ with $\sigma(eH_\gamma) = gH_\beta$ is an isomorphism from γ to β in $\pi_0(G,\mathcal{F};X)$. Thus

$$\chi_\gamma(x) = n_\gamma\chi(G/H_\gamma^K \cap \gamma^{-1}(X_\gamma^K)/\operatorname{Aut}(\gamma)) = n_\gamma = 0.$$

We now verify that the elements $[\alpha]$ in question generate $U(G,\mathcal{F};X)$. This is a formal consequence of the axioms 1.1. Let $f\colon Z \to X$ be a G-map from a finite G-complex Z. We use induction on the number of cells and on the dimension of Z to show that the corresponding element in the Euler group is a linear combination of the $[\alpha]$. Let $Z = W \cup (G/H \times D^n)$ be obtained from W by attaching an n-cell $G/H \times D^n$ of type H. If we restrict f to an orbit in the interior of this n-cell, then we obtain a well-defined element $[\alpha]$. Let $Y = G/H \times D^n(\frac{1}{2})$ be the closed cell in $G/H \times D^n$ of radius $\frac{1}{2}$ about the center. If we remove the interior Y° of Y from Z, then the resulting space is G-homotopy equivalent to W. By additivity,

$$\begin{aligned}
[Z,f] &= [Z \smallsetminus Y^\circ, f] + [Y,f] - [(Z \smallsetminus Y^\circ) \cap Y, f] \\
&= [X,f] + [\alpha] - [G/H \times S^{n-1}, f].
\end{aligned}$$

Here S^{n-1} denotes the boundary of $D^n(\frac{1}{2})$. We show by induction that

$$[G/H \times S^{n-1}] = (1 + (-1)^{n-1})[\alpha], \quad n \geq 0.$$

The induction starts with $n = 0$ and (1.1.3). For the induction step, let D_+ (D_-) be the upper (lower) hemisphere of S^n, respectively. Then

$$\begin{aligned}
[G/H \times S^n, f] &= [G/H \times D_+, f] + [G/H \times D_-, f] - [G/H \times S^{n-1}, f] \\
&= 2[\alpha] - [G/H \times S^{n-1}, f]
\end{aligned}$$

and induction works. $\square$

The previous proof shows that the expansion of $[f\colon Z \to X]$ in terms of a basis can be obtained by counting cells. If $G/H \times E^n \subset Z$ is an open n-cell of Z, then the restriction of f to an orbit of $G/H \times E^n$ defines a basis element $[\alpha]$. We therefore call this cell an n-cell of type α.

(3.3) Corollary. *Let $n(\alpha, i)$ be the number of i-cells of type α in $f\colon Z \to X$ and set $n(\alpha) = \sum_i (-1)^i n(\alpha, i)$. Then $[f\colon Z \to X] = \sum_\alpha n(\alpha)[\alpha]$.* □

(3.4) Proposition. *By universality and lemma* 3.1*, we obtain a homomorphism $\iota\colon U(G, \mathcal{F}; X) \to Eu(G, \mathcal{F}; X)$. It is an isomorphism.*

Proof. The proof of 3.2) shows that the basis elements $[\alpha]$ remain linearly independent in $Eu(G, \mathcal{F}; X)$. By construction, ι is surjective. □

Given $f\colon Z \to X$ and $\beta \in \operatorname{Iso} \pi_0(G; X)$, we denote by $Z(\beta)$ the G-subspace of Z consisting of the orbits C such that $f|C\colon C \to X$ defines β. The latter means the following. Choose a G-isomorphism $\sigma\colon G/H \to C$. Then $f\sigma\colon G/H \to X$ defines an object of $\pi_0(G; X)$, and the isomorphism class is independent of the choice of σ.

Define a partial order on $\operatorname{Iso} \pi_0(G; X)$ by $\alpha \leq \beta$ if and only if there is a morphism in $\pi_0(G; X)$ from a representative of α to a representative of β. We write $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$. Given $f\colon Z \to X$ and α, we let

$$Z^\alpha = \bigcup_{\alpha \leq \beta} Z(\beta), \quad Z^{>\alpha} = \bigcup_{\alpha < \beta} Z(\beta).$$

These are G-subspaces of Z. With this notation, we can express the combinatorial Euler characteristic 3.3 as a relative Euler characteristic $n(\alpha) = \chi(Z^\alpha/G, Z^{>\alpha}/G)$. Since $Z^\alpha \smallsetminus Z^{>\alpha} = Z(\alpha)$, we write $\chi_c(Z(\alpha)/G) = \chi(Z^\alpha/G, Z^{>\alpha}/G)$. The left hand side can be interpreted as a cohomological Euler characteristic of the space $Z(\alpha)/G$ defined by cohomology with compact support. With these notations, we have the following relation in $U(G; X)$

$$[f\colon Z \to X] = \textstyle\sum_\alpha \chi_c(Z(\alpha)/G)[\alpha].$$

If we are willing to admit redundancy, we can write (**??**) in a form which does not depend on X. Two orbits in $Z_{(L)}$ certainly define the same element in $U(G; X)$, if they belong to the same path component of $Z_{(L)}/G$. We could therefore split $Z(\alpha)/G$ into path-components. Thus, if the index β enumerates the basis elements which are represented by orbits in G-path-components of orbit bundles of Z, then

$$[f\colon Z \to X] = \textstyle\sum_\beta \chi_c(Z(\beta)/G)[\beta].$$

It suffices to collect the path components of $Z_{(L)}/G$ which map to the same component of $Z^{(L)}/G$. This is due to the fact that a homotopy of an orbit corresponds to a path in a fixed point set.

(3.5) Remarks.

1. The groups $U(G;X)$ are invariants of the G-homotopy type of X. Let $f_0, f_1: X \to Y$ be G-homotopic. Then the induced maps $(f_j)_*: U(G;X) \to U(G;Y)$ are equal.

By functoriality, it suffices to show that the projection $\mathrm{pr}: X \times [0,1] \to X$ induces an isomorphism pr_*. This is a consequence of the computation, since pr_* induces a bijection between the canonical bases.

One can also give a proof via formal properties.

2. Euler groups can be defined, whenever the Euler characteristic makes sense. A class of spaces which can be used in this context are the finitely dominated spaces. An $\mathcal{F}$-space Z is called ***finitely dominated*** if there exists a finite $\mathcal{F}$-complex E and G-maps $i: Z \to E$, $p: E \to Z$ such that $pi \simeq_G \mathrm{id}(Z)$. In this case we call (i,p) a ***finite domination*** of Z. In a finite domination, the integral homology groups of Z^H are direct summands of those of E^H and therefore finitely generated abelian groups. Hence $\chi(Z^H)$ exists. Moreover, if Z and E are G-spaces over X, then $\chi(Z(f,\alpha)/\mathrm{Aut}(\alpha))$ exists. ◇

4 Burnside Groups

Let $f_i: Z_i \to X$ be finite G-complexes over X. They are called ***Burnside equivalent*** if for all $\alpha: G/H \to X$ the Euler characteristics $\chi(Z(f_i,\alpha))$ are equal. We denote by $A(G;X)$ the set of equivalence classes. As in the case of $U(G;X)$, disjoint union of equivalence classes induces a structure of an abelian group on $A(G;X)$. Let $A[f] \in A(G;X)$ denote the class of $f: Z \to X$. We call groups of this type ***Burnside groups***. From the definition it follows immediately that A is an additive invariant for finite G-complexes. Therefore we obtain a canonical surjection $a: U(G;X) \to A(G;X)$ which is the identity on representatives. We have similar groups for families of isotropy groups.

Composition with G-maps $h: X \to Y$ induces a homomorphism $h_*: A(G;X) \to A(G;Y)$. Suppose $f: Z \to X$ is given. Then $Z(hf,\beta) = (hf)^{-1}(Y_\beta) \cap Z^H$ is a disjoint union of a finite number of $Z(f,\alpha)$ over the α which are mapped to β by h. Hence h_* is compatible with Burnside equivalence.

The next theorem shows the difference between the Euler and the Burnside groups.

(4.1) Theorem. *The group $A(G,\mathcal{F};X)$ is the free abelian group on the set of isomorphism classes of objects $\alpha: G/H \to X$ in $\pi_0(G;X)$ with $H \in \mathcal{F}$ and finite Weyl group WH.* □

If $\mathcal{F}$ is multiplicative, then $A(G,\mathcal{F})$ is a $\mathbb{Z}$-algebra. It has a unit element, if $G \in \mathcal{F}$. If $\mathcal{F}$ is the family of all closed subgroups, then $A(G) = A(G,\mathcal{F})$ is the ***Burnside ring*** of the compact Lie group G.

5 Induction Categories

We construct a $\mathbb{Z}$-category $\Omega(G)$ for each compact Lie group G. The objects of $\Omega(G)$ are the homogeneous spaces. The morphism sets are defined as

$$\mathrm{Mor}_{\Omega(G)}(G/H, G/K) = U(G; G/H \times G/K).$$

We give two descriptions for the composition of morphisms. The composition of morphisms in the diagram is bilinear and we use it to define the composition $(f, g) \mapsto g \circ f$ of morphisms in $\Omega(G)$. We explain the diagram below.

$$\begin{array}{c} U(G; G/H_1 \times G/H_2) \times U(G; G/H_2 \times G/H_3) \\ \downarrow{\scriptstyle (1)} \\ U(H_2; \mathrm{res}_{H_2} G/H_1) \times U(H_2; \mathrm{res}_{H_2} G/H_3) \\ \downarrow{\scriptstyle (2)} \\ U(H_2; \mathrm{res}_{H_2}(G/H_1 \times G/H_3)) \\ \downarrow{\scriptstyle (3)} \\ U(G; G/H_2 \times (G/H_1 \times G/H_3)) \\ \downarrow{\scriptstyle (4)} \\ U(G; G/H_1 \times G/H_3) \end{array}$$

Explanation. For each G-space X we have a canonical G-homeomorphism $G \times_H X \to G/H \times X$, $(g, x) \mapsto (g, gx)$. We apply this in the first line $G/H_1 \times G/H_2 \cong G \times_{H_2} (\mathrm{res}_{H_2} G/H_1)$ and use then the isomorphism **??**. Similarly for the second factor. The morphism (2) is the bilinear map **??**. The morphism (3) is again an application of **??** and the canonical G-homeomorphism above. Finally, (4) is induced by the projection onto the factors **??**.

We now describe the composition of morphisms on the level of representing objects. Let

$$(\alpha, \beta_1)\colon A \to G/H_1 \times G/H_2, \qquad (\beta_2, \gamma)\colon B \to G/H_2 \times G/H_3$$

be given. Set $A(0) = \beta_1^{-1}(eH_2)$ and $B(0) = \beta_2^{-1}(eH_2)$. These are H_2-subspaces of A and B, and we have canonical G-homeomorphisms $A \cong G \times_{H_2} A(0)$, $B \cong G \times_{H_2} B(0)$. Let

$$\alpha(0)\colon A(0) \to \mathrm{res}_{H_2} G/H_1, \qquad \gamma(0)\colon B(0) \to \mathrm{res}_{H_2} G/H_3$$

be the restrictions of α, γ. The image of $((\alpha, \beta_1), (\beta_2, \gamma))$ under (1) is represented by $(\alpha(0), \gamma(0))$, and by $\alpha(0) \times \gamma(0)$ if we apply (2). If we apply the $G \times_{H_2}$-extension process to the pullback

$$\begin{array}{ccc} A(0) \times B(0) & \xrightarrow{\mathrm{pr}} & B(0) \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow \\ A(0) & \longrightarrow & P, \end{array}$$

with P a point, we obtain the pullback

$$\begin{tikzcd}
C = G \times_{H_2} (A(0) \times B(0)) \arrow[r, "\bar\gamma"] \arrow[d, "\bar\alpha"] & G \times_{H_2} B(0) \cong B \arrow[d, "\beta_2"] \\
A \cong G \times_{H_2} A(0) \arrow[r, "\beta_1"] & G/H_2.
\end{tikzcd}$$

Thus, if we apply the $G\times_{H_2}$-extension to $\alpha(0) \times \gamma(0)$ and project away the G/H_2-factor, we obtain $(\alpha\bar\alpha, \gamma\bar\gamma)$ as a representative of the composition of the morphisms represented by **??**. We display the composition of **??** in the diagram

$$\begin{tikzcd}
C \arrow[r, "\bar\gamma"] \arrow[d, "\bar\alpha"] & B \arrow[r, "\gamma"] \arrow[d, "\beta_2"] & G/H_3 \\
A \arrow[r, "\beta_1"] \arrow[d, "\alpha"] & G/H_2 & \\
G/H_1 & &
\end{tikzcd}$$

in which the rectangle is a pullback. From this pullback description of the composition and the transitivity of pullbacks we see that composition is associative. The diagonal $G/H \to G/H \times G/H$ represents the identity of G/H in $\Omega(G)$.

We know that $U(G; G/H \times G/K)$ is a free abelian group. We now specify a basis. For this purpose we consider diagrams $\beta|\alpha)\colon G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$. Two such diagrams with G/H and G/K fixed are called equivalent if there exists a G-isomorphism $\sigma\colon G/L \to G/L'$ making the diagram

$$\begin{tikzcd}
G/H \arrow[d, "\mathrm{id}"] & G/L \arrow[l] \arrow[r] \arrow[d, "\sigma"] & G/K \arrow[d, "\mathrm{id}"] \\
G/H & G/L' \arrow[l] \arrow[r] & G/K
\end{tikzcd}$$

commutative up to G-homotopy.

(5.1) Proposition. *Equivalence classes of diagrams $G/H \leftarrow G/L \to G/K$ represent a $\mathbb{Z}$-basis of $U(G; G/H \times G/K)$.*

Proof. A diagram $G/H \xleftarrow{\alpha} G/L \xrightarrow{\beta} G/K$ is just a map $(\alpha, \beta)\colon G/L \to G/H \times G/K$ and represents an element of $U(G; G/H \times G/K)$. Equivalent diagrams represent the same element. A reformulation of (3.4) shows that the equivalence classes form a basis. $\square$

We verify the axioms of an induction category for $\pi_0 \operatorname{Or}(G), \Omega(G)$, see []. The first one is clear and the second one follows from the definition of morphisms and proposition (**??**). The interchange map $\tau\colon G/H \times G/K \to G/K \times G/H$ induces a bijection $U(G; G/H \times G/K) \cong U(G; G/K \times G/H)$. If we interprete this in terms of morphisms of $\Omega(G)$, we obtain a contravariant functor $D\colon \Omega(G) \to \Omega(G)$ with $D \circ D = \mathrm{id}$. We call this functor the ***self duality*** of $\Omega(G)$.

The composition of some of the basis elements is easy to describe. We have the following relations.

$$\begin{array}{rcl} (\alpha|\,\mathrm{id}) \circ (\beta|\gamma) & = & (\alpha\beta|\gamma) \\ (\beta|\gamma) \circ (\mathrm{id}\,|\delta) & = & (\beta|\delta\gamma) \end{array}$$

This follows immediately from the pullback description of the composition. This yields axiom (3). The fourth axiom follows from the fact that in a relation $(\mathrm{id}\,|\alpha)\circ(\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$ the α_s and β_s are obtained by the restriction of $\bar{\alpha}$ and $\bar{\beta}$ in (**??**) to orbits of C.

We call the relation $(\mathrm{id}\,|\alpha)\circ(\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$ the ***double coset decomposition*** in the category $\Omega(G)$. We make this explicit for the present situation. Express the identity of $G/H \times G/K$ as an element of $U(G; G/H \times G/K)$ in terms of the standard basis $\mathrm{id} = \sum_\alpha n_\alpha[\alpha]$. This decomposition belongs to a decomposition $G/H \times G/K = \coprod_\alpha Z(\alpha)$ into G-subspaces; each $Z(\alpha)$ is a subset of an orbit bundle $Z_{(L)}$ of $G/H \times G/K$ for an isotropy subgroup L. Moreover $n_\alpha = \chi_c(Z(\alpha)/G)$. Thus we have to use a decomposition of the double coset space $H\backslash G/K \cong (G/H \times G/K)/G = \coprod_\alpha Z(\alpha)/G$. With these notations we have the ***double coset formula***

$$(G/H \to G/G|\,\mathrm{id}_{G/H}) \circ (\mathrm{id}_{G/K}\,|G/K \to G/G) = \textstyle\sum_\alpha n_\alpha[\alpha]$$

for the composition of morphisms in $\Omega(G)$.

We remark that, by the discussion after **??**, we can take for the $Z(\alpha)$ the orbits which fill a path component of $Z_{(L)}$. This may be slightly redundant in the sense that some basis element can appear several times, since only the path components in $Z^{(L)}/G$ matter.

6 Functorial Properties of Induction Categories

Let $L \leq G$. There is an induction functor

$$i_L^G\colon \Omega(L) \to \Omega(G).$$

It maps the object L/A to the object $G/A = G \times_L A$ and the morphism $(u,v)\colon L/A \to L/B \times L/B$ to $(G \times_L u, G \times_L v)$. The pullback description of the composition is used to check the functor property. Induction is transitive $i_L^M \circ i_K^L = i_K^M$.

A morphism in $\Omega(G)$ of the type $G/L \to G/H \times G/K$ with $L \leq H, K$ is in the image of this functor. Other basic morphisms are obtained from this one by composing with the conjugation of subgroups.

We can define i_L^G also by universal constructions. We have a G-map

$$G \times_L (X \times Y) \to (G \times_L X) \times (G \times_L Y), \quad (g, x, y) \mapsto (g, x, g, y).$$

This yields

$$\Omega(L; X \times Y) \to \Omega(G; G \times_L (X \times Y) \to \Omega(G; (G \times_L X) \times (G \times_L Y)),$$

and from this one defines i_L^G.

Suppose $p\colon G \to K$ is a surjective homomorphism. This induces a functor

$$p_K^G\colon \Omega(K) \to \Omega(G).$$

It maps the object K/L to $G/p^{-1}L$, i.e., we view K/L via p as a G-space. We can apply the same device to a basic morphism

$$K/L_1 \leftarrow K/L_0 \to K/L_2$$

and obtain the morphism

$$G/p^{-1}L_1 \leftarrow G/p^{-1}L_0 \to G/p^{-1}L_2.$$

If we work with representatives of $U(K; K/L_1 \times K/L_2)$, then we can view the same representative via p as an element of $U(G; p^{-1}G/L_1 \times G/p^{-1}L_2)$. This process is additive and compatible with pullbacks and thus defines a homomorphism

$$\mathrm{Mor}_{\Omega(K)}(K/L_1, K/L_2) \to \mathrm{Mor}_{\Omega(G)}(G/p^{-1}L_1, G/p^{-1}L_2)$$

which is compatible with composition and identities. Thus we have a functor. This construction is transitive, i.e., $p_M^L p_N^M = p_N^L$. We can combine the constructions i_L^G and p_L^G. For this purpose we use a category LIE. The objects are the compact Lie groups. A morphism from H to K is an isomorphism class of diagrams

$$(p|i)\colon H \xleftarrow{i} L \xrightarrow{p} K$$

with an injection i and a surjection p. Another diagram

$$(p'|i')\colon H \xleftarrow{iI} L \xrightarrow{p'} K$$

is called isomorphic to the first one if there exists an isomorphism $\sigma\colon L \to L'$ such that $i'\sigma = i$ and $p'\sigma = p$.

In order to define the composition in LIE we consider the diagram

$$\begin{tikzcd} H & & \\ L \arrow[u, "i"'] \arrow[r, "p"] & K & \\ Q \arrow[u, "j'"'] \arrow[r, "p'"] & M \arrow[u, "j"'] \arrow[r, "q"] & N \end{tikzcd}$$

with a pullback square. We define $(j,q)\circ(i,p)=(ij',qp')$.

We define a contravariant functor from LIE into the category $\mathbb{Z}$-CAT of $\mathbb{Z}$-categories. On object level we set $L\mapsto\Omega(L)$. We map the morphism $H\xleftarrow{i}L\xrightarrow{p}K$ to the functor $i_L^H p_K^L$. We have to verify that this is a functor. This amounts to the verification of the relation $p_K^L j_M^K=(j')_Q^L(p')_M^Q$ in the notation of the diagram above.

(6.1) Remark. We describe the category $\Omega(G)$ for an abelian group G. By the general theory of the previous section it mainly remains to compute the composition of morphisms $(G/M\leftarrow G/H)\circ(G/L\to G/H)$ which appear in the double coset decomposition. We have

$$(G/M\leftarrow G/H)\circ(G/L\to G/H)=\chi(H/LM)[G/L\leftarrow G/(L\cap M)\to G/M].$$

There is a similar formula for the product in the Euler ring $U(G)$:

$$[G/H]\times[G/K]=\chi(G/LM)[G/(L\cap M).$$

Thus, if $G=S^1$ is the circle group, then all products of basis elements $[G/H]$, $H\neq G$ are zero. But for a higher dimensional torus there exist non-zero products if $LM=G$; and if $L\neq G$, then $[G/L]^2=0$. $\diamond$

7 Categories with Families

We consider multiplicative families $\mathcal{F}_2\subset\mathcal{F}_1$ and define a category $\Omega(G,\mathcal{F}_1,\mathcal{F}_2)$. Suppose first that $\mathcal{F}_2$ is empty. Then $\Omega(G,\mathcal{F}_1)$ is the subcategory of $\Omega(G)$ with the same morphisms and with $U(G,\mathcal{F}_1;G/H\times G/K)$ as morphism group from G/H to G/K. Since $\mathcal{F}$ is assumed to be multiplicative, this is compatible with composition. We have a natural inclusion of categories $\mu\colon\Omega(G,\mathcal{F}_2)\to\Omega(G,\mathcal{F}_1)$ and the image is an ideal. Thus we can form the quotient category $\Omega(G,\mathcal{F}_1,\mathcal{F}_2)$, on morphism sets the cokernel of μ. Since morphisms $\mathrm{Mor}(G/H,G/K)$ only use basis elements $G/L\to G/H\times G/K$, we only need the family $\mathcal{F}=\{L\mid(L)\le(H),(L)\le(K)\}$, i.e.,

$$\mathrm{Mor}_{\Omega(G)}(G/H,G/K)=\mathrm{Mor}_{\Omega(G,\mathcal{F})}(G/H,G/K).$$

We call the families (H)-***adjacent*** if $\mathcal{F}_1\smallsetminus\mathcal{F}_2=(H)$.

(7.1) Proposition. *If $\mathcal{F}_1\supset\mathcal{F}_2$ are (H)-adjacent, then*

$$\mathrm{End}_{\Omega(G,\mathcal{F}_1,\mathcal{F}_2)}(G/H)\cong\mathbb{Z}(\pi_0WH).$$

Proof. $\square$

(7.2) Corollary. *We always have a split surjection of rings*

$$\mathrm{End}_{\Omega(G)} \to \mathbb{Z}(\pi_0 WH).$$

Proof. □

Let $\mathcal{F}$ be a multiplicative family and $\mathcal{G}$ an $\mathcal{F}$-modular family. We show that the induction categories under consideration carry an enriched structure.

(7.3) Theorem. *The morphism sets of* $\Omega(G, \mathcal{F}, \mathcal{G})$ *carry a natural structure of an* $U(G, \mathcal{F})$*-module. Composition is bilinear with respect to this modules structure. In this way* $\Omega(G, \mathcal{F}, \mathcal{G})$ *becomes an* $\Omega(G, \mathcal{F})$*-category.*

Proof. We interprete $U(G, \mathcal{F}) = U(G, \mathcal{F}; P)$ with a point P. Then the module structure

$$U(G, \mathcal{F}) \times U(G, \mathcal{F}, \mathcal{G}; G/H \times G/K) \to U(G, \mathcal{F}, \mathcal{G}; G/H \times G/K)$$

is defined as a special case of (??). More generally, we have this module structure on $U(G, \mathcal{F}, \mathcal{G}; X)$. It is given by $[Y] \cdot [f \colon Z \to X] = [\mathrm{pr} \circ f \colon Y \times Z \to X]$. One verifies bilinearity. □

8 Ideals in Ω

In practice, there occur natural quotients of the categories $\Omega(G)$. We describe one such quotient.

Let $\mathrm{Or}_\infty(G)$ denote the subcategory of $\mathrm{Or}(G)$ which consists of morphisms $f \colon G/H \to G/K$ such that $\mathrm{Aut}(f)$ is infinite.

(8.1) Lemma. *If* $\mathrm{Aut}(f)$ *is finite then any composition of* f *in* $\mathrm{Or}(G)$ *has the same property.*

Proof. This is obvious for a composition of the form hf. Up to automorphism, we can assume that f is induced by an inclusion $H \subset K$. In that case $\mathrm{Aut}(f) \cong W_K H$. If $H \subset L \subset K$ and $W_K H$ is finite, then also $W_K L$, since K/L^H consists of a finite number of $W_K H$ orbits and carries a free $W_K L$-action. From this fact it follows that also a composition of the form fh has infinite automorphism group. □

We define $\omega(G; G/H \times G/K)$ as the quotient of $\Omega(G; G/H \times G/K)$ by the subgroup spanned by $(\alpha|\beta)$ with $\mathrm{Aut}(\beta)$ infinite. It is seen as for the lemma that these subgroups of the morphism groups from an ideal in $\Omega(G)$. Hence we can define the quotient category $\omega(G)$ of $\Omega(G)$ by this ideal. We call $\omega(G)$ the *stable induction category* for G. The reason is that this category is isomorphic to the category of homogeneous spaces and stable G-homotopy classes of maps. We return to this point later.

The endomorphism ring in $\omega(G)$ of G/G is the Burnside ring of G. The $U(G)$-module structure on the morphism sets of $\Omega(G)$ passes to an $A(G)$-module structure on the morphism sets of $\omega(G)$.

9 Coherent Systems of Mackey Functors

We relate Mackey functors for different groups. If they are related in a reasonable way, they are called coherent systems.

In the previous section we have constructed a contravariant functor from LIE to $\mathbb{Z}$-CAT which is given on object by $H \mapsto \Omega(H)$. Suppose we are given a family $(M_G\colon \Omega(G) \to R\text{-}\mathrm{Mod})$ of Mackey functors. A *coherence* for this family associates to each morphism $\sigma\colon H \to K$ in LIE a natural transformation $h_\sigma\colon M_H \to M_K \circ \sigma^*$ such that the following holds:

(1) $h_{\tau\sigma} = h_\sigma \tau^* \circ h_\tau$;

(2) If $\sigma = i\colon H \to K$ is an injection, then h_i is a natural isomorphism.

A family (M_G) together with a coherence (h_σ) is called a ***coherent family of Mackey functors***. We call it ***strongly coherent*** if we have the additional property

(3) Let i be an inner automorphism. Then h_i is the identity.

We display in form of a diagram, what condition (1) of a coherence says. For this purpose let the next diagram describe a composition of morphisms in LIE.

$$\begin{tikzcd}
A & & \\
B \arrow[u, "i"] \arrow[r, "p"] & C & \\
Z \arrow[u, "J"] \arrow[r, "P"] & D \arrow[u] \arrow[r, "q"] & E
\end{tikzcd}$$

Then the following diagram is commutative.

$$\begin{tikzcd}
M_E(E/E_0) \arrow[d, "h_q"] \arrow[dr, "h_\tau"] & & \\
M_D(D/q^{-1}E_0) \arrow[r, "h_j"] \arrow[d, "h_P"] & M_C(C/jq^{-1}E_0) \arrow[d, "h_p"] \arrow[dr, "h_\sigma"] & \\
M_Z(Z/p^{-1}q^{-1}E_0) \arrow[r, "h_J"] & M_B(B/p^{-1}jq^{-1}E_0) \arrow[r, "h_i"] & M_A(A/ip^{-1}jq^{-1}E_0)
\end{tikzcd}$$

Moreover, in this diagram

$$h_{\tau\sigma} = h_\sigma h_\tau, \quad h_{qP} = h_P h_q, \quad h_{iJ} = h_i h_J.$$

10 Global Categories

We construct in this section a source category Ω for global Mackey functors. The objects of Ω are the compact Lie groups. The morphism set $\mathrm{Mor}_\Omega(G,H) = \Omega(G,H)$ is defined again as a suitable universal additive invariant.

We consider (H,G)-principal bundles over finite H-complexes [**?**]. These are right G-principal bundles $p\colon X \to Z$ with a left action on H on X and Z such that p is H-equivariant and the actions of H and G on X commute; in other words: H acts as a group of automorphisms of the G-principal bundle $p\colon X \to Z$.

An ***additive invariant*** for the category of these (H,G)-bundles is defined in the usual way. It consists of an abelian group $U(H,G)$ and an element $u[p\colon X \to Z]$ for each bundle $p\colon X \to Z$ such that the following holds:

(1) If the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Phi"] \arrow[d, "p'"] & X \arrow[d, "p"] \\ Z' \arrow[r, "\varphi"] & Z \end{tikzcd}$$

is a bundle map and φ an H-homotopy equivalence, then $u[p'] = u[p]$.

(2) If we have H-subcomplexes $Z_1 \supset Z_0 \subset Z_2$ and $Z = Z_1 \cup Z_2$, and $p\colon X \to Z$ is an (H,G)-bundle with restrictions p_i to Z_i, then

$$u[p] + u[p_0] = u[p_1] + u[p_2].$$

(3) $u[p\colon \emptyset \to \emptyset] = 0$.

We denote by $\Omega(H,G)$ the value group of the universal such invariant.

We also use another view point for (H,G)-bundles: Spaces X with left H-action, right free G-action; the actions commute; and X is a finite (H,G)-complex.

We postulate $\mathrm{Mor}_\Omega(H,G) = \Omega(H,G)$. The category Ω is a $\mathbb{Z}$-category. The composition is defined on representatives

$$\Omega(H,K) \times \Omega(G,H) \to \Omega(G,K), \quad (Y,X) \mapsto X \times_H Y$$

with right K-action coming from Y and left G-action coming from X. Since this construction is compatible with equivariant homotopy equivalences and is additive in both variables, it induces, by the universal property, a well defined bilinear map. The construction is obviously associative. The identity of $\Omega(G,G)$ is represented by the (G,G)-space G with left and right G-action by group multiplication.

We can, of course, define in the same manner a category by working with left principal bundles with right automorphism group.

An (H,G)-bundle over a homogeneous space G/K is called a *local* (H,G)-bundle. Local (H,G)-bundles have the following form:

Let $K \leq H$ and $\rho: K \to G$ a homomorphism. Let $H \times_{(K,\rho)} G$ denote the quotient of $H \times G$ by the relation $(hk, \rho(k^{-1})g) \sim (h,g)$ for $k \in K$, with induced left H- and right G-action by group multiplication.

We consider some special cases of local objects and the corresponding morphisms in Ω.

Let $\sigma: H \to K$ and $\tau: K \to G$ be homomorphisms. The composition of morphisms represented by the corresponding local bundles is given by

$$(H \times_{(H,\sigma)} K) \times_K (K \times_{(K,\tau)} G) \cong H \times_{(H,\tau\sigma)} G.$$

We therefore have a covariant functor from the category of compact Lie groups to Ω which is the identity on objects and maps τ to $K \times_{(K,\tau)} G$.

Let now $i: K \to H$ be an injection and consider $H \times_{(iK,i^{-1})} K$ as a morphism from K to H in Ω. In this way we obtain a contravariant functor from the category of injections of compact Lie groups to Ω.

The two functors above are not injections. Conjugate homomorphisms or injections yield the same local object, hence the same morphism in Ω.

Consider now $H \stackrel{i}{\leftarrow} K \stackrel{\rho}{\to} G$ with an inclusion $i: K \subset H$. We have

$$(H \times_K K) \times_{(K,\rho)} G \cong H \times_{(K,\rho)} G.$$

Hence a general morphism is a composition of the above two special types of morphisms.

The computation of the morphism set is a special case of an Euler group computation. We consider (H,G)-bundles as left $H \times G^\circ$-spaces (G° the opposite group of G) with free G°-action. Thus we use the family

$$\mathcal{F} = \{K \leq H \times G^\circ \mid K \cap (1 \times G^\circ) = 1\}.$$

Then

$$\Omega(H,G) = U(H \times G^\circ, \mathcal{F}).$$

Therefore we obtain from the general theory, in particular:

(10.1) Proposition. *$\Omega(H,G)$ is the free abelian group on the isomorphism classes of local (H,G)-bundles.* $\square$

The local objects which appear in a (K,H)-bundle Y are determined as follows. Fix $y \in Y$. let $K_0 \leq K$ be the group of elements $k \in K$ such that there exists $h \in H$ with $ky = yh$. Define $\rho(k) = h$. Then ρ is a homomorphism $K_0 \to H$ which defines the local object through y.

We now describe the local objects which appear in $X \times_H Y$ for (H,K)-bundles Y and (K,G)-bundles X. Consider $(x,y) \in X \times_H Y$. This is a $K \times G$-space with action $(k,g)(x,y) = (kx, yg^{-1})$. The isotropy group of (x,y) is

$$\{(k,g) \mid \exists h \in H \text{ such that } kx = xh, hy = yg\}.$$

Describe the local objects through x and y by

$$\begin{array}{lll} kx = xg & \rho(k) = g & k \in K_0 \\ hy = yk & \sigma(h) = k & h \in H_0. \end{array}$$

For the isotropy group of (x, y) we then have to use $\rho\circ\sigma\colon \sigma^{-1}(K_0) \to G$. Therefore, in order to define a category of type Ω for certain subclasses of compact Lie groups, it suffices that this class is closed under taking subgroups, composition of morphisms, and restriction of morphisms to subgroups.

11 Local and Global Categories

We relate the categories $\Omega(G)$ to Ω. For this purpose we construct a functor

$$i_G\colon \Omega(G) \to \Omega$$

which is given on object level by $G/H \mapsto H$. Let $G/H_2 \leftarrow G/H_0 \to G/H_1$ be a basic morphism of $\Omega(G)$ with $H_2 \le H_0$ and $\rho\colon H_0 \to H_1$, $h \mapsto g^{-1}hg$ describing the G-map $G/H_0 \to G/H_2$. This is mapped to the local (H_2, H_1)-bundle given by ρ. In order to verify the functor property we describe the effect on morphisms for general representatives. Let

$$G/H \xleftarrow{\alpha} Z \xrightarrow{\beta} G/K$$

represent a morphism from G/H to G/K. We assign to it the (H, K)-bundle $\tilde{Z}_0 \to Z_0$ defined by the following pullback

$$\begin{tikzcd} \alpha^{-1}(eH) = Z_0 \arrow[d, "\beta"] & \tilde{Z}_0 \arrow[l] \arrow[d] \\ G/K & G. \arrow[l] \end{tikzcd}$$

This construction is additive and yields therefore a homomorphism

$$i_G\colon \mathrm{Mor}_{\Omega(G)}(G/H, G/K) \to \mathrm{Mor}_{\Omega}(H, K).$$

(11.1) Proposition. *The assignment i_G is compatible with composition and yields the prescription above for the basic morphisms.* $\square$

(11.2) Proposition. *Let $j\colon H \to G$ be an injection. Then $i_G \circ j_H^G = i_H\colon \Omega(H) \to \Omega$.* $\square$

Let $M\colon \Omega \to \mathbb{Z}$-Mod be a contravariant $\mathbb{Z}$-functor. We obtain from it the family $M_G := M \circ i_G$ of Mackey functors. We also have an associated coherence. Let

$$\sigma\colon A \xleftarrow{i} B \xrightarrow{p} C$$

be a morphism in LIE. A natural transformation h_σ consists of a family of homomorphisms

$$h_\sigma(C/C_0)\colon M(C_0) = M_C(C/C_0) \to M_A\sigma^*(C/C_0) = M(ip^{-1}C_0).$$

We have the surjective homomorphism $pi^{-1}\colon ip^{-1}C_0 \to C_0$. We define $h_\sigma(C/C_0) = M(pi^{-1})$. One verifies from the functor property that the h_σ constitute a natural transformation. Moreover:

(11.3) Proposition. *The h_σ are a strong coherence for the family M_G.* □

We now prove a converse.

(11.4) Theorem. *A strongly coherent family (M_G, h_σ) of Mackey functors arises from a unique $\mathbb{Z}$-functor M by the construction above.*

The functor is given on objects by $M(H) = M_H(H/H)$. Suppose

$$\sigma\colon G \xleftarrow{i} L \xrightarrow{q} Q \xrightarrow{j} K$$

is a morphism in Ω with inclusions i, j and a surjection q. We define

$$M(\sigma)\colon M(K) \to M(G)$$

as the following composition:

$$M_K(K/K) \xrightarrow{\alpha^*} M_K(K/Q) \xleftarrow{h_j} M_Q(Q/Q) \xleftarrow{h_\tau} M_G(G/L) \xrightarrow{\beta_*} M_G(G/G).$$

Here we have used the following notations: $\alpha\colon K/Q \to K/K$ and $\beta\colon G/L \to G/G$ are the canonical quotient maps and $\tau = (q|i)$ is a morphism in LIE. From the fact that the family is ***strongly*** coherent it is verified that an isomorphic diagram (which yields the same morphism in Ω) leads to the same composition, so that $M(\sigma)$ is well-defined.

It remains to show that this definition is compatible with the composition of morphisms.

12 Vector Bundles

We define a global Mackey functor from vector bundles. Let $E_G(n) \to B_G(n)$ denote the universal n-dimensional complex G-vector bundle [**?**]. We set

$$V_n(G) = U(G; B_G(n)), \qquad W_n(G) = A(G; B_G(n)).$$

We construct a functor V_n on Ω. On objects we set $G \mapsto V_n(G)$. Let S be a (K, L)-bundle. We have the map

$$V_n(S)\colon U(K; B_K(n)) \to U(L; S \times_K B_K(n)) \to U(L; B_L(n)).$$

The first homomorphism is induced by applying the functor $S \times_K ?$ from K-spaces to L-spaces, see **??**. The second homomorphism is induced by a map $k_S \colon S \times_K B_K(n) \to B_L(n)$. It is the classifying map of the L-vector bundle $S \times_K E_G(n) \to S \times_K B_K(n)$. By homotopy invariance of the Euler groups, the morphism $V_n(S)$ is well-defined. If T is an (L, M)-space, we have

$$V_n(T) \circ V_n(S) = V_n(T \times_L S).$$

A similar construction works for the Burnside groups.

The case of one-dimensional bundles $V_1(G)$ is interesting, since it is the basis for the so called explicit Brauer induction. We describe $V_1(G)$ in more explicit terms. It is the free abelian group on isomorphism classes of G-line bundles over homogeneous spaces G/H. Such a line bundle corresponds to a one-dimensional character, i.e., a homomorphism $\varphi \colon H \to \mathbb{C}^*$. We denote by $(H, \varphi)^G$ the element in $V_1(G)$ defined by this character. The group is a ring. The product of two characters is computed in the following manner.

13 Explicit Brauer Induction

Let V be a complex representation of G. Associated to it is the canonical G-line bundle $E(V) \to P(V)$ over the complex projective space $P(V)$ of V. Let $k_V \colon P(V) \to B_G(1)$ denote the classifying map of this line bundle. It represents an element $b_G(V) \in W_1(G) = W(G)$.

(13.1) Proposition. *The assignment $V \to b_G(V)$ is additive, $b_G(V \oplus W) = b_G(V) + b_G(W)$, and induces therefore an additive homomorphism*

$$b_G \colon R(G) \to W(G)$$

from the complex representation ring $R(G)$.

Proof. Proof. We have to show the additivity $[P(V \oplus W)] = [P(V) + [P(W)]$

in the Burnside group. The deviation fromm additivity is represented by the classifying map restricted to $P(V \oplus W) \smallsetminus P(V) \smallsetminus P(W)$. This space is, up to equivariant homotopy, the quotient $S(V) \times_{S^1} S(W)$ of the two unit spheres. All fixed point sets are spheres bundles with an odd-dimensional sphere as fibre and have therefore zero Euler characteristic. Thus the deviation space represents the zero element in the Burnside group. □

From the construction we see:

(13.2) Proposition. *The morphisms b_G commute with restriction to subgroups.* □

Let G be finite. We define a homomorphism $c_G\colon W(G) \to R(G)$ by sending a basis element (H, ψ) to the induced representation $\mathrm{ind}_H^G \psi$. From the double coset formula one verifies immediately:

(13.3) Proposition. *The morphisms c_G commute with restrictions to subgroups.* □

(13.4) Proposition. *The composition $c_G b_G$ is the identity for each finite group G.*

Proof. By **??** and **??** it suffices to consider cyclic groups G. From the constructions it is immediately clear that $c_G b_G(V) = V$ for a one-dimensional representation V. □

The theorem says in particular that each element in $R(G)$ is an integral linear combination of representations which are induced from one-dimensional representations of subgroups. But this linear combination is obtained via the map b_G in some canonical form.

We generalize the theorem to compact Lie groups and give a more conceptual interpretation to some ingredients in the constructions.

14 The Kernel-Image Theorem for the Burnside Ring

Let N be a finite set of subgroups of G and $p \in \mathbb{Z}$ a prime. A set $M(p)$ of subgroups is called (N,p)***-hyperelementary*** if it has the following property: For each $L \subset H \in N$ there exists $K \in M(p)$ such that G/K^L is finite and its cardinality non-zero modulo p, and $\chi(X^L) \equiv \chi(X^K) \bmod p$ for each finite G-complex.

The localization at the prime ideal (p) is denoted by an index (p). Let $\mathrm{Ke}(N)$ denote the kernel of the restriction map $A(G)_{(p)} \to \prod_{H \in N} A(H)_{(p)}$ and $\mathrm{Im}(M(p))$ the image of the induction map $\bigoplus_{L \in M(p)} A(L)_{(p)} \to A(G)_{(p)}$. The following theorem is a basic fact about the Burnside ring and has many applications to induction theory.

(14.1) Theorem. $\mathrm{Ke}(N) + \mathrm{Im}(M(p)) = A(G)_{(p)}$.

Proof. The group $\mathrm{Ke}(N) + \mathrm{Im}(M(p)) = A(G)_{(p)}$ is an ideal of $A(G)_{(p)}$, since $\mathrm{Ke}(N)$, as a kernel of a ring homomorphism, is an ideal and the image of the induction map is an ideal for any Green functor.

If this ideal were different from $A(G)_{(p)}$, then we could find a maximal ideal q of $A(G)_{(p)}$ which contains $\mathrm{Ke}(N)+\mathrm{Im}(M(p))$. This ideal has the form $q = q(L,p)$, see [2, p.]. The inclusion $A(G)_{(p)}/\mathrm{Ke}(N) \to \prod_{H \in N} A(H)_{(p)}$ is an integral ring extension, see [2, p.]??. By a basic theorem of commutative algebra [1, 5.10], the ideal q can be extended to a prime ideal of the product $\prod_{H \in N} A(H)_{(p)}$. Any such prime ideal is obtained by lifting a prime ideal of some factor $A(H)_{(p)}$. Therefore, we can assume that $L \subset H$ for some $H \in N$. Let L' be the defining group of $q(L, p)$ and choose $K \in M(p)$ with $L' \subset K$, by hypothesis **??**. Then $G/L' \in A(G)$ is the image of K/L' under induction $A(K) \to A(G)$; thus $G/L' \in \mathrm{Ke}(N) + \mathrm{Im}(M(p)) \subset q = q(L, p)$, a contradiction. □

A similar theorem holds for the universal Euler ring $U(G)$, since the projection $U(G) \to A(G)$ consists of nilpotent elements and induces therefore an isomorphism of prime ideal spectra.

(14.2) Corollary. $\mathrm{Ke}(N) + \mathrm{Im}(M(p)) = U(G)_{(p)}$. □

15 Hyperelementary Induction

A group K is called ***p-hyperelementary*** for the prime p if there exists an exact sequence

$$1 \to S \to K \to P \to 1$$

with a finite p-group P and a topologically cyclic group S such that the group of components S/S_0 of S has order prime to p. A group is called ***hyperelementary*** if it is p-hyperelementary for some prime p.

A closed subgroup S of G is called a ***Cartan subgroup*** of G if it is topologically cyclic and the Weyl group $W_G S$ is finite. It is called p-***regular*** if its group of components has order prime to p.

(15.1) Proposition. *The set of conjugacy classes of Cartan subgroups of a compact Lie group is finite.*

(15.2) Proposition. *A set $M(p)$ for the set N of Cartan subgroups is the set of p-hyperelementary subgroups.*

The next theorem is called the ***hyperelementary induction theorem***.

(15.3) Theorem. *Let M be a Green functor such that $M(G/G)$ is torsion free. Let N be the set of Cartan subgroups of G and suppose that* $\mathrm{Ke}(N) = 0$. *Then the induction*

$$\mathrm{ind}\colon \bigoplus_{L \in H} M(G/L) \to M(G/G)$$

from the set H of hyperelementary subgroups in surjective.

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