The Euler Ring of the Rotation Group

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Abstract

We determine the multiplication table of the homogeneous spaces in the Euler ring of the group SO_3 and relate this to structural data about restriction and induction homomorphisms between Euler rings. Certain specific results use the fact that the Euler ring functor is a Mackey functor or even a Green functor.

1 Introduction

The Euler ring $U(G)$ of a compact Lie group G was studied in [2] in the context of additive invariants and induction categories. An element of $U(G)$ is represented by a compact G-ENR (G-equivariant Euclidean neighbourhood retract) X. The G-action on X induces for each subgroup H of G (always closed, notation $H \le G$) an action of the normalizer $N_G H$ on the H-fixed set X^H. We denote by $\chi(Y)$ the Euler characteristic of a space Y and let $\chi_H(X)$ be the Euler characteristic $\chi(N_G H \backslash X^H)$ of the orbit space $N_G H \backslash X^H$. Two G-ENR's X and Y define the same element of $U(G)$ if and only if for each H the equality $\chi_H(X) = \chi_H(Y)$ holds. Addition and multiplication in $U(G)$ are induced by disjoint union and Cartesian product, respectively. The value $\chi_H(X)$ only depends on the conjugacy class of H in G. Let $[X] \in U(G)$ denote the element represented by X. The assignment $\chi_H \colon U(G) \to \mathbb{Z}$, $[X] \mapsto \chi(N_G H \backslash X^H)$ is an additive homomorphism.

Additively, $U(G)$ is the free abelian group with basis the isomorphism classes of homogeneous spaces $[G/H]$ for closed subgroups H of G.

One purpose of this note is to determine the multiplication table of the homogeneous spaces in $U(\mathrm{SO}_3)$. The result in itself may not be very informative.

In fact, a brute force computation is in some sense straightforward but quite tedious. We rather want to relate the computation to other structural data and the concept of a Mackey functor. Thereby we obtain additional information about $U(\mathrm{SO}_3)$.

There is a canonical surjective ring homomorphism $\pi = \pi_G \colon U(G) \to A(G)$ onto the Burnside ring $A(G)$ of G. The kernel of π_G is the nil-radical $N(G)$ of $U(G)$, see [2, p. 241]. The subgroup $N(G)$ is spanned by the $[G/H]$ for subgroups H which have infinite index in their normalizer $N_G H$; and $A(G)$ has a $\mathbb{Z}$-basis of isomorphism classes $[G/H]$ for subgroups H with finite Weyl group $W_G H = N_G H/H$. We let $\iota \colon A(G) \to U(G)$ denote the additive inclusion which is the identity on the canonical basis elements $[G/H]$. We call $b(x) = \iota\pi(x)$ the Burnside part of $x \in U(G)$. Thus $U(G)$ is additively the direct sum of the subgroups $N(G)$ and $\iota A(G)$. In order to simplify the notation, we also write H for the basis element $[G/H]$ if G is clear from the context.

By the way, it will turn out that the nilpotent elements (which are not present in the Burnside ring) are quite useful. We assume known the group theory and the geometry of the subgroups of SO_3.

In the sequel we describe various methods, apply these to chosen examples and leave other cases to the reader.

2 Products and Mackey functors

A homomorphism $\rho \colon K \to L$ induces ring homomorphisms

$$U(\rho) \colon U(L) \to U(K), \qquad A(\rho) \colon A(L) \to A(K)$$

by viewing an L-space via ρ as K-space. In the case of an inclusion $\rho \colon K \subset L$ we call it the restriction res^L_K. There is also an additive induction homomorphism

$$\mathrm{ind}^L_K \colon U(K) \to U(L), \quad [X] \mapsto [L \times_K X].$$

Restriction and induction are the basic ingredients to make the Euler ring functor into a Mackey functor (Green functor). Here we use the setting of [2, p. 276].

The product $[G/K] \times [G/L]$ in $U(G)$ can be given another interpretation. There exists a canonical G-homeomorphism

$$G/K \times G/L \cong G \times_K \mathrm{res}^G_K G/L, \quad (uK, vL) \mapsto (uK, u^{-1}vL);$$

and $G/L = \mathrm{ind}^G_L(L/L)$. Thus, starting from the unit element $[L/L] \in U(L)$, we obtain the product as the composition

$$\mathrm{ind}^G_K \mathrm{res}^G_K \mathrm{ind}^G_L [L/L].$$

The induction homomorphism has the simple property $\mathrm{ind}^G_K[K/K'] = [G/K']$. The basic remaining problem is therefore to express $\mathrm{res}^G_K \mathrm{ind}^G_L[L/L]$ as a linear combination of the basis elements. This is obtained, in general terms, by the double coset formula (DCF), a main ingredient of a Mackey functor. We do not determine in this note the relevant DCF's, but later we use at least its existence.

The double coset formula has the following shape

$$\mathrm{res}^G_K \mathrm{ind}^G_L = \sum\nolimits_\alpha a_\alpha \mathrm{ind}^K_{K\cap g^{-1}Lg} \circ c(g) \circ \mathrm{res}^L_{gKg^{-1}\cap L}$$

where $a_\alpha \in \mathbb{Z}$, the sum is taken over certain double cosets $KgL \subset G$ and $c(g)$ is induced by conjugation with g. In the case of a finite group the a_α equal 1 and the sum is over the double cosets [1, p. 164]. For the general case see [2, p. 280].

In this context there is also an interesting duality: We can interchange the roles of K and L. Then $\mathrm{res}^G_L \mathrm{ind}^G_K[K/K]$ is relevant. This is an instance of the fact that the induction category $\Omega(G)$ is self-dual, see [2, p. 274].

3 The Euler ring of SO_3

The multiplication table of the $[\mathrm{SO}_3/H]$ for the Burnside $A(\mathrm{SO}_3)$ring was determined by Schwänzl [5]. The result is stated in [1, p.156]. We deduce the result for $U(\mathrm{SO}_3)$ from $A(\mathrm{SO}_3)$; this is not strictly necessary but may be interesting in itself. We use the restriction $r\colon U(\mathrm{SO}_3) \to U(\mathrm{SO}_2)$.

(3.1) Proposition. *The ring homomorphism*

$$(\pi, r)\colon U(\mathrm{SO}_3) \to A(\mathrm{SO}_3) \times U(\mathrm{SO}_2)$$

is injective.

Proof. An element in the kernel of π is a linear combination of the C_m. By the result $r(\mathrm{C}_m) = 2\mathrm{C}_m$ in 3.8 we see that such elements are detected by r. □

The ring $U(\mathrm{SO}_2)$ is trivial. An additive basis consists of SO_2 and the cyclic groups C_m of order $m \geq 1$. All products of basis elements which do not involve the unit element are zero.

We list the conjugacy classes of closed subgroups of SO_3 and their normalizers.

H	SO_3	O_2	SO_2	A_5	S_4	A_4	$\mathrm{D}_m, m \geq 3$	D_2	$\mathrm{C}_n, n \geq 2$	C_1
NH	SO_3	O_2	O_2	A_5	S_4	S_4	D_{2m}	A_4	O_2	SO_3

Here D_m is the dihedral group of order $2m$ and C_n is the cyclic group of order n. Moreover the alternating groups A_5 and A_4 are the icosahedral and the tetrahedral group and the symmetric group S_4 is the octahedral group. The subgroup $N(\mathrm{SO}_3)$ is spanned by the C_n. Let $\nu\colon N(\mathrm{SO}_2) \to N(\mathrm{SO}_3)$ be the homomorphism which is the "identity" $\mathrm{C}_n \mapsto \mathrm{C}_n$ on the basis elements.

(3.2) Proposition. *The product xy of basis elements x, y in $U(\mathrm{SO}_3)$ is obtained via the formula $xy = b(xy) + \frac{1}{2}\nu(r(x)r(y) - rb(xy))$.*

Proof. By construction, the element $xy - b(xy)$ is contained in the kernel of π and therefore a linear combination of the form $\sum_j a_j \mathrm{C}_j$. Then, by 3.8, $r(xy - b(xy)) = 2\sum_j a_j \mathrm{C}_j$ and hence $\frac{1}{2}\nu r(xy - b(xy)) = \sum_j a_j \mathrm{C}_j$. Now use that r is a ring homomorphism. □

In order to obtain from 3.1 and 3.2 explicit results, it is necessary to determine the value of the restriction homomorphism r on the basis elements. The result is displayed in the table 3.8. For the verification of the table we have to express elements $[X] \in U(G)$ in terms of the basis elements $[G/H]$.

(3.3) Remark. Suppose $[X] = \sum_H a_H [G/H]$, $a_H \in \mathbb{Z}$; this is a sum over conjugacy classes of isotropy groups of X. Apply the homomorphism χ_C to this equation. From the set of equations

$$\chi(N_G C \backslash X^C) = \textstyle\sum_H a_H \chi(N_G C \backslash (G/H^C))$$

the a_H can be determined recursively by downward induction. The numbers $\chi_C(G/H)$ can be determined by groups theory, see 3.5. ◇

(3.4) Examples. (1) If we apply 3.3 to the trivial subgroup $C = 1$, we see that the sum $\sum a_H$ of the coefficients equals the Euler characteristic $\chi(G\backslash X)$ of the orbit space.

(2) In the case of the SO_2-space SO_3/L the orbit space $\mathrm{SO}_2\backslash\mathrm{SO}_3/L$ is a closed interval for infinite L and a 2-sphere for finite L. Therefore the sum of the coefficients in 3.8 is 1 or 2, respectively. Note that $\mathrm{SO}_2\backslash\mathrm{SO}_3$ is a 2-sphere with standard L-action.

(3) The orbit space $\mathrm{O}_2\backslash\mathrm{SO}_3/L$ has in each case the Euler characteristic 1. It is a closed interval for infinite L, a 2-disk for finite $L \neq \mathrm{C}_m$ and a projective plane for $L = \mathrm{C}_m$. The sum of the coefficients is therefore in each case 1. See table 4.3.

(4) If we apply 3.3 to a maximal isotropy type $H = C$ of X, then G/H^H consists of a single $N_G H$-orbit, and therefore $a_C = \chi(N_G C \backslash X^C)$. ◇

The use of 3.3 is based on the next general result. It describes the group theoretic situation. (The symbol $\sim_L$ means L-conjugate.)

(3.5) Proposition. *The space* $G/L^K = \{tL \mid t^{-1}Kt \leq L\}$ *consists of a finite number of* $N_G K$*-orbits* [2, p. 41]. Elements sL and tL are in the same $N_G K$-orbit if and only if the subgroups $s^{-1}Ks$ and $t^{-1}Kt$ are L-conjugate. The isotropy group of the $N_G K$-action at tL is $tN_L(t^{-1}Kt)t^{-1}$. The number of $N_G K$-orbits is equal to the number of L-conjugacy classes of subgroups $A \leq L$ which are G-conjugate to K. □

(3.6) Example. Consider $X = \mathrm{SO}_3/L$ as SO_2-space. The isotropy groups are SO_2 or finite cyclic groups. In order to determine X in $U(\mathrm{SO}_2)$ as a linear combination of homogeneous spaces we have to know the values $L|C =$

$\chi(\mathrm{SO}_2\backslash(\mathrm{SO}_3/L^C))$ for $L \le \mathrm{SO}_3$ and $C \le \mathrm{SO}_2$. (The normalizer of C in SO_2 is always SO_2.) An $N_{\mathrm{SO}_3}L = \mathrm{O}_2$-orbit of SO_3/L^C is isomorphic to O_2/N_LC. The quotient by SO_2 therefore consists of one or two points if N_LC is a dihedral or cyclic group, respectively. From this fact one easily reads off the vales in the list 3.7. Let us look at two cases.

The group D_m, $m \equiv 0(2)$ contains three conjugacy classes of subgroups of order two and their normalizers are dihedral groups. This accounts for the value $\mathrm{D}_m|\mathrm{C}_2 = 3$.

The tetrahedral group A_4 has a single conjugacy class of subgroups of order three and they are equal to their normalizers in A_4. This accounts for the value $\mathrm{A}_4|\mathrm{C}_3 = 2$. ◇

(3.7) Proposition. *The data $L|C$ which are relevant for* 3.8*.*

$O_2\|SO_2$	$=$	1		$A_5\|C_k$	$=$	1	$k=5,3,2$
$O_2\|C_2$	$=$	2		$S_4\|C_k$	$=$	1	$k=4,3$
$SO_2\|SO_2$	$=$	2		$S_4\|C_2$	$=$	2	
$D_m\|C_m$	$=$	1	$m>2$	$A_4\|C_3$	$=$	2	
$D_m\|C_2$	$=$	2	$m\equiv 1(2)$	$A_4\|C_2$	$=$	1	
$D_m\|C_2$	$=$	3	$m\equiv 0(2)$	$C_m\|C_m$	$=$	2	

(3.8) Proposition. *The value of the restriction homomorphism $r\colon U(\mathrm{SO}_3) \to U(\mathrm{SO}_2)$ on the basis elements.*

SO_3	SO_2	C_m	$2\,\mathrm{C}_m$
O_2	$\mathrm{SO}_2 + \mathrm{C}_2 - \mathrm{C}_1$	A_5	$\mathrm{C}_5 + \mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$
SO_2	$2\,\mathrm{SO}_2 - \mathrm{C}_1$	S_4	$\mathrm{C}_4 + \mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$
D_m	$\mathrm{C}_m + 2\,\mathrm{C}_2 - \mathrm{C}_1$	A_4	$2\,\mathrm{C}_3 + \mathrm{C}_2 - \mathrm{C}_1$

Proof. We have to consider the spaces SO_3/L as SO_2-spaces. The isotropy groups have the form $\mathrm{SO}_2 \cap gLg^{-1}$.

Consider the case D_m. The isotropy groups are C_m, C_2 and C_1. Let $m > 2$. The isotropy group C_m is maximal. Hence $\mathrm{D}_m|\mathrm{C}_m$=1 is its coefficient. For $m \equiv 1(2)$ also C_2 is maximal, hence $\mathrm{D}_m|\mathrm{C}_2$ is its coefficient. For $m \equiv 0(2)$ the summands $\mathrm{SO}_2/\mathrm{C}_m$ and $\mathrm{SO}_2/\mathrm{C}_2$ contribute to the C_2-fixed points. Hence $\mathrm{D}_m|\mathrm{C}_2 = 3$ is the sum of the coefficients and 2 is again the coefficient of C_2. (The case of D_2 is included.) Finally we use that the sum of all coefficients is 2 and therefore -1 the coefficient of C_1.

Consider the case A_5. The group has elements of order $2, 3, 5$. The related isotropy groups are maximal, hence their coefficient is 1. □

The multiplication table of the homogeneous spaces in $A(\mathrm{SO}_3)$ can be found in [1, p. 156]. From the multiplication in $A(\mathrm{SO}_3)$ and 3.2 one now obtains the multiplication table of $U(\mathrm{SO}_3)$ by a simple computation. Note that $r(x)r(y) = 0$ if x and y are finite groups.

(3.9) Example. From the Burnside ring we know that $\mathrm{O}_2\mathrm{O}_2 = \mathrm{O}_2 + \mathrm{D}_2 + R$, where R is a linear combination of cyclic groups. We now apply the restriction homomorphism r. The left side yields

$$(\mathrm{SO}_2 + \mathrm{C}_2 - \mathrm{C}_1)^2 = \mathrm{SO}_2 + 2\mathrm{C}_2 - 2\mathrm{C}_1,$$

since the product in $U(\mathrm{SO}_2)$ is trivial. The right hand side yields, by 3.8,

$$(\mathrm{SO}_2 + \mathrm{C}_2 - \mathrm{C}_1) + (\mathrm{C}_2 + 2\mathrm{C}_2 - \mathrm{C}_1) + 2R.$$

Thus $R = -\mathrm{C}_2$. ◇

(3.10) Proposition. *We list below the deviation of the multiplication in $U(\mathrm{SO}_3)$ from the multiplication in $A(\mathrm{SO}_3)$ for the cases where no dihedral groups are involved. Let us write $K \cdot L = b(K \cdot L) + R(K, L)$; here $R(K, L)$ is a linear combination of cyclic groups. We know already $R(\mathrm{O}_2, \mathrm{O}_2) = -\mathrm{C}_2$.*

$R(\mathrm{O}_2, \mathrm{A}_5)$	$-3\,\mathrm{C}_2 + \mathrm{C}_1$	$R(\mathrm{A}_5, \mathrm{A}_5)$	$-\mathrm{C}_5 - 2\,\mathrm{C}_3 - 3\,\mathrm{C}_2 + 2\,\mathrm{C}_1$
$R(\mathrm{O}_2, \mathrm{A}_4)$	$\mathrm{C}_3 - \mathrm{C}_2$	$R(\mathrm{A}_4, \mathrm{A}_4)$	$-2\,\mathrm{C}_3 - \mathrm{C}_2 + \mathrm{C}_1$
$R(\mathrm{O}_2, \mathrm{S}_4)$	$-3\,\mathrm{C}_2 + \mathrm{C}_1$	$R(\mathrm{A}_4, \mathrm{A}_5)$	$-2\,\mathrm{C}_3 - \mathrm{C}_2 + \mathrm{C}_1$
$R(\mathrm{O}_2, \mathrm{SO}_2)$	$\mathrm{C}_2 - \mathrm{C}_1$	$R(\mathrm{S}_4, \mathrm{S}_4)$	$-\mathrm{C}_4 - \mathrm{C}_3 - 4\,\mathrm{C}_2 + 2\,\mathrm{C}_1$
$R(\mathrm{SO}_2, \mathrm{A}_5)$	$r(\mathrm{A}_5)$	$R(\mathrm{S}_4, \mathrm{A}_4)$	$-\mathrm{C}_3 - 2\,\mathrm{C}_2 + \mathrm{C}_1$
$R(\mathrm{SO}_2, \mathrm{A}_4)$	$r(\mathrm{A}_4)$	$R(\mathrm{S}_4, \mathrm{A}_5)$	$-2\,\mathrm{C}_3 - 4\,\mathrm{C}_2 + 2\,\mathrm{C}_1$
$R(\mathrm{SO}_2, \mathrm{S}_4)$	$r(\mathrm{S}_4)$	$R(\mathrm{SO}_2, \mathrm{SO}_2)$	$r(\mathrm{SO}_2)$

(3.11) Example. The ring $U(\mathrm{SO}_3)$ contains the orthogonal idempotents

$$\begin{aligned} x &= \mathrm{A}_5 - \mathrm{A}_4 - \mathrm{D}_5 - \mathrm{D}_3 + \mathrm{C}_3 + 2\,\mathrm{C}_2 - \mathrm{C}_1, \\ y &= \mathrm{O}_2 + \mathrm{S}_4 - \mathrm{D}_4 - \mathrm{D}_3 + \mathrm{C}_2. \end{aligned}$$

They lift the idempotents of $A(\mathrm{SO}_3)$ with the same name which are displayed in [1, p. 156]. Although this can be verified from our results in a straightforward (but tedious) manner, we will give another argument in the sequel. ◇

4 Restriction to the orthogonal group

In this section we study in the restriction homomorphism $U(\mathrm{SO}_3) \to U(\mathrm{O}_2)$. In order to compare the result with 3.8 we list the values of the restriction $U(\mathrm{O}_2) \to U(\mathrm{SO}_2)$.

(4.1) Proposition. *The restriction homomorphism $U(\mathrm{O}_2) \to U(\mathrm{SO}_2)$.*

O_2	SO_2	D_m	C_m
SO_2	$2\,\mathrm{SO}_2$	C_m	$2\,\mathrm{C}_m$

Proof. The isomorphism of SO_2-spaces $\mathrm{O}_2/\mathrm{D}_m \cong \mathrm{SO}_2/\mathrm{C}_m$ shows that D_m is sent to C_m. □

(4.2) Proposition. *The multiplication table for $U(\mathrm{O}_2)$. The symbol (m,n) denotes the greatest common divisor of m and n.*

	O_2	SO_2	D_n	C_l
C_k	C_k	$2\mathrm{C}_k$	0	0
D_m	D_m	C_m	$2\mathrm{D}_{(m,n)} - \mathrm{C}_{(m,n)}$	
SO_2	SO_2	$2\mathrm{SO}_2$		
O_2	O_2			

Proof. Consider the product of D_m with D_n. The isotropy groups are $\mathrm{D}_{(m,n)}$ and $\mathrm{C}_{(m,n)}$. Hence the product has the form $a\mathrm{D}_{(m,n)} + b\mathrm{C}_{(m,n)}$. The coefficient a is the order of the Weyl group, hence equals 2. The space of double cosets is a closed interval and has Euler characteristic 1. Hence $b = -1$, see (1) in 3.4. □

(4.3) Proposition. *The restriction homomorphism $U(\mathrm{SO}_3) \to U(\mathrm{O}_2)$.*

O_2	$\mathrm{O}_2 + \mathrm{D}_2 - \mathrm{D}_1$	C_m	C_m
SO_2	$\mathrm{SO}_2 + \mathrm{D}_1 - \mathrm{C}_1$	A_5	$\mathrm{D}_5 + \mathrm{D}_3 + \mathrm{D}_2 - 3\,\mathrm{D}_1 + \mathrm{C}_1$
$\mathrm{D}_m, m \equiv 1(2)$	$\mathrm{D}_m - \mathrm{D}_1 + \mathrm{C}_2$	S_4	$\mathrm{D}_4 + \mathrm{D}_3 + \mathrm{D}_2 - 3\,\mathrm{D}_1 + \mathrm{C}_1$
$\mathrm{D}_m, m \equiv 0(2)$	$\mathrm{D}_m + 2\,\mathrm{D}_2 - 3\,\mathrm{D}_1 + \mathrm{C}_1$	A_4	$\mathrm{D}_2 - \mathrm{D}_1 + \mathrm{C}_3$

One should observe that in SO_3 the equality $\mathrm{D}_1 = \mathrm{C}_2$ holds, but not in O_2 (rotation versus reflection).

Proof. One method of proof combines the information contained in 3.8, 3.4 part (3) and 4.1. In addition, one has to know the isotropy types which occur in SO_3/L as O_2-space. We consider an example. For $m \equiv 0 \bmod 2$ the element $r(\mathrm{D}_m)$ is a linear combination of D_m, D_2, D_1 and C_1. By 3.4 the sum of the coefficients is one. Then we use 3.8 and 4.1 and compare coefficients. □

(4.4) Proposition. *The kernel of the restriction $r\colon U(\mathrm{SO}_3) \to U(\mathrm{O}_2)$ has rank 4 and is spanned by the following elements (called exceptional)*

$$\begin{aligned}
a &= \mathrm{A}_5 - \mathrm{A}_4 - \mathrm{D}_5 - \mathrm{D}_3 + \mathrm{C}_3 + 2\mathrm{C}_2 - \mathrm{C}_1 \\
b &= \mathrm{S}_4 + \mathrm{A}_4 - \mathrm{D}_4 - \mathrm{D}_3 - \mathrm{C}_3 + \mathrm{C}_2 \\
c &= 3\mathrm{A}_4 - \mathrm{D}_2 - 3\mathrm{C}_3 + \mathrm{C}_1 \\
d &= \mathrm{SO}_3 - \mathrm{O}_2 + \mathrm{A}_4 - \mathrm{C}_3.
\end{aligned}$$

The cokernel of r is free of rank one.

Proof. One can verify from 4.3 that these elements are contained in the kernel; we will interpret these elements and their properties in a moment from a more systematic point of view. Let $\eta\colon U(\mathrm{O}_2) \to \mathbb{Z}$ be the surjective homomorphism which sends SO_2 to -1, the D_m to 1 and all other basis elements to zero. Then one verifies from 4.3 that the kernel of η is the image of r.

Suppose α is contained in the kernel of r. Write α in terms of the basis of homogeneous spaces. By comparing coefficients one sees directly that the D_m, $m > 5$ do not occur in α. Moreover, one sees that the coefficients of SO_3 and O_2 must have sum zero. Thus, subtracting a suitable multiple of d, we can eliminate these basis elements. Then we eliminate A_5 and S_4 with the help of a and b. There remains a linear combination of A_4, D_j and C_m in the kernel. From 4.3 we see that D_j, $j = 5, 4, 3$ cannot occur. Since $r(\mathrm{D}_2) = 3\mathrm{D}_2 - 3\mathrm{D}_1 + \mathrm{C}_1$, we can eliminate A_4 and D_2 simultaneously by subtraction of a multiple of c. It remains a linear combination of the C_m. By inspection, r is injective on such elements. □

5 The exceptional elements

We now discuss the exceptional elements a, b, c, d in more detail. We first collect their multiplicative properties.

(5.1) Proposition. *The multiplication table of the exceptional elements.*

	a	b	c	d
d	a	$b+c$	$3c$	$d+c$
c	0	$3c$	$6c$	
b	0	$b+c$		
a	a			

The elements $e_1 = a,\ e_2 = 2b - c,\ e_3 = c,\ e_4 = d - b - a$ satisfy

$$e_1^2 = e_1, \quad e_2^2 = 2e_2, \quad e_3^2 = 6e_3, \quad e_4^2 = e_4, \quad e_ie_j = 0$$

for $i \neq j$. From these relations the multiplication table can be recovered. □

A verification of these relations from the multiplication table of the homogeneous spaces would be quite awkward. We therefore discuss these elements from other view points.

Recall that the Burnside ring $U(G) = A(G)$ of a finite group G is a subring of the ring $C(G)$ of $\mathbb{Z}$-valued functions on the set of conjugacy classes of subgroups of G. The inclusion $A(G) \to C(G)$ assigns to $[X] \in A(G)$ the function $(H) \mapsto \chi(X^H)$ (Burnside marks). The subring $A(G)$ of $C(G)$ can be characterized by a set of congruence relations. From these congruences it is easy to verify that $U(G)$ contains the following elements (see [1, p. 11]).

$$\begin{array}{llll}
\varepsilon_1 \in U(\mathrm{A}_5); & \varepsilon_1(\mathrm{A}_5) = 1, & \varepsilon_1(H) = 0 \text{ for } H \neq \mathrm{A}_5 \\
\varepsilon_2 \in U(\mathrm{S}_4); & \varepsilon_2(\mathrm{S}_4) = 2, & \varepsilon_2(H) = 0 \text{ for } H \neq \mathrm{S}_4 \\
\eta_2 \in U(\mathrm{S}_4); & \eta_2(\mathrm{S}_4) = 1, & \eta_2(\mathrm{A}_4) = 3,\ \eta_2(H) = 0 \text{ for } H \neq \mathrm{S}_4, \mathrm{A}_4 \\
\varepsilon_3 \in U(\mathrm{A}_4); & \varepsilon_3(\mathrm{A}_4) = 3, & \varepsilon_3(H) = 0 \text{ for } H \neq \mathrm{A}_4
\end{array}$$

These elements therefore satisfy

$$\varepsilon_1^2 = \varepsilon_1, \quad \varepsilon_2^2 = 2\varepsilon_2, \quad \varepsilon_3^2 = 3\varepsilon_3.$$

Now recall the restriction and induction homomorphisms res^G_H and ind^G_H and the double coset formula (DCF) for the computation of $\mathrm{res}^G_H \mathrm{ind}^G_K$. The Euler ring functor is a Mackey functor on the induction category $\Omega(G)$; for details see sections IV.8 and IV.9 in [2]. The restrictions are ring homomorphisms and for the induction we have the Frobenius reciprocity (FR)

$$(\mathrm{ind}^G_H x) \cdot y = \mathrm{ind}^G_H(x \cdot \mathrm{res}^G_H y)$$

for $x \in U(H)$ and $y \in U(G)$.

Let L be a finite subgroup of $G = \mathrm{SO}_3$. Suppose $x \in U(L)$ has Burnside marks with value zero on dihedral and cyclic groups. Then $\mathrm{ind}^G_L x$ is contained in the kernel of the restriction to $U(\mathrm{O}_2)$, a consequence of the DCF. The elements ε_j and η_2 just defined have this property.

The elements a, b, c, d are obviously images under induction of elements with the same name but where now the groups in the sum are subgroups of the appropriate L. In the case of S_4 one has to be a little careful, though. Since S_4 has two conjugacy classes of elements of order 2, one has to interpret in this case the symbol C_2 as the group $\mathrm{C}_2^{\#}$ generated by a transposition.

We now claim that $\mathrm{ind}^G_H \varepsilon_1$ is an idempotent element contained in the kernel of r; here $H = \mathrm{A}_5$, $G = \mathrm{SO}_3$. It will turn out that this element equals e_1. The DCF says that $\mathrm{res}^G_{\mathrm{O}_2} \mathrm{ind}^G_H \varepsilon_1$ is an integral linear combination of elements of the form

$$\mathrm{ind}_{g^{-1}Kg} \circ c(g) \circ \mathrm{res}^H_K(\varepsilon_1)$$

where K has the form $H \cap g\mathrm{O}_2 g^{-1}$ and $c(g)$ is a conjugation automorphism. For our present purpose it is not necessary to know the precise form of the DCF. The groups K are dihedral or cyclic groups, and by the very definition of ε_1 the elements $\mathrm{res}^H_K(\varepsilon_1)$ are zero. This shows $r(\mathrm{ind}^G_H \varepsilon_1) = 0$, as already remarked above. Now we use FR:

$$\mathrm{ind}(\varepsilon_1) \cdot \mathrm{ind}(\varepsilon_1) = \mathrm{ind}(\varepsilon_1 \cdot \mathrm{res}\,\mathrm{ind}\,\varepsilon_1) = \mathrm{ind}(\varepsilon_1^2) = \mathrm{ind}(\varepsilon_1).$$

This uses $\mathrm{res}\,\mathrm{ind}\,\varepsilon_1 = \varepsilon_1$, again a consequence of the DCF. For the purpose of finding generators of the kernel of r it would be sufficient to see that ε_1 has the form $A_5 + \rho$ where ρ is a linear combination of basis elements different from A_5, and this is easy. But actually ε_1 is exactly the linear combination displayed for the element a and therefore $\mathrm{ind}\,\varepsilon_1 = e_1$.

The space $P = SO_3/A_5$ is the Poincaré sphere. We consider it as an A_5-manifold. It has a single fixed point. When we excise an open invariant disk about the fixed point we obtain an A_5-manifold D (Poincaré disk), where each $H \neq A_5$ has a homology disk as fixed point set. Hence the Burnside mark is the function $1 - \varepsilon_1$. This gives a topological interpretation for the idempotent element ε_1.

There exists a finite SO_3-CW-complex X with empty fixed set X^H for $H = SO_3, A_5$ and with contractible fixed sets for all other groups. See [4] for the geometric significance of this Oliver disk. From these properties of the fixed sets we see that $[X]$ is an idempotent element of $U(SO_3)$. From [4, p. 234] we see, that $[X]$ is the linear combination $O_2 + S_4 - D_4 - D_3 + C_2$ in terms of the basis elements. Thus $1 - [X]$ is the element which was called $d - a$ above. From the geometry we also see that this element is contained in the kernel of r. (In the Burnside ring it also represents the element $1 - y$ in the notation of [1, p. 156].)

We now turn our attention to A_4. It is not difficult to verify that ε_3 is the element $3A_4 - D_2 - 3C_3 + C_1$. Hence $e_3 = \mathrm{ind}\,\varepsilon_3$. As in the previous case we use FR to verify $e_3^2 = 6e_3$; it uses $\mathrm{res}\,\mathrm{ind}(\varepsilon_3) = 2\varepsilon_3$, a consequence of the DCF.

Finally one computes η_2 as a linear combination of basis elements and obtains the relation $\mathrm{ind}\,\eta_2 = b$ as a consequence.

One can also derive the orthogonality relations for the e_i by using FR. For instance, the relation $e_1 e_4 = 0$ amounts to showing that the restriction of the Oliver disk to A_5 represents the same element as the Poincaré disk.

(5.2) Proposition. *The exists an injective homomorphism of rings*

$$U(SO_3) \to U(O_2) \times \mathbb{Z}^4.$$

The first component is the restriction r. The other components are the fixed point ring homomorphisms $[X] \mapsto \chi(X^H)$ for the groups $H = A_5, A_4, S_4, SO_3$.

Proof. Let x be in the kernel. From 4.4 we know that x is a linear combination of a, b, c, d. Now consider the fixed points of x for SO_3, A_5, S_4 and A_4, in that order, to conclude that $x = 0$. □

Note that the multiplicative structure of the image ring is fairly simple. A similar discussion can be applied to the Burnside ring. Thus one could compute the multiplication table from these homomorphisms.

6 Standard representations in the Euler ring

We now study the situation which is dual to 3.8 and 4.3. Here we have to determine the L-spaces $SO_3/SO_2 \cong S^2$ and $SO_3/O_2 \cong \mathbb{R}P^2$ (standard L-action on the 2-sphere and the projective plane) in the Euler ring $U(L)$ for the subgroups L of SO_3.

(6.1) Proposition. *The standard representations in the Euler ring.*

L	S^2 *in* $U(L)$	$\mathbb{R}P^2$ *in* $U(L)$
O_2	$SO_2 + D_1 - C_1$	$O_2 + D_2 - D_1$
SO_2	$2SO_2 - C_1$	$SO_2 + C_2 - C_1$
$D_m, m \equiv 1(2)$	$C_m + 2D_1 - C_1$	D_m
$D_m, m \equiv 0(2)$	$C_m + D_1 + D_1^{\#} - C_1$	$D_m + D_2 + D_2^{\#} - C_2 - D_1 - D_1^{\#} + C_1$
C_m	$2C_m$	C_m
A_5	$C_5 + C_3 + C_2 - C_1$	$D_5 + D_3 + D_2 - 3C_2 + C_1$
S_4	$C_4 + C_3 + C_2^{\#} - C_1$	$D_4 + D_3 + D_2^{\#} - C_2 - 2C_2^{\#} + C_1$
A_4	$2C_3 + C_2 - C_1$	$D_2 + C_3 - C_2$

We have used the following notation. The group D_m *has a single conjugacy class of subgroups isomorphic to* D_k *if* m/k *is odd and two conjugacy classes if* m/k *is even. In the latter case we denote the conjugacy classes by* D_k *and* $D_k^{\#}$. *We also understand that* $D_k \sim D_k^{\#}$ *if* m/k *is odd.*

The group S_4 *has two conjugacy classes of subgroups of order* 2*: The group* $C_2^{\#}$ *generated by a transposition and* C_2*. The normalizer of* $C_2^{\#}$ *is denoted* $D_2^{\#}$*. The normalizer of* C_2 *is a* D_4 *and contains another conjugacy class* D_2*.*

Proof. The case of infinite L in the table is easily derived from the geometry.

Suppose L is finite. In the case of S^2 the isotropy groups are cyclic. Each element $g \neq 1$ has two fixed points on S^2. All elements in the cyclic subgroup generated by g have the same fixed points. The isotropy group of a fixed point is therefore a maximal cyclic subgroup of L. The non-trivial isotropy groups of the action are therefore the maximal cyclic subgroups. If $C \leq L$ is such a group and a_C the coefficient of C, then

$$2 = \chi(S^2) = \chi((S^2)^C) = a_C|L/C^C| = a_C|W_L C|.$$

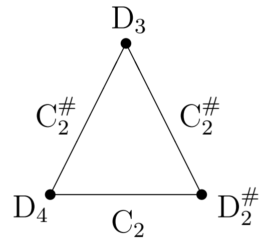
Hence $a_C = 1, 2$ and $|W_L C| = 2, 1$, respectively. From this information and the knowledge of the sum of the coefficients we obtain the data of the table.

Now we turn our attention to the projective plane. Each element $g \neq 1$ has either a unique fixed point (order of g not 2) or a fixed point and a fixed S^1 (g of order 2); in each case the Euler characteristic is 1. The general formula

$$\chi(X/L) = \tfrac{1}{|L|}\left(\sum_{g\in L}\chi(X^g)\right),$$

see [1, p. 96], thus yields in our case that $\chi(X/L) = 1$. A closer study of the topology gives the result stated in part (3) of 3.4. Note that if an isotropy group of a fixed point is a dihedral group, then the orbit space is a manifold with boundary.

The most complicated case is perhaps the group S_4. The orbit space is a disk. The next figure indicates the subsets of a given orbit type.



In order to visualize the situation the reader may start with a standard cube in 3-space with vertices $(\gamma_1, \gamma_2, \gamma_3)$, $\gamma_j \in \{\pm 1\}$ and its symmetry group S_4.

The orbit space in the case A_5 has a similar shape. The vertices represent D_j for $j = 2, 3, 5$ and the edges C_2. The interior of the triangle corresponds in both cases to the free orbits.

By the way, in order to determine the coefficients of the groups one can take the Euler characteristic minus 1 of the one-point compactification of the subsets with given type in the orbit space. For example, in the S_4-case the $\mathrm{C}_2^{\#}$-set consists of two open intervals; the one-point compactification is a wedge of two circles with reduced Euler characteristic -2. □

When we pass via induction to SO_3 then 6.1 yields the same result as 3.8 and 4.3 (by the duality mentioned earlier).

7 The Euler ring of SU_2

We have the double covering $p\colon \mathrm{SU}_2 \to \mathrm{SO}_3$. It induces ring homomorphisms $A(p) = p^*$ and $U(p) = p^*$. Let $H^* = p^{-1}(H)$. Then $p^*[G/H] = [G^*/H^*]$. The homomorphism $U(p)$ is injective. The only basis elements of SU_2 which are not contained in the image of $U(p)$ are the C_m with $m \equiv 1 \bmod 2$. It is easy to investigate their product behaviour. By the way, $\mathrm{C}_m^* = \mathrm{C}_{2m}$. In $U(\mathrm{SU}_2)$ we have

$$\mathrm{O}_2^* \cdot \mathrm{C}_n = \mathrm{C}_n, \qquad \mathrm{SO}_2^* \cdot \mathrm{C}_n = 2\,\mathrm{C}_n, \qquad \mathrm{SU}_2 \cdot \mathrm{C}_n = \mathrm{C}_n.$$

The remaining products with C_n are zero. Thus the Euler ring of SU_2 is basically the same as the Euler ring of SO_3. The Euler ring $U(\mathrm{SU}_2)$ was investigated by Hoffmann [3].

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