Hyperbolic modifications and acyclic affine foliations

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In this note we present a new method for the construction of manifolds and varieties, called hyperbolic modification. By this method we find new acyclic affine varieties and affine space forms. Space forms are algebraic varieties which are diffeomorphic to a Euclidean space. There has been some interest recently in affine space forms (see e. g. Dimca [7], Kaliman [11], Zaidenberg [22]). Only a few methods are known for the construction of affine space forms. In particular, it is usually not easy to verify that the varieties are not isomorphic to the standard affine space. As a typical application of our methods we show:

Theorem A. *For each integer $n \geq 4$ there exist complex polynomial functions $q\colon \mathbb{C}^n \to \mathbb{C}$ with the following properties:*

(1) *The polynomial has no singularities.*

(2) *Each fibre $q^{-1}(c)$ is diffeomorphic to Euclidean space $\mathbb{C}^{n-1}$.*

(3) *The polynomial is not isomorphic (by an algebraic coordinate change) to a projection $V \times \mathbb{C} \to \mathbb{C}$ of a product.*

(4) *Some fibres of q carry an exotic algebraic structure, i. e. are not isomorphic to an affine space.*

We express the situation of Theorem A by saying that q yields an acyclic affine foliation of $\mathbb{C}^n$. The proof of Theorem A is based on the following construction and Theorems B and C below.

Let $h\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial such that the origin is a regular point of the affine hypersurface $L = h^{-1}(0)$. Then there exists a unique polynomial

$$q_h = q\colon \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}$$

such that

$$uq(u,x) = h(ux), \quad u \in \mathbb{C},\, x \in \mathbb{C}^n.$$

We call q_h the *hyperbolic modification* of h.

Theorem B. *Suppose $n \geq 3$. Let $h\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial such that $h^{-1}(0)$ is a regular contractible hypersurface containing the origin. Then the hyperbolic modification $q_h\colon \mathbb{C}^{n+1} \to \mathbb{C}$ is a regular polynomial such that each fibre is diffeomorphic to a Euclidean space.*

We find explicit polynomials h to which Theorem B can be applied by starting with Brieskorn polynomials. The topological fact will be important that a neighbourhood of the isolated singularity has a spherical boundary. A simple example of a polynomial with the properties of Theorem A is

$$q(u,x,y,z) = uz^2(ux+1)^3 + 3z(ux+1)^2 + 3x - z(uy+1)^2 - 2y.$$

The algebraic nature of the fibres is studied with the help of group actions and the logarithmic Kodaira dimension $\bar{\kappa}$. The basic input into proving part (4) of Theorem A is the addition theorem of KAWAMATA [12]. It yields in our case:

Theorem C. *Under the assumptions of* Theorem B*, if* $\bar{\kappa}(h^{-1}(0)) \geq k$*, then* $\bar{\kappa}q^{-1}(0)) \geq k$*.*

We can apply Theorem C to the polynomial

$$h(x,y,z) = z^{-1}\left((xz+1)^a - (yz+1)^b\right)$$

for coprime integers a, b, since it was shown in [6] that $V = h^{-1}(0)$ is a contractible variety with $\bar{\kappa}(V) = 1$. The explicit example above is the case $(a,b) = (3,2)$.

Once we have a situation where Theorems A and B hold we obtain Theorem A(3) from the cancellation theorem of IITAKA and FUJITA [10].

We point out that it is very easy to construct affine space forms X of general type, i. e. with $\bar{\kappa}(X) = \dim X$. There are plenty of homology planes W of general type. By compatibility of $\bar{\kappa}$ with products, $\bar{\kappa}(V) + \bar{\kappa}(W) = \bar{\kappa}(V \times W)$. It is a topological fact (compare the proof of (3.14)) that the product of two homology planes is diffeomorphic to $\mathbb{C}^4$.

1 Hyperbolic modifications

The hyperbolic modification produces new manifolds from a given manifold. The modified manifold carries a group action. If we start with an acyclic manifold, then the resulting modification is again acyclic. This section is devoted to the topological part of the definition.

Let A be a finite-dimensional algebra over the field F of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. We assume that the group of units $G = A^*$ in A is open and dense in the vector space A. We consider G as Lie group over F. Suppose $\rho\colon A \times V \to V$ is a continuous ($F = \mathbb{R}$) or holomorphic ($F = \mathbb{C}$) map such that $\rho_a\colon v \mapsto \rho(a,v)$ is linear, and $\rho_a\rho_b = \rho_{ab}$, $\rho_0 = 0$.

Let U be a star-shaped open neighbourhood of the origin in V. Then the open subset

$$\tilde{U} = \{(a,v) \mid av \in U\} \subset A \times V$$

is contractible (by linear connection of v with 0). Set

$$b\colon G \times (U \setminus 0) \to \tilde{U}, \quad (a,u) \mapsto (a, a^{-1}u).$$

Suppose $\varphi\colon (U,0) \to (L,x)$ is a chart of the smooth manifold L which is centered at x. We consider the pushout diagram

$$\textbf{(1.1)}\qquad \begin{array}{ccc} G\times(U\setminus 0) & \xrightarrow{\mathrm{id}\times\varphi} & G\times(L\setminus x) \\ \big\downarrow b & & \big\downarrow B \\ \tilde U & \xrightarrow{\Phi} & X. \end{array}$$

The maps $\mathrm{id}\times\varphi$ and b are embeddings onto open subsets. Moreover $(\mathrm{id}\times\varphi, b)$ is an embedding onto a closed submanifold of $G\times(L\setminus x)\times\tilde U$. Therefore ([4], I(1.8)) X carries the structure of a smooth manifold such that B and Φ are embeddings onto open subsets.

In case $F = \mathbb{C}$ and φ a holomorphic chart of a complex manifold L, the resulting manifold X is again a complex manifold. The manifold character of L is only used via the chart φ. Apart from this the object could have singularities.

(1.2) Definition. Any manifold X constructed in the manner above from L is called a *hyperbolic modification* of L. ♡

We list some topological properties of hyperbolic modifications. A space is called acyclic if its reduced integral homology is zero.

(1.3) Proposition. *Let X be a hyperbolic modification of the n–dimensional manifold L. Then the following holds:*
(1) *If L is acyclic, the X is acyclic.*
(2) *If $n > 2$ and $\pi_1(L) = 0$, the $\pi_1(X) = 0$.*
(3) *If $n > 2$ and L is contractible, then X is contractible.*

Proof. We apply the Mayer-Vietoris sequence of homology to the defining diagram (1.1). Then (1) follows, because $\tilde U$ is contractible and $\mathrm{id}\times\varphi$ a homology isomorphism.

In order to prove (2), we apply the theorem of Seifert and van Kampen ([4], II(5.7)) to (1.1). Since $n > 2$ and $\pi_1(L) = 0$, we have $\pi_1(L\setminus 0) = 0$.

Finally, (3) follows from (1) and (2) and general results of algebraic topology ([4], II(5.11) and V(6.3)). □

The hyperbolic modification X carries a smooth right action of G. We let $g \in G$ act on $G\times(U\setminus 0)$ and $G\times(L\setminus x)$ by right multiplication on the first factor and by $g\cdot(a,u) = (ag, g^{-1}u)$ on $\tilde U$ and postulate (1.1) as a pushout in the category of G–spaces. In the complex case the action is holomorphic.

We have a canonical map

$$\textbf{(1.4)}\qquad t\colon X \to L$$

which, in terms of (1.1), forgets the first component of $G\times(L\setminus 0)$ and is given by $(a,u)\mapsto \varphi(au)$ on $\tilde U$. The map t is invariant, hence factors over the orbit space. The fibres of t over points in $L\setminus x$ are closed free orbits. The fibre $t^{-1}(x)$ can be identified with $\{(a,v) \mid av = 0\} \subset A\times V$. It contains the origin as a closed orbit.

All other orbits in $A \times 0 \cup 0 \times V$ have the origin in its closure This latter fact is the reason for the name hyperbolic modification: The G–fixed point $(0,0)$ of the action $g \cdot (a,v) = (ag, g^{-1}v)$ on $A \times V$ is of hyperbolic type.

Let $X_0 = t^{-1}(L \setminus x) \cup b(0,0)$. This is a G–stable subset. The map $t\colon X_0 \to L$ has a section. Therefore the following holds:

(1.5) Proposition. *The map t induces a homeomorphism of the orbit space X_0/G with L.* $\square$

(1.6) Example. The simplest, but typical, example is $A = V = \mathbb{C}$ and $G = \mathbb{C}^*$ with map $\rho\colon A \times V \to V$, $(\lambda, z) \mapsto \lambda^l z$ for an integer $l \geq 1$. $\heartsuit$

(1.7) Example. Similarly, $A = M_n(\mathbb{C})$, the complex (n,n)–matrices, $G = GL(n,\mathbb{C})$, $V = \mathbb{C}^n$ and $\rho\colon A \times V \to V$ the matrix multiplication. $\heartsuit$

(1.8) Example. One can consider direct products of the previous examples, e. g.: $A = \mathbb{C} \times \ldots \times \mathbb{C}$, $G = \mathbb{C}^* \ldots \times \mathbb{C}^*$, $V = \mathbb{C}^n$, and

$$\rho\colon A \times \mathbb{C}^n \to \mathbb{C}^n,\ ((\lambda_1, \ldots, \lambda_n), (x_1, \ldots, x_n)) \mapsto (\lambda_1^{l(1)} x_1, \ldots, \lambda_n^{l(n)} x_n).$$

One can show that this case also arises from an n–fold iteration of (1.6). $\heartsuit$

In the case of example (1.6) and $l = 1$ we note:

(1.9) Lemma. *The complex manifold X is independent of the choice of the holomorphic chart φ about x_0.*

Proof. It is easy to see that shrinking U leads to the same manifold. Therefore it suffices to consider charts $\varphi, \psi : U \to L$ which differ by a holomorphic automorphism $\alpha : U \to U$. In this case there exists a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] \arrow[d, "\mathrm{id} \times \alpha"] & \tilde{U} \arrow[d, "A"] \\
\mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] & \tilde{U}
\end{tikzcd}$$

with a holomorphic automorphism A. This uses the fact that $(u,x) \mapsto u^{-1}\alpha(ux)$ has a holomorphic extension to $u = 0$ by the derivative of α. $\square$

One can also consider parametrized versions of the hyperbolic modification. We only treat the case of $\mathbb{C}^*$–actions. Let L be a complex manifold, $M \subset L$ a closed complex submanifold, and $\tau : U \to L$ a holomorphic tubular map from an open star-shaped neighbourhood U of the zero section in the normal bundle $E \to M$ of M in L. Let

$$\tilde{U} = \{(\lambda, u) \mid \lambda u \in U\} \subset \mathbb{C} \times E.$$

Define the complex manifold X with $\mathbb{C}^*$–action to be the pushout of the diagram

$$\begin{tikzcd} \mathbb{C}^* \times (U \setminus M) \arrow[r, "\mathrm{id} \times \tau"] \arrow[d, "b"] & \mathbb{C}^* \times (L \setminus M) \arrow[d] \\ \tilde{U} \arrow[r] & X. \end{tikzcd}$$

As for (1.3) one verifies:

(1.10) Proposition. *Suppose L and M are acyclic. Then X is acyclic.* □

2 Affine varieties

We apply the hyperbolic modification with the data of (1.6) to a regular point of an affine variety L.

Suppose the affine variety $L = L(I)$ is the zero set of the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. Let $0 \in L$ be a regular point of L. Consider the ideal

$$J = q(I) \subset \mathbb{C}[u, x_1, \ldots, x_n]$$

which is generated by all polynomials q_h such that

(2.1) $$u q_h(u, x) = h(ux), \quad h \in I.$$

Since $0 \in L$, the equality (2.1) yields for each $h \in I$ a well defined polynomial q_h. The polynomial q_h has the equivariance property

(2.2) $$q_h(\lambda^{-1} u, \lambda x) = \lambda q_h(u, x), \quad \lambda \in \mathbb{C}^*.$$

Therefore the affine variety $V = V(J) \subset \mathbb{C} \times \mathbb{C}^n$, the zero locus of J, is stable under the $\mathbb{C}^*$–action $\lambda \cdot (u, x) = (\lambda^{-1} u, \lambda x)$ on $\mathbb{C} \times \mathbb{C}^n$.

(2.3) Proposition. *The $\mathbb{C}^*$–variety $V(J)$ coincides with the hyperbolic modification of L at 0, as a complex space.*

Proof. We select a holomorphic chart $\varphi \colon U \to L$ centered at 0 and consider the following diagram

$$\begin{tikzcd} \mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] \arrow[d, "a"] & \tilde{U} \arrow[d, "B"] \\ \mathbb{C}^* \times (L \setminus 0) \arrow[r, "A"] & V \end{tikzcd}$$

with maps

$$\begin{aligned} a(u, x) &= (u, \varphi(x)) \\ b(u, x) &= (u, u^{-1} x) \\ A(u, x) &= (u, u^{-1} x) \\ B(u, x) &= (u, u^{-1} \varphi(ux)). \end{aligned}$$

The map B has to be interpreted as a holomorphic map: For $u = 0$ we have to set $B(0, x) = (0, D\varphi(x))$ with the differential $D\varphi$ of φ at 0. By construction, the diagram is commutative.

We have to show that the diagram is a pushout. This amounts to the following verifications: The maps A and B are holomorphic embeddings onto open subsets $\tilde{A}$ and $\tilde{B}$ of V. The intersection $\tilde{A} \cap \tilde{B}$ is as predicted by the diagram.

The image of A is $V \cap \{(u, x) \mid u \neq 0\}$, hence open in V. An inverse of A is induced by the morphism $A_1 \colon (u, x) \mapsto (u, ux)$.

Suppose $r = \dim L$. Let $\Phi \colon W_1 \to W_2$ be a holomorphic isomorphism between open neighbourhoods of zero in $\mathbb{C}^n$ such that Φ restricts to

$$\varphi \colon 0 \times U = W_1 \cap (0 \times \mathbb{C}^r) \to L \cap W_2.$$

The isomorphism Φ exists since $0 \in L$ was assumed to be a regular point. Let $Z_2 = \{(u, x) \mid ux \in W_2\} \subset \mathbb{C} \times \mathbb{C}^n$.

One verifies that $V \cap Z_2$ is the image of B. An inverse of B is induced by the morphism $(u, x) \mapsto (u, \mathrm{pr}(u^{-1}\Phi^{-1}(ux)))$ with pr: $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ the projection; again, for $u = 0$ one has to use the differential of Φ^{-1}. □

We consider the group action from an algebraic point of view. The coordinate rings of L and V are $\mathbb{C}[x]/I$ and $\mathbb{C}[u, x]/J$. The $\mathbb{C}^*$–action $\lambda \cdot (u, x) = (\lambda u, \lambda^{-1}x)$ induces an action on $\mathbb{C}[u, x]/J$ because J is invariant. Let R denote the ring of invariants.The homomorphism

$$j \colon \mathbb{C}[x] \to \mathbb{C}[u, x],\ \ x_i \mapsto ux_i$$

induces a homomorphism

$$j_* \colon \mathbb{C}[x]/I \to R.$$

(2.4) Proposition. *The homomorphism j_* is an isomorphism.*

Proof. Suppose p is in the kernel of j_*. Then there exists a relation of the type

$$p(ux) = \sum_i a_i(u, x) q_i(u, x).$$

We multiply by u, set $u = 1$ and see that $p \in I$.

Let S be the ring of $\mathbb{C}^*$–invariants in $\mathbb{C}[u, x]$. Then it is easy to see that j induces an isomorphism $j : \mathbb{C}[x] \to S$. Since $\mathbb{C}^*$ is a reductive group, the surjection $C[u, x] \to \mathbb{C}[u, x]/(q)$ induces a surjection $S \to R$ (compare [21], II).□

One expresses (2.4) by saying that L is the algebraic quotient of V under the $\mathbb{C}^*$–action (compare [14], p. 96, and [21], p. 14). The map $\pi \colon V \to L,\ (u, x) \mapsto ux$ is the quotient map.

If L is a regular hypersurface, then V is again regular. Therefore we can iterate the hyperbolic modification. Since V carries a $\mathbb{C}^*$–action with fixed point 0 the hyperbolic modification of V at 0 carries a $\mathbb{C}^* \times \mathbb{C}^*$–action. The n–fold modification $L(n)$ of L carries an action of the n–dimensional torus $T(n) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$. The hyperbolic modification is transitive: The n–fold modification can be obtained as a single modification by using (1.8).

(2.5) Proposition. *The algebraic quotient $L(n)//T(n)$ is isomorphic to L.* $\square$

The general pattern of the hyperbolic modification in this algebraic setting is: Let $\pi\colon X \to Y$ be a morphism and $L \subset Y$ a subvariety. Exclude some singular set $S \subset Y$ of π, take $\pi^{-1}(L \setminus S)$ and form the (Zariski) closure of this pre-image.

3 Acyclic affine foliations

In this section we construct polynomials $q\colon \mathbb{C}^{n+1} \to \mathbb{C}$ with the property that all fibres $q^{-1}(c)$ are diffeomorphic to Euclidean space. The construction is based on the hyperbolic modification (section 2). We give a proof of Theorem B of the introduction.

Let $h\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $h(0) = 0$ such that $0 \in h^{-1}(0) = L$ is a regular point of L. Let $q = q_h\colon \mathbb{C}^{n+1} \to \mathbb{C}$ denote the hyperbolic modification of h.

Notation. In this section, X denotes the Hopf bundle (recalled below) and $V = q^{-1}(1)$ the general fibre of q.

We blow up the point $0 \in L$ and consider the complement of the proper transform. This construction will also clarify the conceptual meaning of q_h.

Denote by $X = \mathbb{C} \times_{\mathbb{C}^*} (\mathbb{C}^n \setminus 0)$ the quotient of $\mathbb{C} \times (\mathbb{C}^n \setminus 0)$ under the relation $(u, x) \sim (\lambda^{-1}u, \lambda x)$ for $\lambda \in \mathbb{C}^*$ (the variety X is the Hopf bundle). The morphism $p\colon X \to \mathbb{C}^n,\ (u, x) \mapsto ux$ blows up the point $0 \in \mathbb{C}^n$ and $p\colon X \setminus E \to \mathbb{C}^n \setminus 0$, $E = p^{-1}(0)$ is an isomorphism. Moreover we have:

(3.1) $p\colon X \setminus p^{-1}(L) \to \mathbb{C}^n \setminus L$ is an isomorphism.

(3.2) $p^{-1}(L) = L_1 \cup E$ with $L_1 = \{(u, x) \mid q(u, x) = 0\}$.

We let $V = q^{-1}(1)$. The next result shows that V is the general fibre of q.

(3.3) Lemma. *The restriction $q\colon q^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ is a trivial fibration with typical fibre V, a regular affine hypersurface.*

Proof. The regularity of V follows easily from the identity

$$u\frac{\partial q(u, x)}{\partial u} + q(u, x) = \sum_{i=1}^{n} x_i \frac{\partial h(ux)}{\partial x_i}.$$

The map

$$\varphi\colon \mathbb{C}^* \times V \to q^{-1}(\mathbb{C}^*),\ (\lambda, (u, x)) \mapsto (\lambda^{-1}u, \lambda x)$$

satisfies $q \circ \varphi = \mathrm{pr}$. An inverse ψ of φ is given by

$$\psi(u, x) = (q(u, x); q(u, x)u, q(u, x)^{-1}x).$$

Therefore φ is an algebraic bundle isomorphism. $\square$

(3.4) Lemma. *Suppose $h^{-1}(0) \subset \mathbb{C}^n$ is a regular variety. Then q is a regular polynomial.*

PROOF. (3.3) shows that q is regular in all point of $q^{-1}(\mathbb{C}^*)$. If $q(u,x) = 0$, then $h(ux) = 0$. We have

$$\frac{\partial q}{\partial x_j}(u,x) = \frac{\partial h}{\partial x_j}(ux) \tag{3.5}$$

and since $h^{-1}(0)$ is regular there exists j such that (3.5) is non zero. $\square$

(3.6) Proposition. *The complement $X \setminus L_1$ is isomorphic to the regular affine hypersurface V.*

PROOF. We have seen in (3.3) that V is regular. Moreover $V \subset \mathbb{C} \times (\mathbb{C}^n \setminus 0)$, as seen from (3.3) and (3.4). The morphism $V \subset \mathbb{C} \times (\mathbb{C}^n \setminus 0) \to X$ induces the desired isomorphism. The pre-image of $X \setminus L_1$ in $\mathbb{C} \times (\mathbb{C}^n \setminus 0)$ is the open subvariety $W = \{(u,x) \mid q(u,x) \neq 0\}$. The morphism $W \to V$, $(u,x) \mapsto (q(u,x)u,\ q(u,x)^{-1}x)$ factors over $X \setminus L_1$ and yields an inverse. $\square$

Because of (3.3), we call $q^{-1}(c)$, $c \neq 0$, the general fibre of q and $q^{-1}(0)$ the singular fibre. The singular fibre was investigated in section 2 and studied under the name of hyperbolic modification. We now deal with the topology of the general fibre V.

(3.7) Proposition. *The differentiable manifold V is obtained from $\mathbb{C}^n \setminus L$ by attaching an open 2–handle. The homotopy type Y of V is obtained by attaching a 2–cell D^2 to $\mathbb{C}^n \setminus L$ along a small normal 1–sphere about L.*

PROOF. Set $E_0 = E \setminus (E \cap L_1)$. Set theoretically we have $X \setminus L_1 = (X \setminus p^{-1}(0)) \cup E_0$ and $X \setminus p^{-1}(0) \cong \mathbb{C}^n \setminus L =: W$. Therefore we have to describe the way E_0 is attached to W. This requires a tubular neighbourhood of E_0.

The normal bundle of the exceptional divisor $E \cong \mathbb{P}^{n-1}$ in X is the line bundle

$$\pi \colon X \to \mathbb{P}^{n-1}, \quad (u,x) \mapsto [x].$$

We have $L_1 = \{(u,x) \mid q(u,x) = 0\}$ and $[x] \in E_0$ is equivalent to $q(0,x) \neq 0$. If we fix x, then there exists a neighbourhood U_x of zero, such that $q(u,x) \neq 0$ for $u \in U_x$, i. e. $(u,x) \notin L_1$ for $u \in U_x$. In other words: Let $\pi_0 \colon X_0 = \pi^{-1}(E_0) \to E_0$ denote the restriction of π; then there exists an open cell subbundle $U \subset X_0 \to E_0$ such that $(u,x) \in U \setminus E_0$ implies $q(u,x) \neq 0$.

The complex manifold $X \setminus L_1$ can therefore be defined by the pushout diagram

$$\begin{tikzcd} U \setminus E_0 \arrow[r, "c"] \arrow[d, "j"] & U \arrow[d] \\ \mathbb{C}^n \setminus L \arrow[r] & X \setminus L_1 \end{tikzcd}$$

where j is the embedding $(u, x) \mapsto ux$.

Let U_z denote the fibre of $\pi_0 : U \to E_0$ over $z \in E_0$. Then $j : U_z \to \mathbb{C}^n$, $(u, x) \mapsto ux$ is transverse to L in 0. Let $D_z \subset U_z$ be a closed cell with boundary S_z. The circle S_z is the normal sphere about L which appears in the statement of (3.7).

The pushout diagram

$$\begin{tikzcd} S_z \arrow[r] \arrow[d, "j"] & D_z \arrow[d] \\ \mathbb{C}^n \setminus L \arrow[r] & Y \end{tikzcd}$$

defines the attachment of the 2–cell to $\mathbb{C}^n \setminus L$. The space Y is homotopy equivalent to $X \setminus L_1$. In order to see this, consider the commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \setminus L & U \setminus E_0 \arrow[l, "j"'] \arrow[r] & U \\ \mathbb{C}^n \setminus L \arrow[u, "="] & S_z \arrow[l] \arrow[r] \arrow[u, "\alpha"] & D_z. \arrow[u, "\beta"] \end{tikzcd}$$

The inclusions α and β are homotopy equivalences. The set E_0 is the complement of the projectivized tangent space $E_0 = \mathbb{P}(\mathbb{C}^n) \setminus \mathbb{P}(T_0L)$, hence an affine space. Therefore the cell bundle U is contractible. The inclusion α is a morphism of fibrations which is a homotopy equivalence in the base and in the fibre and therefore a homotopy equivalence ([5], 9.1). Now apply a general result of homotopy theory ([3], Lemma 1). □

Proposition (3.7) has some consequences for the homotopy and homology of V.

(3.8) Corollary. *Let $N \subset \pi_1(\mathbb{C}^n \setminus L)$ be the normal subgroup generated by a normal sphere of L. Then $\pi_1(V) \cong \pi_1(\mathbb{C}^n \setminus L)/N$.*

Proof. The theorem of Seifert and van Kampen implies that attaching a 2–cell factors out exactly the subgroup N. □

(3.9) Corollary. *$H_i(V) \cong H_i(\mathbb{C}^n \setminus L)$ for $i \neq 1$.*

Proof. Let $W = \mathbb{C}^n \setminus L$. Since $H_i(Y, W) = 0$ for $i \neq 2$, the exact homology sequence of (Y, W) shows that the homology groups of Y and W can only differ for $i = 1, 2$. In this case we have the exact sequence

$$0 \to H_2(W) \to H_2(Y) \to H_2(Y, W) \xrightarrow{\partial} H_1(W) \to H_1(Y) \to 0$$

with $H_2(Y, W) \cong \mathbb{Z}$. The map ∂ is injective. In order to see this, consider

$$s\colon H_1(\mathbb{C}^n \setminus L) \cong H_2(\mathbb{C}^n, \mathbb{C}^n \setminus L) \xrightarrow{\tau} \mathbb{Z},$$

where τ gives the intersection number with L (see [4], V.5). Then $s \circ \partial$ is an isomorphism, since $H_2(Y, W)$ is generated by a normal disk which has intersection number one with L. □

(3.10) Proposition. *Suppose L has only isolated singularities and is a topological manifold. Then $H_i(\mathbb{C}^n \setminus L) \cong H_{i-i}(L)$ for $i > 0$.*

PROOF. If L has a tubular neighbourhood in $\mathbb{C}^n$ we can apply the exact homology sequence and the Thom-isomorphism ([20], p. 259) $H_i(\mathbb{C}^n \setminus L) \cong H_{i+1}(\mathbb{C}^n, \mathbb{C}^n \setminus L) \cong H_{i-1}(L)$ and deduce the claim.

Another argument uses the fact that the sphere $S^{2n-1}(r) = \{z \in \mathbb{C}^n \mid |z| = r\}$ intersects L transversely for all sufficiently large r. For such r, the space $\mathbb{C}^n \setminus L$ is homotopy equivalent to the intersection with the disk $D^{2n}(r) \setminus D^{2n}(r) \cap L$. We therefore study the following situation: D is an m–disk, $L \subset D$ a topological submanifold with $\partial D \cap L = \partial L$ and transverse intersection of L and ∂D. Let $S = D \cup D'$ be the double of $D = D'$; this is an m–sphere. Now we have the following chain of isomorphisms for $i > 0$:

$$\begin{aligned} H_i(D \setminus L) &\underset{(1)}{\cong} H_{i+1}(D, D \setminus L) \\ &\underset{(2)}{\cong} H^{m-i-1}(D' \cup L,\, D') \\ &\underset{(3)}{\cong} H^{m-i-1}(\partial D \cup L,\, \partial D) \\ &\underset{(4)}{\cong} H^{m-i-1}(L, \partial L) \\ &\underset{(5)}{\cong} H_{i-1}(L). \end{aligned}$$

Explanation:
(1) comes from the exact homology sequence.
(2) is Poincaré–Lefschetz duality in S, see [8], VIII.7.2.
(3) and (4) is excision.
(5) is duality in L. □

(3.11) Theorem. (1) *Suppose L is an acyclic topological manifold with isolated singularities. Then V is acyclic.*

(2) *If, moreover, $\pi_1(\mathbb{C}^n \setminus L)$ is normally generated by a normal sphere (e. g. $\pi_1(\mathbb{C}^n \setminus L) \cong \mathbb{Z}$), then V is contractible.*

PROOF. From (3.10) we see that $H_i(\mathbb{C}^n \setminus L)$ is zero except for $i = 1$. For $i = 1$ the argument with intersection numbers in the proof of (3.9) shows that $H_1(\mathbb{C}^n \setminus L) \cong \mathbb{Z}$, generated by a normal sphere. The result now follows from (3.7), (3.9), and (3.10). □

Altogether we can now deduce the next result (Theorem B).

(3.12) Theorem. *Suppose $n \geq 3$. Let $h\colon \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $h(0) = 0$ such that $h^{-1}(0)$ is a regular contractible hypersurface. Then the hyperbolic modification $q : \mathbb{C}^{n+1} \to \mathbb{C}$ is a regular polynomial such that each fibre $q^{-1}(c)$, $c \in \mathbb{C}$ is diffeomorphic to Euclidean space.*

For the proof we need another Lemma.

(3.13) Lemma. *Suppose L is regular. Then:*

(1) *If $\pi_1(L) = 0$, then $\pi_1(V) = 0$.*

(2) *If L is contractible, then V is contractible.*

Proof. (1) Let U be an open tubular neighbourhood of L in $\mathbb{C}^n$. By the theorem of Seifert and van Kampen, the diagram

$$\begin{tikzcd}
\pi_1(U \setminus L) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\
\pi_1(\mathbb{C}^n \setminus L) \arrow[r] & \pi_1(\mathbb{C}^n)
\end{tikzcd}$$

is a pushout diagram. We have homotopy equivalences $L \simeq U$ and $U \setminus L \simeq L \times S^1$. Therefore $\pi_1(U) = 0$, $\pi_1(U \setminus L) \cong \pi_i(L \times S^1) \cong \pi_1(S^1)$, and the pushout shows that $\pi_1(\mathbb{C}^n \setminus L)$ is normally generated by a normal sphere of L. Now we apply (3.8).

(2) follows from the proof of (1) and (3.11). □

Proof of (3.12). By (3.4), q is regular. By (3.13), the general fibre of q is contractible. By (1.3), the singular fibre is contractible. Now apply the next (well known) theorem. □

(3.14) Theorem. *Let V be a contractible regular affine variety over $\mathbb{C}$ of dimension $m \geq 3$. Then V is diffeomorphic to $\mathbb{C}^m$.*

Proof. The statement is obtained by combining the next three results. □

The first one is a consequence of Smale's h-cobordism theorem, see [16], p. 108.

(3.15) Proposition. *Let D be a contractible smooth k–manifold with simply connected boundary. Then D is diffeomorphic to the k–dimensional unit disk D^k, provided $k \geq 6$.* □

There are important cases in which a simply connected manifold will have a simply connected boundary. In order to state the result we use the handle decomposition of a smooth manifold which follows from the existence of a smooth Morse function.

(3.16) Proposition. *Suppose the compact connected m–manifold M has only handles of index i for $i \leq r \leq \frac{m}{2}$. Then the relative homotopy groups $\pi_j(M, \partial M)$ are zero for $j < m - r$.*

Proof. In order to prove this proposition we investigate what happens when a single handle is attached.

In order to attach an i–handle one has to choose an embedding $C = S^{i-1} \times D^{m-i} \subset \partial M$ and form the adjunction space $M' = M \cup_C (D^i \times D^{m-i})$. The space M' has the homotopy type of $M \cup D^i$, an i–cell D^i attached to M along $S^{i-1} \times 0$. Let us consider the following diagram.

$$\begin{tikzcd}
(M', \partial M') \arrow[r, "\alpha"] & (M', \partial M' \cup D^{m-i}) \arrow[d, "\simeq"] \\
& (M \cup (D^i \times D^{m-i}), \partial M \times I \cup (D^i \times D^{m-i})) \\
(M, \partial M) \arrow[r, "\beta"] & (M \cup D^i,\ \partial M \cup D^i) \arrow[u, "\simeq"]
\end{tikzcd}$$

The vertical maps are the obvious homotopy equivalences, whereas α and β are inclusions. We consider the induced maps on homotopy groups π_j. The exact homotopy sequence shows that α induces an isomorphism for $j < m-i$. Suppose $\pi_j(M, \partial M) = 0$ for $j < m - r$. Then homotopy excision ([4], p. 178) shows that β induces an isomorphism for $j < m - r + i - 2$.

Now one uses this information inductively, starting with (D^m, S^{m-1}), and attaching successively 1–handles, 2–handles etc. □

Affine varieties have the following remarkle property; see [15], §7 for a proof.

In order to deal with regular affine varieties V from a topological point of view it is useful to know that there exists a compact manifold B with boundary such that V is diffeomorphic to the interior of B. If one realizes V as a regular subvariety of some $\mathbb{C}^N$, then there exists an $R > 0$ such that for all $r > R$ the sphere $S(r) = \{z \in \mathbb{C}^N \mid |z| = r\}$ is transverse to V. This is proved in [17], Cor. 2.6. Now one can take as B the intersection of V with a large disk in $\mathbb{C}^N$.

(3.17) Proposition. *Let V be an m–dimensional regular affine variety over $\mathbb{C}$. Then B has a decomposition into i–handles, $i < m$.* □

We call regular polynomials $p\colon \mathbb{C}^n \to \mathbb{C}$ such that all fibres are diffeomorphic to Euclidean space *slice polynomials.* Linear forms are trivial slice polynomials. We now collect some results which imply that q is not isomorphic to a product projection.

For the next three Propositions we assume that we are in the situation of (3.12).

(3.18) Proposition. *Suppose* $\dim L = 2$ *and let $L(n)$ denote the n–fold hyperbolic modification. If $L \not\cong \mathbb{C}^2$, then $L(n) \not\cong \mathbb{C}^{n+2}$.*

Proof. We use (2.5) and apply the characterization of $\mathbb{C}^2$ due to Miyanishi and Sugie [18]. Compare also [13]. 4.2. □

(3.19) Proposition. *Suppose the logarithmic Kodaira dimension* $\bar{\kappa}(L) \geq k$. *Then* $\bar{\kappa}(L(n)) \geq$.

Proof. This follows from (2.5) by repeated application of the addition theorem of Kawamata [12] which says: Let $f : X \to Y$ be a dominant morphism of algebraic varieties over $\mathbb{C}$; let the general fibre $X_y = f^{-1}(y)$ be an irreducible curve; then $\bar{\kappa}(X) \geq \bar{\kappa}(Y) + \bar{\kappa}(X_y)$. □

(3.20) Proposition. *Suppose* $\bar{\kappa}(q^{-1}(c)) \geq 0$ *for some* $c \in \mathbb{C}$. *Then* q *is not isomorphic to a product projection.*

Proof. By the cancellation theorem of Iitaka and Fujita [10] there is no isomorphism $\mathbb{C}^n \cong \mathbb{C} \times q^{-1}(c)$. □

If remains to find nontrivial examples to which (3.12) applies. This will be the subject of the next section.

4 Brieskorn varieties

The Brieskorn polynomials are among the simplest polynomials to which the considerations of the previous section can be applied. In particular, we find explicit examples which prove Theorem A of the introduction.

Let $a(1), \ldots, a(n)$ be positive integers. The associated Brieskorn polynomial is

(4.1) $$h(x_1, \ldots, x_n) = x_1^{a(1)} + \ldots + x_n^{a(n)}.$$

These polynomials have an isolated singularity at the origin. The Brieskorn manifold is the intersection

(4.2) $$B = B(a(1), \ldots, a(n)) = h^{-1}(0) \cap S^{2n-1}.$$

Brieskorn [2] investigated under which condition this manifold is a topological sphere. The result is as follows (4.3).

Define the graph Γ of the family $(a(j))$: The vertices are $\{1, \ldots, n\}$. There is an edge connecting i and j if and only if the greatest common divisor $(a(i), a(j)) > 1$.

(4.3) Theorem. $B(a(1), \ldots, a(n))$ *is a homology sphere if and only if one of the following conditions holds:*

1. Γ *has at least two isolated points.*

2. Γ *has an isolated point and another connected component* Γ' *with an odd number of vertices such that for different* $i, j \in \Gamma'$ *always* $(a_i, a_j) = 2$. □

The Brieskorn varieties are simply connected for $n \geq 4$. A simple consequence of Smale's h-cobordism theorem asserts that they are homeomorphic to the sphere S^{2n-3}, provided they are homology spheres, [19], [16], p. 109. For further information about Brieskorn manifolds see also [9], [1], and [17].

(4.4) Theorem. *Let $q : \mathbb{C}^{n+1} \to \mathbb{C}$ be a hyperbolic modification of h, applied to a regular point of $h^{-1}(0)$. Suppose the manifold B in* (4.2) *is homeomorphic to a sphere. Then $V = q^{-1}(1)$ is a contractible affine variety* ($n \geq 2$).

PROOF. By homogeneity of h, the space $L = h^{-1}(0)$ is homeomorphic to the open cone over B. Therefore, L is homeomorphic to Euclidean space if B is a sphere. By (3.11.1), V is acyclic.

In order to show that V is contractible we derive another topological construction of V. The space $\mathbb{C}^n \setminus L$ is homeomorphic to the product of $S^{2n-1} \setminus B$ with an open interval J. The inclusion $B \subset S^{2n-1}$ can be considered as a (generalized) knot. In order to obtain the homotopy type of V we have to attach a 2–cell along a normal sphere (3.7). Let $U \subset S^{2n-1}$ be an open tubular neighbourhood of B. Up to homotopy, attaching a 2–cell amounts to adding a fibre over $x \in B$ of the tubular neighbourhood to $S^{2n-1} \setminus U$. The result is the sphere S^{2n-1} with the tubular neighbourhood W of $B \setminus x$ deleted. Since B is assumed to be a sphere, W is an open cell and its complement is a disk. The resulting homotopy type is therefore contractible. □

Instead of Brieskorn polynomials one can use other weighted homogeneous polynomials with appropriate topological properties.

The simplest case of (4.4) arises for the polynomial $h(x, y) = x^a - y^b$ for coprime integers a, b. If we apply the hyperbolic modification to the regular point (1,1) we obtain the polynomial

(4.5) $$P(x, y, z) = z^{-1}\left((xz + 1)^a - (yz + 1)^b\right).$$

These polynomials have been studied in [6]. There it is shown that the affine variety $P(x, y, z) = 1$ has logarithmic Kodaira dimension one. Therefore we can apply the results of the previous section. The case $(a, b) = (3, 2)$ leads to the contractible hypersurface in $\mathbb{C}^3$

(4.6) $$z^2x^3 + 3zx^2 + 3x - zy^2 - 2y = 1.$$

We can use this polynomial as input for (3.12). If we apply the hyperbolic modification at the point (1,1,0) we obtain the slice polynomial

(4.7) $$q(u, x, y, z) = uz^2(ux + 1)^3 + 3z(ux + 1)^2 + 3x - (uy + 1)^2 - 2y$$

which we mentioned in the introduction. Other simple Brieskorn polynomials to which (4.4) can be applied are

(4.8) $$x^p + y^q + z_2^2 + \ldots + z_n^2$$

for coprime odd integers p, q and $z_0^d + z_1^2 + \ldots + z_n^2$ for d and n odd. These polynomials are interesting because they have large symmetry groups.

(4.9) Example. The group $O(n-1)$ acts by matrix multiplication on the coordinates $(z_1, \ldots, z_n)$ and leaves (4.8) invariant. We apply the hyperbolic modification to the point $(1, -1, 0, \ldots, 0)$. The resulting variety X carries an induced $O(n-1)$–action. The fixed point set is a surface of type (4.6) for $(a, b) = (p, q)$.

Since the fixed point set is not homeomorphic to a Euclidean space, the action on X is topologically non linear. $\heartsuit$

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