Induction Categories

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Preliminary Version November 22, 2011

Abstract

Induction categories are the sourse categories of Mackey functors. We define such categories in an axiomatic setting and show that standard results of axiomatic representation theory can be deduced in this more general setting.

1 Induction categories

Let $\mathcal{C}$ be a category with a set of isomorphism classes of objects. We denote by $\mathrm{Ob}(\mathcal{C})$ its class of objects. The set of morphisms from A to B is denote by $\mathcal{C}(A,B)$, $\mathrm{Mor}_{\mathcal{C}}(A,B)$, $\mathrm{Hom}_{\mathcal{C}}(A,B)$, or without the index $\mathcal{C}$. The identity (of the object A) is id_A, 1_A, id, 1. Let R be a commutative ring. An R-category is a category where the set of morphisms $\mathrm{Hom}(A,B)$ between any two objects A,B carries the structure of a left R-module and where composition

$$\mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C), \quad (g,f) \mapsto g \circ f$$

is R-bilinear. An R-functor $F\colon \mathcal{C} \to \mathcal{D}$ between R-categories is a functor which is R-linear on the morphism modules $F\colon \mathrm{Hom}(A,B) \to \mathrm{Hom}(FA,FB)$. We denote by R-MOD the R-category of left R-modules.

We consider diagrams in $\mathcal{C}$

$$(\beta|\alpha)\colon A \xleftarrow{\alpha} X \xrightarrow{\beta} B.$$

The diagram $(\beta|\alpha)$ is isomorphic to the diagram

$$(\beta'|\alpha')\colon A \xleftarrow{\alpha'} X' \xrightarrow{\beta'} B$$

if there exists an isomorphism $\sigma\colon X \to X'$ in $\mathcal{C}$ such that $\alpha'\sigma = \alpha$, $\beta'\sigma = \beta$. Let $[\beta|\alpha]$ denote the isomorphism class of the diagram $(\beta|\alpha)$; thus $[\beta|\alpha] = [\beta\sigma|\alpha\sigma]$ for an isomorphism σ. If $\mathcal{C}$ has products, then a diagram $(\beta|\alpha)$ corresponds to a morphism $X \to B\times A$, and isomorphism of diagrams corresponds to isomorphism of objects in the category of objects over $B \times A$.

(1.1) Definition. An ***induction category*** $I\mathcal{C}$ for $\mathcal{C}$ is an R-category with the following properties:

(1) $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(I\mathcal{C})$.

(2) For $A, B \in \mathrm{Ob}(\mathcal{C})$ the morphism set $I\mathcal{C}(A,B)$ is the free R-module on the set of isomorphism classes of diagrams in $\mathcal{C}$

$$(\beta|\alpha)\colon A \xleftarrow{\alpha} X \xrightarrow{\beta} B.$$

The identity of A is $[1_A|1_A]$.

(3) The following rules hold for the composition in $I\mathcal{C}$:

$$\begin{aligned}
[\alpha|\,\mathrm{id}] \circ [\beta|\,\mathrm{id}] &= [\alpha\beta|\,\mathrm{id}],\\
[\,\mathrm{id}\,|\gamma] \circ [\mathrm{id}\,|\delta] &= [\mathrm{id}\,|\delta\gamma],\\
[\beta|\,\mathrm{id}] \circ [\mathrm{id}\,|\alpha] &= [\beta|\alpha].
\end{aligned}$$

(4) Suppose $[\mathrm{id}\,|\alpha]\circ[\beta|\,\mathrm{id}] = \sum_s n_s[\alpha_s|\beta_s]$ with $n_s \in R$ and certain morphisms $\beta_s\colon Z_s \to B$ and $\alpha_s\colon Z_s \to B$. Then for each s the equality $\alpha\alpha_s = \beta\beta_s$ holds; it does not depend on the choice of the representatives.

If the assignment $[\alpha|\beta] \mapsto [\beta|\alpha]$ extends to an R-functor D from $I\mathcal{C}$ into the dual category $I\mathcal{C}^{op}$ we call $I\mathcal{C}$ an ***induction category with self-duality***. $\diamond$

For the moment the ground ring R will be fixed and is therefore not recorded in the notation of the category. We discuss the axioms.

(1.2) The assignment $\alpha \mapsto [\alpha|\,\mathrm{id}]$ is a covariant functor $\iota_*\colon\mathcal{C} \to I\mathcal{C}$ which is the identity on objects. $\diamond$

(1.3) The assignment $\beta \mapsto [\mathrm{id}\,|\beta]$ is a contravariant functor $\iota^*\colon\mathcal{C} \to I\mathcal{C}$ which is the identity on objects. $\diamond$

(1.4) From 1.1 and associativity of composition we obtain the rules

$$[\alpha|\,\mathrm{id}] \circ [\beta|\gamma] = [\alpha\beta|\gamma], \quad [\beta|\gamma] \circ [\mathrm{id}\,|\delta] = [\beta|\delta\gamma].$$

Diagrams $(\alpha|\,\mathrm{id})$ and $(\alpha'|\,\mathrm{id})$ are isomorphic if and only if $\alpha = \alpha'$. Therefore ι_* is an embedding of $\mathcal{C}$. We identify $\mathcal{C}$ via ι_* with a subcategory of $I\mathcal{C}$. Similarly, ι^* yields an embedding of the dual category $\mathcal{C}^{op}$ into $I\mathcal{C}$. Since, by part (3) of 1.1, $[\beta|\,\mathrm{id}]\circ[\mathrm{id}\,|\alpha] = [\beta|\alpha]$, we see that the images of ι_* and ι^* span $I\mathcal{C}$. We call β the ***covariant*** and α the ***contravariant component*** of $[\beta|\alpha]$. $\diamond$

(1.5) For an isomorphism σ in $\mathcal{C}$ special relations hold:

$$\begin{array}{rcl} [\beta|\alpha] & = & [\beta\sigma|\alpha\sigma] \\ [\sigma|\,\mathrm{id}] & = & [\mathrm{id}\,|\sigma]^{-1} \\ [\,\mathrm{id}\,|\beta]\circ[\sigma|\,\mathrm{id}] & = & [\mathrm{id}\,|\sigma^{-1}\beta] \\ [\,\mathrm{id}\,|\sigma]\circ[\alpha|\,\mathrm{id}] & = & [\sigma^{-1}\alpha|\,\mathrm{id}]. \end{array}$$

We display an identity of the type

$$[\mathrm{id}\,|\alpha]\circ[\beta|\,\mathrm{id}] = \textstyle\sum_s n_s[\alpha_s|\beta_s] = \sum_s n_s[\alpha_s|\,\mathrm{id}]\circ[\mathrm{id}\,|\beta_s]$$

from part (4) in 1.1 in the form of a diagram

$$\begin{tikzcd} \sum_s n_s Z_s \arrow[r, "\alpha_s"] \arrow[d, "\beta_s"] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & X. \end{tikzcd}$$

It is not really a diagram in our category but rather a formal pictorial notation. We think of it as a replacement for a pullback diagram of (β,α). In the sequel we refer to this diagram as a ***pullback*** and call $(n_s, Z_s, \beta_s, \alpha_s)$ the ***pull back data*** of (β,α). The ***transitivity of pullbacks*** is implicitly contained in the associativity of composition in a category, if applied to $(\mathrm{id}\,|\alpha)\circ(\beta|\,\mathrm{id})\circ(\beta'|\,\mathrm{id})$. Explicitly, it amounts to the following: Let

$$\begin{tikzcd} \sum_t m_{st} Y_{st} \arrow[r, "\beta_{st}"] \arrow[d, "\beta'_{st}"] & Z_s \arrow[d, "\beta_s"] \\ B' \arrow[r, "\beta'"] & B \end{tikzcd}$$

be the pullback data for (β', β_s); in the summation, the second index t runs through some set $I(s)$ which depends on the first index s. Then the diagram

$$\begin{tikzcd} \sum_{s,t} n_s m_{st} Y_{st} \arrow[r, "\alpha_s\beta_{st}"] \arrow[d, "\beta'_{st}"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\beta\beta'"] & X \end{tikzcd}$$

displays the pullback data for $(\beta\beta', \alpha)$. In fact, these relations are the basic ones:

(1.6) Proposition. *Suppose for each pair $\alpha\colon A\to X$ and $\beta\colon B\to X$ a composition*

$$(1|\alpha)\circ(\beta|1) = \sum_s n_s(\alpha_s|\beta_s)$$

with $n_s \in R$ *and* $\alpha\alpha_s = \beta\beta_s$ *is given such that for each isomorphism* σ

$$(1|\sigma) \circ (\beta|1) = (\sigma^{-1}\beta|1), \quad (1|\alpha) \circ (\sigma|1) = (1|\sigma^{-1}\alpha)$$

and such that

$$\begin{aligned} ((1|\alpha_1) \circ (1|\alpha_2)) \circ (\beta|1) &= (1|\alpha_1) \circ ((1|\alpha_2) \circ (\beta|1)) \\ (1|\alpha) \circ ((\beta_1|1) \circ (\beta|1)) &= ((1\alpha) \circ (\beta_1|1)) \circ (\beta_2|1) \end{aligned}$$

whenever these expressions make sense. Then

$$(\alpha_1|\alpha) \circ (\beta|\beta_1) = \sum_s n_s(\alpha_1\alpha_s|\beta_1\beta_s)$$

is a well-defined associative composition and thus yields the structure of an induction category $I\mathcal{C}$. □

We can also specify the preceding proposition in terms of coefficient matrices.

(1.7) Proposition. *Write the composition in the form*

$$(1|a)(b|1) = \sum_{(c,d)} \lambda^{a,b}_{c,d}(c|d)$$

where the sum is taken over pairs of morphisms (c, d) *such that* $ac = bd$. *Then these data define an induction category if and only if the following holds:*

$$\begin{aligned} \lambda^{m,ab}_{cd,n} &= \sum_w \lambda^{m,a}_{c,w}\lambda^{w,b}_{d,n} \\ \lambda^{ab,m}_{n,cd} &= \sum_w \lambda^{b,w}_{n,d}\lambda^{a,m}_{w,c} \end{aligned}$$

Let s, t *be isomorphisms. Then*

$$\begin{aligned} \lambda^{a,b}_{cs,ds} &= \lambda^{a,b}_{c,d} \\ \lambda^{s,b}_{s^{-1}b,1} = 1 &= \lambda^{a,t}_{1,at^{-1}} \end{aligned}$$

and $\lambda^{s,b}_{m,d} = 0 = \lambda^{a,t}_{c,n}$ *if* c *and* d *are not isomorphisms.* □

Suppose the induction category has a self-duality D. Then we obtain: Suppose $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$. Then $(\mathrm{id}\,|\beta) \circ (\alpha|\,\mathrm{id}) = \sum_s n_s(\beta_s|\alpha_s)$.

The commutativity (**??**) implies that the diagram (**??**) is equivalent to the diagram

$$\begin{tikzcd} \sum_s n_s Z_s \arrow[r, "\beta_s"] \arrow[d, "\alpha_s"] & B \arrow[d, "\beta"] \\ A\alpha \arrow[r] & X. \end{tikzcd}$$

2 Pullback Categories

Let C be a category and let C_1, C_2 be subcategories. The three categories have the same objects. We assume that the isomorphisms of C are contained in C_j. The diagrams

$$(b|a)\colon A \xleftarrow{a} X \xrightarrow{b} B$$

with $a \in C_1$ and $b \in C_2$ are the objects of a category $C|(A,B)$. The morphisms $(b|a) \to (b'|a')$ are the morphisms $\sigma \in C$ such that $a'\sigma = a$ and $b'\sigma = b$. We assume that C has (strictly transitive functorial) pullbacks such that in a pullback

$$\begin{tikzcd}
Z \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{a}"] & A \arrow[d, "a"] \\
B \arrow[r, "b"] & X
\end{tikzcd}$$

with $b \in C_2$ and $a \in C_1$ the morphisms $\tilde{a} \in C_1$ and $\tilde{b} \in C_2$. We define a category $PC = P(C; C_1, C_2)$ with the same objects as C. The class of diagrams $(b|a)$ as above is the class $\mathrm{Mor}_{PC}(A, B)$. Composition is defined as

$$(c|d) \circ (a|b) = (a\tilde{c}|d\tilde{b})$$

where in the diagram

$$\begin{tikzcd}
Z \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{c}"] & Y \arrow[r, "d"] \arrow[d, "c"] & C \\
X \arrow[r, "b"] \arrow[d, "a"] & B & \\
A & &
\end{tikzcd}$$

the square is a pullback. By assumption (**??**) this is again an allowable diagram. This category structure and the category structure on the diagrams induce on PC the structure of a 2-category. It is called a ***pullback category***.

In most cases the vertical structure of the 2-category is not relevant. In that case we define $P_1(C; C_1, C_2)$. The morphisms from A to B are the isomorphism classes of diagrams above. Composition is again defined by the pullback construction, but one can now dispense with the strict transitivity of pullbacks.

3 Mackey Functors

Let $\mathcal{C}$ be a category with induction category $I\mathcal{C}$. A ***Mackey functor*** an $I\mathcal{C}$ is a contravariant R-functor from $I\mathcal{C}$ into R-Mod. A morphism between Mackey functors is a natural transformation. Let $\mathcal{M}(I\mathcal{C})$ denote the R-category of Mackey functors on $I\mathcal{C}$.

A ***bifunctor*** $M = (M^*, M_*)$ on $\mathcal{C}$ with values in the category $\mathcal{A}$ consists of a covariant functor M_* and a contravariant functor M^* from $\mathcal{C}$ into $\mathcal{A}$ which have the same value on objects. A bifunctor is called ***compatible with isomorphisms*** if for each isomorphism α the relation $M_*(\alpha)M^*(\alpha) = \mathrm{id}$ holds. A morphism $M \to N$ between bifunctors consists of a family of linear maps $M(S) \to N(S)$, $S \in \mathrm{Ob}(\mathcal{C})$, which constitute a natural transformation of the covariant and the contravariant part. We thus obtain the category of bifunctors.

A Mackey functor M is completely determined by the bifunctor $(M^* = M\iota_*, M_* = M\iota^*)$. This is due to the fact that the images of ι_* and ι^* span $I\mathcal{C}$. By **??**, the bifunctor of a Mackey functor is compatible with isomorphisms. Let $f\colon S \to T$ be a morphism in $\mathcal{C}$. We write $f^* = M\iota_*(f)$ and $f_* = M\iota^*(f)$. The upper index is for contravariant morphisms as in cohomology. Morphisms f_* are sometimes called ***restriction maps***, morphisms f^* ***transfer maps*** or ***induction maps***. This terminology comes from representation theory.

An example of a Mackey functor is the contravariant Hom-functor in $I\mathcal{C}$.

(3.1) Proposition. *Let M be a Mackey functor and (M_*, M^*) the associated bifunctor. Then this bifunctor is compatible with isomorphisms. If $(\mathrm{id}\,|\alpha)(\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$ in $I\mathcal{C}$, then*

$$\beta^*\alpha_* = \sum_s n_s(\beta_s)_*\alpha_s^*$$

*for any two morphisms $\alpha\colon A \to X$ and $\beta\colon B \to X$ in $\mathcal{C}$. We sometimes call (**??**) the* **double coset formula**. □

(3.2) Proposition. *Let (M_*, M^*) be a bifunctor which is compatible with morphisms and satisfies* **??** *for each pair of morphisms. Then there exists a unique Mackey functor M with associated bifunctor (M_*, M^*).*

Proof. We define $M(\beta|\alpha) = \alpha_*\beta^*$. Since the bifunctor is compatible with isomorphisms, this is well-defined on isomorphism classes of diagrams. We extend this definition by R-linearity to the morphism modules of $I\mathcal{C}$. The double coset formula is used to verify that M is compatible with composition. □

Let M, N, and L be Mackey functors for $I\mathcal{C}$. A ***bilinear map*** or a ***pairing*** $M \times N \to L$ between Mackey functors is a family of R-bilinear maps

$$M(S) \times N(S) \to L(S), \quad (x, y) \mapsto x \cdot y,$$

one for each object S of $\mathcal{C}$, such that for each morphism $f\colon S \to T$ in $\mathcal{C}$ the following holds:

$$\begin{array}{rcll} L^*f(x \cdot y) & = & (M^*fx) \cdot (N^*fy), & x \in M(T),\, y \in N(T) \\ x \cdot (N_*fy) & = & L_*f((M^*fx) \cdot y), & x \in M(T),\, y \in N(S) \\ (M_*fx) \cdot y & = & L_*f(x \cdot (N^*fy)), & x \in M(S),\, y \in N(T). \end{array}$$

A universal bilinear map $M \times N \to M \Box N$ is called a ***tensor product*** (or, because of the notation, a ***box product***) of M, N. (Universal means, of course, that any other pairing $M \times N \to L$ is obtained from the universal one by composing with a unique morphism $M \Box N \to L$.)

In order to establish the canonical associativity of the box product we define a ***trilinear*** map $M \times N \times P \to Q$ between Mackey functors as a family of trilinear map

$$M(S) \times N(S) \times P(S) \to Q(S), \quad (x, y, z) \mapsto x \cdot y \cdot z$$

such that

$$f^*(x \cdot y \cdot z) = f^*x \cdot f^*y \cdot f^*z$$

and

$$f_*(f^*x \cdot f^*y \cdot z) = x \cdot y \cdot f_*z$$

and similarly if the two contravariant maps appear at other places. In the same way one defines n-linear maps between Mackey functors.

A ***Green functor*** A is a Mackey functor $A\colon I\mathcal{C} \to R\text{-}\mathrm{Mod}$ together with a pairing $A \times A \to A$ such that for each object S the pairing map $A(S) \times A(S) \to A(S)$ turns $A(S)$ into an associative R-algebra with unit such that the morphisms $A^*(f)$ preserve the units.

A ***left module*** over the Green functor A is a Mackey functor M together with a pairing $A \times M \to M$ such that for each object S the pairing map $A(S) \times M(S) \to M(S)$ equips $M(S)$ with the structure of a left unital $A(S)$-module.

4 Canonical Pairings

We fix a category $\mathcal{C}$ and an associated induction category $I\mathcal{C}$. For $S \in \mathcal{C}$ let $U(S)$ be the free abelian group on isomorphism classes of objects $\alpha\colon X \to S$ over S. We denote by $[\alpha] \in U(S)$ the element represented by α. We make the assignment $S \mapsto U(S)$ into a Mackey functor. Let $f\colon S \to T$ in $\mathcal{C}$ be given. Then $f_*\colon U(S) \to U(T)$ is defined as composition with f; functoriality $(gf)_* = g_*f_*$ is obvious. Suppose $(\mathrm{id}\,|f) \circ (\alpha|\,\mathrm{id}) = \sum_s n_s(f_s|\alpha_s)$; then we define $f^*[\alpha] = \sum_s n_s[f_s]$. The functoriality $(gf)^* = f^*g^*$ is a direct consequence of the transitivity of pullbacks. Thus we have defined a bifunctor. The double coset formula is again a direct consequence of the transitivity of pullbacks. By **??** we have a Mackey functor U. The following results show its universal character.

(4.1) Proposition. *Let M be any Mackey functor. There exists a canonical pairing $U \times M \to M$. If $u = [f\colon X \to S] \in U(S)$ and $x \in M(S)$, then $u \cdot x$ is defined as f_*f^*x.*

Proof. We have to verify the axioms of a pairing. Let $h\colon S \to T$ in $\mathcal{C}$ be given. Then

$$h_*(u \cdot h^*x) = h_*f_*f^*h^*x = h_*u \cdot x$$

since $h_* u = [hf]$. Let

$$\begin{tikzcd} \sum_s n_s Z_s \arrow[r, "h_s"] \arrow[d, "f_s"] & S \arrow[d, "h"] \\ X \arrow[r, "f"] & T \end{tikzcd}$$

be the pullback data in $I\mathcal{C}$. Then $h^* u = \sum n_s [h_s]$ and therefore

$$h_*(h^* u \cdot x) = \sum_s n_s h_* h_{s*} h_s^* x = \sum_s n_s f_* f_{s*} h_s^* x = f_* f^* h_* x = u \cdot h_* x.$$

The computation

$$h^*(u \cdot x) = h^* f_* f^* x = \sum_s n_s h_{s*} f_s^* f^* x = \textstyle\sum_s n_s h_{s*} h_s^* h^* x = h^* u \cdot h^* x$$

shows the second axiom of a pairing. □

(4.2) Proposition. *The pairing of the previous proposition, applied to $M = U$, makes U into a Green functor and M into a left U-module.*

Proof. The relation $1 \cdot x = x \cdot 1 = x$ is easily seen. We have to verify associativity of the multiplication. Let $u\colon X \to S$ and $v\colon Y \to S$ be given. On the one hand $u \cdot (v \cdot x) = f_* f^* g_* g^* x$. On the other hand

$$(u \cdot v) \cdot x = \sum_t m_t (f f_t)_* (g g_t)^* x = \sum_t f_* f_{t*} g_t^* g^* x = f_* f^* g_* g^* x.$$

Here we have used the pullback data m_t, f_t, g_t of f, g.

The same proof shows that M is a U-module. □

The multiplication in $U(S)$ has the following description. Suppose $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s (\alpha_s|\beta_s)$. Then $[\alpha][\beta] = \sum_s n_s [\alpha\alpha_s] = \sum_s n_s [\beta\beta_s]$ (compare axiom **??** of an induction category). The identity is represented by id_S. The next proposition is easily verified from the definitions.

(4.3) Proposition. *The ring $U(S)$ is canonically isomorphic to a subring of the endomorphism ring $\mathrm{End}_{I\mathcal{C}}(S)$, namely as the subring generated by morphisms of the type $(\alpha|\alpha)$ under the map $\alpha \mapsto (\alpha|\alpha)$.* □

(4.4) Proposition. *The pairings $\Sigma\colon U \times M \to N$ correspond bijectively to the morphisms $\sigma\colon M \to N$ of Mackey functors.*

Proof. Given a morphism σ, we obtain a pairing $U \times M \to N$ by composing the canonical pairing $\Lambda\colon U \times M \to M$ of (**??**) with σ.

Given a pairing $\Sigma\colon U \times M \to N$ we define

$$\sigma_\Sigma(S)\colon M(S) \to N(S), \quad x \mapsto 1 \cdot x.$$

From the axioms of a pairing it is verified that the $\sigma(S)$ constitute a morphism of Mackey functors.

The two constructions are inverse to each other. □

(4.5) Proposition. *Let A be a Green functor. The morphisms*

$$\lambda(S)\colon U(S) \to A(S), \quad [f] \mapsto f_* f^*(1_S)$$

are ring homomorphisms and constitute a morphism of Mackey functors.

Proof. Let n_s, f_s, g_s be the pullback data for f, g. We compute

$$\begin{aligned}
\lambda(S)([f][g]) &= \lambda(S)(\sum_s n_s[ff_s]) \\
&= \sum_s n_s f_* f_{s*} g_s^* g^*(1_S) \\
&= f_* f^* g_* g^*(1_S) \\
&= f_* f^* \lambda(S)(1_s) \\
&= (f_* f^*)(1 \cdot \lambda(S)(g)) \\
&= f_*(f^* 1 \cdot f^* \lambda(S)(g)) \\
&= f_* f^* 1 \cdot \lambda(S)(g) \\
&= \lambda(S)(f) \cdot \lambda(S)(g).
\end{aligned}$$

Moreover $\lambda(S)(1) = \mathrm{id}_* \, \mathrm{id}^*(1) = 1$. The following two computations verify the compatibility with morphisms. Let $h\colon T \to S$ be given, and let (m_t, f_t, h_t) denote the pullback data for (f, h). The contravariant case

$$\begin{aligned}
\lambda(T)(h^*[f]) &= \lambda(T)(\sum_t m_t[h_t]) \\
&= \sum_t m_t h_{t*} h_t^* 1_T = \sum_t m_t h_{t*} h_t^* h^* 1_S \\
&= \sum_t m_t h_{t*} f_s^* f^* 1_S = \sum_t h^* f_* f^* 1_S \\
&= h^* \lambda(S)[f].
\end{aligned}$$

And finally

$$\lambda(T)(k_*[f]) = \lambda(T)([kf]) = k_* f_* f^* k^* 1_T = k_* f_* f^* 1_S = h_* \lambda(S)[f]$$

settles the covariant case. □

5 The Projective Induction Theorem

Let $\mathcal{C}$ be a category and $I\mathcal{C}$ an associated induction category. We call any family $\Sigma = (S_j \mid j \in J)$ of objects an ***induction system***. We assume that $\mathcal{C}$ ccontains a ***point*** P. This is a terminal object: Each object S of $\mathcal{C}$ has a unique morphism $p(S)\colon S \to P$.

Let M be a Mackey functor. An induction system Σ leads to a homomorphism $p(\Sigma)$, called ***induction morphism***, and $i(\Sigma)$, called ***restriction morphism***:

$$p(\Sigma)\colon \bigoplus_{j\in J} M(S_j) \to M(P), \quad (x_j \mid j \in J) \mapsto \sum_{j \in J} p(S_j)_* x_j$$

$$i(\Sigma)\colon M(P) \to \prod_{j\in J} M(S_j), \quad x \mapsto (p(S_j)^* x \mid j \in J).$$

We call the induction system Σ ***projective***, if $p(\Sigma)$ is surjective, and ***injective***, if $i(\Sigma)$ is injective. Suppose $S, T \in \mathrm{Ob}(\mathcal{C})$. The corresponding pullback will be denoted

$$\begin{tikzcd}
\sum_s n_s(S,T) Z_s \arrow[r, "b_s(S,T)"] \arrow[d, "a_s(S,T)"] & T \arrow[d, "p(T)"] \\
S \arrow[r, "p(S)"] & P
\end{tikzcd}$$

with $s \in I(S,T)$. Let $\Sigma = (S_j \mid j \in J)$ be an induction system. We have morphisms

$$p(\Sigma, T)\colon \bigoplus_{j,s} M(Z_s) \to M(T), \quad (x(j,s)) \mapsto \sum_{j,s} n_s(S,T) b_s(S,T)_* x(j,s)$$

$$i(\Sigma, T)\colon M(T) \to \bigoplus_{j,s} M(Z_s), \quad x \mapsto (a_s(S,T)^* x \mid j, s).$$

The sums are double sums and $s \in I(S_j, T)$.

Theorem 5.1 *Let A be a Green functor and M a left A-module. Let Σ be a projective induction system for A. Then $p(\Sigma, T)$ is split surjective and $i(\Sigma, T)$ is split injective.*

Proof. Since $p(\Sigma)$ is surjective for A, we can find $x_j \in A(S_j)$ such that

$$\sum_{j \in J} p(S_j)_* x_j = 1 \in A(P).$$

Of course, the sum is essentially finite, so that we can assume without essential restriction, that J is finite. We define a map

$$q(\Sigma, T)\colon M(T) \to \bigoplus_{j,s} M(Z_s), \quad x \mapsto (a_s^* x_j \cdot b_s^* x \mid j, s).$$

Here $a_s^* x_j \in A(Z_s)$, $b_s^* x \in M(Z_s)$ and the dot denotes the pairing $A \times M \to M$. We claim

$$p(\Sigma, T) \circ q(\Sigma, T) = \mathrm{id}_{M(T)}\,.$$

For the proof we use the basic identity

$$\sum_{s \in I(S_j, T)} n_s b_{s*} a_s^* = p(T)^* p(S_j)_*,$$

valid for any Mackey functor, and the properties of a pairing.

$$\begin{aligned} p(\Sigma,T)q(\Sigma,T)x &= \textstyle\sum_{j,s} n_s b_{s*}(a_s^* x_j \cdot b_s^* x) \\ &= \sum_{j,s} n_s b_{s*} a_s^* x_j \cdot x \\ &= \sum_j p(T)^* p(S_j)_* x_j \cdot x \\ &= p(T)^* (\sum_j p(S_j)_* x_j) \cdot x \\ &= p(T)^* 1 \cdot x \\ &= x. \end{aligned}$$

Thus $q(\Sigma, T)$ is a splitting for $p(\Sigma, T)$.

A splitting $j(\Sigma, T)$ for $i(\Sigma, T)$ is defined in a dual fashion

$$j(\Sigma, T)\colon \bigoplus_{j,s} M(Z_s) \to M(T), \quad x(j,s) \mapsto \sum_{j,s} n_s b_{s*}(a_s^* x_j \cdot b_s^* x(j,s)).$$

A similar proof as above yields the identity $j(\Sigma, T) i(\Sigma, T) = \mathrm{id}_{M(T)}$. □

An induction theorem for a Mackey functor consists in the determination of a projective induction system. The significance of (**??**) is that a projective induction system for A is also a projective induction system for any A-module. Since A is a module over itself, a projective system for A is also an injective system for A.

We remark that the image of $p(\Sigma)$ in $A(P)$ is always an ideal. This is a general property of Green functors (see ??). Therefore $p(\Sigma)$ is surjective if and only if 1 is in the image of $p(\Sigma)$. If this is the case, then a finite subfamily of Σ suffices. Therefore, if A has a projective system, then also a finite one.

It is not really necessary that the category has a point. Let P be any object of $\mathcal{C}$. We can consider the category $\mathcal{C}/P$ of objects over P. This inherits an induction category $I\mathcal{C}/P$ from $I\mathcal{C}$. More explicitly: An induction system for P consists in a family of morphisms $(p(S_j)\colon S_j \to P \mid j \in J)$. All that matters in the previous proof is another morphism $p(T)\colon T \to P$. Note that $\mathcal{C}/P$ now has a terminal object id_P.

The induction theorem (**??**) has a second part. In it we describe the kernel of $p(\Sigma)$ and the image of $i(\Sigma)$. Let M be a A-module and $\Sigma = (S_j \mid j \in$

J) a projective induction system for A. We write $I(i,j) = I(S_i, S_j)$. In a sum over i,j,s we understand $s \in I(i,j)$; similarly, in a sum over j,s. We have two homomorphisms

$$p_1, p_2 \colon \bigoplus_{i,j,s} M(Z_s) \to \bigoplus_{k \in J} M(S_k)$$

defined by

$$p_1(x(i,j,s)) = (\sum_{j,s} n_s a_{s*} x(i,j,s) \mid i \in J)$$

$$p_2(x(i,j,s)) = (\sum_{i,s} n_s b_{s*} x(i,j,s) \mid j \in J).$$

Theorem 5.2 *The sequence*

$$\bigoplus_{i,j,s} M(Z_s) \xrightarrow{p_2 - p_1} \bigoplus_k M(S_k) \xrightarrow{p(\Sigma)} M(P) \to 0$$

is exact.

Proof. We use (**??**) and the notation of its proof. We know already that $p = p(\Sigma)$ is surjective. In this case a splitting is given by $q(\Sigma, P) = q$, defined as $q(z) = (x_j \cdot p(S_j)^* z \mid j \in J)$. We construct a homomorphism q_1 which satisfies

$$(p_2 - p_1) q_1 + qp = \mathrm{id}\,.$$

This identity yields that the kernel of $p(\Sigma)$ is contained in the image of p_2, p_1. Since, by construction, $p(p_2 - p_1) = 0$, we have exactness. We define q_1 as the direct sum $\bigoplus_k q(\Sigma, S_k)$. Note that p_2 is defined as $\bigoplus_k p(\Sigma, S_k)$. Hence, by the proof of (**??**), $p_2 q_1 = \mathrm{id}$. Thus it remains to verify $p_1 q_1 = qp$. This is a computation as in the proof of the previous theorem. □

We also have a dual exact sequence. Its statement uses the following homomorphisms

$$i_1, i_2 \colon \bigoplus_k M(S_k) \to \bigoplus_{i,j,s} M(Z_s)$$

defined as

$$i_1(z_k) = (n_s b_s(i,k)^* z_k \mid i,s), \quad i_2(z_i) = (n_s a_s(i,k)^* z_i \mid k,s).$$

Theorem 5.3 *The sequence*

$$0 \to M(P) \xrightarrow{i(\Sigma)} \bigoplus_k M(S_k) \xrightarrow{i_1 - i_2} \bigoplus_{i,j,s} M(Z_s)$$

is exact.

Proof. We already know that i is injective and has a splitting j. Dually to the previous proof we construct a homomorphism j_1 which satisfies the identity $ij + j_1(i_1 - i_2) = \mathrm{id}$. □

6 The n-universal Groups

We generalize the universal functor U to a functor in n variables.

Let $S_1, \dots, S_n$ be objects of $\mathcal{C}$. We consider the category $\mathcal{C}/(S_1, \dots, S_n)$ of objects $(a_j : X \to S_j)$ over the family (S_j). A morphism from (X, a_j) to (Y, b_j) is a morphism $f : X \to Y$ such that $b_j f = a_j$ for all j. Let $U(S_1, \dots, S_n)$ denote the free abelian group on the isomorphism classes of objects in $\mathcal{C}/(S_1, \dots, S_n)$. Up to canonical isomorphism, this group is invariant under permutation of the S_j. We make this construction into an n-variable functor in $I\mathcal{C}$, given on objects by $(S_1, \dots, S_n) \mapsto U(S_1, \dots, S_n)$. Fix the first variable. Suppose $f : S \to T$ in $\mathcal{C}$ is given. Then $f_* : U(S, S_j) \to U(T, S_j)$ is given by composition with f in the first component. We clearly have $f_* g_* = (fg)_*$. In order to define $f^* : U(T, S_j) \to U(S, S_j)$ we consider the pullback

$$\begin{tikzcd} S \arrow[d, "f"] & \sum_s n_s Z_s \arrow[l, "f_s"'] \arrow[d, "b_s"] & \\ T & X \arrow[l, "b"] \arrow[r, "a_j"] & S_j \end{tikzcd}$$

and set

$$f^*(b, a_j) = \sum_s n_s (f_s, a_j b_s).$$

The relation $(fg)^* = g^* f^*$ follows from the transitivity of the pullback. In general we define $(\beta|\alpha)^* = \alpha_* \beta^*$. In order to see that this assignment defines a contravariant R-functor on $I\mathcal{C}$ and that these functors in different variables commute, one uses the transitivity of pullbacks.

The n-universal groups allow a characterization of pairings and n-linear maps. Suppose $P(S) \otimes Q(S) \to R(S)$, $x \otimes y \to x \cdot y$ is a pairing. For objects S_1, S_2, S_3 in $\mathcal{C}$ we define a homomorphism

$$\pi(S_1, S_2, S_3) : U(S_1, S_2, S_3) \to \mathrm{Hom}(P(S_1) \otimes Q(S_2), R(S_3)),$$

which maps the basis element (y_1, y_2, y_3) of $U(S_1, S_2, S_3)$ to

$$x_1 \otimes x_2 \mapsto y_{3*}(y_1^* x_1 \cdot y_2^* x_2).$$

(6.1) Proposition. *The homomorphisms $\pi(S_1, S_2, S_3)$ form a natural transformation of functors on $I\mathcal{C}$ in three variables.*

Proof. In order to read the proposition correctly, we have to interprete the variance of the functor in an appropriate manner. When we use the bifunctor language this means: Let $f_3 : S_3 \to S_3'$ be a morphism in $\mathcal{C}$. It induces f_{3*} and f_3^* in the third variable of U. Similarly, it induces homomorphisms when R is applied and then the Hom-functor. In the first and second variable we have to compare

f_* with the Hom-maps induced by f^*. The compatibility of the π-morphisms with the f_* on the left follows directly from the definitions. The compatibility with the f^* uses the pairing axioms and the double coset formula. □

Conversely, we can characterize pairings by natural transformations. Let a natural transformation $\pi(S_1, S_2, S_3)$ as above be given. Let

$$\pi_S \colon P(S) \otimes Q(S) \to R(S), \quad x \otimes y \mapsto x \cdot y$$

denote the homomorphism which is the image of $(\mathrm{id}, \mathrm{id}, \mathrm{id}) \in U(S, S, S)$.

(6.2) Proposition. *The π_S form a pairing $P \times Q \to R$ of Mackey functors.* □

Via (**??**) and (**??**) we obtain a bijection between pairings and natural transformations. We have a similar situation for n-linear maps $P_1 \times \cdots \times P_n \to L$. They correspond to natural transformations

$$U(S_1, \ldots, S_{n+1}) \to \mathrm{Hom}(P_1(S_1) \otimes \cdots \otimes P_n(S_n), L(S_{n+1})).$$

7 Tensor Products

In this section we make the category of Mackey functors into a symmetric tensor category[1]. We begin with the construction of the box-product.

Suppose $M_1, \ldots, M_n$ are Mackey functors. We consider $U(S_1, \ldots, S_n, T)$ as a covariant functor in the S_j by using the self duality of $I\mathcal{C}$. We form the tensor product N of this covariant functor over $(I\mathcal{C})^n$ with the contravariant functor $(S_j) \mapsto M(S_1) \otimes \cdots \otimes M(S_n)$. This is, by construction, a Mackey functor. We show that it gives the universal n-linear map.

Let $M_1 \times \cdots \times M_n \to L$ be an n-linear map between Mackey functors. Note that $N(T)$ can be defined as a quotient of

$$\tilde{N}(T) = \bigoplus_{(S_j)} U(S_j, T) \otimes M_1(S_1) \otimes \cdots \otimes M_n(S_n).$$

Let $(a_j, b) \in U(S_j, Z)$ denote a basis element. We map $(a_j, b) \otimes x_1 \otimes \cdots \otimes x_n$ to $b_*(a_1^* x_1 \cdot \ldots \cdot a_n^* x_n) \in L(T)$. Here the dots refer to the given n-linear morphism. One verifies

(7.1) Proposition. *The linear maps $\tilde{N}(T) \to L(T)$ factor over the quotient $N(T)$ and the resulting maps $N(T) \to L(T)$ constitute a morphism of Mackey functors.* □

We have seen in the previous section that the pairing $M_1 \times \cdots \times M_n \to L$ corresponds to a natural transformation. When we take the adjoint of $\pi(S_1, \ldots, S_n)$ we obtain a homomorphism

$$U(S_j, T) \otimes M_1(S_1) \otimes \cdots \otimes M_n(S_n) \to L(T).$$

[1] Also called symmetric monoidal category.

The set of these homomorphisms yields the homomorphisms $\tilde{N}(T) \to L(T)$ above; and the fact that the $\pi(S_j)$ form a natural transformation is equivalent to the fact (**??**) that these homomorphisms factor over $N(T)$.

(7.2) Proposition. *The canonical maps*

$$M_1(S) \otimes \cdots \otimes M_n(S) \to U(S, S, \ldots, S) \otimes M_1(S) \otimes \cdots \otimes M_n(S) \to N(S)$$

which send $x_1 \otimes \cdots \otimes x_n$ to the class of $(\mathrm{id}, \ldots, \mathrm{id}) \otimes x_1 \otimes \cdots \otimes x_n$ form an n-linear morphism. This is a universal such morphism. $\square$

The last proposition says that N is an n-fold box-product $M_1 \square \cdots \square M_n$. One verifies that the two canonical maps of $M(S) \otimes N(S) \otimes P(S)$ into $((M \square N) \square P)(S)$ and $(M \square (N \square P))(S)$ are both universal trilinear maps. This gives the canonical isomorphism $(M \square N) \square P \cong M \square (N \square P)$ which satisfies the pentagon axiom for tensor categories.

The functor U is a neutral element for this tensor product. This follows from **??** and **??**.

The symmetric pairing in this tensor category is simply given by the canonical morphism $\tau \colon M \square N \to N \square M$ which makes the diagrams

$$\begin{tikzcd}
M(S) \otimes N(S) \arrow[r, "\tau(S)"] \arrow[d] & N(S) \otimes M(S) \arrow[d] \\
M \square N \arrow[r, "\tau"] & N \square M
\end{tikzcd}$$

with the twist maps $\tau(S)(x \otimes y) = y \otimes x$ commutative. One verifies the axioms of a braiding.

Problems

1. Here is a slightly more elementary construction of the tensor product (= box-product). The tensor product of Mackey functors M and N is constructed as follows. Let $R(S, T)$ denote the free R-module om the set of morphisms $S \to T$ in $\mathcal{C}$. The group $(M \square N)(T)$ is a quotient of $\bigoplus_S R(S, T) \otimes M(S) \otimes N(S)$ (tensor products always over R): We factor out the submodule generated by the elements

$$gh \otimes h^* z \otimes y - g \otimes z \otimes h_* y, \quad g \in R(S, T), h \in R(U, S), z \in M(S), y \in N(U)$$

$$g \otimes h_* z \otimes y - gh \otimes z \otimes h^* y, \quad g \in R(U, T), h \in R(S, U), z \in M(S), y \in N(U).$$

This becomes a covariant functor in T from the covariant Hom-functor $R(S, \text{-})$. In order to make it into a contravariant functor, let $\varphi \colon Z \to U$ be given. Suppose $\langle \varphi, h \rangle = \sum_t n_t(\alpha_t, \beta_t)$. Then φ^* maps $h \otimes a \otimes b \in R(S, T) \otimes M(S) \otimes N(S)$ to $\sum_t n_t \alpha_t \otimes \beta_t^* a \otimes \beta_t^* b$. It is shown with the transitivity relation (4) of the double coset decomposition that this is compatible with the equivalence relation. The same rule is used to show functoriality and the double coset formula, so that we have obtained a Mackey functor.

8 Internal Hom-Functors

An internal Hom-functor for the category of Mackey functors in $I\mathcal{C}$ assigns to each pair P, Q of Mackey functors another Mackey functor $\mathrm{HOM}(P,Q)$ which is right adjoint to the box product

$$\mathrm{Hom}_{I\mathcal{C}}(N \Box P, Q) \cong \mathrm{Hom}_{I\mathcal{C}}(N, \mathrm{HOM}(P,Q)).$$

Given P, Q we let

$$\Lambda(T) = \mathrm{Nat}_{S_1,S_2}(U(S_1,S_2,T), \mathrm{Hom}_R(P(S_1), Q(S_2))$$

denote the R-module of natural transformations of Mackey functors in the variables S_1, S_2. These modules form a Mackey functor in the variable T. We call this Mackey functor $\mathrm{HOM}(P,Q)$. In order to establish the adjunction (**??**) we take another Mackey functor N and consider a natural transformations $N \to \Lambda$. By adjunction, the natural transformation consists of a family of homomorphisms

$$N(T) \otimes U(S_1,S_2,T) \otimes P(S_1) \to Q(S_2)$$

with properties which ensure a factorization over a natural transformation $N \Box P \to Q$. This assignment is the starting point for the construction of (**??**). A formal consequence of (**??**) is the formal adjunction

$$\mathrm{HOM}(N \Box P, Q) \cong \mathrm{HOM}(N, \mathrm{HOM}(P,Q)).$$