Rooted cylinder ribbons

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This paper deals with the interrelation between the following areas

(1) Braids, knots, tangles, ribbons, and graphs. Their invariants.
(2) Tensor categories with braiding and duality.
(3) Quantum groups and their representations.
(4) Towers of algebras. Traces.

It is based on the general idea that there (should) exist parallel theories modelled (at least) on series of classical or more general root system data. The Artin braid group belongs to the A-series. It is easy to define braid groups, Coxeter groups, Hecke algebras for general Coxeter matrices. For those Coxeter matrices which give raise to finite reflection groups the associated braid groups are, by a theorem of Brieskorn [2], fundamental groups of the regular orbits of the reflection representation of the Coxeter group (Weyl group). Elements of the fundamental groups can in some cases be interpreted as braids. The Coxeter graph A_{n-1} yields the Artin braid group ZA_{n-1} of braids with n strings. The graph B_n yields the braid group ZB_n of braids in the cylinder with n strings. Some aspects of (1) – (4) above can be extended from the usual A-case to quite general Coxeter graphs. It is the purpose of this paper to demonstrate that at least the B-case leads to an extension of the classical situation in many respects. The material has been chosen such that specific results can be obtained without extensive technical work.

The additional structure of the B-case is in one way or the other related to the axis of the cylinder. (The term “rooted” refers to tangle components which end on the axis.) This is obvious for the geometrical part. The main point here is to find analogous structures in tensor categories, representations of quantum groups, and suitable algebras. One of the main notions is that of a cylinder braiding.

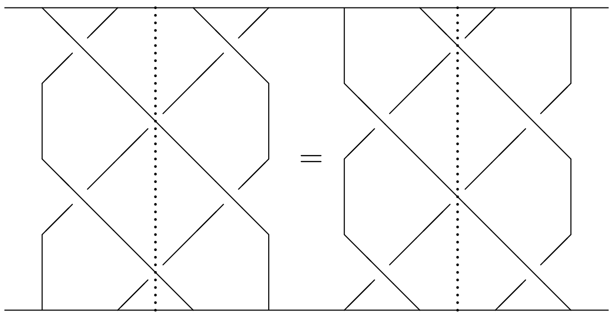
We first describe geometrical categories of tangles and ribbons. The presentation is partly heuristic based on geometric evidence. Formal proofs would lead too far afield. Thereafter we collect the categorical notions behind the geometry. We then treat the extension of the Kauffman bracket and skein theory (Kauffman polynomial). Most of the applications are concerned with the view point of representations of categories by modules over quantum groups. For simplicity we restrict mainly to the sl_2-case. This choice of material should be most convincing since (at first sight apparently) innocent geometrical ideas lead finally to subtle algebraic identities. Among other things we obtain an algebraic description (= a faithful representation) of the B-type Temperley-Lieb category.

1 Categories of ribbons

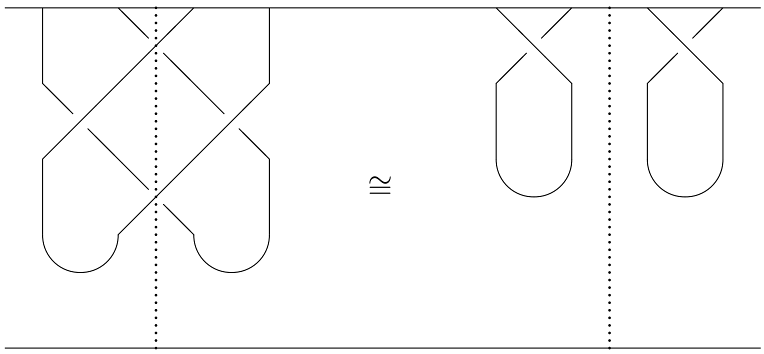
We recall some standard notions about tangles, ribbons and the associated tensor categories. A (k,l)-tangle is (a smooth isotopy class relative to the boundary of) a compact one-dimensional submanifold of $\mathbb{C} \times [0,1]$ such that the set of its boundary points is $\{1,\ldots,k\} \times 0 \cup \{1,\ldots,l\} \times 1$. A ribbon is a tangle with a normal framing; the framing vector at the boundary always points to $-\infty$. In the graphical calculus a tangle is represented by a generic immersion of a one-manifold into the strip $]0,\infty[\times[0,1]$ together with overcrossing information at the double points. (For our applications it is convenient not to use immersions into $\mathbb{R} \times [0,1]$.) The tangles or ribbons and their graphical analogues form a tensor category. Objects are the natural numbers and the morphisms from k to l are the (k,l)-tangles or ribbons. There are oriented versions. For simplicity we mainly work with unoriented objects in this paper. We refer to Turaev [20] and [21, Ch.I] for detailed back ground information about these tensor categories and their presentation by generators and relations. The graphical category of ribbons will be denoted by RA.

In this paper we are concerned with tangles and ribbons in the cylinder $\mathbb{C}^* \times [0,1]$. The definition of tangles and ribbons is completely analogous to the ordinary case recalled above. The only difference is that now everything takes place in the cylinder $\mathbb{C}^* \times [0,1]$.

We use two versions of a graphical calculus for cylinder ribbons. The first one is based on generic immersions into $\mathbb{R}\times[0,1]$ which are symmetric with respect to the axis $0 \times [0,1]$. This setting was already used in [3]. There are two additional Reidemeister type moves. They are represented graphically as follows (the axis is dotted).

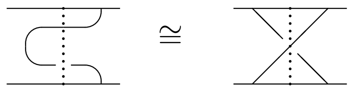


This first relation is called the *four braid relation.* In order to understand the twist in the right part of the next figure observe that the untwisting of the left part is by a rotation about 180°.



The second relation also has an upside-down version. We call these the *untwist relation.*

Because of the $\mathbb{Z}/2$-symmetry of the figures it suffices to consider essentially the part in $[0,\infty[\times[0,1]$. This leads to the second version of the graphical calculus (compare [17]). A symmetric crossing of the axis will then be represented by the left part of the next figure.



The first version is obtained from the second one by taking the two-fold covering ramified along the axis. One could also pass to the universal covering of the cylinder; this would yield infinite but periodic tangles.

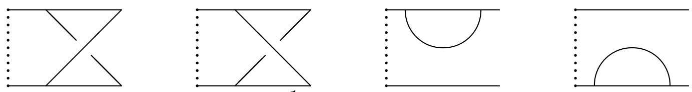
The trefoil has a symmetric picture with three crossings on the axis. In the second version this becomes an unknotted circle which winds two times about the axis. The symmetric Hopf link corresponds to an unknotted circle which winds about the axis just once. The figure eight knot has a symmetric representative with the axis passing the knot twice. This is not allowed at present but later when we consider rooted tangles.

The category of these cylinder ribbons will be denoted by RB. The letters A and B in RA, RB refer to Coxeter graphs of type A, B. The reason is that the corresponding braid groups are part of these ribbon categories.

We can place an ordinary graph to the right of a cylinder ribbon graph without producing new double points. This process makes RB into a tensor module category over RA; see the formal definitions in the next section. Actually, by placing one cylinder into a second one we can make RB into a tensor category. It turns out that this is not suitable for our purposes. Again there are oriented versions. The natural framing of the strand in version 2 above (intendend in the drawing) is the one where the normal vector always points to the axis. If we intend to draw this with the black board framing, then we have to add a twist. But the natural framing of the components which do not touch the axis is the black board framing. This convention has to be kept in mind when the figures in this text are interpreted.

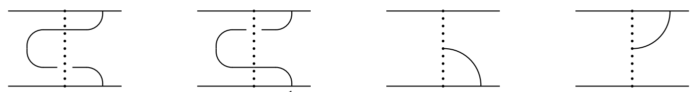
If one wants to develop skein invariants for cylinder ribbons (as we will do), one is led to consider more general ribbons. They will be called *rooted cylinder ribbons*. These are framed tangles represented by embeddings of compact one-manifolds in $\mathbb{C} \times [0,1]$ where the circle components are contained in $\mathbb{C}^* \times]0,1[$ and the interval components have their boundary points in $]0,\infty[\times\{0,1\} \cup \{0\}\times]0,1[$. Thus, some components may have one or two boundary points on the interior of the axis. An isotopy is allowed to move the points on the axis, the isotopy respects the axis setwise but not poinwise. The graphical calculus uses immersions into $[0,\infty[\times]0,1[$, except that there may be some crossings of the axis as for RB. Let RRB denote the graphical category of rooted cylinder ribbons. Again this is a tensor module category over RA. We point out that the objects of the categories under consideration are the natural numbers $n \in \mathbb{N}_0$ and a morphism from k to l is a ribbon graph Γ with $\Gamma \cap (\mathbb{R} \times [0,1]) = \{1,\dots,k\} \times 0 \cup \{1,\dots,l\} \times 1$. The symbol 1_n denotes the identity of the object n of RRB.

The basis of this paper is the description of RRB as a tensor module category over RA by generators and relations. The generators of RA are



X $\quad X^{-1}$ $\quad k$ $\quad f$

and the additional generators of RRB are



F $\quad F^{-1}$ $\quad \kappa$ $\quad \varphi$

For RB one only needs the additional generators F, F^{-1}. The relations for the generators of RA are known [20, 21]. The following version suffices for the unoriented category.

(1.1) Relations for RA.

(1) $XX^{-1} = 1_2 = X^{-1}X$

(2) $(X \otimes 1_1)(1_1 \otimes X)(X \otimes 1_1) = (1_1 \otimes X)(X \otimes 1_1)(1_1 \otimes X)$

(3) $(1_1 \otimes f)(k \otimes 1_1) = 1_1 = (f \otimes 1_1)(1_1 \otimes k)$

(4) $(f \otimes 1_1) = (1_1 \otimes f)(X^{\pm 1} \otimes 1_1)(1_1 \otimes X^{\pm 1})$
$(k \otimes 1_1) = (X^{\pm 1} \otimes 1_1)(1_1 \otimes X^{\pm 1})(1_1 \otimes k)$

The additional relations involving $F, F^{-1}, \kappa, \varphi$ are as follows. For RB one only needs (1), (2), and (3).

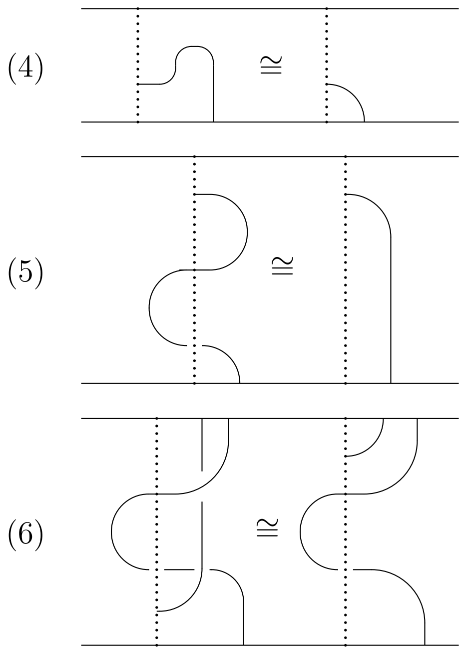
(1.2) Additional relations for RRB.

(1) $FF^{-1} = 1_1 = F^{-1}F$

(2) $X_2 := X(F \otimes 1_1)X(F \otimes 1_1) = (F \otimes 1_1)X(F \otimes 1_1)X$

(3) $X_2 k = k, \quad f X_2 = f$
(4) $(\varphi \otimes 1_1)k = \kappa, \quad f(\kappa \otimes 1_1) = \varphi$
(5) $F\kappa = \kappa, \quad \varphi F = \varphi$
(6) $X(F \otimes 1_1)X(\kappa \otimes 1_1) = (\kappa \otimes 1_1)F, \quad (\varphi \otimes 1_1)X(F \otimes 1_1)X = F(\varphi \otimes 1_1)$

We do not prove in this paper that (1.2) contains a complete set of additional relations since our interest is in algebraic realizations of the relations. We have already illustrated (2) and (3) as the four braid relation and the untwist relation. Here are figures for (4), (5), and (6). There are also upside-down versions.



A skein invariant for $(0,0)$-ribbons in RRB with values in a commutative ring $\mathfrak{K}$ in the spirit of the Kauffman polynomial [14] introduces additional local relations (written in terms of generators) as follows. The symbols $C_1, \dots, C_9$ are suitable parameters in $\mathfrak{K}$; but they cannot be chosen arbitrarily.

(1.3) Skein relations.
(1) $X - X^{-1} = C_1(1_2 - kf)$
(2) $fX = C_2 f, \quad Xk = C_2 k$
(3) $fk = C_3 1_0$
(4) $C_4^{-1}F + C_4 F^{-1} = C_5(\kappa\varphi - 1_1)$
(5) $\varphi\kappa = C_6 1_0$
(6) $C_7((\varphi \otimes 1_1)X(\kappa \otimes 1_1) - \kappa\varphi) = C_8 1_1 + C_9 F$

A version of relation (1.3.6) is due to Häring-Oldenburg [9]. The local modifications of the Kauffman polynomial are (1.3.1), (1.3.2), and (1.3.3). One can, of course, contemplate other skein relations; compare sections 4 and 10.

(1.4) Proposition. *Relations of the type above suffice to compute the value of a rooted cylinder $(0,0)$-ribbon.*

Proof. By the ordinary theory and relation (1.3.6) we can remove all crossings which do not lie on the axis. What remains are circles which wind around the axis, say n times. The relation (1.3.6) is now used to evaluate such ribbons by induction on n, in the presence of the other axioms. The value of $f(F \otimes 1)k$ is $C_9^{-1} C_3((C_2^{-1} - 1)C_7 - C_8)$. □

Later we shall construct examples of skein invariants via representations of categories. It would be interesting to extend the geometric methods of [14].

In order to discuss the role of the parameters one can immitate the construction of the Birman-Wenzl algebras [1] in our situation. The detailed analysis of these and more general algebras is carried out in [9]. Coherence of the axioms and geometry tell that the following is a reasonable set of relations between the parameters. We defer a justification to section 10.

(1.5) Relations between parameters.

(1) $C_2 - C_2^{-1} = C_1(1 - C_3)$
(2) $C_4 + C_4^{-1} = C_5(C_6 - 1)$
(3) $C_2 C_4^2 = 1$
(4) $C_5 = C_7$
(5) $C_8 = C_1 C_4$
(6) $C_9 = -C_1 C_4^{-1}$

In section 3 we construct a certain representation of RRB which yields

(1.6) Theorem. *There exists a skein invariant of* RRB *with parameters* $C_1 = q^2 - q^{-2}$, $C_2 = q^{-4}$, $C_4 = q^2$, $C_5 = \rho^2 + \rho^{-2}$. *The other values are given by the relations above. (Here* q *and* ρ *are suitable elements in a commutative ring.)*

2 Tensor categories with cylinder braiding

This section collects the categorical notions which are behind the geometry of the previous section.

Let $\mathfrak{A} = (\mathfrak{A}, \otimes, I, a, r, l)$ be a tensor category. In this notation, $\otimes$ is the tensor product functor, I the neutral object, $a\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ the associativity isomorphism, $l\colon I \otimes X \to X$ the left unit and $r\colon X \otimes I \to X$ the right unit isomorphism. For the axioms of a tensor category (pentagon, triangle) see [12, XI.2]. Let $\mathfrak{B}$ be another category.

We use functors $*\colon \mathfrak{B} \times \mathfrak{A} \to \mathfrak{B}$ in two variables and denote them by $(Y, X) \mapsto Y * X$ and $(f, g) \mapsto f * g$ on objects and morphisms. We shall also use the $\otimes$-sign instead of $*$ and call the functor $*$ a tensor product.

Definition. A *right action* $(*, \alpha, \rho)$ of $\mathfrak{A}$ on $\mathfrak{B}$ consists of a functor $*\colon \mathfrak{B} \times \mathfrak{A} \to \mathfrak{B}$, a natural isomorphism in three variables

$$\alpha = \alpha_{U,V,W}\colon (U * V) * W \to U * (V \otimes W), \qquad U \in \mathrm{Ob}(\mathfrak{B}), V, W \in \mathrm{Ob}(\mathfrak{A}),$$

and a natural isomorphism $\rho = \rho_X\colon X * I \to X$, $X \in \mathrm{Ob}(\mathfrak{B})$, such that the following axioms hold (pentagon (2.1), triangle (2.2)):

(2.1) Given four objects $U \in \mathrm{Ob}(\mathfrak{B})$ and $V, W, X \in \mathrm{Ob}(\mathfrak{A})$, the diagram

$$\begin{array}{ccccc}
((U*V)*W)*X & \xrightarrow{\alpha} & (U*V)*(W\otimes X) & \xrightarrow{\alpha} & U*(V\otimes(W\otimes X)) \\
\downarrow{\scriptstyle \alpha\otimes \mathrm{id}} & & & & \uparrow{\scriptstyle \mathrm{id}\otimes a} \\
(U*(V\otimes W))*X & & \xrightarrow{\qquad\qquad\alpha\qquad\qquad} & & U*((V\otimes W)\otimes X)
\end{array}$$

is commutative.

(2.2) Given two objects $U \in \mathrm{Ob}(\mathfrak{B})$, $V \in \mathrm{Ob}(\mathfrak{A})$, the diagram

$$\begin{array}{ccc}
(U*I)*V & \xrightarrow{\alpha} & U*(I\otimes V) \\
\downarrow{\scriptstyle \rho_U * \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes l_V} \\
U*V & \xrightarrow{\mathrm{id}} & U*V
\end{array}$$

is commutative.
The action is called *strict*, if $\mathfrak{A}$ is a strict tensor category and α and ρ are the identity.

A category $\mathfrak{B}$ together with a right action of $\mathfrak{A}$ on $\mathfrak{B}$ is called a right *tensor module category* over $\mathfrak{A}$, or *right* $\mathfrak{A}$*-module* for short. The tensor module is called strict, if the action is strict. $\diamond$

There is, of course, an analogous definition of a left action. But note that in this case α changes brackets from right to left. Either one uses the same convention for a or one has to work with a^{-1}. A tensor category acts on itself by $* = \otimes$, $\alpha = a$, and $\rho = r$.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be tensor categories and $T = (T, \varphi, i)\colon \mathfrak{A} \to \mathfrak{A}'$ a tensor functor, consisting of a functor $T\colon \mathfrak{A} \to \mathfrak{A}'$, a natural isomorphism $\varphi_{A,B}\colon TA \otimes TB \to T(A \otimes B)$, and an isomorphism $i\colon T(I_C) \to T_D$, see [12, XI.4]. Let $\mathfrak{B}$ be a right $\mathfrak{A}$-module and $\mathfrak{B}'$ a right $\mathfrak{A}'$-module.

Definition. A T*-tensor module functor* $(U, \omega)\colon \mathfrak{B} \to \mathfrak{B}'$ consists of a functor $U\colon \mathfrak{B} \to \mathfrak{B}'$ and a natural isomorphism $\omega\colon U(X) * T(A) \to U(X * A)$ such that the diagrams

$$\begin{array}{ccc}
(UA*TB)*TC & \xrightarrow{\alpha_{UX,TB,TC}} & UA*(TB\otimes TC) \\
\downarrow{\scriptstyle \omega\otimes 1} & & \downarrow{\scriptstyle 1\otimes\varphi} \\
U(A*B)*TC & & UA*T(B\otimes C) \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \omega} \\
U((A*B)*C) & \xrightarrow{U(\alpha_{A,B,C})} & U(A*(B\otimes C))
\end{array}$$

$$\begin{array}{ccc} U(V) * TI_C & \xrightarrow{1 \otimes i} & U(V) * I_D \\ \downarrow \omega & & \downarrow \rho' \\ U(V * I_C) & \xrightarrow{U(\rho)} & U(V) \end{array}$$

are commutative. $\diamond$

Recall the notion of a braided tensor category [12, XIII.1] From now on we work with the following data:

(1) $(\mathfrak{A}, z)$ is a braided tensor category. The braiding z consists of natural isomorphisms $z_{X,Y} \colon X \otimes Y \to Y \otimes X$.

(2) $(\mathfrak{B}, *, \alpha, \rho)$ is a right $\mathfrak{A}$-module.

(3) $\mathfrak{A}$ is a subcategory of $\mathfrak{B}$ with $\mathrm{Ob}(\mathfrak{A}) = \mathrm{Ob}(\mathfrak{B})$.

(4) $*, \alpha, \rho$ restrict to $\otimes, a, r$ on $\mathfrak{A} \times \mathfrak{A}$.

A $\mathfrak{B}$-*endomorphism* t of $\mathfrak{A}$ is a family of morphisms $t_X \in \mathrm{Mor}_{\mathfrak{B}}(X, X)$ such that for $f \in \mathrm{Mor}_{\mathfrak{A}}(X, Y)$ the naturality $t_Y f = f t_X$ holds.

Definition. A *cylinder twist* for $(\mathfrak{B}, \mathfrak{A})$ consists of a $\mathfrak{B}$-automorphism t of $\mathfrak{A}$ such that for each pair X, Y of objects the following relations hold:

(2.3) $$z_{Y,X}(t_Y \otimes 1_X) z_{X,Y}(t_X \otimes 1_Y) = (t_X \otimes 1_Y) z_{Y,X}(t_Y \otimes 1_X) z_{X,Y}$$

(2.4) $$(t_X \otimes 1_Y) z_{Y,X}(t_Y \otimes 1_X) z_{X,Y} = t_{X \otimes Y}.$$

The first equation is called the *four braid relation* for X, Y. A pair $(\mathfrak{B}, \mathfrak{A})$ as above together with a cylinder twist t is called a *tensor pair* with *cylinder braiding.* $\diamond$

Recall the notion of a left duality (b, d) and a right duality (a, c) in a tensor category $\mathfrak{A}$ [12, XIV.2]. We use V^* for the dual object of V in a left duality and $V^\#$ for the dual object in a right duality.

Definition. Suppose that $\mathfrak{A}$ is provided with a left duality (b, d). A cylinder twist is *compatible with the left duality* if the following holds:

(2.5) $$\begin{array}{rcl} d_X(t_{X^*} \otimes 1) z_{X,X^*}(t_X \otimes 1) z_{X^*,X} & = & d_X \\ z_{X^*,X}(t_{X^*} \otimes 1) z_{X,X^*}(t_X \otimes 1) b_X & = & b_X. \end{array}$$

Note that these relations involve terms which appear in (2.3). Thus, by (2.3), we could require instead

(2.6) $$d_X \, t_{X^* \otimes X} = d_X, \qquad t_{X \otimes X^*} \, b_X = b_X.$$

Similarly, compatibility with a right duality (a, c) is defined by the relations

(2.7) $$c_X \, t_{X \otimes X^\#} = c_X, \qquad t_{X^\# \otimes X} \, a_X = a_X.$$

The relation

(2.8) $$t_I = \mathrm{id}$$

is called *compatibility of t with the neutral object.* ◇

(2.9) Note. *If t is compatible with the neutral object, then also with duality.*

PROOF. Since $b_X: I \to X \otimes X^*$ is a morphism in $\mathfrak{A}$ and t a $\mathfrak{B}$-automorphism, we have $t_{X\otimes X^*} b_X = b_X t_I = b_X$ which is one of the relations in (2.6). Similarly for the other assertions. □

(2.10) Proposition. *The relation $(t_X \otimes 1) z_{Y,X} (t_Y \otimes 1_X) z_{X,Y} = t_{X\otimes Y}$ implies the four braid relation.*

PROOF. Since $z_{X\otimes Y}$ is a morphism of $\mathfrak{A}$ we have the naturality of t

$$t_{Y\otimes X} z_{X,Y} = z_{X,Y} t_{X\otimes Y}.$$

We compose both sides of the hypothesis (2.10) from the left with $z_{X,Y}$ and from the right with $z^{-1}_{X,Y}$, use the naturality of t, and obtain

$$t_{Y\otimes X} = z_{X,Y}(t_X \otimes 1) z_{Y,X} (t_Y \otimes 1).$$

We interchange X and Y in this relation and compare it with the hypothesis. The four braid relation drops out. □

One can use (2.5) to express t_{X^*} in terms of t_X. The result is as follows:

$$\begin{aligned} t_{X^*} &= ((d_x z^{-1}_{X^*,X}(t^{-1}_X \otimes 1) z^{-1}_{X,X^*}) \otimes 1) \circ (1 \otimes b_X) \\ t^{-1}_{X^*} &= ((d_x z_{X,X^*}(t_X \otimes 1) z_{X^*,X}) \otimes 1) \circ (1 \otimes b_X) \end{aligned}$$

In a similar manner one can express $t^{\pm 1}_X$ in terms of $t^{\pm 1}_{X^*}$.

Definition. A *left duality* for $(\mathfrak{B}, \mathfrak{A})$ consists of

(1) A left duality (b, d) for $\mathfrak{A}$.
(2) A pair of morphisms in $\mathfrak{B}$ $\beta_X: I \to X^*$, $\delta_X: X \to I$ for each object X in $\mathfrak{A}$.

These data are assumed to satisfy the following axioms:

(2.11) $$d_X(\beta_X \otimes 1_X) = \delta_X, \qquad (\delta_X \otimes 1_{X^*}) b_X = \beta_X$$

(2.12) $$\beta_{X\otimes Y} = (\beta_X \otimes 1_Y)\beta_Y, \qquad \delta_{X\otimes Y} = \delta_Y(\delta_X \otimes 1_Y).$$

We call δ_X a *rooting* and β_X a *corooting* of X. ◇

Definition. A left duality for $(\mathfrak{B}, \mathfrak{A})$ satisfying

(2.13) $$\delta_X t_X = \delta_X, \qquad t_X \beta_X = \beta_X$$

(2.14) $$\begin{aligned} z_{Y,X}(t_Y \otimes 1) z_{X,Y}(\beta_X \otimes 1) &= (\beta_X \otimes 1) t_Y \\ (\delta_X \otimes 1) z_{Y,X}(t_Y \otimes 1) z_{X,Y} &= t_Y(\delta_X \otimes 1). \end{aligned}$$

is called *compatible with a cylinder braiding.* ◇

(2.15) Proposition. *In the presence of the other axioms* (2.14) *is equivalent to*

$$t_{X\otimes Y}(\beta_X\otimes 1)=(\beta_X\otimes 1)t_Y, \quad t_Y(\delta_X\otimes 1)=(\delta_X\otimes 1)t_{X\otimes Y}.$$

PROOF. We apply $t_X\otimes 1$ to the first equation in (2.14) and use (2.4) and (2.13). This yields the first identity in the proposition. And conversely. □

With the help of (2.15) we see that (2.12) and (2.13) are compatible.

There are similar axioms for a right duality $(a, c, \alpha\colon I\to X, \gamma\colon X^*\to I)$. We replace the axioms (2.11) by

(2.16) $$c_X(\alpha_X\otimes 1_{X^*})=\gamma_X, \qquad (\gamma_X\otimes 1_X)a_X=\alpha_X.$$

If the category has a left duality and a twist, we can take in this case the associated right duality [12, XIV.3].

One can use the notions of this section to define coloured ribbons based on RRB in the sense of [21]. The geometry of the first section leads to a graphical calculus for the categories. In general one has to incorporate orientations.

3 A representation of rooted cylinder ribbons

We construct a tensor module functor from the category of unoriented rooted cylinder ribbons. As an application we obtain a skein invariant.

Let $U=U_q(sl_2)$ be the quantum enveloping algebra over the field $\mathfrak{K}$ of characteristic zero (for simplicity) generated by K, K^{-1}, E, F with relations $KK^{-1}=K^{-1}K=1$, $KE=q^2EK$, $KF=q^{-2}FK$, $EF-FE=(K-K^{-1})/(q-q^{-1})$. We assume that $q\neq 0$ is not a root of unity. See [11] for more information.

Let V_n be the left U-module with basis $x_0, x_1, \dots, x_n$ and action $F(x_i)=[i+1]x_{i+1}$, $E(x_i)=[n-i+1]x_{i-1}$, $Kx_i=q^{n-2i}x_i$. Here $[k]=(q^k-q^{-k})/(q-q^{-1})$. The universal R-matrix is the operator

$$\mathcal{R}=q^{H\otimes H/2}\sum_{n\geq 0} q^{n(n-1)/2}\frac{(q-q^{-1})^n}{[n]!}F^n\otimes E^n$$

on finite-dimensional U-modules. It makes the category of these modules into a braided tensor category. The operator $q^{H\otimes H/2}$ acts on the tensor product of weight spaces $M^m\otimes N^n$ as multiplication by $q^{mn/2}$. (The weight space M^m of M is the K-eigenspace of M for the eigenvalue q^m.)

The representation of the category RRB is based on the module $V=V_2$. We use the basis $w_0=x_0$, $w_1=(1+q^{-2})^{1/2}x_1$, $w_2=x_2$. In general, we use on $V\otimes W$ the antilexicographical basis in order to display matrices. The universal R-matrix then gives the following R-matrix X on $V \otimes V$ in the antilexicographical basis

$$\left(\begin{array}{ccc|ccc|ccc}
q^2 & & & & & & & & \\
 & 0 & & 1 & & & & & \\
 & & 0 & & & & q^{-2} & & \\
\hline
 & 1 & & \delta^* & & & & & \\
 & & & & 1 & & \lambda & & \\
 & & & & & 0 & & 1 & \\
\hline
 & & q^{-2} & & \lambda & & \mu & & \\
 & & & & & 1 & & \delta^* & \\
 & & & & & & & & q^2
\end{array}\right)$$

with $\delta^* = q^2 - q^{-2}$, $\mu = \delta^*(1 - q^{-2})$, $\lambda = q^{-1}\delta^*$. From the properties of the universal R-matrix we see that X is a Yang-Baxter operator, i. e. it satisfies (1.1.2). It consists of blocks of size 1, 2, and 3 and satisfies the characteristic equation $(X - q^{-4})(X - q^2)(X + q^{-2}) = 0$. An eigenvector for the eigenvalue q^{-4} is $(0, 0, -q, 0, 1, 0, -q^{-1}, 0, 0)$. We therefore define a linear map $f\colon V \otimes V \to \mathfrak{K}$ with this matrix and a linear map $k\colon \mathfrak{K} \to V \otimes V$ with the transposed matrix ($1 \in \mathfrak{K}$ basis). We have a decomposition $V \otimes V = V_4 \oplus V_2 \oplus V_0$ of U-modules [12, VII.7]. Here V_4, V_2, V_0 are the eigenspaces of X for the eigenvalues $q^2, -q^{-2}, q^{-4}$. In particular, f and k are morphisms of U-modules. They satisfy the duality relations (1.1.3). Since f and k are U-linear, the naturality of the braiding yields the relations (1.1.4). Therefore we see:

(3.1) Proposition. *We obtain a tensor representation of the category* RA *of unoriented ribbon tangles if we map the generators X, f, k to the linear maps above with the same names. This representation leads to the Kauffman polynomial with local modifications* (1.3) *and parameters $C_1 = q^2 - q^{-2}$, $C_2 = q^{-4}$, $C_3 = [3]$.* □

Our aim is to extend this representation to a representation of RRB. Let F be the following 3×3-matrix

$$F = \begin{pmatrix} 0 & 0 & -q \\ 0 & -q^2 & -p^3\omega\theta \\ -q & -p^3\omega\theta & 1 - q^2 - q^2\theta^2 \end{pmatrix}$$

with $\omega = \sqrt{[2]} = \sqrt{q + q^{-1}}$, $\theta = \rho - \rho^{-1}$, and $p = q^{1/2}$. The inverse of F is the matrix

$$F^{-1} = \begin{pmatrix} 1 - q^{-2} - q^{-2}\theta^2 & p^{-3}\omega\theta & -q^{-1} \\ p^{-3}\omega\theta & -q^{-2} & 0 \\ -q^{-1} & 0 & 0 \end{pmatrix}$$

and is obtained from F by reflection in the codiagonal and $p \mapsto p^{-1}$, $\rho \mapsto \rho^{-1}$. The matrix F satisfies the equation $(F - 1)(F + q^2\rho^2)(F + q^2\rho^{-2}) = 0$.

An eigenvector of F for the eigenvalue 1 is $(p\omega, \theta, -p^{-1}\omega)$. We therefore define linear maps $\kappa\colon \mathfrak{K} \to V$ and $\varphi\colon V \to \mathfrak{K}$ with this matrix divided by $\sqrt{\rho^2 + \rho^{-2}}$ and its transpose. These normalizations yield the identities (1.2.5).

The basic property of the matrix F is the four braid relation (1.2.2) (with $Y = F \otimes 1$)

$$\textbf{(3.2)} \qquad XYXY = YXYX.$$

This can be verified by computation. The amount of computation can be reduced by the machinery of [8]. We point out that F is not an endomorphism of the U-module $V = V_2$. One verifies the duality relations (1.2.4).

As an aid for further computations we display the matrix of XYX. For its general structure see [5]. We use the abbreviation $\gamma = 1 - q^2 - q^2\theta^2$. The four braid relation (3.2) is equivalent to the fact that each block in the following matrix commutes with F.

$$XYX = \begin{pmatrix} 0 & 0 & -qI \\ 0 & -q^2 I & M \\ -qI & M & N \end{pmatrix}$$

with

$$M = \begin{pmatrix} -p^{-1}\omega\theta & -q\delta^* & 0 \\ -q\delta^* & -p^3\omega\theta & -q^2\delta^* \\ 0 & -q^2\delta^* & -p^7\omega\theta \end{pmatrix}$$

$$N = \begin{pmatrix} q^{-4}\gamma - q^2\lambda^2 & -p^3\lambda\omega\theta & -q^3\mu \\ -p^3\lambda\omega\theta & \gamma - q^2\delta^{*2} & -p^7\omega\theta\delta^* \\ -q^3\mu & -p^7\omega\theta\delta^* & q^4\gamma \end{pmatrix}.$$

Using this matrix the reader can verify (1.2.6). One can also use this matrix to verify (1.2.3) in the form $XYXk = Y^{-1}k$. These relations are quite unlikely from the computational point of view. Therefore we explain the structure of this result in a moment. If we collect the results obtained so far we see:

(3.3) Proposition. *If we map the generators F, κ, and φ to the linear maps with the same name we obtain a tensor representation Φ of* RRB *into the category of $\mathfrak{K}$-vector spaces which extends the representation* (3.1). □

Finally, we explain the skein relations. We define a matrix $\mathcal{E}$ by

$$(\rho^2 + \rho^{-2})(\mathcal{E} - I) = q^{-2}F + q^2F^{-1},$$

see (1.3.4). Then

$$\textbf{(3.4)} \qquad (\rho^2 + \rho^{-2})\mathcal{E} = \begin{pmatrix} q[2] & p\omega\theta & -[2] \\ p\omega\theta & \theta^2 & -p^{-1}\omega\theta \\ -[2] & -p^{-1}\omega\theta & q^{-1}[2] \end{pmatrix}.$$

We have the following identities, in particular (1.3.5),

$$\textbf{(3.5)} \qquad \mathcal{E} = \kappa\varphi, \quad \varphi\kappa = \left(\frac{q^2 + q^{-2}}{\rho^2 + \rho^{-2}} + 1\right) 1_0, \quad \mathcal{E}F = F\mathcal{E} = \mathcal{E}.$$

One computes that the operator $(\rho^2+\rho^{-2})(\varphi\otimes 1)X(\kappa\otimes 1)$ has the following matrix

$$Z = \left(\begin{array}{ccc} q^3[2] & p\omega\theta & -q^{-2}[2] \\ p\omega\theta & q[2]\delta^* + \theta^2 & p\omega\lambda\theta - p^{-1}w\theta \\ -q^{-2}[2] & p\omega\lambda\theta - p^{-1}\omega\theta & q[2]\mu + \theta^2\delta^* + q[2] \end{array} \right).$$

Using this, the following skein relation of type (1.3.6) is easily verified

(3.6) $$(\rho^2+\rho^{-2})(Z-\mathcal{E}) = (q^4-1)I + (q^{-4}-1)F = \delta^*(q^2 I - q^{-2}F).$$

We now give some information about the validity of the relations (1.2.3) and (1.2.6).

The cylinder twist $t_{V\otimes V} = YXYX$ on $V\otimes V$ commutes with X and $Y = F\otimes 1$. Therefore the eigenspaces of X and Y are stable under $t_{V\otimes V}$. Thus, if we consider the eigenspace for the eigenvalue 1 of Y, we see that there exists a linear map $\tilde{F}$ which satisfies

$$t_{V\otimes V}(\kappa\otimes 1) = XYX(\kappa\otimes 1) = (\kappa\otimes 1)\tilde{F}.$$

It is therefore not too surprising that $F = \tilde{F}$ does the job. For completeness we communicate the eigenspace structure. In section 5 we consider eigenspace structures in general.

The first table gives the information for $t_{V\otimes V}$.

Eigenvalue	$q^8\rho^4$	$q^8\rho^{-4}$	1	$-q^2\rho^2$	$-q^2\rho^{-2}$
Multiplicity	1	1	3	2	2
Module	V_1^+	V_1^-	V_3	V_2^+	V_2^-

The third row gives names to the eigenspaces. The decomposition

$$V\otimes V = V_1^+ \otimes V_1^- \oplus V_2^+ \oplus V_2^- \oplus V_3$$

is the decomposition into irreducible representations of the braid group

$$ZB_2 = \langle X, Y \mid XYXY = YXYX\rangle.$$

Although the eigenspaces have multiplicities there is a canonical decomposition into one-dimensional eigenspaces. This comes from the action of $F\otimes 1$. We assume here that F has three different eigenvalues (generic case). The next table gives the eigenvalue of X and Y on the modules above.

	V_1^+	V_1^-	V_3	V_2^+	V_2^-
X	q^2	q^2	$q^{-4}, q^2, -q^{-2}$	$q^2, -q^{-2}$	$q^2, -q^{-2}$
Y	$-q^2\rho^2$	$-q^2\rho^{-2}$	$1, -q^2\rho^2, -q^2\rho^{-2}$	$1, -q^2\rho^2$	$1, -q^2\rho^{-2}$

The spaces V_4, V_2, V_0 of the Clebsch-Gordan decomposition are invariant under $t_{V\otimes V}$. They split into different eigenspaces with eigenvalues as in the following table.

V_4	$q^8\rho^4, q^8\rho^{-4},$	$1,$	$-q^2\rho^2, -q^2\rho^{-2}$
V_2		$1,$	$-q^2\rho^2, -q^2\rho^{-2}$
V_0		1	

We mention the following values of the representation

$$\Phi(f(F^{\pm 1} \otimes 1)k) = q^{\pm 1}(1 - \rho^2 - \rho^{-2}).$$

Since Φ is a tensor representation it maps the object n to $M = V^{\otimes n}$. The iteration of k and f yield the duality maps $k(n)\colon \mathfrak{K} \to M \otimes M$ and $f(n)\colon M \otimes M \to \mathfrak{K}$. The quantum trace Trq of a $\mathfrak{K}$-linear map $l\colon M \to M$ is defined by the composition

$$f(n) \circ (l \otimes 1) \circ k(n)\colon 1 \mapsto \operatorname{Trq}(l).$$

This trace can be computed as a linear algebra trace Sp. Let $u\colon V \to V$ denote the morphism with diagonal $\operatorname{Dia}(q^{-2}, 1, q^2)$. Then $\operatorname{Trq}(l) = \operatorname{Sp}(l \circ u^{\otimes n})$. The proof is by linear algebra. Suppose M has basis (v_i) and let in general

$$k(1) = \sum k_{ij} v_i \otimes v_j, \quad f(v_i \otimes v_j) = f_{ij}.$$

Let $u\colon V \to V$ have matrix $(u_{rs}) = (\sum_j f_{rj} k_{sj})$. Then

$$\operatorname{Sp}(f(l \otimes 1)k) = \operatorname{Sp}((l \otimes 1) \circ kf) = \operatorname{Sp}(l \circ u).$$

4 The Kauffman functor

In this section we extend the Kauffman functor (Kauffman bracket) [13] to cylinder ribbons. This will be a tensor module functor $\mathcal{K}\colon \mathrm{RRB} \to \mathrm{TB}$ into the category TB of symmetric bridges (for the latter see [3]). The results of this section complement those of [3].

In the graphical calculus the functor $\mathcal{K}$ is defined by the following local modifications. The parameter a is the usual one for the the Kauffman bracket; the parameters x, y, D have to be determined yet.

(4.1) Local modifications.
(1) $\mathcal{K}(X) = a1_2 + a^{-1}kf$
(2) $\mathcal{K}(fk) = (-a^2 - a^{-2})1_0$
(3) $\mathcal{K}(F) = x1_1 + y\kappa\varphi$
(4) $\mathcal{K}(\varphi\kappa) = D1_0$.

The first two are the standard moves in the definition of the Kauffman bracket. The parameters x, y, and D have to be chosen correctly so as to be compatible with the relations of the category RRB.

A computation as in [3] shows that the parameters are compatible with the four braid relation (1.1.2) if and only if $x(a^{-2} - 1) = yD$. The relation (1.1.5) is satisfied if and only if $x + yD = 1$. These two conditions give

(4.2) $$x = a^2, \quad y = D^{-1}(1 - a^2).$$

We assume (4.2) from now on.

One verifies with these parameters that also the relation (1.1.6) holds. Therefore we obtain a well-defined functor RRB $\to$ TB which extends the classical Kauffman functor RA $\to$ TA and is compatible with tensor products.

We now construct a tensor representation of RRB which factors over the Kauffman functor $\mathcal{K}$: RRB $\to$ TB and induces a module-theoretic description of TB.

The tensor representation is based on the fundamental two-dimensional module $V = V_1$ of $U = U_q(sl_2)$. Here $q = a^2$ is again not a root of unity.

We map X to the standard R-matrix (in the antilexicographical basis)

$$X = \begin{pmatrix} a & & & \\ & 0 & a^{-1} & \\ & a^{-1} & a - a^{-3} & \\ & & & a \end{pmatrix}.$$

We assume that a square root i of -1 is contained in $\mathfrak{K}$ in order to produce the most symmetric representation. The morphisms

$$f: V \otimes V \to \mathfrak{K}, \qquad k: \mathfrak{K} \to V \otimes V$$

are defined by the matrix $(0, ia, (ia)^{-1}, 0)$ and its transpose. The following proposition is known. The proof is analogous to the proof for RRB given below in section 6.

(4.3) Proposition. *The values for X, k, f above yield a tensor functor from* RA *into the category of U-modules which factors over the Kauffman functor* $\mathcal{K}$: RA $\to$ TA *and induces a bijection*

$$\mathrm{Hom}_{TA}([m],[n]) \cong \mathrm{Hom}_U(V^{\otimes m}, V^{\otimes n})$$

for $m, n \in \mathbb{N}_0$. □

We extend this functor as follows. The morphism F is sent to

$$F = \begin{pmatrix} 0 & v \\ u & a^2+1 \end{pmatrix} \quad \text{with} \quad uv = -a^2.$$

We make the choice $u = v = ia$. We define

$$\varphi: V \to \mathfrak{K}, \quad \kappa: \mathfrak{K} \to V$$

by $\sqrt{\frac{D}{1-a^2}}(ia, 1)$ and its transpose.

(4.4) Proposition. *With the data above the relations* (1.2) *and* (4.1) *hold. Thus we obtain a representation of* RRB *which factors over the Kauffman functor for* RRB. □

The Kauffman functor assumes the following values on "the symmetric unknots" (unknotted circles which wind about the axis once):

$$\mathcal{K}(f(F \otimes 1)k) = -a^3(a + a^{-1}), \qquad \mathcal{K}(f(F^{-1} \otimes 1)k) = -a^{-3}(a + a^{-1}).$$

These values differ slightly from those in [3] since the category theory dictates a different choice of parameters. By closing ribbons one obtaines, as usual, invariants of framed links in the cylinder. The Kauffman calculus uses the parameter D. But as long as we consider invariants of links in RB we don't need D since we can work with the representation of RB obtained from X and F.

Using the representation above, these invariants are obtained as quantum traces in the sense of the theory of tensor categories. The quantum trace can be computed as an ordinary trace in the following manner.

(4.5) Proposition. *Let $u: V \to V$ be the morphism with diagonal matrix $\mathrm{Dia}(-a^{-2}, -a^2)$. Suppose a framed link L is obtained as a closure of an (n,n)-ribbon with value $\alpha_L: V^{\otimes n} \to V^{\otimes n}$ of the representation. Then the invariant $\mathcal{K}(L) \in \mathfrak{K}$ is the ordinary linear algebra trace of the linear map $\alpha_L \circ u^{\otimes n}$.* $\square$

5 The structure of the cylinder twist

In this section V denotes the U-module V_1. We study the eigenspace structure of the cylinder twist t_n on $V^{\otimes n}$ based on the matrices

$$t = t(\rho) = \begin{pmatrix} 0 & 1 \\ 1 & \theta \end{pmatrix}, \qquad g = g(p) = \begin{pmatrix} p & & & \\ & 0 & p^{-1} & \\ & p^{-1} & p - p^3 & \\ & & & p \end{pmatrix}$$

with $\theta = \rho - \rho^{-1}$, $p^2 = q$, $\delta = q - q^{-1}$. We only consider the generic case that the $\rho^a q^b$, $(a,b) \in \mathbb{Z}^2$ are pairwise different. We use the basis $v_{-1} = x_0$, $v_1 = x_1$ of V and have written g in the antilexicographical basis. We will construct in the generic case 2^n eigenvectors of t_n and compute the eigenvalues.

We define inductively $t(1) = t$ and

$$t(n) = (1_{n-2} \otimes g)(t(n-1) \otimes 1_1)((1_{n-2} \otimes g).$$

We also abbreviate $t(j) = t(j) \otimes 1_{n-j}$. The automorphisms $t(j)$ of $V^{\otimes n}$ pairwise commute. We set

$$t_n = t(1)t(2)\cdots t(n).$$

If we map the j-th standard generator of $\mathbb{Z}^n$ to $t(j)$ we obtain a representation of $\mathbb{Z}^n$ on $V^{\otimes n}$. Let ZB_n be the braid group with generators $t, g_1, \dots, g_{n-1}$ and relations $tg_1tg_1 = g_1tg_1$, $tg_j = g_jt$ for $j > 1$, $g_ig_j = g_jg_i$ for $|i-j| \geq 2$, $g_ig_jg_i = g_jg_ig_j$ for $|i-j| = 1$. We obtain a representation of ZB_n on $V^{\otimes n}$ by the assignment $t \mapsto t(\rho) \otimes 1 \cdots \otimes 1$ and $g_j \mapsto 1 \otimes \cdots \otimes g \otimes \cdots \otimes 1$. The morphisms $t(j)$ are values of elements in ZB_n with similar definition and notation.

We need some notation in order to state the result. Let $P(n)$ be the set of all functions $\{1, 2, \dots, n\} \to \{\pm 1\}$. We associate to $e \in P(n)$ another function $e^* \in P(n)$ defined by

$$e^*(j) = \prod_{k=1}^{j} e(k).$$

The assignment $e \mapsto e^*$ is a bijection of $P(n)$. For $e \in P(n)$ we denote by $e' \in P(n-1)$ the restriction of e to $\{1,\ldots,n-1\}$.

Given $e \in P(n)$, we define inductively

$$\lambda(e) = \alpha(e)q^{\beta(e)}\rho^{\gamma(e)}, \quad \alpha(e) \in \{\pm 1\}, \quad \beta(e) \in \mathbb{Z}, \quad \gamma(e) \in \mathbb{Z}$$

as follows: For $e \in P(1)$ we set

$$\alpha(e) = e(1), \quad \beta(e) = 0, \quad \gamma(e) = e(1),$$

i. e. $\lambda(e) = \rho$ in case $e(1) = 1$ and $\lambda(e) = -\rho^{-1}$ in case $e(1) = -1$. For $e \in P(n)$, $n > 1$, we set

$$\lambda(e) = e(n)(q\lambda(e'))^{e(n)}$$

hence

$$\begin{aligned} \alpha(e) &= e(n)\alpha(e') \\ \beta(e) &= e(n)(\beta(e') + 1) \\ \gamma(e) &= e(n)\gamma(e'). \end{aligned}$$

This recursive definition yields

$$\alpha(e) = e^*(n), \quad \gamma(e) = e^*(n), \quad \beta(e) = e^*(n) \cdot \sum_{j=1}^{n-1} e^*(j) \quad n > 1.$$

We define inductively $x(e) \in V^{\otimes n}$ by

$$x(e) = x(e') \otimes (v_{-1} + \lambda(e)v_1), \qquad n \geq 1$$

(in case $n = 1$ the term $x(e')$ does not appear).

(5.1) Theorem. *The $\mathbb{Z}^n$-module $V^{\otimes n}$ decomposes into 2^n pairwise different one-dimensional modules. The vectors $x(e)$ are eigenvectors of $t(n)$ with eigenvalue $\lambda(e)$ and simultaneous eigenvectors for the $\mathbb{Z}^n$-action.*

We set

$$k_e = |\{j \mid e^*(j) = 1\}|, \qquad \ell_e = 2k_e - n.$$

Then we have:

(5.2) Theorem. *The vector $x(e)$ is an eigenvector of t_n with eigenvalue*

$$\mu(e) = (-1)^{n-k_e}\rho^{\ell_e}p^{\ell_e^2 - n}.$$

The vectors $x(e)$ are formally definable with suitable parameters q, ρ in an integral domain $\mathfrak{K}$. We use on $V^{\otimes n}$ a symmetric bilinear form which makes the $v_{e(1)} \otimes \cdots \otimes v_{e(n)} =: v_e$, $e \in P(n)$ into an orthonormal basis. Then we have:

(5.3) Theorem. *The vectors $x(e)$ are pairwise orthogonal. They are a basis of $V^{\otimes n}$ provided*

$$(1+\rho^2)\prod_{j=1}^{n-1}(1+q^{2j}\rho^2)(1+q^{2j}\rho^{-2})$$

is invertible in $\mathfrak{K}$.

From (5.2) be see that t_n has $n+1$ different eigenvalues (generic case), namely according to the value of k_e. The module $V^{\otimes n}$ decomposes into $n+1$ irreducible ZB_n-modules $M_j(n)$. The element t_n is contained in the center of ZB_n. Thus t_n acts as a scalar on $M_j(n)$. The eigenspaces of t_n are the modules $M_j(n)$. The dimension of $M_j(n)$ is $\binom{n}{j}$. There are $\binom{n}{j}$ functions $e \in P(n)$ with $k_e = j$. We choose the indexing such that $M_j(n)$ belongs to $k_e = j$.

Proof of (5.1). Induct over n. A simple computation shows that $v_{-1} + \rho v_1$ and $v_{-1} - \rho^{-1} v_1$ are eigenvectors of $t(\rho)$ with eigenvalues ρ and $-\rho^{-1}$, respectively. We also need an explicit computation in the case $n = 2$. The operator $t(2)$ has in the basis $v_{-1} \otimes v_{-1}$, $v_1 \otimes v_{-1}$, $v_{-1} \otimes v_1$, $v_1 \otimes v_1$ the matrix

$$t(2,\rho) = t(2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \theta q^{-1} & \delta \\ 0 & 1 & \delta & \theta q \end{pmatrix}.$$

A direct computation gives the following eigenvectors and eigenvalues in accordance with (5.1).

eigenvector	eigenvalue
$(v_{-1} + \rho v_1) \otimes (v_{-1} + q\rho v_1)$	$q\rho$
$(v_{-1} + \rho v_1) \otimes (v_{-1} - q^{-1}\rho^{-1} v_1)$	$-q^{-1}\rho^{-1}$
$(v_{-1} - \rho^{-1} v_1) \otimes (v_{-1} - q\rho^{-1} v_1)$	$-q\rho^{-1}$
$(v_{-1} - \rho^{-1} v_1) \otimes (v_{-1} + q^{-1}\rho v_1)$	$q^{-1}\rho$

For the induction step we decompose $V^{\otimes n} = V^{\otimes(n-2)} \otimes V^2$ and use the defining relation

$$t(n) = (1_{n-2} \otimes g)(t(n-1) \otimes 1_1)(1_{n-2} \otimes g).$$

The morphisms $t(n-j) \otimes 1_j$, $0 \le j \le n-1$ pairwise commute. Thus, for each simultaneous eigenvector $u \in V^{\otimes(n-1)}$ of the $t(j)$, $1 \le j \le n-1$, the subspace $u \otimes V$ is $t(n)$-stable. The reason is that in the generic case the simultaneous eigenspaces have multiplicity one, as follows easily by induction. By induction, we assume that u has the form $w \otimes z$, $w \in V^{\otimes(n-2)}$, $z \in V$. By induction again, the map $t(n-1)$ acts on $w \otimes V$ with eigenvectors of the form

$$w \otimes (v_{-1} + \lambda v_1), \quad w \otimes (v_{-1} - \lambda^{-1} v_1),$$

i. e. as $t(\lambda)$ in the basis $w \otimes v_{-1}, w \otimes v_1$. Therefore $t(n)$ acts on $w \otimes V \otimes V$ as $t(2,\lambda)$ with the following eigenvectors and eigenvalues.

eigenvector	eigenvalue
$w \otimes (v_{-1} + \lambda v_1) \otimes (v_{-1} + q\lambda v_1)$	$q\lambda$
$w \otimes (v_{-1} + \lambda v_1) \otimes (v_{-1} - q^{-1}\lambda^{-1} v_1)$	$-q^{-1}\lambda^{-1}$
$w \otimes (v_{-1} - \lambda^{-1} v_1) \otimes (v_{-1} - q\lambda^{-1} v_1)$	$-q\lambda^{-1}$
$w \otimes (v_{-1} - \lambda^{-1} v_1) \otimes (v_{-1} + q^{-1}\lambda v_1)$	$q^{-1}\lambda$

This gives the induction step. □

Proof of (5.2). By definition $t_n = t(1)t(2)\cdots t(n)$, where $t(j)$ also denotes the action $t(j) \otimes 1_{n-j}$ on $V^{\otimes j} \otimes V^{\otimes(n-j)}$. Inductively we see that the $x(e)$ are eigenvectors of each $t(j)$. The eigenvalue belongs to $e|\{1,\dots,j\} =: e^{(j)}$. We have to multiply the eigenvalues in order to obtain the eigenvalue $\mu(e)$ of t_n

$$\begin{aligned} \mu(e) &= q^{b(e)} \cdot \rho^{c(e)} \cdot \prod_{j=1}^{n} e^*(j) \\ b(e) &= \sum_{j=1}^{n} \beta(e^{(j)}) \\ c(e) &= \sum_{j=1}^{n} \gamma(e^{(j)}). \end{aligned}$$

With our definition of $k_e = k$ we have $c(e) = 2k - n$. We note that $b(e)$ is the second elementary symmetric function $\sum_{i<j} e^*(i)e^*(j)$. In order to compute it we determine the coefficient of x^{n-2} in

$$\prod_{j=1}^{n}(x - e^*(j)) = (x-1)^k(x+1)^{n-k}.$$

We obtain

$$-k(n-k) + \binom{k}{2} + \binom{n-k}{2} = \frac{1}{2}(\ell^2 - n).$$

Similarly, by considering the constant term,

$$\prod_{j=1}^{n} e^*(j) = (-1)^{n-k}.$$

This proves (5.2). □

Proof of (5.3). Induct over n. The vectors $(1,\rho)$ and $(1,-\rho^{-1})$ are orthogonal. Set

$$x(e) = \sum_{f \in P(n)} \lambda(e,f) v_f$$

with $v_f = v_{f(1)} \otimes \cdots \otimes v_{f(n)}$. In the induction step we have to consider two vectors of the form

$$x(e_1) \otimes (v_{-1} + \lambda_1 v_1), \qquad x(e_2) \otimes (v_{-1} + \lambda_2 v_1).$$

We have

$$\sum_f \lambda(e_1,f)\lambda(e_2,f) + \lambda_1\lambda_2 \sum_f \lambda(e_1,f)\lambda(e_2,f) = 0,$$

since, by induction, the first sum ist zero. For the second assertion we have to study the transition matrix from the v_e to the $x(e)$. For $n = 1$ we have

$$\begin{vmatrix} 1 & \rho \\ 1 & -\rho^{-1} \end{vmatrix} = -\rho^{-1}(1+\rho^2).$$

We assume inductively that the $x(e)$, $e \in P(n-1)$ are a basis of $V^{\otimes(n-1)}$. Then the $x(e) \otimes v_{-1}$ and $x(e) \otimes v_1$ are a basis of $V^{\otimes n}$. If $x(e)$ has eigenvalue $\lambda(e)$, then the transition matrix to the vectors

$$x(e) \otimes (v_{-1} + q\lambda(e)v_1), \qquad x(e) \otimes (v_{-1} - q^{-1}\lambda(e)^{-1}v_1)$$

consists of 2×2-blocks with determinant

$$\begin{vmatrix} 1 & q\lambda(e) \\ 1 & -q^{-1}\lambda(e) \end{vmatrix} = -q^{-1}\lambda(e)^{-1}(1 + q^2\lambda(e)^2).$$

Thus we require that the $1 + q^2\lambda(e)^2$ be invertible. We have $\lambda(e)^2$ of the form $q^{2\alpha}\rho^{\pm 2}$. Without essential restriction we can assume $\alpha \geq 0$. Thus the invertibility of the product in (5.3) suffices. $\square$

In the last proof we have assumed that we have a basis for $V^{\otimes 1}, \dots, V^{\otimes n}$. But the transition determinant for $V^{\otimes j}$ is a factor of the determinant for $V^{\otimes(j+1)}$.

The preceding results will also be used to obtain information about the eigenspace structure of the cylinder twist on the irreducible U-modules V_n.

The proof of (5.1) also yields the following result.

(5.4) Theorem. *Let $x(e) \in V^{\otimes m}$ be as above and $v \in V^{\otimes n}$. Then*

$$t_{m+n}(\rho)(x(e) \otimes v) = \mu(e)x(e) \otimes t_n(q\lambda(e))(v).$$

There are reasons [8] to consider instead of $t(\rho)$ matrices and the cylinder twist based on

$$\begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix} = t(\alpha, \beta, \theta).$$

We can reduce formally to the previously considered case $\alpha = \beta = 1$ as follows. Let D be the diagonal matrix $\mathrm{Dia}(\lambda_1, \lambda_2)$. Then $D \otimes D$ commutes with g. We can therefore make the basis change with D. This leads to $F(\lambda_1\lambda_1^{-1}\alpha, \lambda_1\lambda_2^{-1}\beta, \theta)$. Thus set $\mu = \lambda_2\lambda_1^{-1}$ and determine μ by $\mu^2 = \beta\alpha^{-1}$. Then we are reduced to $F(\gamma, \gamma, \theta)$, $\gamma = \mu\alpha = \mu^{-1}\beta$. Finally consider $\gamma^{-1}F(\gamma, \gamma, \theta)$.

We write $t(n, \rho)$ for the map $t(n)$ in order to show its dependend on ρ. The maps $t(j)$ commute. Let $W \subset V^{\otimes m}$ denote an eigenspace of $t(m)$. Then the subspace $W \otimes V^{\otimes n} \subset V^{\otimes(m+n)}$ is $t(m+n)$-stable.

Suppose $\alpha\beta = -q$ and $\theta = ip(\rho - \rho^{-1})$. In that case the eigenvalues of t_n in (5.2) have to be multiplied by $(ip)^n$. If we further specialize to the setting of section 4, then $\rho = -ip = -ia$ and the eigenvalues become

(5.5) $$p^{l(l+1)}, \qquad l = 2k - n,\ 0 \leq k \leq n.$$

These eigenvalues are still pairwise different.

6 An algebraic model for TB

In section 4 we have constructed a representation $\mathcal{K}$ which yields on morphisms

$$\mathcal{K}\colon \mathrm{Hom}_{\mathrm{TB}}(r,s) \to \mathrm{Hom}(V^{\otimes r}, V^{\otimes s}).$$

By construction, the morphisms in the image of $\mathcal{K}$ commute with the twists $\mathcal{K}(\alpha) \circ t_r = t_s \circ \mathcal{K}(\alpha)$. Let $\mathrm{Hom}_t(V^{\otimes r}, V^{\otimes s})$ be the subspace of $\mathcal{K}$-linear maps $h\colon V^{\otimes r} \to V^{\otimes s}$ with $h \circ t_r = t_s \circ h$. The algebraic model for TB is given by

(6.1) Theorem. *The linear map*

$$\mathcal{K}\colon \mathrm{Hom}_{\mathrm{TB}}(r,s) \to \mathrm{Hom}_t(V^{\otimes r}, V^{\otimes s})$$

is an isomorphism.

Proof. Let $\{k\}$ denote the largest integer below k. It was shown in [3] that $\mathrm{Hom}_{\mathrm{TB}}(r,s)$ has dimension $\binom{r+s}{\{(r+s)/2\}}$. We first verify that $\mathrm{Hom}_t(r,s)$ has the same dimension. The eigenvector $x(e)$ in (5.2) has multiplicity $\binom{n}{k}$. Since the eigenvalues are pairwise different the dimension in question equals

$$\sum_k \binom{r}{k}\binom{s}{\tilde{k}}$$

with $2k - r = 2\tilde{k} - s$ in case $r+s$ is even. A well-known formula for binomial coefficients shows the claim to be correct. A similar argument works if $s+r$ is odd. (Remark: For $r+s$ even a similar proof works for general parameters ρ.)

By the dimension count above it suffices to show injectivity. By dualization, it suffices to consider the case $s = 0$ (or $r = 0$). By the results of [3] about the Markov trace, the composition of morphisms

$$\mathrm{Hom}(0,r) \times \mathrm{Hom}(r,0) \to \mathrm{Hom}(0,0) = \mathfrak{K}$$

is a perfect pairing. Since the Markov trace is a quantum trace this pairing can be computed from the corresponding bilinear form via $\mathcal{K}$. Therefore $\mathcal{K}$ has to be injective. □

7 The cylinder twist on irreducible modules

We consider the representation of the braid groups ZB_n on $V^{\otimes n}$ given by the four braid pair (X, F) with the standard R-Matrix g from section 5 and

$$t = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix}, \quad \alpha\beta = -q.$$

The cylinder twist t_n is compatible with the Clebsch-Gordan decomposition and induces on the unique irreducible component $V_n \subset V^{\otimes n}$ a morphism τ_n. A matrix

$(F_{k,\ell})$ for τ_n in the standard basis $x_0,\dots,x_n$ of the U-module V_n was computed in [8]. The result is

$$F_{k,\ell} = \alpha^k \beta^{n-k} q^{k(n-k)} \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j$$

with $k+\ell = n+j$. These entries are zero for $j<0$. The γ_j are polynomials determined by the recursion relation $\gamma_{-1}=0$, $\gamma_0 = 1$ and

$$\alpha\gamma_{k+1} = q^k\theta\gamma_k + \beta q^{k-1}(q^k - q^{-k})\gamma_{k-1}$$

for $k>0$.

We want to work with symmetric matrices. For this purpose we make the following assumptions about the base field $\mathfrak{K}$. It has characteristic zero and q is transcendental over $\mathbb{Q}$. We assume given square roots

$$p^2 = q, \quad \gamma^2 = \alpha\beta = -q, \quad \sigma^2 = \alpha\beta^{-1}.$$

Then there exists $\varepsilon = \pm 1$ such that $\sigma\beta = \varepsilon\gamma$. We set $\alpha(\ell) = \sigma^{-\ell}$. Then $\alpha(\ell)\alpha(k)^{-1}\alpha^\ell\beta^{n-\ell} = \varepsilon^n\sigma^j\gamma^n$. We assume given square roots $\sqrt{[n]}$ of the quantum numbers and use these to define the square roots of the quantum binomial coefficients

$$[n]!^{1/2} = [1]^{1/2}[2]^{1/2}\cdots[n]^{1/2}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}^{1/2} = \frac{[n]!^{1/2}}{[k]!^{1/2}[n-k]!^{1/2}}.$$

We choose the basis $z_0,\dots,z_n$ defined by

$$x_k = \sigma^{-k}p^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}^{1/2} z_k.$$

In this basis the operator τ_n has the symmetric matrix

$$\textbf{(7.1)} \qquad F_{k,\ell} = \varepsilon^n\sigma^j\gamma^n p^{k(n-k)+\ell(n-\ell)} \begin{bmatrix} k \\ j \end{bmatrix}^{1/2} \begin{bmatrix} \ell \\ j \end{bmatrix}^{1/2} \gamma_j$$

with $k+\ell = n+j$. In this basis also the R-matrix (= braiding) on $V_n \otimes V_n$ obtained from the universal R-matrix has a symmetric matrix. It is independent of σ.

Guided by the Kauffman calculus of section 4 we specialize to the case $\theta = q+1 = 1-\gamma^2$. In that case we use the renormalized polynomials β_k defined by

$$\gamma_k = \sigma^{-k}p^{k(k-1)}\beta_k.$$

They satisfy the recursion relation

$$\beta_{k+1} = (\gamma^{-1}-\gamma)\beta_k + (1-q^{-2k})\beta_{k-1}.$$

(7.2) Proposition. *The β-polynomials have the following product decomposition*

$$\beta_k = (-\gamma)^k \prod_{j=1}^{k}(1+q^{-j}).$$

PROOF. By definition $\beta_{-1} = 0$ and $\beta_0 = 1$.
We verify that the right hand side satisfies the recursion formula for the β_k

$$\begin{aligned}
& (\gamma^{-1}-\gamma)(-\gamma)^k \prod_{j=1}^{n}(1+q^{-k}) + (1-q^{-2k})(-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= \; & \left[(\gamma^{-1}-\gamma)(-\gamma)(1+q^{-k}) + (1-q^{-2k})\right](-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= \; & (1+q^{-k})\left[-1-q+1-q^{-k}\right](-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= \; & (-\gamma)^2(1+q^{-k})(1+q^{-k-1})(-\gamma)^{k-1}\prod_{j=1}^{k-1}(1+q^{-j}) \\
= \; & \beta_{k+1}.
\end{aligned}$$

This finishes the proof by induction. □

The quantum binomial coefficients are Laurent polynomials in q. In the sequel we have to use the same polynomials with q replaced by p. We use the notation $\begin{bmatrix} n \\ k \end{bmatrix}_p$ for these binomial coefficients.

(7.3) Theorem. *The vector*

$$\varepsilon^n \sum_{k=0}^{n} \gamma^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2} z_k$$

is an eigenvector of the matrix (7.1) *for the eigenvalue* 1.

PROOF. By matrix multiplication the claim is equivalent to the identities

$$\gamma^n \sum_{j=0}^{k} F_{k,n-k+j}\gamma^{k-j-n} \begin{bmatrix} n \\ k-j \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2} = \gamma^{-k}\begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2}.$$

We insert the value (7.1) and see that these identities are equivalent to

$$\sum_{j=0}^{k}(-1)^j \begin{bmatrix} n-k+j \\ j \end{bmatrix} \begin{bmatrix} n \\ n-k+j \end{bmatrix}_p p^{\bullet} \prod_{\nu=1}^{j}(1+q^{-\nu}) = (-1)^k q^{-k} \begin{bmatrix} n \\ k \end{bmatrix}_p$$

with $\bullet = k(n-k) + \ell(n-\ell) + j(j-1)$ and $\ell = n-k+j$. We set

$$\prod_{\nu=1}^{j}(1+q^{-\nu}) = p^{-j(j+1)/2}\pi_j \quad \text{with} \quad \pi_j = \prod_{\nu=1}^{j}(p^j + p^{-j}).$$

We compute

$$\left[\begin{array}{c} n-k+j \\ j \end{array}\right] \left[\begin{array}{c} n \\ n-k+j \end{array}\right]_p \left[\begin{array}{c} n \\ k \end{array}\right]_p^{-1} = \frac{\pi_{n-k+j}}{\pi_j \pi_{n-k}} \left[\begin{array}{c} k \\ j \end{array}\right]_p$$

and use this to put the claimed identities into the form ($t = n-k$)

$$\sum_{j=0}^{k} (-1)^j p^{\#} \left[\begin{array}{c} k \\ j \end{array}\right]_p \pi_{t+j} = (-1)_q^{k-k(t+1)} \pi_t$$

with $\# = j(k-t-1) - j(j+1)/2$.

It is now possible to verify these identities by induction over k. For $k = 0$ it reduces to $\pi_t = \pi_t$ and thus holds for all t.

We rewrite the left hand side of the identity in question for $k+1$ by using the Pascal formula

$$\left[\begin{array}{c} k+1 \\ j \end{array}\right]_p = p^{-j} \left[\begin{array}{c} k \\ j \end{array}\right] + p^{j-k+1} \left[\begin{array}{c} k \\ j-1 \end{array}\right].$$

We obtain

$$\sum_{j=0}^{k} (-1)^j p^{j(k-t)-j(j+3)/2} \left[\begin{array}{c} k \\ j \end{array}\right] \pi_{t+j} + \sum_{j=1}^{k+1} (-1)^j p^{j(k-t)-j(j-1)/2-k+1} \left[\begin{array}{c} k \\ j-1 \end{array}\right] \pi_{t+j}.$$

By induction, the first sum equals $(-1)^k q^{-k(t+1)} \pi_t$. In the second sum we replace j by $j-1$. Then we see, again by induction, that it equals

$$-q^{-k(t+1)+k} p^{-t-1} \pi_{t+1} = -q^{-k(t+1)} (1 + q^{-t-1}) (-1)^k \pi_t.$$

Altogether we obtain the correct result. □

We symmetrize the vector (7.3). Suppose $\bar{\gamma}^2 = \gamma$. We use

$$\kappa_k = \sum_{k=0}^{n} \bar{\gamma}^{n-2k} \left[\begin{array}{c} n \\ k \end{array}\right]_p \left[\begin{array}{c} n \\ k \end{array}\right]^{-1/2} z_k.$$

It is sensible to consider (z_k) as an orthogonal basis of V_n. In that case the norm-square of κ_k is

$$A(n) = \sum_{k=0}^{n} \gamma^{n-2k} \left[\begin{array}{c} n \\ k \end{array}\right]_p^2 \left[\begin{array}{c} n \\ k \end{array}\right]^{-1} = \pi_n^{-1} \sum_{k=0}^{n} \gamma^{n-2k} \pi_j \pi_{n-j} \left[\begin{array}{c} n \\ k \end{array}\right]_p.$$

It turns out that $A(n)$ also has the following product decomposition.

(7.4) Theorem. *The following identity holds for $n \geq 1$*

$$[2]_p \sum_{k=0}^{n} \gamma^{n-2k} \pi_j \pi_{n-j} \left[\begin{array}{c} n \\ k \end{array}\right]_p = (\gamma + \gamma^{-1})^n [n+1]_p^!.$$

A direct verification of this identity does not seem easy. We shall obtain it from the representation theory of the Temperley-Lieb category and the structure of the Jones-Wenzl idempotent. For later use we mention already at this point a q-analogue of a well known formula

(7.5)
$$\sum_{j=1}^{n}[j] = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_p .$$

There is no difficulty to prove this by induction.

By (7.4), we can also write

$$A(n) = \frac{(\gamma+\gamma^{-1})^n}{[n]!}\prod_{j=1}^{n}\begin{bmatrix} j+1 \\ 2 \end{bmatrix}_p .$$

8 The category of coloured cylinder ribbons

We consider unoriented rooted cylinder ribbons with components coloured by the irreducible U-modules V_n, $n \in \mathbb{N}$. The meaning of a colouring is as in [21, Ch. I]. There is an associated category. The objects of this category are sequences $(j_1, \dots, j_r)$ with $j_k \in \mathbb{N}$ (the empty sequence for $r = 0$). The morphisms from $(j_1, \dots, j_r)$ to $(k_1, \dots, k_k)$ are the coloured rooted (r, s)-ribbons; a component which ends in $(a, 0)$ carries the colour V_a; a component which ends in $(b, 1)$ carries a colour V_b. We call this category RRB($\mathbb{N}$). It is a tensor module category over the tensor category RA($\mathbb{N}$) of coloured ordinary ribbons. The category RRB($\mathbb{N}$) has the following presentation by generators and relations.

(8.1) Generators.

(1) $X_{m,n}\colon (m,n) \to (n,m)$, $\quad X^{-1}_{m,n}\colon (n,m) \to (m,n)$
(2) $k_m\colon \emptyset \to (m,n)$, $\quad f_m\colon (m,n) \to \emptyset$
(3) $t_m\colon (m) \to (m)$, $\quad t^{-1}_m\colon (m) \to (m)$
(4) $\kappa_m\colon \emptyset \to (m)$, $\quad \varphi_m\colon (m) \to \emptyset$

The relations are coloured versions of (1.1) and (1.2).

(8.2) Relations.

(1) $X_{m,n}X^{-1}_{m,n} = 1_{(m,n)} = X^{-1}_{n,m}X_{n,m}$
(2) $(X_{n,p} \otimes 1_{(m)})(1_{(n)} \otimes X_{m,p})(X_{m,n} \otimes 1_{(p)}) = (1_{(p)} \otimes X_{m,n})(X_{m,p} \otimes 1_{(n)})(1_{(m)} \otimes X_{n,p})$
(3) $(1_{(m)} \otimes f_m)(k_m \otimes 1_{(m)}) = 1_{(m)} = (f_m \otimes 1_{(m)})(1_{(m)} \otimes k_m)$
(4) $(f_m \otimes 1_{(p)}) = (1_{(p)} \otimes f_m)(X_{m,p} \otimes 1_{(p)})(1_{(m)} \otimes X_{m,p})$
$(k_p \otimes 1_{(m)}) = (1_p \otimes X_{m,p})(X_{m,p} \otimes 1_p)(1_{(m)} \otimes k_p)$
and similar relations with $X_{m,p}$ replaced by $X^{-1}_{p,m}$
(5) $t_m t^{-1}_m = 1_{(m)} = t^{-1}_m t_m$
(6) $t_{(m,n)} := X_{n,m}(t_n \otimes 1_{(m)})X_{m,n}(t_m \otimes 1_{(n)}) = (t_m \otimes 1_{(n)})X_{n,m}(t_n \otimes 1_{(m)})X_{m,n}$
(7) $t_{(m,m)}k_m = k_m$, $\quad f_m t_{(m,n)} = f_m$
(8) $(\varphi_m \otimes 1_{(m)})k_m = \kappa_m$, $\quad f_m(\kappa_m \otimes 1_{(m)}) = \varphi_m$

(9) $t_m \kappa_m = \kappa_m, \quad \varphi_m t_m = \varphi_m$

(10) $t_{(m,n)}(\kappa_m \otimes 1_{(n)}) = (\kappa_m \otimes 1_{(n)}) t_n, \quad (\varphi_m \otimes 1_{(n)}) t_{(m,n)} = t_n(\varphi_m \otimes 1_{(n)})$

Our aim is to construct a representation of this category. On object level it maps $(j_1, \dots, j_r)$ to $V_{j_1} \otimes \cdots \otimes V_{j_r}$. We now specify the values of the generators and verify the relations. The values of the generators are denoted with the same symbol.

We take for $X_{m,n}: V_m \otimes V_n \to V_n \otimes V_m$ the braiding (R-matrix) which is induced from the universal R-matrix. The operator $t_m: V_m \to V_m$ is the cylinder twist. These data satisfy the relations (1), (2), (5), (6).

We have a Clebsch-Gordan decomposition of U-modules

$$V_m \otimes V_m = V_{2m} \otimes V_{2m-2} \oplus V_{2m-2} \oplus \cdots \oplus V_0.$$

The morphism $k_m: \mathfrak{K} \to V_m \otimes V_m$ corresponds to the inclusion of V_0 and the morphisms $f_m: V_m \otimes V_m \to \mathfrak{K}$ to the projection onto V_0. These conditions determine them up to a scalar multiple. The naturality of the braiding yields relation (4). We have to normalize k_m and f_m. A normalization of k_m yields, by (3), a normalization of f_m. In order to specify the normalization, we use the basis (z_k) of section 7, set

(8.3) $$k_m(1) = \sum_{j=0}^{m} \gamma^{m-2j} z_j \otimes z_{m-j},$$

and define f_m by the transposed matrix, i. e.

$$f_m(z_j \otimes z_{m-j}) = \gamma^{m-2j} \quad \text{and} \quad f_m(z_k \otimes z_\ell) = 0$$

otherwise. Then (3) holds. We note

(8.4) $$f_m k_m(1) = [m+1]_{\gamma^2}.$$

In order to satisfy (9) we have to take for κ_m and φ_m eigenvalues of t_m for the eigenvalue 1. We set

(8.5) $$\kappa_m(1) = \varepsilon^m \sum_{k=0}^{m} \bar{\gamma}^{m-2k} \begin{bmatrix} m \\ k \end{bmatrix}_p \begin{bmatrix} m \\ k \end{bmatrix}^{-1/2} z_k.$$

Since the matrix for t_m is symmetric, we define φ_m by the transposed matrix

$$\varphi_m(z_k) = \varepsilon^n \bar{\gamma}^{n-2k} \begin{bmatrix} n \\ k \end{bmatrix}_p \begin{bmatrix} n \\ k \end{bmatrix}^{-1/2} z_k.$$

These choices yield the relations (9). We have already discussed $\varphi_m \kappa_m(1)$ in the previous section.

If $\kappa_m(1) = \sum_{k=0}^{m} a_k z_k$, then the second relation in (8) gives

$$f_m(\kappa_m \otimes 1_{(m)})(z_\ell) = \gamma^{2\ell - m} a_{m-\ell}.$$

With our choices this equals $\varphi_m(z_\ell)$. Similarly for the first relation in (8).

The relations (7) hold, since the cylinder twist commutes with U-linear maps, in particular $t_{(m,m)}$ is compatible with the Clebsch-Gordan decomposition.

It remains to verify (10). These relations do not depend on the normalization of κ_m and φ_m. The second one is the transposition of the first one. The first relation is a consequence of (5.4). There we have shown that a similar result holds if the V_k are replaced by $V^{\otimes k}$ throughout. Hence we have verified all relations.

The connection between $V^{\otimes m}$ and V_m is as follows. Let f_m be the Jones-Wenzl idempotent in the Temperley-Lieb algebra $\mathrm{Hom}_{\mathrm{RA}}(m,m) = T_m$. Via the representation of RA the element f_m yields a projection operator on $V^{\otimes m}$ with image isomorphic to V_m. From section 5 we know that the m-fold tensor product

$$z := (\bar\gamma v_{-1} + \bar\gamma^{-1} v_1) \otimes \cdots \otimes (\bar\gamma v_{-1} + \bar\gamma^{-1} v_1)$$

is an eigenvector with eigenvalue 1 for the cylinder twist. The model $V_m \subset V^{\otimes m}$ has the standard basis $z_0, \ldots, z_m$ with $z_0 = v_{-1} \otimes \cdots \otimes v_{-1}$. We assume the value $\varepsilon = 1$.

(8.6) Proposition. *The projection $f_m(z)$ is the vector* (8.5) *with $\varepsilon = 1$.*

Proof. It certainly is an eigenvector for the eigenvalue 1, by naturality of the cylinder twist. Thus $f_m(z) = \lambda \kappa_m(1)$ for some scalar λ. In order to determine the scalar we consider the coefficient of z_0 in $f_m(z)$. By the structure of the Jones-Wenzl idempotent this coefficient is $\bar\gamma^n$. In order to see this, express f_m as a linear combination of the standard graphical basis of the Temperley-Lieb algebra T_m and observe that all basis elements except 1 map $z_0 \in V^{\otimes m}$ to zero. □

The following recursion formula for f_m is due to Hermisson [10]. Let $e_1, \ldots, e_{m-1}$ be the standard generators of the Temperley Lieb algebra. Write

$$e(m,n) = e_{m-1} e_{m-2} \ldots e_n, \qquad e(m,m) = 1.$$

Then

(8.7) $$f_m = f_{m-1} \cdot \frac{1}{[m]} \sum_{j=1}^{m} [j] e(m,j).$$

Suppose b is a bridge in T_m and also the corresponding morphism in $V^{\otimes m} \to V^{\otimes m}$. Then

(8.8) $$[b\rangle := \varphi_{V^{\otimes m}} b \kappa_{V^{\otimes m}} = (\gamma + \gamma^{-1})^m.$$

Thus, by (8.7),

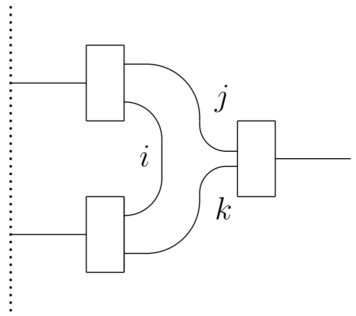
(8.9) $$[f_m\rangle = (\gamma + \gamma^{-1})^m \prod_{k=1}^{m} \frac{1}{[k]} \left(\sum_{j=1}^{k} [j] \right).$$

On the other hand this value equals $\varphi_m \kappa_m(1)$. If we use (7.5) we see that the equality of these values yields the identity (7.6).

9 Trivalent graphs

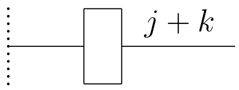
The ℕ-coloured Temperley-Lieb category TA(ℕ) has a natural extension to a category ot trivalent graphs [16], [15], [19], [21]. There is a correcponding extension of TB. A trivalent vertex in TA(ℕ) is defined in graphical notation as Figure 8.1 in [21, p. 552]. See the figure below for $\omega_{i,j,k}$; the boxes in that figure represent Jones-Wenzl idempotents.

We here consider trivalent graphs where some edges end on the axis. The evaluation of such graphs by the method of [19], say, reduces to the determination of



$$= \omega_{i,j,k}$$

in terms of



$$= \omega_{j+k}.$$

(9.1) Theorem. *There exists a scalar* $[i,j,k\,\rangle$ *such that*

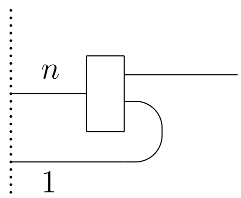
$$\omega_{i,j,k} = [i,j,k\,\rangle \omega_{j+k}.$$

Let $\alpha_n = (\gamma+\gamma^{-1})[n]^{-1}([1]+[2]+\cdots+[n])$. *We have the recursion relations*

$$\omega_{i,j,k} = \alpha_{i+k}\omega_{i-1,j,k} + \frac{[j]}{[i+j]}\alpha_{i+j-1}\omega_{i-1,j-1,k+1}$$

and $[0,j,k\,\rangle = 1$, $[i,0,k\,\rangle = \alpha_{i+k}[i-1,0,k\,\rangle$.

Proof. We use the method of [19] and our earlier results. A first step evaluates



We apply the second recursion formula (8.7) for the Jones-Wenzl idempotent f_n and obtain the value $\alpha_n\omega_{n-1}$. At the same time we obtain

$$(*) \qquad \omega_{i,0,k} = \alpha_{i+k}\omega_{i-1,0,k}$$

and

$$\omega_{i,0,k} = [i,0,k\,\rangle \omega_{i+k}.$$

The value $[0,j,k\,\rangle = 1$ comes from the definitions; similarly $\omega_{0,j,k} = \omega_{j+k}$.

We now consider the case $i > 0, j > 0$ and apply the standard recursion formula [21, (4.2.a) on p. 531] for f_{i+j}. Then $(*)$ above in conjunction with [19, Lemma 2] yield the recursion formula for $\omega_{i,j,k}$ as stated in (9.1). $\square$

10 Parameters

We have already mentioned in the first section relations between skein parameters. This section is devoted to an algebraic discussion of these parameters. The analysis is based on a certain associative algebra $\mathcal{S}$ with 1 over $\mathfrak{K}$ generated by $X, X^{-1}, E, Y, Y^{-1}, e$. The algebra is derived from rooted $(2,2)$-tangles. The connection with the geometric generators of section one is as follows: X is as in section one, $E = fk$, $e = \varphi\kappa \otimes 1$, $Y = F \otimes 1$. We work with the following relations.

(1) $XX^{-1} = 1 = X^{-1}X, \quad YY^{-1} = 1 = Y^{-1}Y$
(2) $X - X^{-1} = (q - q^{-1})(1 - E)$
(3) $XE = EX = \lambda E$
(4) $E^2 = DE$
(5) $\mu^{-1}Y + \mu Y^{-1} = (\rho + \rho^{-1})(e - 1)$
(6) $eY = Ye = e$
(7) $e^2 = de$
(8) $XYXY = YXYX$
(9) $EXYXY = E$
(10) $EY^{\pm 1}E = \alpha_\pm E$

Here $q, \lambda, D, \mu, \rho, d, \alpha_\pm \in \mathfrak{K}$ are parameters. We assume that q, λ, μ, ρ are invertible. We set $\delta = q - q^{-1}$ and $\varepsilon = \rho + \rho^{-1}$. For simplicity we assume that δ and ε are invertible. Our aim is to deduce relation between the parameters. We use geometric assumptions and assume certain non-degeneracies for the algebra. Set $\mathrm{Ann}(x) = \{z \in \mathcal{S} \mid zx = 0\}$. We multiply (2) by E and use (3) and (4). If $\mathrm{Ann}(E) = 0$, then

(10.1) $$D = 1 - \delta^{-1}(\lambda - \lambda^{-1}), \quad E = 1 - \delta^{-1}(X - X^{-1}).$$

In a similar manner we obtain from (5), (6), and (7) in case $\mathrm{Ann}(e) = 0$

(10.2) $$d = 1 + \varepsilon^{-1}(\mu + \mu^{-1}), \quad e = \varepsilon^{-1}(\mu^{-1}Y + \mu Y^{-1}) + 1.$$

We use this to compute $EeE = (\varepsilon^{-1}(\mu^{-1}\alpha_+ + \mu\alpha_-) + D)E$. For geometric reasons we require $EeE = dE$. This forces us to postulate

(10.3) $$\varepsilon^{-1}(\mu^{-1}\alpha_+ + \mu\alpha_-) + D = d.$$

From (3), (6), and (9) we obtain $\lambda^{-1}EY^{-1} = EYX$ and therefore

$$\alpha_- E = EY^{-1}E = \lambda EYXE = \lambda^2 EYE = \lambda^2\alpha_+ E.$$

This gives

(10.4) $$\beta := \lambda^{-1}\alpha_- = \lambda\alpha_+.$$

From (10.1) to (10.4) we derive

$$\varepsilon^{-1}(\mu^{-1}\lambda^{-1}+\mu\lambda)\beta = \varepsilon^{-1}(\mu+\mu^{-1}) + \delta^{-1}(\lambda-\lambda^{-1}).$$

Later we shall see that we must have

(10.5) $$\mu^2\lambda = 1.$$

If we use this we obtain

(10.6) $$\beta = 1 - \delta^{-1}\varepsilon(\mu-\mu^{-1}).$$

In the generic situation the relations (2) – (4) are equivalent to the cubic relation $(X-\lambda)(X-q)(X+q^{-1}) = 0$ and the relations (5) – (7) to the cubic relation $(Y-1)(Y+\mu\rho)(Y+\mu\rho^{-1}) = 0$. The further analysis is based on the right ideal of $\mathcal{S}$ generated by E. We will derive from the relations that it is generated by E, Ee, EY. We then postulate that this is a basis for a simple $\mathcal{S}$-module ($\mathfrak{K}$ a field). We compute the right products of E, Ee, EY by e, E, Y, X and state the result as if E, Ee, EY were a basis.

We obtain fairly directly

$$e = \begin{pmatrix} 0 & 0 & 0 \\ 1 & d & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} D & e & \lambda\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & \mu^2 \\ 0 & 1 & \varepsilon\mu \\ 1 & 0 & -\varepsilon\mu \end{pmatrix}.$$

The products with X are more complicated. We have

$$\begin{aligned} EX &= EX. \\ EYX &= \lambda^{-1}EY^{-1} = \lambda^{-1}(-\varepsilon\mu^{-1}E + \varepsilon\mu^{-1}Ee - \mu^2 EY). \\ EeX &= E(\varepsilon^{-1}(\mu^{-1}Y+\mu Y^{-1})+1)X \\ &= \lambda E + \varepsilon^{-1}\mu^{-1}EYX + \varepsilon^{-1}\mu EY^{-1}X. \\ EY^{-1}X &= \lambda EYX^2 = \lambda EYX(1+\delta(X-\lambda E)). \end{aligned}$$

We insert the last relation in the previous one and obtain

$$EeX = c_1E + c_2Ee + c_3EY$$

with

$$\begin{aligned} c_1 &= -\lambda\delta\mu\varepsilon^{-1} + \mu^2\lambda - \mu^{-1}\lambda^{-2} - \delta \\ c_2 &= \delta + \lambda^{-1}\mu^{-2} \\ c_3 &= \mu^{-1}\varepsilon^{-1}(\lambda\mu^2 - \lambda^{-1}\mu^{-2} - \delta). \end{aligned}$$

We derive $c_2 + \varepsilon\mu c_3 = \mu^2\lambda$ and $c_1 - \varepsilon\mu c_3 = -\lambda\delta\mu\varepsilon^{-1}$. A computation now gives

$$YXYX = \begin{pmatrix} 1 & -\lambda\mu^4c_3 - \delta\mu\varepsilon^{-1} & 0 \\ 0 & \lambda^2\mu^4 & 0 \\ 0 & -\lambda\mu^2c_3 - \delta\lambda^2\mu^3\varepsilon^{-1} & 1 \end{pmatrix}.$$

Since $YXYX$ is contained in the center of $\mathcal{S}$, it should act on a simple module as a scalar. It is seen that this is the case if and only if (10.5) holds. One could now use the matrices above to define a module of rank three. From the geometric view point it is sensible to look for the quotient $\mathcal{T}$ of $\mathcal{S}$ by relations of the type

$$\begin{aligned} eXe - eEe &= u_+e + v_+eXYX \\ eX^{-1}e - eEe &= u_-e + v_-eX^{-1}Y^{-1}X^{-1}. \end{aligned}$$

The first relation holds in the three-dimensional module above if and only if $u_+ + v_+ = \varepsilon^{-1}\delta(\mu - \mu^{-1})$. By subtraction of the relations we should arive at

$$\mu^{-1}EXYX + \mu EX^{-1}Y^{-1}X^{-1} = \varepsilon(eEe - e).$$

The only sensible choice then is

(10.7)
$$\begin{array}{ll} u_+ = \varepsilon^{-1}\delta\mu & v_+ = -\varepsilon^{-1}\mu^{-1}\delta \\ u_- = -\varepsilon^{-1}\delta\mu^{-1} & v_- = \varepsilon^{-1}\delta\mu \quad . \end{array}$$

This finishes the discussion of the derisable relations between the parameters. In the generic case the algebra $\mathcal{S}$ is semi-simple of dimension 27 with 6 one-dimensional, 3 two-dimensional and 1 three-dimensional module. The quotient $\mathcal{T}$ has dimension 25. See [9] for more details.

A variant of the algebra $\mathcal{S}$ replaces the relations for Y by those modelled on the Kauffman bracket. Thus we replace (5) by the relation $Y = x+ye$. We deduce $e = eY = xe + yde$ and hence $1 = x + yd$ as in section 4. Also $Y^2 = xY + ye$ and (5) yield the quadratic relation $Y^2 - (1 + x)Y + x = (Y - 1)(Y - x) = 0$. The relation (10) leads to conditions for the parameters. This is based on the following computation.

$$\begin{aligned} EYE &= (1 - \delta^{-1}(X - X^{-1}))YE \\ &= YE - \delta^{-1}XYE + \delta^{-1}X^{-1}YE \\ &= YE - \delta^{-1}\lambda^{-1}Y^{-1}E + \delta^{-1}X^{-1}(-xY^{-1} + (1 + x))E \\ &= YE + \delta^{-1}\lambda^{-1}(x^{-1}Y - x^{-1} - 1)E - \delta^{-1}\lambda xYE + \delta^{-1}\lambda^{-1}(1 + x)E \end{aligned}$$

The coefficient of YE should be zero, i. e. $\lambda x - \lambda^{-1}x^{-1} = \delta$. Therefore $\lambda x = q$ or $\lambda x = -q^{-1}$. The computation also yields

$$\alpha_+ = \delta^{-1}\lambda^{-1}(x - x^{-1}), \quad \alpha_- = -\delta^{-1}\lambda(x - x^{-1}).$$

This algebra has dimension 12 and in the generic case 4 one-dimensional and 2 two-dimensional modules. The right ideal spanned by E and EY is one such module.

References

1. Birman, S. J., and H. Wenzl: Braids, link polynomials and a new algebra. Trans. Amer. Math. Soc. 313, 249 – 273 (1989).

2. Brieskorn, E.: Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe. Inv. Math. 12, 57 – 61 (1971).

3. tom Dieck, T.: Symmetrische Brücken und Knotentheorie zu den Dynkin-Diagrammen vom Typ B. J. reine angew. Math. 451, 71 – 88 (1994).

4. tom Dieck, T.: Tensor representations of knot algebras. To appear Manuscripta Math. 1997.

5. tom Dieck, T.: Cylinder braiding for quantum linear groups. To appear.

6. tom Dieck, T.: Knotentheorien und Wurzelsysteme. Teil II. Mathematica Gottingensis 44 (1993).

7. tom Dieck, T.: On tensor representations of knot algebras. Mathematica Gottingensis 45 (1995).

8. tom Dieck, T., and R. Häring-Oldenburg: Quantum groups and cylinder braiding. To appear.

9. Häring-Oldenburg, R.: Dissertation. Göttingen 1997.

10. Hermisson, U.: Diplomarbeit. Göttingen 1996.

11. Jantzen, J. C.: Lectures on quantum groups. Graduate studies in Math. Vol 6. Amer. Math. Soc. 1996.

12. Kassel, C.: Quantum groups. New York, Springer 1995.

13. Kauffman, L. H.: State models and the Jones polynomial. Topology 26, 395 – 407 (1987).

14. Kauffman, L. H.: An invariant of regular isotopy. Trans. Amer. Math. Soc. 318, 417 – 471 (1990).

15. Kauffman, L. H.: Spin networks and knot polynomials. Int. J. of Modern Physics A5, 93 – 115 (1990).

16. Kirillov, A. N., and N. Yu. Reshetikhin: Representations of the algebra $U_q(sl_2)$, q-orthogonal polynomials, and invariants of links. In: Infinite dimensional Lie algebras and groups, (ed. V. G. Kac), 285 – 339. Adv. Ser. in Math. Phys. 7, World Scientific, Singapore 1988.

17. Lambropoulou, S. S. F.: Solid torus links and Hecke algebras of B-type. Proc. of quantum topology (D. Yetter ed.), 225 – 245. World Scientific, Singapore 1994.

18. Lusztig, G.: Introduction to quantum groups. Basel, Birkhäuser 1993.

19. Masbaum, G., and P. Vogel: 3-valent graphs and the Kauffman bracket. Pacific J. of Math. 165, 361 – 381 (1994).

20. Turaev, V. G.: Operator invariants of tangles and R-matrices. USSR Izvestia 35, 411 – 444 (1990).

21. Turaev, V. G.: Quantum invariants of knots and 3-manifolds. Berlin, W. de Gruyter 1994.

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