Algebraic Space Forms

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1 Algebraic space forms

An n–dimensional *affine hypersurface* V over the complex numbers $\mathbb{C}$ is the zero set of a polynomial $f \in \mathbb{C}[z_0, \dots, z_n]$:

$$V = V(f) = \{z \in \mathbb{C}^{n+1} \mid f(z) = 0\}.$$

The variety $V(f)$ is called *regular* if 0 is a regular value of the holomorphic map $f : \mathbb{C}^{n+1} \to \mathbb{C}$, i. e. if for each $z \in V(f)$ some partial derivative $\partial f / \partial z_i$ is non zero.

A classical problem is: Relate the geometry and topology of $V(f)$ with the algebraic properties of f. Here are two view points:

1. Given f, derive the topological properties of $V(f)$.

2. Given a topological type, find f such that $V(f)$ has this type.

Here "topological type" can refer to different categories: homology or homotopy type, diffeomorphism type etc.

Similar questions can be asked for affine varieties or algebraic varieties in general.

A distinguished situation occurs when the underlying geometrical object is Euclidean space. We therefore consider the following properties of an n–dimensional algebraic variety V over the complex numbers:

(0.1) V is diffeomorph or homeomorphic to $\mathbb{C}^n$.

(0.2) V is contractible.

(0.3) For a subring R of the rational numbers $\mathbb{Q}$ we have a homology isomorphism $H_*(V;R) \cong H_*(\mathbb{C}^n;R)$.

In case (1.1) we call V an (algebraic) *space form*. In case (1.3) we call V an *R–homology space form* or *R–acyclic* (in case $R = \mathbb{Z}$ we do not specify R). In the case (1.3) of complex surfaces ($n = 2$) the term *R–homology plane* was introduced in TOM DIECK–PETRIE [1989].

(0.4) Example. Let m, n be coprime natural numbers. The affine variety

$$V = \{(x,y) \mid x^m = y^n\} \subset \mathbb{C}^2$$

is homeomorphic to $\mathbb{C}$. The map $t \mapsto (t^n, t^m)$ is a homeomorphism. But V is not regular at the origin and not a smooth submanifold of $\mathbb{C}^2$.

The following examples were studied in TOM DIECK – PETRIE [1990]. The corresponding abstract surfaces were first constructed by GURJAR – MIYANISHI [1987].

(0.5) Theorem. *Let $a \geq b > 0$ be coprime integers. Define a polynomial $P = P_{a,b}$ by*

$$P(x,y,z) = z^{-1}\left((xz+1)^a - (yz+1)^b\right).$$

Then the variety

$$V(a,b) = \{(x,y,z) \mid P(x,y,z) = 1\}$$

is a regular contractible affine hypersurface in $\mathbb{C}^3$. If $V(a,b)$ and $V(a',b')$ are homeomorphic, then $(a,b) = (a',b')$. □

Later we come back to these surfaces and study natural generalizations. In dimension two one cannot find exotic affine space forms. This is a consequence of a result of RAMANUJAM [1971]:

(0.6) Theorem. *If the regular affine variety V is homeomorphic to $\mathbb{C}^2$, then the variety is isomorphic to the standard affine plane $\mathbb{C}^2$.* □

Ramanujam also gave the first example of a contractible affine surface which is not homeomorphic to $\mathbb{C}^2$. A basic tool for the study of affine varieties are compactifications. One has two different (but related) possibilities:

(0.7) An *algebraic conpactification* of V is a projective variety X such that V is isomorphic to the complement $X \setminus D$ of a divisor D in X. We call D a *boundary divisor* or *compactification divisor* of V. The compactification divisor is called a *normal crossing* divisor if it is the sum (= union) of irreducible regular subvarieties $D_1, \ldots, D_r$ with transverse intersection.

(0.8) A *topological compactification* of the (regular) variety V is a compact smooth manifold B with boundary ∂B such that V is diffeomorphic to the interior of B. The fundamental group $\pi_1(\partial B)$ is called the *fundamental group at infinity* of V. The topology of ∂B is called the topology ov V at infinity. If D is a normal crossing divisor as in (1.7), then B can be taken to be the complement of a suitable tubular neighbourhood of D. Thus knowledge of a nice algebraic compactification implies knowledge of a topological compactification.

Here is a trivial example to illustrate these concepts.

(0.9) Example. Affine space $\mathbb{C}^2$ is compactified by projective space $\mathbb{P}^2$. The compactification divisor is a projective line L in $\mathbb{P}^2$. Also $\mathbb{C}^2$ is diffeomorphic to the interior of the 4–dimensional unit disk $D^4 = \{z \in \mathbb{C}^2 \mid |z| \leq 1\}$.

Because of Ramanujam's theorem (1.6), the contractible surfaces are a natural class of varieties to be studied in this context. They can be used to produce space forms in higher dimensions as we shall demonstrate later. The basic input from differential topology is the h–cobordism theorem of Smale [1962] which leads to a characterization of the unit disk as a differentiable manifold in terms of algebraic topology (see also Milnor [1965], p. 108).

(0.10) Theorem. *Let D be a k–dimensional compact contractible smooth manifold with contractible boundary ∂D. Then D is diffeomorphic to the unit disk D^k in $\mathbb{R}^k$, provided $k \geq 6$.* □

By the Hurewicz and Whitehead theorems of algebraic topology, the contractibility of a manifold can be detected by algebraic invariants:

(0.11) Proposition. *Let D be a simply connected, acyclic manifold. Then D is contractible.* □

A contractible manifold can have a boundary which is not simply connected (see Mazur [1961], Gordon [1975] for examples). A similar phenomenon cannot happen for homology:

(0.12) Proposition. *Suppose the compact oriented $(n+1)$–manifold B is R–acyclic. Then ∂B is an R–homology n–sphere.*

Proof. The exact homology sequence (coefficient ring R) gives $H_{i+1}(B, \partial B) \cong H_i(\partial B)$ for $i > 0$. Poincaré duality $H_{i+1}(B, \partial B) \cong H^{n-i}(B)$ then shows $H_i(\partial B) = 0$ for $n > i > 0$. Similarly, we see $H_n(\partial B) \cong R$ and $H_0(\partial B) \cong R$ (for $n > 0$). □

It is an interesting topological problem to determine which homology spheres are boundaries of contractible or acyclic manifolds (compare Casson–Harer [1981]). More generally, one wants to find the "smallest" manifold which bounds a given homology sphere.

In view of (1.11) and (1.12) we can say:

(0.13) Proposition. *Let D be a contractible smooth k–manifold with simply connected boundary. Then D is diffeomorphic to D^k, provided $k \geq 6$.* □

There are important cases in which a simply connected manifold will have a simply connected boundary. In order to state the result we use the handle decomposition of a smooth manifold which follows from the existence of a smooth Morse function, see Milnor [19??].

(0.14) Proposition. *Suppose the compact connected m–manifold M has only handles of index i for $i \leq r \leq \frac{m}{2}$. Then the relative homotopy groups $\pi_j(M, \partial M)$ are zero for $j < m - r$.*

Proof. In order to prove this proposition we investigate what happens when a single handle is attached.

In order to attach an i–handle one has to choose an embedding $C = S^{i-1} \times D^{m-i} \subset \partial M$ and form the adjunction space $M' = M \cup_C (D^i \times D^{m-i})$. The space M' has the homotopy type of $M \cup D^i$, an i–cell D^i attached to M along $S^{i-1} \times 0$. Let us consider the following diagram.

$$\begin{tikzcd}
(M', \partial M') \arrow[r, "\alpha"] & (M', \partial M' \cup D^{m-i}) \arrow[d, "\simeq"] \\
 & (M \cup (D^i \times D^{m-i}), \partial M \times I \cup (D^i \times D^{m-i})) \\
(M, \partial M) \arrow[r, "\beta"] & (M \cup D^i,\ \partial M \cup D^i) \arrow[u, "\simeq"']
\end{tikzcd}$$

The vertical maps are the obvious homotopy equivalences, whereas α and β are inclusions. We consider the induced maps on homotopy groups π_j. The exact homotopy sequence shows that α induces an isomorphism for $j < m - i$. Suppose $\pi_j(M, \partial M) = 0$ for $j < m - r$. Then homotopy excision (tom Dieck [19??], p. 178) shows that β induces an isomorphism for $j < m - r + i - 2$.

Now one uses this information inductively, starting with (D^m, S^{m-1}), and attaching successively 1–handles, 2–handles etc. □

Affine varieties have the following remarkle property (1.15); see Milnor [1963], §7 for a proof.

In order to deal with regular affine varieties V from a topological point of view it is useful to know that there exists a compact manifold B with boundary such that V is diffeomorphic to the interior of B. If one realizes V as a regular subvariety of some $\mathbb{C}^N$, then there exists an $R > 0$ such that for all $r > R$ the sphere $S(r) = \{z \in \mathbb{C}^N \mid |z| = r\}$ is transverse to V. This is proved in MILNOR [1968], Cor 2.6. Now one can take as B the intersection of V with a large disk in $\mathbb{C}^N$.

(0.15) Proposition. *Let V be an m–dimensional regular affine variety over $\mathbb{C}$. Then B has a decomposition into i–handles, $i < m$.* □

Combining (1.13), (1.14), and (1.15) we obtain:

(0.16) Theorem. *Let V be a contractible regular affine variety over $\mathbb{C}$ of dimension $m \geq 3$. Then V is diffeomorphic to $\mathbb{C}^m$.*

PROOF. From (1.14) and (1.15) we obtain that ∂B is simply connected. Now apply (1.13). □

The h–cobordism theorem has another interesting consequence.

(0.17) Proposition. *Let B denote a compact contractible manifold of dimension $n \geq 4$. Then $B \times D^2$ is diffeomorphic to D^{n+2}.*

2 The embedding problem

Suppose the affine variety $V \subset \mathbb{C}^n$ is isomorphic to $\mathbb{C}^k$. Can V be transformed into a linear subspace by an algebraic automorphism of $\mathbb{C}^n$?

The answer is yes in case $(n, k) = (2, 1)$. This was proved by ABHYANKAR and MOH [1975] and SUZUKI [1974]; compare also LIN–ZAĬDENBERG [1983] and NEUMANN [1989a]. The answer is also yes for high codimension (??), see KALIMAN [19??]. Apparently no exotic ambedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ is known.

In the case of hypersurfaces the problem can be rephrased in the following way. Given a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$. Suppose $V(f)$ is isomorphic to $\mathbb{C}^{n-1}$. Does there exist a polynomial automorphism $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ such that $f = p_1 \circ \varphi$, with $p_1(z_1, \ldots, z_n) = z_1$?

Here is a view point: Suppose φ as above exists. Then there exists a large automorphism group of $\mathbb{C}^n$ under which the polynomial f is invariant. In order to find φ try to construct invariants of f. We demonstrate this view point at length by an example which has also been considered by DIMCA [1990].

Let $d \geq 1$ and $k \geq 1$ be natural numbers. Consider the polynomial

(0.18) $$p(x, y, z) = x + x^d y + y^k z.$$

For $c \in \mathbb{C}$ and $(x, y, z) \in \mathbb{C}^3$ we set

(0.19) $$c \cdot (x, y, z) = (x + cy^k, y, z - c - y^{-k}((x + cy^k)^d y - x^d y))$$

The right most term is a polynomial in (x, y, z). One verifies:

(0.20) Lemma. (1) *The map*

$$\mathbb{C} \times \mathbb{C}^3 \to \mathbb{C}^3,\ \ (c, (x, y, z)) \mapsto c \cdot (x, y, z)$$

is a free action of the additive group $\mathbb{C}$ *on* $\mathbb{C}^3$.

(2) *The polynomial* p *is invariant under this action:* $p(c\cdot(x, y, z)) = p(x, y, z)$.$\square$

By construction, the second component of $\mathbb{C}^3$ is also invariant under the action (2.3).

(0.21) Proposition. *The map*

$$\gamma : \mathbb{C}^3 \to \mathbb{C}^3,\ \ (x, y, z) \mapsto (p(x, y, z), y)$$

is the orbit map of the action (2.3).

Proof. Each orbit through a point with $y \neq 0$ contains a unique point with first coordinate zero. We can then compute z from the value $p(x, y, z)$. We have $p(x, y, z) = x$, hence $\gamma(x, 0, z) = (x, 0)$. In this case $c \cdot (x, 0, z) = (x, 0, z - c)$. Therefore $\{(x, 0, z) \mid z \in \mathbb{C}\}$ is an orbit. This shows that γ induces a bijection of the orbit space onto $\mathbb{C}^2$. The full statement will follow from the further investigations. $\square$

We want to exhibit γ as a trivial principal bundle. This requires to find a polynomial section. (We remark that locally trivial $\mathbb{C}$–bundles over affine varieties are trivial.) If we set

$$\textbf{(0.22)} \qquad \gamma(x, y, z) = (x + x^d y + y^k z, y) =: (a, b),$$

then we have

$$\textbf{(0.23)} \qquad z = \frac{a - x - x^d b}{b^k}.$$

Therefore we show:

(0.24) Proposition. *There exist polynomials* $x(a, b) \in \mathbb{Z}[a, b]$ *such that the polynomial* $a - x(a, b) - (x(a, b))^d b$ *is divisible by* b^k.

Proof. Induction over k. For $k = 1$ we can set $a - x = b$. For the induction step we show: Let $q \in \mathbb{Z}[u, a, b],\ r \in \mathbb{Z}[a]$. There exists $u = u(a, b) \in \mathbb{Z}[a, b]$ such that

$$\textbf{(0.25)} \qquad u - r(a) - bq(u, a, b)$$

is divisible by b^k if we substitute $u = u(a, b)$. For the proof of this statement we set $u = r(a) = vb$. We substitute into (2.8) and obtain

$$b^{-1}(vb - bq(vb + r(a), a - b)) = v - q(vb + r(a), a, b).$$

The right side is a polynomial of the form (2.8), if v is replaced with u. By induction, we can find $v(a, b) \in \mathbb{Z}[a, b]$ such that this polynomial is divisible by b^{k-1}. Then $u = r(a) + bv(a, b)$ has the desired property. $\square$

Using (2.7), we obtain a section s of γ

$$s : \mathbb{C}^2 \to \mathbb{C}^3, \quad (a,b) \mapsto (x, b, b^{-k}(a - x - x^d b))$$

with $x = x(a,b)$. This finally leads to a bijective morphism

$$\Phi : \mathbb{C} \times \mathbb{C}^2 \to \mathbb{C}^3, \ (c,(a,b)) \mapsto c \cdot s(a,b).$$

In order to exhibit Φ as an algebraic isomorphism we begin by showing that the action (2.3) is proper, in an algebraic sense. The relation of the action

$$R = \{(x,y,z), \, c \cdot (x,y,z)\} \subset \mathbb{C}^3 \times \mathbb{C}^3$$

is contained in the variety W of those $((x,y,z),(x_1,y_1,z_1)) \in \mathbb{C}^3 \times \mathbb{C}^3$ which satisfy

(0.26)
$$\begin{aligned} y &= y_1 \\ x_1 - x &= (z - z_1)y^k - y(x_1^d - x^d). \end{aligned}$$

(0.27) Lemma. *The map $\tau : R \to \mathbb{C}, \ (\alpha, c \cdot \alpha) \mapsto c$ is a morphism.*

Proof. If $y \neq 0$, then we can compute c

(0.28)
$$c = y^{-k}(x_1 - x).$$

If $1 + y(x_1^{d-1} + x_2^{d-2}x + \ldots + x^{d-1}) \neq 0$, then we can again compute c

(0.29)
$$c = \frac{z - z_1}{1 + y(x_1^{d-1} + \ldots + x^{d-1})}.$$

Because of (2.9), the right hand sides of (2.11) and (2.12) define the same regular function on the intersection of their domain of definition. Therefore τ is regular on Zariski open sets which cover the variety. $\square$

An inverse to Φ is now given by

$$\Psi : \mathbb{C}^3 \to \mathbb{C} \times \mathbb{C}^2, \ \alpha \mapsto (\tau(s\gamma(\alpha)), \alpha), s\gamma(\alpha)).$$

Finally, we remark that Φ is defined over $\mathbb{Z}$ and can therefore be used to study the diophantine properties of (2.1).

3 Hyperbolic modifications.

In this section we present an inductive construction of acyclic varieties. The construction is called hyperbolic modification. It produces manifolds (varieties) with an action of the multiplicative group $\mathbb{C}^*$.

We start whith an n–dimensional complex manifold L. Let $U \subset \mathbb{C}^n$ be a connected open neighbourhood of the origin and let $\varphi : U \to L$ be a holomorphic chart. Set $\varphi(0) = x_0$. We also use the open set

$$\tilde{U} = \{(u,x) \in \mathbb{C} \times \mathbb{C}^n \mid ux \in U\}.$$

It contains $\mathbb{C} \times 0 \cup 0 \times \mathbb{C}^n$. The holomorphic map

$$b : \mathbb{C}^* \times (U \setminus 0) \to \tilde{U},\ (u, x) \mapsto (u, u^{-1}x)$$

is a holomorph embedding onto an open subset of $\tilde{U}$.

We define the complex manifold V to be the pushout of the diagram

$$\mathbb{C}^* \times (L \setminus x_0) \xleftarrow{\ a\ } \mathbb{C}^* \times (U \setminus 0) \xrightarrow{\ b\ } \tilde{U}$$

with $a(u, x) = (u, \varphi(x))$. Since a and b are holomorphic embeddings onto open subspaces and the pushout is seen to be a Hausdorff space V is a complex manifold. It is called the *hyperbolic modification* of L at x_0. If we let $\mathbb{C}^*$ act on $\mathbb{C}^* \times (L \setminus x_0)$ and $\mathbb{C}^* \times (U \setminus 0)$ by scalar multiplication on the first component and on $\tilde{U}$ by $\lambda \cdot (u, x) = (\lambda u, \lambda^{-1}x)$, then V inherits a holomorphic $\mathbb{C}^*$–action.

There is a canonical holomorphic projection $t : V \to L$ which forgets the first component of $\mathbb{C}^* \times (L \setminus x_0)$ and is given by $(u, y) \mapsto \varphi(uy)$ on $\tilde{U}$. It is invariant under the $\mathbb{C}^*$–action. The fibres of t over $L \setminus x_0$ are closed free orbits. The fibre $t^{-1}(x_0)$ is isomorphic to $\mathbb{C} \times 0 \cup 0 \times \mathbb{C}^n$ in $\tilde{U}$. It contains the origin as a single closed orbit: A fixed point of hyperbolic type. All other orbits of this fibre have the origin in its closure. The manifold L is the orbit space of the subset of closed orbits. In the algebraic category this is called the *algebraic quotient* $V//\mathbb{C}^*$, compare KRAFT [1989], p. 96, and SPRINGER [1989], p. 14. We explain this with (3.9).

(0.30) Lemma. *The complex manifold V is independent of the choice of the holomorphic chart φ about x_0.*

PROOF. It is easy to see that shrinking U leads to the same manifold. Therefore it suffices to consider charts $\varphi, \psi : U \to L$ which differ by a holomorphic automorphism $\alpha : U \to U$. In this case there exists a commutative diagram

$$\begin{tikzcd}
\mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] \arrow[d, "\mathrm{id} \times \alpha"] & \tilde{U} \arrow[d, "A"] \\
\mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] & \tilde{U}
\end{tikzcd}$$

with a holomorphic automorphism A. This uses the fact that $(u, x) \mapsto u^{-1}\alpha(ux)$ has a holomorphic extension to $u = 0$ by the derivative of α. □

(0.31) Remark. The definition of V uses only that $x_0 \in L$ is a regular point. Apart from x_0 the object L could have singularities. Moreover the construction itself is meaningful for differentiable manifolds. But then there is no analogue of (3.1). ♡

(0.32) Definition. Let $h : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with the following property: $0 \in h^{-1}(0) =: L$ is a regular point of the affine hypersurface L. There exists a unique polynomial

$$q_h = q : \mathbb{C} \times \mathbb{C}^{n+1} \to \mathbb{C}$$

such that:

(0.33) $uq(u,x) = h(ux), \quad u \in \mathbb{C}^*, x \in \mathbb{C}^n.$

(0.34) $q(0,x)$ is a nonzero linear polynomial.

(0.35) $q(\lambda^{-1}u, \lambda x) = \lambda q(u,x), \quad \lambda \in \mathbb{C}^*.$

We call q_h the *hyperbolic modification* of h. ♡

(0.36) Proposition. *The affine variety* $q^{-1}(0) =: V$ *together with the* $\mathbb{C}^*$*– action induced by* (3.6) *is the hyperbolic modification of* $(L,0)$.

PROOF. We have to show that V, as a complex space, arises by the pushout construction described above.

We select a holomorphic chart $\varphi : U \to L$ centered at 0 and consider the following diagram

$$\begin{tikzcd} \mathbb{C}^* \times (U \setminus 0) \arrow[r, "b"] \arrow[d, "a"] & \tilde{U} \arrow[d, "B"] \\ \mathbb{C}^* \times (L \setminus 0) \arrow[r, "A"] & V \end{tikzcd}$$

with maps

$$\begin{aligned} a(u,x) &= (u, \varphi(x)) \\ b(u,x) &= (u, u^{-1}x) \\ A(u,x) &= (u, u^{-1}x) \\ B(u,x) &= (u, u^{-1}\varphi(ux)). \end{aligned}$$

The map B has to be interpreted as a holomorphic map — compare the proof of (3.1). By construction, the diagram is commutative.

We have to show that the diagram is a pushout. This amounts to the following verifications: The maps A and B are holomorphic embeddings onto open subsets $\tilde{A}$ and $\tilde{B}$ of V. The intersection $\tilde{A} \cap \tilde{B}$ is as predicted by the diagram.

The image of A is $V \cap \{(u,x) \mid u \neq 0\}$, hence open in V. An inverse of A is induced by the morphism $A_1 : (u,x) \mapsto (u, ux)$.

Let $\Phi : W_1 \to W_2$ be a holomorphic isomorphism between open neighbourhoods of zero in $\mathbb{C}^{n+1}$ such that Φ restricts to $\varphi : 0 \times U = W_1 \cap (0 \times \mathbb{C}^n) \to L \cap W_2$. Let $Z_2 = \{(u,x) \mid ux \in W_2\} \subset \mathbb{C} \times \mathbb{C}^n$.

One verifies that $V \cap Z_2$ is the image of B. An inverse of B is induced by the morphism $(u,x) \mapsto (u, \mathrm{pr}(u^{-1}\Phi^{-1}(ux)))$ with pr: $\mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ the projection.□

(0.37) Proposition. (1) *Let* V *be a hyperbolic modification of the* n*–dimensional manifold* L*. Then the following holds: If* L *is acyclic, then* V *is acyclic.*
(2) *If* $n > 2$ *and* $\pi_1(L) = 0$*, then* $\pi_1(V) = 0$.
(3) *If* $n > 2$ *and* L *is contractible, then* V *is contractible.*

PROOF. (1) follows by applying the Mayer–Vietoris sequence to the defining pushout. (2) follows similarly from the theorem of Seifert and van Kampen. (3)

follows from (1) and (2) and general results of algebraic topology [tD], III (5.11) and V (6.3)). □

The hyperbolic modification can be applied to a regular point of an arbitrary affine variety. The result is again an affine variety with $\mathbb{C}^*$–action.

Suppose the affine variety $L = L(I)$ is the zero set of the ideal $I \subset \mathbb{C}[x_1, \ldots, x_n]$. Let $0 \in \mathbb{C}^n$ be a regular point of L. Let $J \subset \mathbb{C}[u, x_1, \ldots, x_n]$ be the ideal generated by all polynomials q_h such that $q_h(u,x) = uh(u,x)$ for $h \in I$. Then the variety $V = V(J)$ is the hyperbolic modification of L at zero. The proof is similar to the proof of (3.7). It uses the following fact. Suppose $L \subset \mathbb{C}^n$ has codimension k. Since $0 \in L$ is a regular point there exist k polynomials $h_1, \ldots, h_k \in I$ such that the Jacobi matrix of $(h_1, \ldots, h_k)$ has rank k at the origin. Let

$$\pi : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n, \quad (u,x) \mapsto ux.$$

Then $\pi^{-1}(L \setminus 0) \subset V$ and $\pi^{-1}(L \setminus 0)$ is isomorphic to $\mathbb{C}^* \times (L \setminus 0)$ as a $\mathbb{C}^*$–variety. Let W be an open neighbourhood of zero such that

$$L \cap W = \{x \in W \mid h_1(x) = \ldots = h_k(x) = 1\}.$$

Finally, let $\varphi : U \to L \cap W$ be a holomorphic chart of L about 0. Then the complex space V is isomorphic to the pushout of

$$\mathbb{C}^* \times (L \setminus 0) \xleftarrow{\mathrm{id} \times \varphi} \mathbb{C}^* \times (U \setminus 0) \xrightarrow{\quad b \quad} \pi^{-1}(U)$$

with $b(u,x) = (u, u^{-1}x)$.

There are several natural generalizations of the hyperbolic modifications which yield acyclic manifolds when applied to acyclic manifolds.

Firstly, one can introduce hyperbolic fixed points with different weights. This replaces the map $b : (u,x) \mapsto (u, u^{-1}x)$ by $b_{k,l} : (u,x) \mapsto (u^k, u^{-l}x)$ for a pair (k,l) of coprime natural numbers. The proof of (3.8) also applies to this case. Later we will deal with this more general construction.

Secondly, one can contemplate using other groups than $\mathbb{C}^*$. Consider $GL(n, \mathbb{C})$ as open subset of the vector space $M_n(\mathbb{C})$ of complex (n,n)–matrices. Let $\pi : M_n(\mathbb{C}) \times \mathbb{C}^n \to \mathbb{C},\ (A,x) \mapsto Ax$ and $\tilde{U} = \pi^{-1}(U)$. Define $b : (A,x) \mapsto (A, A^{-1}x)$ and consider the pushout of

$$\mathrm{GL}(n,\mathbb{C}) \times (L \setminus 0) \xleftarrow{\mathrm{id} \times \varphi} \mathrm{GL}(n,\mathbb{C}) \times (U \setminus 0) \xrightarrow{\quad b \quad} \tilde{U}.$$

The maps $\mathrm{id} \times \varphi$ and b are holomorphic embeddings onto open subsets. Moreover $(\mathrm{id} \times \varphi, b)$ is a closed embedding. Therefore the pushout is a Hausdorff complex manifold with holomorphic $\mathrm{GL}(n,\mathbb{C})$–action.

We return to the hyperbolic modification V of an affine hypersurface L defined by h. The coordinate ring of V is $\mathbb{C}[u,x]/(q)$. It carries the $\mathbb{C}^*$–action $\lambda \cdot (u,x) = (\lambda u, \lambda^{-1}x)$. Let $R \subset \mathbb{C}[u,x]/(q)$ denote the ring of invariants. The homomorphism

$$j : \mathbb{C}[x] \to \mathbb{C}[u,x],\ x_i \mapsto ux_i$$

induces a homomorphism

$$J : \mathbb{C}[x]/(h) \to R.$$

(0.38) Proposition. *The homomorphism J is an isomorphism.*

PROOF. Suppose p is in the kernel of J. Then there exists a relation of the type $p(ux) = a(u,x)q(u,x)$. We multiply by u and set $u = 1$. This yields $p(x) = \alpha(x)h(x)$. This shows J to be injective.

Let S be the ring of $\mathbb{C}^*$–invariant in $\mathbb{C}[u,x]$. Then it is easy to see that j induces an isomorphism $j : \mathbb{C}[x] \to S$. Since $\mathbb{C}^*$ is a reductive group, the surjection $C[u,x] \to \mathbb{C}[u,x]/(q)$ induces a surjection $S \to R$ (compare SPRINGER [1989], II). □

One expresses (3.9) by saying that L is the algebraic quotient of V under the $\mathbb{C}^*$–action. The map $\pi : V \to V,\ (u,x) \mapsto ux$ is the quotient map.

If L is a regular hypersurface, then V is again regular. Therefore we can iterate the hyperbolic modification. Since V carries a $\mathbb{C}^*$–action with fixed point 0 the hyperbolic modification of V at 0 carries a $\mathbb{C}^* \times \mathbb{C}^*$–action. The n–fold modification $L(n)$ of L carries an action of the n–dimensional torus $T(n) = \mathbb{C}^* \times \ldots \times \mathbb{C}^*$.

(0.39) Proposition. *The algebraic quotient $L(n)//T(n)$ is isomorphic to L.*□

The general pattern of the hyperbolic modification is: Let $\pi : X \to Y$ be a morphism and $L \subset Y$ a subvariety. Exclude some singular set $S \subset Y$ of π, take $\pi^{-1}(L \setminus S)$ and form the closure of this pre–image.

4 Acyclic affine foliations

In this section we construct polynomials $q : \mathbb{C}^{n+1} \to \mathbb{C}$ with the property that all fibres $q^{-1}(c)$ are diffeomorphic to Euclidean space. The construction is based on the hyperbolic modification (section 3).

Let again $h : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $h(0) = 0$ such that $0 \in h^{-1}(0) = L$ is a regular point of L. Let $q = q_h : \mathbb{C}^{n+1} \to \mathbb{C}$ denote the hyperbolic modification of h.

We blow up the point $0 \in L$ and consider the complement of the proper transform. This construction will also clarify the conceptual meaning of q_h.

Denote by $X = \mathbb{C} \times_{\mathbb{C}^*} (\mathbb{C}^n \setminus 0)$ the quotient of $\mathbb{C} \times (\mathbb{C}^n \setminus 0)$ under the relation $(u,x) \sim (\lambda^{-1}u, \lambda x)$ for $\lambda \in \mathbb{C}^*$. The morphism $p : X \to \mathbb{C}^n,\ (u,x) \mapsto ux$ blows up the point $0 \in \mathbb{C}^n$ and

$$p : X \setminus E \to \mathbb{C}^n \setminus 0, \quad E = p^{-1}(0)$$

is an isomorphism. Moreover we have:

(0.40) $p : X \setminus p^{-1}(L) \to \mathbb{C}^n \setminus L$ is an isomorphism.

(0.41) $p^{-1}(L) = L_1 \cup E$ with $L_1 = \{(u,x) \mid q(u,x) = 0\}$.

We let $V = q^{-1}(1)$.

(0.42) Lemma. *The restriction $q : q^{-1}(\mathbb{C}^*) \to \mathbb{C}^*$ is a trivial fibration with typical fibre V.*

PROOF. The map

$$\varphi : \mathbb{C}^* \times V \to q^{-1}(C^*),\ (\lambda; u, x) \mapsto (\lambda^{-1}u, \lambda x)$$

satisfies $q \circ \varphi = \mathrm{pr}$. An inverse ψ of φ is given by

$$\psi(u,x) = (q(u,x); q(u,x)u, q(u,x)^{-1}x).$$

Therefore φ is an algebraic bundle isomorphism. $\square$

(0.43) Lemma. *Suppose $h^{-1}(0) \subset \mathbb{C}^n$ is a regular variety. Then q is a regular polynomial.*

PROOF. (4.3) shows that q is regular in all point of $q^{-1}(\mathbb{C}^*)$. If $q(u,x) = 0$, then $h(ux) = 0$. We have

$$\textbf{(0.44)} \qquad \frac{\partial q}{\partial x_j}(u,x) = \frac{\partial h}{\partial x_j}(ux)$$

and since $h^{-1}(0)$ is regular there exists j such that (4.5) is non zero. $\square$

(0.45) Proposition. *The complement $X \setminus L_1$ is isomorphic to the regular affine hypersurface V.*

PROOF. We have seen in (4.3) that V is regular. Moreover $V \subset \mathbb{C} \times (\mathbb{C}^n \setminus 0)$, as seen from (3.3) and (3.4). The morphism $V \subset \mathbb{C} \times (\mathbb{C}^n \setminus 0) \to X$ induces the desired isomorphism. The pre-image of $X \setminus L_1$ in $\mathbb{C} \times (\mathbb{C}^n \setminus 0)$ is the open subvariety $W = \{(u,x) \mid q(u,x) \neq 0\}$. The morphism $W \to V$, $(u,x) \mapsto (q(u,x)u,\ q(u,x)^{-1}x)$ factors over $X \setminus L_1$ and yields an inverse. $\square$

Because of (4.3), we call $q^{-1}(c)$, $c \neq 0$, the general fibre of q and $q^{-1}(0)$ the singular fibre. The singular fibre was investigated in section 3 and studied under the name of hyperbolic modification. We now deal with the topology of the general fibre V.

(0.46) Proposition. *The differentiable manifold V is obtained from $\mathbb{C}^n \setminus L$ by attaching an open 2–handle. The homotopy type Y of V is obtained by attaching a 2–cell D^2 to $\mathbb{C}^n \setminus L$ along a small normal 1–sphere about L.*

PROOF. Set $E_0 = E \setminus (E \cap L_1)$. Set theoretically we have $X \setminus L_1 = (X \setminus p^{-1}(0)) \cup E_0$ and $X \setminus p^{-1}(0) \cong \mathbb{C}^n \setminus L =: W$. Therefore we have to describe the way E_0 is attached to W. This requires a tubular neighbourhood of E_0.

The normal bundle of the exceptional divisor $E \cong \mathbb{P}^{n-1}$ in X is the line bundle

$$\pi : X \to \mathbb{P}^{n-1}, \quad (u,x) \mapsto [x].$$

We have $L_1 = \{(u,x) \mid q(u,x) = 0\}$ and $[x] \in E_0$ is equivalent to $q(0,x) \neq 0$. If we fix x, then there exists a neighbourhood U_x of zero, such that $q(u,x) \neq 0$ for $u \in U_x$, i. e. $(u,x) \notin L_1$ for $u \in U_x$. In other words: Let $\pi_0 : X_0 = \pi^{-1}(E_0) \to E_0$ denote the restriction of π; then there exists an open cell subbundle $U \subset X_0 \to E_0$ such that $(u,x) \in U \setminus E_0$ implies $q(u,x) \neq 0$.

The complex manifold $X \setminus L_1$ can therefore be defined by the pushout diagram

$$\begin{tikzcd} U \setminus E_0 \arrow[r, "c"] \arrow[d, "j"] & U \arrow[d] \\ \mathbb{C}^n \setminus L \arrow[r] & X \setminus L_1 \end{tikzcd}$$

where j is the embedding $(u,x) \mapsto ux$.

Let U_z denote the fibre of $\pi_0 : U \to E_0$ over $z \in E_0$. Then $j : U_z \to \mathbb{C}^n$, $(u,x) \mapsto ux$ is transverse to L in 0. Let $D_z \subset U_z$ be a closed cell with boundary S_z. The circle S_z is the normal sphere about L which appears in the statement of (4.7).

The pushout diagram

$$\begin{tikzcd} S_z \arrow[r] \arrow[d, "j"] & D_z \arrow[d] \\ \mathbb{C}^n \setminus L \arrow[r] & Y \end{tikzcd}$$

defines the attachment of the 2–cell to $\mathbb{C}^n \setminus L$. The space Y is homotopy equivalent to $X \setminus L_1$. In order to see, this consider the commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \setminus L & U \setminus E_0 \arrow[l, "j"'] \arrow[r] & U \\ \mathbb{C}^n \setminus L \arrow[u, "="] & S_z \arrow[l] \arrow[r] \arrow[u, "\alpha"] & D_z. \arrow[u, "\beta"] \end{tikzcd}$$

The inclusions α and β are homotopy equivalences. The set E_0 is the complement of the projectivized tangent space $E_0 = \mathbb{P}(\mathbb{C}^n) \setminus \mathbb{P}(T_0 L)$, hence an affine space. Therefore the cell bundle U is contractible. The inclusion α is a morphism of fibrations which is a homotopy equivalence in the base and in the fibre and therefore a homotopy equivalence. Now apply a general result of homotopy theory (tom Dieck [1971]). □

Proposition (4.7) has some consequences for the homotopy and homology of V.

(0.47) Corollary. *Let $N \subset \pi_1(\mathbb{C}^n \setminus L)$ be the normal subgroup generated by a normal sphere of L. Then $\pi_1(V) \cong \pi_1(\mathbb{C}^n \setminus L)/N$.*

Proof. The theorem of Seifert and van Kampen implies that attaching a 2–cell factors out exactly the subgroup N. □

(0.48) Corollary. *$H_i(V) \cong H_i(\mathbb{C}^n \setminus L)$ for $i \neq 1$.*

Proof. Let $W = \mathbb{C}^n \setminus L$. Since $H_i(Y, W) = 0$ for $i \neq 2$, the exact homology sequence of (Y, W) shows that homology groups of Y and W can only differ for $i = 1, 2$. In this case we have the exact sequence

$$0 \to H_2(W) \to H_2(Y) \to H_2(Y, W) \xrightarrow{\partial} H_1(W) \to H_1(Y) \to 0$$

with $H_2(Y, W) \cong \mathbb{Z}$. The map ∂ is injective. In order to see this, consider

$$s : H_1(\mathbb{C}^n \setminus L) \cong H_2(\mathbb{C}^n, \mathbb{C}^n \setminus L) \xrightarrow{\tau} \mathbb{Z},$$

where τ gives the intersection number with L (see tom Dieck [1991], V.5). Then $s \circ \partial$ is an isomorphism, since $H_2(Y, W)$ is generated by a normal disk which has intersection number one with L. □

(0.49) Proposition. *Suppose L has only isolated singularities and is a topological manifold. Then $H_i(\mathbb{C}^n \setminus L) \cong H_{i-i}(L)$ for $i > 0$.*

Proof. If L has a tubular neighbourhood in $\mathbb{C}^n$ we can apply the exact homology sequence and the Thom–Isomorphism (Spanier [1966]), $H_i(\mathbb{C}^n \setminus L) \cong H_{i+1}(\mathbb{C}^n, \mathbb{C}^n \setminus L) \cong H_{i-1}(L)$ and deduce the claim.

Another argument uses the fact that the sphere $S^{2n-1}(r) = \{z \in \mathbb{C}^n \mid |z| = r\}$ intersects L transversely for all sufficiently large r. For such r, the space $\mathbb{C}^n \setminus L$ is homotopy equivalent to the intersection with the disk $D^{2n}(r) \setminus D^{2n}(r) \cap L$. We therefore study the following situation: D is an m–disk, $L \subset D$ a topological submanifold with $\partial D \cap L = \partial L$ and transverse intersection of L and ∂D. Let $S = D \cup D'$ be the double of $D = D'$; this is an m–sphere. Now we have the following chain of isomorphisms for $i > 0$:

$$\begin{aligned}
H_i(D \setminus L) &\underset{(1)}{\cong} H_{i+1}(D, D \setminus L) \\
&\underset{(2)}{\cong} H^{m-i-1}(D' \cup L,\ D') \\
&\underset{(3)}{\cong} H^{m-i-1}(\partial D \cup L,\ \partial D) \\
&\underset{(4)}{\cong} H^{m-i-1}(L, \partial L) \\
&\underset{(5)}{\cong} H_{i-1}(L).
\end{aligned}$$

Explanation:

(1) comes from the exact homology sequence.

(2) is Poincaré–Lefschetz duality in S, see Dold [1972], VIII.7.2.

(3) and (4) is excision.

(5) is duality in L. $\square$

(0.50) Theorem. (1) *Suppose L is an acyclic topological manifold with isolated singularities. Then V is acyclic.*

(2) *If, moreover, $\pi_1(\mathbb{C}^n \setminus L)$ is normally generated by a normal sphere (e. g. $\pi_1(\mathbb{C}^n \setminus L) \cong \mathbb{Z}$), then V is contractible.*

PROOF. From (4.10) we see that $H_i(\mathbb{C}^n \setminus L)$ is zero except for $i = 1$. For $i = 1$ the argument with intersection numbers in the proof of (4.9) shows that $H_1(\mathbb{C}^n \setminus L) \cong \mathbb{Z}$, generated by a normal sphere. The result now follows from (4.7), (4.9), and (4.10). $\square$

Altogether we can now deduce the next result.

(0.51) Theorem. *Suppose $n \geq 3$. Let $h : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with $h(0) = 0$ such that $h^{-1}(0)$ is a regular contractible hypersurface. Then the hyperbolic modification $q : \mathbb{C}^{n+1} \to \mathbb{C}$ is a regular polynomial such that each fibre $q^{-1}(c)$, $c \in \mathbb{C}$ is diffeomorphic to Euclidean space.*

For the proof we need another Lemma.

(0.52) Lemma. *Suppose L is regular. Then:*

(1) *If $\pi_1(L) = 0$, then $\pi_1(V) = 0$.*

(2) *If L is contractible, then V is contractible.*

PROOF. (1) Let U be an open tubular neighbourhood of L in $\mathbb{C}^n$. By the theorem of Seifert and van Kampen, the diagram

$$\begin{tikzcd} \pi_1(U \setminus L) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \pi_1(\mathbb{C}^n \setminus L) \arrow[r] & \pi_1(\mathbb{C}^n) \end{tikzcd}$$

is a pushout diagram. We have homotopy equivalences $L \simeq U$ and $U \setminus L \simeq L \times S^1$. Therefore $\pi_1(U) = 0$, $\pi_1(U \setminus L) \cong \pi_i(L \times S^1) \cong \pi_1(S^1)$, and the pushout shows that $\pi_1(\mathbb{C}^n \setminus L)$ is normally generated by a normal sphere of L. Now we apply (4.8).

(2) follows from the proof of (1) and (4.11). $\square$

Proof of (4.12). By (4.4), q is regular. By (4.13), the general fibre of q is contractible. By (3.8), the singular fibre is contractible. Now apply (1.??). $\square$

We call regular polynomials $p : \mathbb{C}^n \to \mathbb{C}$ such that all fibres are diffeomorphic to Euclidean space *slice polynomials*. Linear forms are trivial slice polynomials. If remains to find nontrivial examples to which (4.12) applies. This will be the subject of the next section.

5 Brieskorn varieties

The Brieskorn polynomials are among the simplest polynomials to which the considerations of the previous section can be applied.

Let $a(1), \dots, a(n)$ be positive integers. The associated Brieskorn polynomial is

(0.53) $$h(x_1, \dots, x_n) = x_1^{a(1)} + \dots + x_n^{a(n)}.$$

These polynomials have an isolated singularity at the origin. The Brieskorn manifold is the intersection

(0.54) $$B = B(a(1), \dots, a(n)) = h^{-1}(0) \cap S^{2n-1}.$$

Brieskorn [1966] investigated under which condition this manifold is a topological sphere. The result is as follows (5.3).

Define the graph Γ of the family $(a(j))$: The vertices are $\{1, \dots, n\}$. There is an edge connecting i and j if and only if the greatest common divisor $(a(i), a(j)) > 1$.

(0.55) Theorem. *$B(a(1), \dots, a(n))$ is a homology sphere if and only if one of the following conditions holds:*

1. *Γ has at least two isolated points.*

2. *Γ has an isolated point and another connected component Γ' with an odd number of vertices such that for different $i, j \in \Gamma'$ always $(a_i, a_j) = 2$.* □

The Brieskorn varieties are simply connected for $n \geq 4$. A simple consequence of the h–cobordism theorem asserts that they are homeomorphic to the sphere S^{2n-3}, provided they are homology spheres, Smale [1956], Milnor [1968], p. 109. For further information see also Hirzebruch–Mayer [1968], Bredon [1972], and Milnor [1968].

(0.56) Theorem. *Let $q : \mathbb{C}^{n+1} \to \mathbb{C}$ be a hyperbolic modification of h, applied to a regular point of $h^{-1}(0)$. Suppose the manifold B in (5.2) is homeomorphic to a sphere. Then $V = q^{-1}(1)$ is a contractible affine variety $(n \geq 2)$.*

Proof. By homogeneity of h, the space $L = h^{-1}(0)$ is homeomorphic to the open cone over B. Therefore, L is homeomorphic to Euclidean space if B is a sphere. By (4.11.1), V is acyclic.

In order to show that V is contractible we derive another topological construction of V. The space $\mathbb{C}^n \setminus L$ is homeomorphic to the product of $S^{2n-1} \setminus B$ with an open interval J. The inclusion $B \subset S^{2n-1}$ can be considered as a (generalized) knot. In order to obtain the homotopy type of V we have to attach a 2–cell along a normal sphere (4.7). Let $U \subset S^{2n-1}$ be an open tubular neighbourhood of B. Up to homotopy, attaching a 2–cell amounts to adding a fibre over $x \in B$ of the tubular neighbourhood to $S^{2n-1} \setminus U$. The result is the sphere S^{2n-1} with the tubular neighbourhood W of $B \setminus x$ deleted. Since B is assumed to be a sphere,

W is an open cell and its complement is a disk. The resulting homotopy type is therefore contractible. $\square$

Instead of Brieskorn polynomials one can use other weighted homogeneous polynomials with appropriate topological properties.

The simplest case of (5.4) arises for the polynomial $h(x,y) = x^a - y^b$ for coprime integers a, b. If we apply the hyperbolic modification to the regular point (1,1) we obtain the polynomial (1.5)

$$P(x,y,z) = z^{-1}\left((xz+1)^a - (yz+1)^b\right).$$

The case $(a,b) = (3,2)$ leads to the contractible hypersurface in $\mathbb{C}^3$

$$z^2x^3 + 3zx^2 + 3x - zy^2 - 2y = 1.$$

We can use this polynomial as input for (4.12). If we apply the hyperbolic modification at the point (1,1,0) we obtain the slice polynomial

$$q(u,x,y,z) = uz^2(ux+1)^3 + 3z(ux+1)^2 + 3x - (uy+1)^2 - 2y.$$

Other simple Brieskorn polynomials to which (5.4) can be applied are

$$x^p + y^q + z_2^2 + \ldots + z_n^2$$

for coprime odd integers p, q and

$$z_0^d + z_1^2 + \ldots + z_n^2$$

for d and n odd. These polynomials are interesting because they have large symmetry groups.

We now verify that some of the contractible varieties carry an exotic algebraic structure, i. e. are not isomorphic as varieties to affine space.

6 Ramified coverings

Ramified coverings can be used to construct acyclic varieties.

We assume given an action

$$\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n, \quad (\lambda, x) \mapsto \lambda \cdot x$$

of $\mathbb{C}^*$ on $\mathbb{C}^n$. For the time being, the action is just assumed to be continuous, if we are just dealing with topological properties. But in an algebraic context it seems more reasonable to assume that left translations $x \mapsto \lambda \cdot x$ are polynomial automorphisms.

A polynomial $q : \mathbb{C}^n \to \mathbb{C}$ (or any function) is called *quasi– invariant* of *weight* $l \in \mathbb{Z}$ with respect to the given action if

$$q(\lambda \cdot x) = \lambda^l q(x), \quad x \in \mathbb{C}^n, \ \lambda \in \mathbb{C}^*$$

holds.

We assume given a quasi–invariant polynomial q of weight l. We denote the fibre $q^{-1}(c)$ by F_c. The k–fold *ramified covering* of $\mathbb{C}^n$ along $q^{-1}(0)$ is the affine variety

$$X = \{(u,x) \in \mathbb{C} \times \mathbb{C}^n \mid u^k + q(x) = 0\}, \quad k \geq 1.$$

We shall exhibit conditions on q under which X will be diffeomorphic to Euclidean space.

We begin with some elementary remarks about ramified coverings.

(0.57) Lemma. *If F_0 is regular, then X is regular.*

Proof. Compute the partial derivatives of $p(u,x) = u^k + q(x)$. □

(0.58) Lemma. *The map $\sigma : X \to \mathbb{C}^n$, $(u,x) \mapsto x$ is a k–fold ramified covering with ramification locus F_0.*

Proof. Given $x \in \mathbb{C}^n$, there exists u such that $u^k + q(x) = 0$. Therefore σ is surjective. Consider the $\mathbb{Z}/k$–action $\mu \cdot (u,x) = (\mu u, x)$ on X. Then σ induces a homeomorphism of the orbit space $X/(\mathbb{Z}/k)$ with $\mathbb{C}^n$. □

(0.59) Lemma. *$F_0 = \{(u,x) \mid u = 0,\ q(x) = 0\}$ is a regular submanifold of X. The intersection of X with $\{u = 0\}$ in $\mathbb{C}^{n+1}$ is transverse.* □

(0.60) Proposition. *The map $q : \mathbb{C}^n \setminus F_0 \to \mathbb{C}^*$ is locally trivial with typical fibre F_1 and structure group $\mathbb{Z}/l$.*

Proof. Let $F_c = q^{-1}(c)$. The quasi–invariance of q implies that $(\omega, x) \mapsto \omega \cdot x$ is a $\mathbb{Z}/l$–action on F_c. We use this action to form $\mathbb{C}^* \times_G F_1$, $(\lambda, x) \sim (\lambda\omega, \omega^{-1} \cdot x)$, $\omega \in G = \mathbb{Z}/l$. The map $\varphi : \mathbb{C}^* \times_G F_1 \to \mathbb{C}^n \setminus F_0$, $(\lambda, x) \mapsto \lambda \cdot x$ is well–defined and satisfies $q \circ \varphi = \mathrm{pr}_1$. Let $Y = \{(\lambda, x) \mid \lambda^l = q(x)\} \subset \mathbb{C}^* \times (\mathbb{C}^n \setminus F_0)$. Then $\mathrm{pr}_2 : Y \to \mathbb{C}^n \setminus F_0$ is a quotient map. The morphism $Y \to \mathbb{C}^* \times_G F_1$, $(\lambda, x) \mapsto (\lambda, \lambda^{-1} \cdot x)$ induces an inverse to φ. □

(0.61) Proposition. *Suppose $\pi_1(\mathbb{C}^n \setminus F_0) \cong \mathbb{Z}$ and F_1 is connected. Then $\pi_1(F_c) = 0$.*

Proof. Consider the exact homotopy sequence of the fibration $q : \mathbb{C}^n \setminus F_0 \to \mathbb{C}^*$

$$0 \to \pi_1(F_0) \to \pi_1(\mathbb{C}^n \setminus F_0) \xrightarrow{q_*} \pi_1(\mathbb{C}^*) \to 0.$$

The hypothesis implies that q_* is an isomorphism. □

We study the topology of X via $\gamma : X \to \mathbb{C}, \quad (u,x) \mapsto u$. Let $\Gamma_c = \gamma^{-1}(c)$.

(0.62) Proposition. *The morphism $\gamma : X \setminus \Gamma_0 \to \mathbb{C}^*$ is locally trivial with typical fibre F_1 and structure group $\mathbb{Z}/l$.*

Proof. The fibre $\Gamma_c = \{(u,x) \mid q(x) = -u^k, u = c\}$ is isomorphic to F_{c^k}. We have the $\mathbb{Z}/l$–action $\omega \cdot (1,x) = (1, \omega \cdot x)$ on Γ_1. The map $\varphi' : \mathbb{C}^* \times \Gamma_1 \to X \setminus \Gamma_0$, $(\lambda, (1,x)) \mapsto (\lambda^l, \lambda^k \cdot x)$ is well–defined. Let $jk \equiv 1 \bmod l$. Then $(\lambda, (1,x))$ and

$(\omega^j\lambda, (1, \omega^{-1}\cdot x))$ have the same image under φ' for $\omega \in \mathbb{Z}/l$. We therefore use the $\mathbb{Z}/l$–action $(\omega, \lambda) \mapsto \omega^j\lambda$ on $\mathbb{C}^*$ in order to form $\mathbb{C}^* \times_G \Gamma_1$ and obtain from φ' a morphism $\varphi : \mathbb{C}^* \times_G \Gamma_1 \to X \setminus \Gamma_0$ which satisfies $\gamma \circ \varphi = 0$ with $p : \mathbb{C}^* \times_G \Gamma_1 \to \mathbb{C}^*$, $(\lambda, x) \mapsto \lambda^l$. We obtain an inverse to φ as follows. Write $X \setminus \Gamma_0$ as quotient of

$$Z = \{(v, u, x) \mid v^l = u,\ (u, x) \in X \setminus \Gamma_0\}$$

and define

$$Z \to \mathbb{C}^* \times_G \Gamma_1, \quad (v, u, x) \mapsto (v, (1, v^{-k} \cdot x)).$$

If we replace v by $\omega^j v$, $\omega \in G$, the resulting element has the same image under this map. Thus we obtain an induced map $\psi : X \setminus \Gamma_0 \to \mathbb{C}^* \times_G \Gamma_1$ which is the inverse of φ. □

(0.63) Proposition. *Under the hypothesis of* (6.5) *we have* $\pi_1(X) = 0$.

PROOF. The exact homotopy sequence of γ gives

$$0 \to \pi_1(\Gamma_1) \to \pi_1(X \setminus \Gamma_0) \xrightarrow{\gamma_*} \pi_1(\mathbb{C}^*) \to 0,$$

because $\Gamma_1 \cong F_1$ is connected. By (4.5) $\pi_1(\Gamma_1) = 0$, hence γ_* is an isomorphism. By (6.3) and the transversality theorem (?.?) the inclusion $X \setminus \Gamma_0 \to X$ is generated by a normal sphere about Γ_0. In order to see, this consider the covering $\sigma : X \setminus \Gamma_0 \to \mathbb{C}^n \setminus F_0$. A normal sphere of Γ_0 is mapped to k times a normal sphere of F_0. The diagram

$$\begin{array}{ccc} X \setminus \Gamma_0 & \xrightarrow{\sigma} & \mathbb{C}^n \setminus F_0 \\ \downarrow \gamma & & \downarrow q \\ \mathbb{C}^* & \xrightarrow{\tau} & \mathbb{C}^* \end{array}$$

with $\tau(z) = -z^k$ is commutative. Let U be a tubular neighbourhood of F_0 in $\mathbb{C}^n$. The theorem of Seifert and van Kampen, applied to $U, D^n \setminus F_0$, shows that $\pi_1(\mathbb{C}^n \setminus F_0)$ is generated by a normal sphere of F_0.

A normal sphere of γ_0 is contractible in X. □

The following Theorem is the main result of this section.

(0.64) Theorem. *We assume the following:*

(1) $q : \mathbb{C}^n \to \mathbb{C}$ *is quasi–invariant of weight* l.

(2) $F_0 = q^{-1}(0)$ *is regular and* $\mathbb{Z}/k$*–acyclic.*

(3) $\pi_1(\mathbb{C}^n \setminus F_0) \cong \mathbb{Z}$,

(4) $F_c = q^{-1}(c)$ *for* $c \neq 0$ *is connected. Then* $X = \{(x, u) \in \mathbb{C}^n \times \mathbb{C} \mid u^k + q(x) = 0\}$ *is contractible.*

In the sequel we work under the hypotheses of (6.8). The proof of (6.8) uses (6.9) – (6.11).

(0.65) Lemma. *The groups $H_i(X;\mathbb{Z})$, $i>0$, are annihilated by k.*

PROOF. We have the orbit map $\sigma : X \to \mathbb{C}^n$ of the $\mathbb{Z}/k$–action $\mu \cdot (u,x) = (\mu u, x)$, see (6.2). This action can be exended to an S^1–action, up to an automorphism of $\mathbb{Z}/k$. The S^1–action $\lambda(u,x) := (\lambda^l u, \lambda^k \cdot x)$ restricts a $\mathbb{Z}/k$–action, which coincides with the original action up to $\lambda \mapsto \lambda^l$. This uses that k and l are coprime. As a consequence of this S^1–action we see that $\mathbb{Z}/k = K$ acts trivially on $H_i(X;\mathbb{Z})$. We use the transfer $t : H_i(X/K) \to H_i(X)$. In our case $t \circ \sigma_*$ and $\sigma_* \circ t$ are multiplication by k, see ??. Since $X/K \cong \mathbb{C}^n$ is acyclic we obtain the desired result. $\square$

(0.66) Lemma. *The inclusion $X \setminus \Gamma_0 \to X$ induces an isomorphism*

$$H_i(X \setminus \Gamma_0; \mathbb{Z}/k) \to H_i(X;\mathbb{Z}/k)$$

for $i \geq 2$.

PROOF. We use the exact sequence, the Thom isomorphism and the acyclicity of $\Gamma_0 \cong F_0$. The diagram (coefficients in $\mathbb{Z}/k$)

$$\begin{tikzcd} H_{i+1}(X, X\setminus\Gamma_0) \arrow[r] & H_i(X\setminus\Gamma_0) \arrow[r] & H_i(X) \arrow[r] & H_i(X, X\setminus\Gamma_0) \\ H_{i-1}(\Gamma_0) \arrow[u, "\cong"] & & & H_{i-2}(\Gamma_0) \arrow[u, "\cong"] \end{tikzcd}$$

gives the claim for $i \geq 3$. For $i=2$ we use in addition the isomorphism

$$\partial : \mathbb{Z} \cong H_2(X, X\setminus\Gamma_0;\mathbb{Z}) \to H_1(X\setminus\Gamma_0;\mathbb{Z}) \cong \mathbb{Z}$$

which follows from the fact that $H_1(X\setminus\Gamma_0)$ is generated by a normal sphere. $\square$

In order to show that $H_i(X\setminus\Gamma_0;\mathbb{Z}/k)$ is zero, we use the Wang sequences of the bundles q and γ, or rather their restrictions to the part over S^1:

$$q : A \to S^1, \qquad \gamma : B \to S^1.$$

The Wang sequence of $q : A \to S^1$ has the form

$$\ldots \to H_i(F) \xrightarrow{\alpha_* - \mathrm{id}} H_i(F) \to H_i(A) \to \ldots$$

where $\alpha : F \to F$ is the monodromy of a typical fibre F. Similarly for γ with monodromy $\beta : \Gamma \to \Gamma$. The monodromy $\alpha : F \to F$ is defined by a pullback diagram

$$\begin{tikzcd} A \arrow[d, "q"] & {[0,1]\times F} \arrow[l, "H"'] \arrow[d, "\mathrm{pr}"] \\ S^1 & {[0,1]} \arrow[l, "e"'] \end{tikzcd}$$

with $H(0,x) = x$, $H(1,x) = \alpha(x)$, $F = q^{-1}(1)$, $e(t) = \exp(2\pi i\, t)$.

(0.67) Lemma. *The monodromies α and β are given by*

$$\alpha : F_1 \to F_1, \quad x \mapsto \omega \cdot x$$

$$\beta : \Gamma_1 \to \Gamma_1, \quad x \mapsto \omega^k \cdot x$$

with $\omega = \exp(2\pi i/l)$.

PROOF. Define $H : I \times F_1 \to S^1 \times_G F_1$ by $(t,x) \mapsto (\exp(2\pi it/l), x)$, and similarly for γ. (Compare the proof of (6.4) and (6.6).) $\square$

Proof of (6.8). We show that X is acyclic. By (6.7), (6.9) and (6.10) it suffices to show that $H_i(X \setminus \Gamma_0; \mathbb{Z}/k) \cong 0$ for $i \geq 2$. We use the Wang sequences with $\mathbb{Z}/k$–coefficients.

For $q : A \to S^1$ we know the structure of the Wang sequence because:

$$H_i(A) \cong H_i(\mathbb{C}^n \setminus F_0) \cong H_{i+1}(\mathbb{C}^n, \mathbb{C}^n \setminus F_0) \cong H_{i-1}(F_0);$$

the left most isomorphism is the Thom isomorphism. By hypothesis these groups are zero. Hence $\alpha_* - \mathrm{id}$ is an isomorphism for $i \geq 2$. Since we are dealing with finite groups, bijectivity of $\beta_* - \mathrm{id}$ is equivalent to injectivity. By (6.11) $\beta_* = \alpha_*^k$. Moreover α_* has order l. Suppose $(\beta_* - \mathrm{id})(x) = 0$. Let $kj \equiv 1 \bmod l$. Then $\beta_*(x) = \alpha_*^k(x) = x$ and $\alpha_*(x) = \alpha_*^{kj}(x) = \alpha_*^k \circ \ldots \circ \alpha_*^k(x) = x$. Since $\alpha_* - \mathrm{id}$ is an isomorphism, $x = 0$. Therefore $\beta_* - \mathrm{id}$ is an isomorphism and the Wang sequence implies $H_i(X \setminus \Gamma_0) = 0$ for $i \geq 2$. $\square$

(0.68) Remark. The proof of (6.8) does not really use that q is a polynomial. We leave it to the reader to formulate the corresponding result in the differentiable or holomorphic category.

7 Complements of divisors

This section is devoted to the homological properties of affine surfaces. We study the surfaces together with their compactifications.

Let X be a closed, connected, oriented, smooth 4–manifold. We consider the complement $V = X \setminus D$ of a compact, non–empty subset D. By Poincaré duality, we have for each subring T of the rational numbers $\mathbb{Q}$ an isomorphism

(0.69) $$H_i(V; T) \cong H^{4-i}(X, D; T);$$

see DOLD [1972], VIII.7.2. We are mainly interested in the case when D is a union of two–dimensional submanifolds (with singularities). Therefore we make the following assumptions.

(0.70) Assumptions.

(1) $H_4(D) \cong H^4(D) \cong 0$.

(2) $H^3(D) \cong H^3(D) \cong 0$.

(3) $H^*(D)$ is finitely generated. The coefficient ring for co–homology is $\mathbb{Z}$, unless otherwise specified.

(0.71) Proposition. *Under the assumptions* (8.2) *the following hold*

(1) $H_0(V) \cong H^0(V) \cong \mathbb{Z}$

(2) $H_4(V) \cong H^4(V) \cong 0$.

(3) $H_3(V)$ *and* $H^1(V)$ *are free abelian.*

(4) $H_*(V)$ *and* $H^*(V)$ *are finitely generated.*

Proof. We use exact sequences of the pair (X, D), Poincaré duality and universal coefficient formulas. For the purposes of duality one has to use Čech cohomology.

The exact sequence

$$H^3(D) \to H^4(X, D) \to H^4(D) \to H^4(X)$$

and (8.2) yield $H^4(X, D) \cong H^4(X)$. Since X is oriented and connected, $H^4(X) \cong \mathbb{Z}$. By duality $H^4(X, D) \cong H_0(V) \cong \mathbb{Z}$. Hence V is connected. It is a general fact that for a connected, non–compact 4–manifold V the groups $H_j(V)$ are zero for $j \geq 4$, see Dold [1972], VIII.3. The exact sequence

$$H^0(X) \to H^0(D) \to H^1(X, D) \to H^1(X)$$

and duality $H^1(X, D) \cong H_3(V)$ shows $H_3(V)$ to be free abelian. The universal coefficient isomorphism yields $H^0(V) \cong \mathrm{Hom}(H_0(V), \mathbb{Z}) \cong \mathbb{Z}$.

The cohomology $H^*(X)$ of a compact manifold X is finitely generated, since X is a retract of a finite simplicial complex, see Dold [1972], IV.8.10. The exact sequence of the pair (X, D) and (8.2.3.) shows $H^*(X, D)$ to be finitely generated. By duality (8.1), $H_*(V)$ is finitely generated and then, by the universal coefficient formula for V, the groups $H^*(V)$ are finitely generated. A manifold has the homotopy type of a CW–complex. Therefore singular and Čech cohomology for V agree. Since $H^1(V) \cong \mathrm{Hom}(H_1(V), \mathbb{Z})$, the group $H^1(V)$ is free abelian. □

We are interested in T–acyclic complements V, this means:

(0.72) $\qquad \tilde{H}_*(V; T) = 0 \quad$ or equivalently: $\quad \tilde{H}^*(V; T) = 0)$.

Proposition (8.3.3) has the following consequence:

(0.73) Corollary. *Under the assumptions of* (8.3) *the following holds:*

(1) $H_3(V; T) = 0$ *implies* $H_3(V) = 0$.

(2) $H^1(V;T) = 0$ *implies* $H^1(V) = 0$, $H_1(V)$ *finite.* $\square$

(0.74) Proposition. *Suppose* (8.2) *and* (8.4) *hold. Then we have:*

(1) $H_3(X) \cong H^1(X) \cong 0$.

(2) $H_1(X;T) \cong H^3(X;T) \cong 0$.

(3) $H^1(D;T) \cong 0$.

PROOF. By duality, $H^j(X,D;T) = 0$ for $j < 4$. The exact sequence $H^3(X,D;T) \to H^3(X;T) \to H^3(D;T)$ and (8.2) show $H^3(X;T) = 0$ and duality then gives $H_1(X;T) \cong 0$.

By universal coefficients we conclude $H^1(X;T) = 0$ and, since $H^1(X) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$ is free and $H^1(X;T) \cong H^1(X) \otimes_{\mathbb{Z}} T$, we obtain $H^1(X) = 0$ and finally $H_3(X) = 0$ by duality. The exact cohomology sequence of (X,D) and $H^j(X,D;T) = 0$ imply $H^1(D;T) \cong H^1(X;T) = 0$. $\square$

Affine varieties have special properties.

(0.75) Proposition. *Let V be diffeomorphic to a regular affine surface over* $\mathbb{C}$. *Then the following holds:*

(1) *V has the homotopy type of a finite two–dimensional CW–complex.*

(2) $H_i(V) = 0$ *for* $i > 2$.

(3) $H_2(V)$ *is free abelian.*

(4) D *is connected.*

PROOF. (2), (3) and (4) follow immediately from (1). For (1) see MILNOR [1963], §7. $\square$

We now assume that D is a union of closed, connected, oriented, two–dimensional, smooth submanifolds $D_1, \ldots, D_r$. We say that $D_1, \ldots, D_r$ have *normal crossings*, if the following holds:

(0.76)

(1) D_i and D_j have transverse intersection for $i \neq j$.

(2) For $i \neq j \neq h \neq i$ the intersection $D_i \cap D_j \cap D_k$ is empty.

If $D_1, \ldots, D_r$ have normal crossings we associate to D the *dual graph*

$$\Gamma(D) = (\Gamma_0(D), \Gamma_1(D)).$$

The set $\Gamma_0(D)$ of *vertices* is $\{D_1, \ldots, D_r\}$. Each intersection point in $D_i \cap D_j$, $i \neq j$, is an *edge* connecting the vertices D_i and D_j; thus $\Gamma_1(D)$ is $\coprod_{i \neq j} D_i \cap D_j$. We do not distinguish notationally between a graph Γ and its geometric realization. We apply a geometric terminology to Γ, like: cycle, subdivision, Euler characteristic.

(0.77) Lemma. *Let $D = D_1 \cup \ldots \cup D_r$ have normal crossings. Then $H_1(D;\mathbb{Q}) = 0$ (or $H^1(D;\mathbb{Q}) = 0$) if and only if the D_j are spheres and $\Gamma(D)$ is a tree.*

Proof. Suppose $H_1(D;\mathbb{Q}) = 0$. If D_j is not a sphere, then $H_1(D_j;\mathbb{Q})$ is non–zero of rank twice the genus of D_j. Moreover, $H_1(D_j;\mathbb{Q})$ is a direct summand of $H_1(D;\mathbb{Q})$. Therefore the space D is obtained from a collection of spheres $\coprod D_j$ by identifying isolated points of different spheres. By a geometric construction one shows that $\Gamma(D)$ is a retract of D. Therefore $H_1(\Gamma(D);\mathbb{Q})$ is a direct summand of $H_1(D;\mathbb{Q})$. Since the rank of $H_1(\Gamma(D);\mathbb{Q})$ is the number of (independent) cycles of $\Gamma(D)$ we conclude that $\Gamma(D)$ is a tree.

Conversely, if $\Gamma(D)$ is a tree of spheres one shows by induction on r that $H_1(D) = 0$. □

(0.78) Theorem. *Suppose $D = D_1 \cup \ldots \cup D_r$ has normal crossings. Then the following are equivalent:*

(1) *$V = X \setminus D$ is T–acyclic.*

(2) *The D_j are spheres and $\Gamma(D)$ is a tree. The inclusion $j : D \to X$ induces an isomorphism $j_* : H_2(D;T) \to H_2(X;T)$ and $H_1(X;T) = 0$.*

Proof. (1) $\Rightarrow$ (2). By (8.6.3) $H^1(D;T) = 0$ and hence, by (8.9), the D_j are spheres and $\Gamma(D)$ is a tree.

The exact sequence

$$H^2(X,D;T) \to H^2(X;T) \xrightarrow{j^*} H^2(D;T) \to H^1(X,D;T),$$

duality $H^j(X,D;T) \cong H_{4-j}(V;T)$ and (1) show, that j^* is an isomorphism.

From the universal coefficient theorem Spanier [1966], p. 248, Theorem 12, we conclude that $j_* : H_2(D;T) \to H_2(X;T)$ is an isomorphism.

(2) $\Rightarrow$ (1). The condition $H_1(X;T) = 0$ implies $H^3(X;T) \cong 0$ and $H^1(X;T) \cong 0$. Moreover j_* is an isomorphism on $H_2(-;T)$ if and only if j^* is an isomorphism on $H^2(-;T)$. From (8.3) we know already that $H_0(V) \cong H^0(V)$ and $H_4(V) \cong H^4(V) \cong 0$. The sequence (cohomology with coefficients in T)

$$H^1(D) \to H^2(X,D) \to H^2(X) \to H^2(D) \to H^3(X,D) \to H^3(X)$$

shows $H^3(X,D;T) \cong H_1(V;T) \cong 0$ and $H^2(X,D;T) \cong H_2(V;T) \cong 0$. The sequence

$$H^0(X;T) \to H^0(D;T) \to H^1(X,D;T) \to H^1(X;T)$$

shows $H^1(X,D;T) \cong H_3(V;T) \cong 0$. □

We now collect the information about integral co–homology in the case when $V = X \setminus D$ and D is a union of spheres with normal crossings.

(0.79) Proposition. *Suppose V is T–acyclic. Then the following holds:*

(1) $H_0(V) \cong H^0(V) \cong \mathbb{Z}$.

(2) $H_4(V) \cong H^4(V) \cong 0$.

(3) $H_3(V) \cong H^1(V) \cong 0$.

(4) $H_1(V) \cong H^2(V) \cong \mathrm{coker}(H_2(D) \to H_2(X))$. *This is a finite T–torsion group.*

(5) $H_2(V) \cong H^3(V)$ *is a finite T–torsion group. If V is an affine variety, this group is zero.*

Proof. (1) and (2) are given by (8.3), and (3) is given by (8.5). Since V is T–acyclic, we have $H_1(D) = 0$ by (8.9) and the exact sequence and duality show

$$\mathrm{coker}(H_2(X) \to H_2(X)) \cong H_2(X, D) \cong H^2(V).$$

Also, by (8.5), this is a finite T–torsion group. By universal coefficients, $H^2(V) \cong \mathrm{Ext}(H_1(V), \mathbb{Z}) \cong H_1(V)$.

Since V is T–acyclic, $H_2(V)$ and $H^3(V)$ are finite T–torsion groups. Exact sequence and duality shows $H^3(V) \cong H_1(X)$ and universal coefficients $H^3(V) \cong \mathrm{Ext}(H_2(V), \mathbb{Z}) \cong H_2(V)$. The final statement follows from (8.7.1). □

8 Homology spheres

Let B denote a compact, connected, oriented 4–manifold with boundary $S = \partial B \neq \emptyset$.

(0.80) Proposition. *Suppose B is $\mathbb{Q}$–acyclic. Then S is $\mathbb{Q}$–homology sphere. More precisely, $H_2(S) = 0$ and*

$$|H_1(S)| = |\ker i : H_1(B) \to H_1(B, S)|^2.$$

Proof. Consider the exact sequence

$$H_2(B, S; \mathbb{Q}) \to H_1(S; \mathbb{Q}) \to H_1(B; \mathbb{Q}).$$

By duality $H_2(B, S; \mathbb{Q}) \cong H^2(B; \mathbb{Q}) = 0$. Hence $H_1(S; \mathbb{Q}) = 0$ and $H_1(S)$ is therefore a finite group. This gives, by duality and universal coefficients,

$$H_2(S) \cong H^1(S) \cong \mathrm{Hom}(H_1(S), \mathbb{Z}) \cong 0.$$

Duality and universal coefficients again yield the commutative diagram

$$\begin{tikzcd}
\mathrm{Ext}(H_1(B,S),\mathbb{Z}) \arrow[r, phantom, "\cong"] \arrow[d, "i^*"] & H^2(B,S) \arrow[r, phantom, "\cong"] \arrow[d] & H_2(B) \arrow[d, "j"] \\
\mathrm{Ext}(H_1(B),\mathbb{Z}) \arrow[r, phantom, "\cong"] & H^2(B) \arrow[r, phantom, "\cong"] & H_2(B,S).
\end{tikzcd}$$

By algebra, $\operatorname{coker} i^* \cong \ker i$ and the exact sequence of the pair (B, S) yields the short exact sequence $0 \to \operatorname{coker} j \to H_1(S) \to \ker i \to 0$. $\square$

Now suppose we are in a situation of the previous section: D is a union of embedded spheres $D_1 \dots, D_r$ in X with normal crossing and $V = X \setminus D$. There exists a suitable tubular neighbourhood U of D in X such that $X \setminus U$ is diffeomorphic to V and $B = X \setminus U^\circ$ is a smooth manifold with boundary S and interior $V = B \setminus S$. In this case $H_1(B, S) \cong H^3(V)$ by duality. In case $H^3(V) = 0$ we obtain from (9.1)

(0.81) $$|H_1(S)| = |H_1(V)|^2.$$

We recall the *plumbing construction* for the tubular neighbourhood U of D. We assume that $D_1, \dots, D_r$ has normal crossings and that $\Gamma(D)$ is a tree. One chooses closed tubular neighbourhoods U_i of D_i as smooth images $t_i : E_i \to U_i$ of 2–cell bundles $E_i \to D_i$. The sets U_i and U_j intersect if and only if $D_i \cap D_j \neq \emptyset$. Fix (i, j) with $D_i \cap D_j = \{x\}$. There exist embeddings $\tau_i : D^2 \to D_i$ and $\tau_j : D^2 \to D_j$ about x and trivializations $\sigma_i : D^2 \times D^2 \to t_i^{-1}(U_i \cap U_j) =: E_{ij}$ over τ_i such that

$$D^2 \times D^2 \xrightarrow{\sigma_i} E_{ij} \xrightarrow{t_i} U_i \cap U_j \xleftarrow{t_j} E_{ji} \xleftarrow{\sigma_j} D^2 \times D^2$$

is the switching map $(u, v) \mapsto (v, u)$. Thus U can be obtained from the bundles E_i by a suitable gluing process along the E_{ij}, called plumbing.

The exact sequence

$$H^1(S) \to H^2(U, S) \xrightarrow{s} H^2(U) \to H^2(S) \to H^3(U, S)$$

shows that $H^2(S) \cong H_1(S)$ is isomorphic to the cokernel of s. Let $e_i \in \mathbb{Z}$ denote the Euler number of the oriented normal bundle E_i. The graph Γ becomes a *weighted graph* if we assign to each vertex $D_i \in \Gamma_0(D)$ the weight e_i. The Euler number e_i is also called the self *intersection number* of D_i in U or in X. The matrix $w : \Gamma_0(D) \times T_0(D) \to \mathbb{Z}$ with $w(D_i, D_i) = e_i$ and $w(D_i, D_j) = |D_i \cap D_j|$ is called the *intersection matrix* of the weighted tree $\Gamma(D)$.

The inclusion $D \to U$ and $S \to U \setminus D$ are homotopy equivalences (tom Dieck [1991], II.1.5). Let $x_i' \in H^2(D_i)$ be dual to the fundamental class.

(0.82) Lemma. *$H^2(U)$ is free abelian with basis $x_1, \dots, x_r$ such that under the map induced by the inclusion $D_i \to U$ the element x_j is mapped to $\delta_{ij} x_i'$.*

Proof. This is seen by calculating $H^2(D)$ inductively from the $H^2(D_i)$ via the Mayer–Vietoris sequence. $\square$

We use $H^2(U, U \setminus D)$ for the calculation of $H^2(U, S)$. By excision we have

$$H^2(U, U \setminus D_i) \cong H^2(U_i, U_i \setminus D_i)$$

and, by the Thom Isomorphism, the latter group is generated by the Thom class Φ_i. Let $y_i \in H^2(U, U \setminus D)$ be the image of Φ_i under the canonical map $H^2(U, U \setminus D_i) \to H^2(U, U \setminus D)$.

(0.83) Lemma. *$H^2(U,S)$ is a free abelian group with basis $y_1,\dots,y_r$.*

PROOF. The proof is by induction on r using the relative Mayer–Vietoris sequence (DOLD [1972], III.8.).

(0.84) Proposition. *The matrix of $s : H^2(U,S) \to H^2(U)$ with respect to the bases* (8.3) *and* (8.4) *is the intersection matrix of the weighted tree $\Gamma(D)$.*

PROOF. The Euler number e_i can be defined as follows: The image of the Thom class Φ_i under the map

$$H^2(U_i, U_i \setminus D_i) \to H^2(U_i) \to H^2(D_i)$$

is $e_i x_i'$. The construction of the elements x_i, y_i now shows directly, that the diagonal elements of the matrix are the e_i.

We now suppose $i \neq j$ and $D_i \cap D_j = \{x\}$. The fibre F_i of U_i over x is then part of D_j. We therefore have a commutative diagram induced by inclusions

$$\begin{tikzcd}
y_i \in H^2(U, U \setminus D) \arrow[r] \arrow[dd] & H^2(U) \arrow[d] \\
& H^2(D_j) \ni x_j \arrow[d] \\
H^2(F_i, F_i \setminus x) & H^2(D_j, D_j \setminus x). \arrow[l, "\cong"']
\end{tikzcd}$$

The Thom class Φ_i has the characteristic property that its restriction to a fibre $H^2(F_i, F_i \setminus x)$ is the canonical generator. Similarly, the fundamental class of $H^2(D_j)$ is mapped to the canonical generator of $H^2(D_j, D_j \setminus x)$. Since the orientations of F_i and D_j agree, we see that the image of y_i in $H^2(D_j)$ is x_j.

Finally, we have to show that y_i maps to zero in $H^2(D_j)$ if $D_i \cap D_j = \emptyset$. But in this case we have a commutative diagram

$$\begin{tikzcd}
H^2(U, U \setminus D_i) \arrow[r] \arrow[d] & H^2(U) \arrow[d] & \\
0 = H^2(U_j, U_j) \arrow[r] & H^2(U_j) \arrow[r] & H^2(D_j)
\end{tikzcd}$$

which yields the claim. □

We make the following general remarks. Let U be a compact, connected, oriented 4–manifold with boundary S. Assume $H_2(S) = 0$ and $H_1(U) = 0$. Then we have an exact sequence

$$0 \to H_2(U) \xrightarrow{s} H_2(U,S) \to H_1(S) \to 0.$$

By duality and universal coefficients $H_2(U,S) \cong H^2(U) \cong \mathrm{Hom}(H_2(U), \mathbb{Z})$ is free abelian. The map s translates into a map $H_2(U) \to \mathrm{Hom}(H_2(U), \mathbb{Z})$ and thus into a bilinear form $H_2(U) \times H_2(U) \to \mathbb{Z}$ on the free abelian group $H_2(U)$. This bilinear form is known as the intersection form. In (9.3)–(9.5) above we have made explicit what this means in our case.

From (9.2) and (9.5) we obtain the following:

(0.85) Corollary. *Let $V = X \setminus D$ be $\mathbb{Q}$–acyclic. The order of $H_1(S)$ is the absolute value of the determinant of the intersection matrix. This order is always the square of an integer.* □

If (Γ, w) is any weighted tree with weight function $w : \Gamma_0 \to \mathbb{Q}$ we have the matrix (w_{xy}) with $w_{xx} = w(x,x)$ and $w_{xy} = 1$ for $x \neq y$ if and only if x, y are connected by an edge. Call $\det(w_{xy})$ the determinant $\det(\Gamma) = \det(\Gamma, w)$ of the weighted tree. One computes $\det(\Gamma, w)$ inductively via a generalized continued fraction algorithm as follows. Assume the following situation: Let $e \in \Gamma_1$ be an edge with weights b and c at its boundary vertices X and Y. Let the vertex with weight $b \neq 0$ be a terminal vertex of the tree Γ. We obtain a new weighted tree Γ' by removing e and X from Γ and replacing the weight c of Y by $c - b^{-1}$. Then the following holds:

(0.86) Proposition. $\det(\Gamma) = b \det(\Gamma')$.

Proof. We expand the determinant

$$\left| \begin{array}{c|c} \begin{matrix} b & 1 \\ 1 & c \end{matrix} & 0 \\ \hline 0 & A \end{array} \right|$$

with respect to the first row and get

$$b \begin{vmatrix} c & 0 \\ 0 & A \end{vmatrix} - |A| = b \begin{vmatrix} c - b^{-1} & 0 \\ 0 & A \end{vmatrix}.$$

□

It is also easy to compute what happens if $b = 0$ (compare Eisenbud–Neumann [1985], p. 153).

9 Combinatorial properties

Let M be a closed, connected, oriented, differentiable 4–manifold. Let $(C_1, \ldots, C_n)$ be a set of immersed closed, connected, oriented surfaces C_i in M.

(0.87) Definition. A point $x \in C := \cup C_i$ is called *tidy*, if the following holds:

(1) There exists an oriented C^∞–chart $\varphi : (U, 0) \to (M, x)$, $U \subset \mathbb{C}^2$ open, about x such that $\varphi^{-1}(M \cap C)$ is a union of complex submanifolds $B_1, \ldots, B_r$ wich have pairwise transverse intersection in 0.

(2) Each $\varphi(B_j)$ is contained in exactly one C_i and $\varphi|B_j : B_j \to C_i$ is orientation preserving.

Here U and the B_j carry the canonical orientation induced from the complex structure. We can express (10.1) by saying that the intersection pattern of C in M is locally holomorphic. We call $C \subset M$ tidy if all points of C are tidy. A chart $\varphi : (U, 0) \to (M, x)$ exhibiting x as a tidy point is called *adapted* to C.

A particularly interesting example arises from a non–singular projective surface M over $\mathbb{C}$ and algebraic curves $C_1, \dots, C_n$ in M such that $C = \cup C_i$ has only ordinary multiple point singularities (number of tangents = multiplicity of the point). An arrangement of projective lines $L_1, \dots, L_n$ in projective space $\mathbb{P}^2$ over $\mathbb{C}$ is always tidy.

The main point for giving this definition is that the process of blowing up a point, known from algebraic geometry or complex analysis, still makes sense in this context.

Let $C = (C_1, \dots, C_n)$ be a tidy arrangement of immersed oriented spheres C_i in the complex projective plane $\mathbb{P}^2$. Let M be the set of its singular points. We call the number $m(x)$ of local branches through $x \in M$ the *valence* of x. We denote by t_r the number of points in M with valence r. We also use the abbreviations

(0.88)
$$f_0 = \sum_{r \geq 2} t_r = |M|, \quad f_1 = \sum_{r \geq 2} r t_r.$$

We have $H_2(\mathbb{P}^2) \cong \mathbb{Z}$ and the generator 1 corresponds to a cycle represented by a complex projective line with its natural orientation. Since each sphere C_i is oriented it defines an element $z(C_i) \in H_2(\mathbb{P}^2)$ and a corresponding integer d_i, called the *degree* of C_i. Set $d = \sum\limits_{i=1}^{n} d_i$.

(0.89) Proposition. *For the tidy arrangement $C = (C_1, \dots, C_n)$ the following identity holds:*

$$\sum_{r \geq 2} \binom{r}{2} t_r = \sum_{i<j} d_i d_j + \sum_i \binom{d_i - 1}{2} = \binom{d-1}{2} + n - 1.$$

If all $d_i = 1$, then the right hand side equals $\binom{n}{2}$.

Proof. The left hand side counts the intersection points (with multiplicity) geometrically and the right hand side homologically.

Let $f_i : S^2 \to \mathbb{P}^2$ be an immersion with image C_i. If we deform the immersion

$$\langle f_j \rangle : S = \coprod_j S^2 \to \mathbb{P}^2$$

by a regular homotopy in a neighbourhood of a multiple point into an immersion with transverse intersections in the usual sense of differential topology, then a point of valence r unfolds into $\binom{r}{2}$ points of valence 2. The right hand side of

(3.1) is invariant under regular homotopies. So let us assume that $f : S \to \mathbb{P}^2$ is an immersion which has only transverse intersections with intersection number $+1$. Then the number of intersection points equals

$$\textbf{(0.90)} \qquad \frac{1}{2}(s^2 - e(\nu_f)).$$

Here ν_f is the normal bundle of f and $e(\nu_f)$ its Euler class (= the sum of the Euler classes of the bundles over the components); and s^2 is the self–intersection number of the cycle s defined by f. The counting (10.4) is a general fact of differential topology, see WALL [1970], Theorem 5.2 (iii). We evaluate (10.4) in the case at hand. We obtain

$$\textbf{(0.91)} \qquad \sum_{i<j} C_i C_j + \frac{1}{2}\sum_i (C_i^2 - e(\nu_i)).$$

The intersection number $C_i C_j$ equals $d_i d_j$. Let $D = C_i$ and let $f : D \to \mathbb{P}^2$ have degree k. Let η_s be the canonical line bundle over $\mathbb{P}^s$. Then ν_f is isomorphic to

$$f^*(T\mathbb{P}^2 \oplus \varepsilon)/f^*(T\mathbb{P}^2 \oplus \varepsilon) = f^*(3\eta_2)/2\eta_1 = (3k-2)\eta_1.$$

(T = tangent bundle, ε = trivial line bundle.)

Hence $e(\nu_f) = 3k - 2$. We arrive at

$$\frac{1}{2}(D^2 - e(\nu_f)) = \frac{1}{2}(k^2 - (3k-2)) = \binom{k-1}{2}.$$

□

In the case of algebraic curves (10.3) can also be proved by a counting argument and a formula of M. Noether (BRIESKORN–KNÖRRER [1986]).