Cylinder algebras

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Preliminary version, February 9, 2009*

Cylinder algebras generalize algebras of Birman-Wenzl [??] [??] and Murakami [??]. Geometrically, they are related to tangles and ribbons in the cylinder. For generic parameters, we define algebras $\mathfrak{Z}_n(k)$ of height k which are quotients of the group algebra of the braid group ZB_n. Therefore we call these algebras the algebras with n strings. This article assembles some foundational material for the general investigation of these algebras.

In the first part, we present a thorough discussion of the skein relations. This is based on the algebras with 2 strings.

In the second part, we describe a combinatorial model with the aid of coloured balanced Brauer graphs for certain specializations of the algebras.

1. The algebras of height ∞

We define the cylinder algebras $\mathfrak{Z}(\infty) = \mathfrak{Z}_2(\infty)$ on two strings and of height ∞. The algebras depend on a choice of parameters in the ground ring. We suppress the dependence on the parameters in the notation.

We need some notation. Let $\mathfrak{K}$ be a commutative ring. We fix a family

(1.1) $$P = (\lambda, q, A_j \mid j \in \mathbb{Z})$$

of parameters in the ring $\mathfrak{K}$. We write $A_j = a_j + \varepsilon_j$ where $\varepsilon_j = 1$ if j is even and $\varepsilon_j = 0$ otherwise. We assume that λ and q are invertible and set $\delta = q - q^{-1}$. We call

(1.2) $$R_0 := \lambda - \lambda^{-1} + \delta a_0 = 0$$

(1.3) $$R_n := \lambda a_n - \lambda^{-1} a_{-n} - \delta \sum_{j=1}^{n-1} a_j a_{j-n} + \delta \varepsilon_n \frac{n}{2} = 0, \qquad n \geq 1.$$

the *basic relations* between the parameters. We see from (1.3) that the a_n, $n < 0$ can be computed from the remaining parameters. If $\delta = 0$, then $a_j = a_{-j}$.

(1.4) Definition. The algebra $\mathfrak{Z}(\infty)$ is the associative unital $\mathfrak{K}$-algebra generated by invertible elements X, Y and another element E with the following relations

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(1) $X - X^{-1} = \delta(1-E)$
(2) $EX = \lambda E = XE$
(3) $XYXY = YXYX$
(4) $EXYXY = E = XYXYE$
(5) $EY^jE = A_jE$, $j \in \mathbb{Z}$.
The parameters are assumed to satisfy the basic relations (1.2) and (1.3). ♡

We make some remarks concerning this definition.

We multiply (1) by X^2 and λX, use (2), and eliminate E. This gives the relation

(1.5) $$(X-\lambda)(X-q)(X+q^{-1}) = 0$$

as a consequence of (1) and (2).

(1.6) Remark. Suppose the ring $\mathfrak{K}$ carries an involution φ (called conjugation, also denoted $x \mapsto x^*$) such that

$$\varphi(\lambda) = \lambda^{-1}, \quad \varphi(q) = q^{-1}, \quad \varphi(A_j) = A_{-j}.$$

We can certainly define an involution with these properties on the ring

$$\mathbb{Z}[\lambda^{\pm}, q^{\pm}, A_j \mid j \in \mathbb{Z}].$$

We use this to apply φ formally to any integral polynomial in the parameters. The involution φ can be extended to a conjugate-linear involution φ on $\mathfrak{Z}(\infty)$ such that

$$\varphi(X) = X^{-1}, \quad \varphi(Y) = Y^{-1}, \quad \varphi(E) = E.$$

The algebra always has a $\mathfrak{K}$-linear antiinvolution ι which fixes X, Y, E. ♡

(1.7) Remark. The generic ring for the definition of $\mathfrak{Z}(\infty)$ is

$$\mathbb{Z}[\lambda^{\pm}, q^{\pm}, A_j \mid j \geq 0] \cong \mathbb{Z}[\lambda^{\pm}, q^{\pm}, A_j \mid j \in \mathbb{Z}]/(R_0, R_1, \ldots).$$

Since $\varphi(R_n) = -R_{-n}$, this ring carries an involution as in (1.6). This involution is complicated in terms of the A_j, $j \geq 0$ alone. ♡

(1.8) Remark. Suppose $\delta \in \mathfrak{K}$ is invertible. Then

$$E = 1 - \delta^{-1}(X - X^{-1}).$$

We see that $\mathfrak{Z}(\infty)$ is generated by invertible elements X and Y. The algebra is therefore a quotient of the group algebra of the braid group $ZB_2 = \langle X, Y \mid XYXY = YXYX \rangle$. ♡

The basic relations are justified by the next result.

(1.9) Proposition. *Suppose X, Y, E are elements in a $\mathfrak{K}$-algebra which satisfy* (1.4.1) – (1.4.5). *Suppose the annihilator ideal of E is zero. Then the basic relations hold.*

A further justification will be given later when we construct a certain module for $\mathfrak{Z}(\infty)$. A proof of (1.9) is based on some consequences of the defining relations (1.4).

(1.10) Proposition. *Suppose X, Y, E are elements in a $\mathfrak{K}$-algebra which satisfy* (1.4.1) – (1.47.5). *Then the following relations hold ($n \geq 1$):*

(1) $EY^n X = EY^{n-1}XY^{-1} - \delta EY^{n-2} + \delta A_{n-1}EY^{-1}$

(2) $EY^{-n}X = EY^{-n+1}XY + \delta EY^{-n} - \delta A_{-n}E$

(3) $EY^n X = \lambda^{-1}EY^{-n} - \delta E\sum_{j=1}^{n-1}(Y^{-n+2j} - A_{n-j}Y^{-j})$

(4) $EY^{-n}X = \lambda EY^n + \delta E\sum_{j=0}^{n-1}(Y^{-n+2j} - A_{-n+j}Y^j)$.

There are analogous relations for left multiplication (formally apply the antiinvolution ι).

Proof. Note that, by (1.4.3) and (1.4.4), $XYXY$ is contained in the center. Relation (1.4.4) yields $EY^nXYX = EYXYXY^{n-1} = EY^{n-1}$ and hence the first line in

$$\begin{aligned} EY^nX &= EY^{n-1}X^{-1}Y^{-1} \\ &= EY^{n-1}(X + \delta(E-1))Y^{-1} \\ &= EY^{n-1}XY^{-1} + \delta A_{n-1}EY^{-1} - \delta EY^{n-2}. \end{aligned}$$

(2) Relation (1.4.4) yields $EY^{-n}X^{-1} = EY^{-n+1}XY$. Now eliminate X^{-1} with (1.4.1).

(3) and (4) follow by induction from (1) and (2). □

Proof of (1.9). We multiply (1.4.1) by E and use (1.4.2). This gives $(\lambda - \lambda^{-1} + \delta a_0)E = 0$ and hence (1.2). We multiply (1.10) from the right with E and obtain $0 = \lambda A_n - \lambda^{-1}A_{-n} + \delta(\sum_{j=1}^{n-1} A_{2j-n} - A_{n-j}A_{-j})$. A rewriting in terms of the a_j gives the result for $n \geq 1$. □

(1.11) Corollary. *The ideal $\langle E \rangle$ of $\mathfrak{Z}(\infty)$ generated by E is spanned over $\mathfrak{K}$ by the elements Y^iEY^j ($i, j \in \mathbb{Z}$). The right ideal $[E\,\rangle$ generated by E is spanned by the elements EY^j ($j \in \mathbb{Z}$). Similarly for the left ideal $\langle E\,]$ generated by E.* □

(1.12) Corollary. *Under the hypothesis of* (1.10), *right multiplication by X and YXY maps*

$$I(n) = \mathrm{Span}\{EY^{-n}, EY^{-n+1}, \ldots, EY^{n-1}, EY^n\}$$

into itself. □

Suppose the $(EY^j \mid j \in \mathbb{Z})$ are linearly independent over $\mathfrak{K}$. We can express the right multiplication by X, Y, E in terms of by matrices. We now formally construct this right module by specifying matrices.

Let $\tilde{X}$ denote the following $\mathbb{N} \times \mathbb{N}$-matrix with entries in $\mathfrak{K}$. The first row is

$$(\lambda, 0, -\delta a_{-1}, -\delta, -\delta a_{-2}, 0, -\delta a_{-3}, -\delta, -\delta a_{-4}, \ldots).$$

The remaining part of the matrix, except the first row and column, is decomposed into 2×2-blocks. As such it is an upper triangular block matrix; beginning with the diagonal term, all rows are equal to $\Lambda, \alpha_1, \alpha_2, \dots$ with

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & \delta \end{pmatrix}, \qquad \alpha_j = \delta \begin{pmatrix} -\varepsilon_j & -a_{-j} \\ a_j & \varepsilon_j \end{pmatrix}.$$

Let $\tilde{Y}$ be the shift matrix with 1's below the diagonal and zeros otherwise. Finally, let $\tilde{E}$ be the matrix with first row

$$(A_0, A_1, A_{-1}, A_2, A_{-2}, \dots).$$

If we order the EY^n according to the exponents $(0, 1, -1, 2, -2, 3, -3, \dots)$, then $\tilde{X}$, $\tilde{Y}$, and $\tilde{E}$ describe the right multiplication by X, Y, and E, respectively. For X, this follows from (1.10); the cases Y and E are clear. Since we work with right ideals, $\tilde{Y}\tilde{X}$ gives the action of XY. The matrix $\tilde{Y}\tilde{X}$ is an upper triangular matrix of 2×2-blocks. The rows are, beginning with the diagonal term,

$$\left(\begin{array}{cc|cc|cc|cc|c} 0 & \lambda^{-1} & \delta & \delta a_1 & 0 & \delta a_2 & \delta & \delta a_3 & \cdots \\ \lambda & 0 & -\delta a_{-1} & -\delta & -\delta a_{-2} & 0 & -\delta a_{-3} & -\delta & \cdots \end{array}\right).$$

The square $(\tilde{Y}\tilde{X})^2$ is therefore again a matrix of the same shape. The block on the diagonal is the unit matrix. The block j positions to the right is the diagonal matrix with diagonal $(\delta R_j, \delta R_j)$. If the relations $R_j = 0$ hold, then we can compute $\tilde{X}^{-1}$ as $\tilde{Y}\tilde{X}\tilde{Y}$. The result is the following matrix. The first row is

$$(\lambda^{-1}, \delta a_1, 0, \delta a_2, \delta, \delta a_3, 0, \dots).$$

The rest is again 2×2 upper triangular; the rows are as for $\tilde{X}$, except that the diagonal block is replaced by its inverse. These data and results are used in the following construction.

(1.13) The basic module. Suppose the basic relations (1.2) and (1.3) hold. We construct a right $\mathfrak{Z}(\infty)$-module with basis the symbols $(ey^j \mid j \in \mathbb{Z})$. We order the ey^j as we did for the EY^j and let X, Y, and E act through $\tilde{X}$, $\tilde{Y}$, and $\tilde{E}$. We have to verify the relations (1.4). By construction, $YXYX$ acts as identity. Hence (3) and (4) hold. From statements of the last paragraph we see that (1) holds. Also (2) and (5) are immediate from the definitions. The assignment $ey^j \mapsto EY^j$ is now a $\mathfrak{Z}$-linear epimorphism onto the right ideal generated by E. We shall see in a moment that this is an isomorphism. ♡

(1.14) Theorem. *The algebra $\mathfrak{Z}(\infty)$ is additively a free $\mathfrak{K}$-module with basis*

$$Y^iXY^j, \quad Y^iXY^jX^{-1}, \quad Y^iEY^j \qquad (i, j \in \mathbb{Z}).$$

Proof. It is known that the factor algebra by the ideal generated by E is a free $\mathfrak{K}$-module with basis consisting of the first two families in (1.14). This is shown in [**??**] by a reduction to Ariki-Koike algebras [**??**]. Another proof in [**??**] reduces it to Hecke algebras of affine root systems. We sketch here an independent proof along the lines of Hecke algebra theory (see e. g. [??]). We construct a right module over $A = \mathfrak{Z}(\infty)/\langle E \rangle$ in the free $\mathfrak{K}$-module with basis $y(i,j), z(i,j)$ (such that the assignement $y(i,j) \mapsto Y^iXY^j$, $z(i,j) \mapsto Y^iXY^jX^{-1}$ is a morphism of $\mathfrak{A}$-modules. We have to define the action of X and Y on the basis elements and verify the relations $X - X^{-1} = \delta$ and $XYXY = YXYX$. We define

$$\begin{aligned} z(i,j)X &= y(i,j) \\ z(i,j)Y &= z(i+1,j) + \delta y(i+j,1) - \delta y(i,j+1) \\ y(i,j)X &= z(i,j) + \delta y(i,j) \\ y(i,j)Y &= y(i,j+1) \end{aligned}$$

We leave the verification of the relations to the reader. If we apply a linear combination of the Y^iXY^j, $Y^iXY^jX^{-1}$ to $z(0,0)$, we obtain the analogous linear combination of the $y(i,j)$, $z(i,j)$. Hence we see that the elements in question are linearlx independent.

This shown, it remains to verify that the Y^iEY^j are linearly independent. For this purpose we let them act on the basic module (1.13). We have

$$(ey^a)(\sum_{i,j} \lambda_{ij} Y^iEY^j) = \sum_{i,j} \lambda_{ij} A_{i+a} ey^j.$$

If $c = \sum_{i,j} \lambda_{ij} Y^iEY^j = 0$, we can assume that i,j are taken from the interval $1 \le i,j \le n$. If the determinant of the $n \times n$-matrix $(u,v) \mapsto A_{u+v}$ is not a zero divisor in $\mathfrak{K}$, then $(ey^a)c = 0$ implies $\lambda_{ij} = 0$. The hypothesis on the determinant certainly holds in the generic ring. Hence in this case, $\mathfrak{Z}(\infty)$ has a basis as indicated. Over any other ring, the algebra is obtained by a change of scalars from the generic case, hence has the same basis. $\Box$

(1.15) Remark. If we work over a field such that all the determinants of the matrices $(u,v) \mapsto A_{u+v}$ are non-zero, then the basic module is simple. This holds for the quotient field of the generic ring. Thus, in this generic case, we have an infinite-dimensional irreducible representation of the braid group ZB_2. ♡

2. The algebras of height k

The algebras $\mathfrak{Z}_2(k)$ for $2 \le k < \infty$ are defined as for $k = \infty$. We only require that Y satisfies an equations of degree k. Fix, in addition to (1.1), parameters

(2.1) $$q_0 = 1, q_1, \dots, q_k$$

in the ground ring $\mathfrak{K}$. We assume that q_k is invertible.

(2.2) Definition. The algebra $\mathfrak{Z}(k) = \mathfrak{Z}_2(k)$ on two strings and of height k is the associative unital $\mathfrak{K}$-algebra generated by invertible elements X, Y and another element E with the relations

(1) $X - X^{-1} = \delta(1 - E)$

(2) $EX = \lambda E = XE$
(3) $XYXY = YXYX$
(4) $EXYXY = E = XYXYE$
(5) $EY^jE = A_jE$, $j \in \mathbb{Z}$
(6) $\sum_{j=0}^k q_j Y^{k-j} = 0$.

This is a tentative definition. The final definition will add basic relations for the parameters (1.1) and (2.1). ♡

It takes some time to define and justify the basic relations for (2.2). To begin with we combine (2.2.5) and (2.2.6) as follows. Suppose we have $(A_j \mid j \in \mathbb{Z})$ such that for all $l \in \mathbb{Z}$

(2.3)
$$\sum_{j=0}^k q_j A_{l+k-j} = 0.$$

Given A_{l+i}, $0 \leq i < k$, l fixed, we can define A_j for all $j \in \mathbb{Z}$ uniquely such that (2.3) holds. We often do this by fixing l suitably. If (5) holds for $l \leq j \leq l+k$, then it holds for all $j \in \mathbb{Z}$. Thus we assume (2.3) and (5) for all $j \in \mathbb{Z}$ in the sequel.

Before we discuss and justify these relations, we consider the examples $k = 2, 3$ in the next section.

Occcasionally it is convenient to split (2.2) into linear factors. This uses parameters $p_j \in \mathfrak{K}$. We set

(2.4)
$$\prod_{j=1}^k (Y - p_j) = \sum_{j=0}^k q_j Y^{k-j} =: Q(Y).$$

Hence $(-1)^j q_j$ is the j-th elementary symmetric function in the p_i. Since q_k is invertible, all p_i are invertible. If we assume that $\Delta = \prod_{i<j}(p_i - p_j)$ is invertible in $\mathfrak{K}$, then the algebra $\mathfrak{Z}_1(k) = \mathfrak{K}[Y]/(Q(Y))$ has the pairwise orthogonal idempotents

(2.5)
$$Y_j = \prod_{i, i \neq j} \frac{Y - p_i}{p_j - p_i}$$

which are also a $\mathfrak{K}$-basis. In terms of the involution, we require $p_j^* = p_j^{-1}$. This is then compatible with the relation for q_j^* above.

3. The examples for $k = 2$ and $k = 3$.

The case $k = 2$. We assume given an algebra (2.2) such that the elements E and EY are linearly independent. We deduce from this the basic relations. The right ideal generated by E is spanned by these elements. This follows from

$$EYX = \lambda^{-1} EY^{-1} = -\lambda^{-1} q_2^{-1} q_1 E - \lambda^{-1} q_2^{-1} EY.$$

The right multiplications by X, Y, E have, with respect to E, EY, the matrices

$$X = \begin{pmatrix} \lambda & -\lambda^{-1}q_2^{-1}q_1 \\ 0 & -\lambda^{-1}q_2^{-1} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -q_2 \\ 1 & -q_1 \end{pmatrix}, \quad E = \begin{pmatrix} A_0 & A_1 \\ 0 & 0 \end{pmatrix}.$$

These matrices satisfy $YXYX = XYXY = 1_2$, and the relations (2.2) are satisfied if and only if (with $\mu = \lambda q_2$)

(3.1)
$$\begin{aligned} R_0 &:= \lambda - \lambda^{-1} + \delta a_0 = 0 \\ R_1 &:= \lambda a_1 - \lambda^{-1} a_{-1} = 0 \\ S_0 &:= \mu - \mu^{-1} - \delta = 0 \\ S_1 &:= \delta\lambda a_1 + q_1 - q_1^* = 0. \end{aligned}$$

As to the second relation, we remark that a_{-1} is defined by $A_1 + q_1 A_0 + q_2 A_{-1} = 0$. Therefore this relation is equivalent to

(3.2)
$$(1 + \lambda\mu)A_1 + q_1 A_0 = 0.$$

We could define the algebra over the generic ring

$$\mathbb{Z}[\lambda^{\pm}, q^{\pm}, A_0, A_1, q_1, q_2]/(R_0, R_1, S_0, S_1).$$

Since the second relation has the form $(\lambda q_2 - q)(\lambda q_2 + q^{-1}) = 0$, this is not an integral domain. In an integral domain, we must have $\lambda q_2 = q$ or $\lambda q_2 = -q^{-1}$. These two cases are essentially isomorphic. We prefer the relation

(3.3)
$$\lambda q_2 = -q^{-1},$$

because it is compatible with $\lambda = q = 1$ and $Y^2 = 1$.

If δ is not a zero divisor, then (3.2) is a consequence of R_0, S_0, S_1, and, similarly, R_0 is a consequence of R_1, S_0, S_1. Under the conditions (3.1) we can, as in (1.13), construct a right $\mathfrak{Z}(2)$ module with basis ey^0, ey^1, where X, Y, E act through the matrices above.

Under the conditions (3.1) and (3.3), the generic ring would be

$$\mathbb{Z}[\lambda^{\pm}, q^{\pm}, A_0, A_1, q_1]/(R_0, R_1, S_1).$$

Is this an integral domain?

The case $k = 3$. We assume that EY^{-1}, E, EY are linearly independent. Then these elements span the right ideal generated by E, and the matrices of X, Y, and E in this basis are

$$X = \begin{pmatrix} \delta & 0 & \lambda^{-1} \\ -\delta A_{-1} & \lambda & 0 \\ \lambda & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 & -q_3 \\ 1 & 0 & -q_2 \\ 0 & 1 & -q_1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 0 & 0 \\ A_{-1} & A_0 & A_1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A computation with these matrices shows the relation $YXYX = 1_3$, and (2.2) is satisfied if an only if the following relations hold (with $\mu = \lambda q_3$)

(3.4)
$$\begin{aligned} R_0 &:= \lambda - \lambda^{-1} + \delta a_0 = 0 \\ R_1 &:= \lambda a_1 - \lambda^{-1} a_{-1} = 0 \\ S_0 &:= \mu^2 - 1 = 0 \\ S_1 &:= \delta\lambda a_1 + \delta\lambda q_3 + q_1 - q_1^* = 0. \end{aligned}$$

Again, in an integral domain we have a choice for λq_3. We prefer the choice

(3.5)
$$\lambda q_3 = -1,$$

since it is compatible with the specialization $\lambda = q = 1$ and $Y^3 = 1$. Under the relations (3.4), we can define a free module of rank three by letting X, Y, E act through the matrices above. (We think formally of the basis EY^{-1}, E, EY.)

Later we discuss conditions on the parameters under which the ideal generated by E splits off as an algebra factor. In the case $k = 2$, this is the condition that

$$(p_1 A_0 + A_1)(p_2 A_0 + A_1) = q_2(A_0 - \lambda^2 A_1^2)$$

be invertible in $\mathfrak{K}$.

4. The basic relations for $k < \infty$

In this section we investigate the basic relations for algebras of height k. We write $k = 2n$ or $k = 2n + 1$, as the case may be. We set $\mu = \lambda q_k$ and call it the basic parameter. We set $S_0 = \mu^2 - 1$ for k odd and $S_0 = \mu - \mu^{-1} - \delta$ for k even. We use for $1 \le j \le n$ the following polynomial in the parameters (the dependance on k will not be denoted):

(4.1)
$$S_j = \delta\lambda\Big(\sum_{\nu=0}^{j-1} q_\nu a_{j-\nu} + \sum_{\nu=0}^{j-1} \varepsilon_{k+j-\nu} q_{k-\nu}\Big) + q_j - q_j^*.$$

We call S_j^* the value of S_j under the formal involution (1.6).

(4.2) Theorem. *Let an algebra as in* (2.2) *be given. Suppose $E, EY, \dots, EY^{k-1}$ are linearly independent over $\mathfrak{K}$. Then the relations $R_j = 0$ and $S_j = 0$ hold for $0 \le j \le n$.*

Proof. We know that the elements $(EY^j \mid 0 \le j \le k-1)$ span the right ideal generated by E. We write the matrices for the right multiplication by X, Y, E in terms of this basis and state the conditions under which the relations (2.2) hold.

We begin with the case $k = 2n + 1$. We assume that the elements $EY^{-n}, \dots, EY^n$ are linearly independent. They are then a basis for the right ideal generated by E. We compute the matrices for the right multiplication by X, Y, and E in this basis. For simplicity, we denote the matrices again by the same symbol. The matrices for Y and E are clear. The matrix for X is the upper-left $k \times k$-block of the matrix $\tilde{X}$ in section one, if we order the basis according to the exponents $(0, 1, -1, \dots, n, -n)$. The matrix YX has the following structure:

Take the upper-left $k \times k$-block of $\tilde{Y}\tilde{X}$ in section one and add to the last column the transpose of

$$-\lambda(q_{n-1}, q_n, q_{n+2}, \dots, q_{2n}, q_1, q_{2n+1} + 1).$$

The relation $EX = XE = \lambda E$ holds if and only if the relations $R_j = 0$ hold for $0 \le j \le n$ hold. If these $R_j = 0$, then, be the results of section one, $(YX)^2$ differs from the unit matrix formally only in the last column. A computation shows that the formal structure of the last column is, now written with exponents in the natural order,

$$(R_1 - S_1, \dots, R_n - S_n, -\lambda\mu S_n^*, \dots, \lambda\mu S_1^*, \mu^2).$$

Hence $(YX)^2 = 1_k$ if and only if $\mu^2 = 1$ and $S_j = S_j^* = 0$ for $1 \le j \le n$. We shall see later that the relations $S_j^* = 0$ are implied by the other relations.

The case $k = 2n$ is handled similarly. We work with the basis $EY^{1-n}, \dots, EY^n$ for the right ideal generated by E. The matrix YX is the upper-left $k \times k$-block of the corresponding matrix in section one. Under the assumptions $R_j = 0$ for $1 \le j \le n$ we have therefore $(YX)^2 = 1_k$. In this case, the relation $X(X + \delta(E-1)) = 1_k$ is the problem. Again, only the last column can differ from the identity, if we assume that the $R_j = 0$. This last cloumn is, written in the natural order of the exponents,

$$(-\lambda^{-1}\mu^{-1}S_1, \dots, -\lambda^{-1}\mu^{-1}S_{n-1}, -S_n^*, \dots, -S_1^*, -\mu^{-1}(\delta + \mu^{-1})).$$

This is the last column of the unit matrix if and only if $S_j = S_j^* = 0$. Again we do not need the S_j^*. □

We now discuss the basic relations. We introduce the formal power series in an indeterminate t

(4.3) $$A(t) = 1 + \delta\lambda \sum_{j=1}^{\infty} A_j t^j, \qquad a(t) = 1 + \delta\lambda \sum_{j=1}^{\infty} a_j t^j.$$

They are related by

(4.4) $$A(t) = a(t) + \delta\lambda \frac{t^2}{1 - t^2}.$$

We set $A_j^* = A_{-j}$, $a_j^* = a_{-j}$, and $a^*(t) = 1 - \delta\lambda^{-1} \sum_{j=1}^{\infty} a_j^* t^j$. The identity

(4.5) $$a(t)a^*(t) = \frac{(1 - q^{-2}t^2)(1 - q^2t^2)}{(1 - t^2)^2}$$

is equivalent to $\delta R_j = 0$ for $j \ge 1$.

Under the hypothesis (2.3), the power series $A(t)$, $a(t)$, $A^*(t)$, and $a^*(t)$ are rational functions. We set $q(t) = \sum_{j=0}^{k} q_j t^j$, $q_j^* = q_k^{-1} q_{k-j}$, and $q^*(t) = \sum_{j=0}^{k} q_j^* t^j$. We extend the formal involution (1.6) by $\varphi(q_j) = q_j^*$. We write

$$r(t) = q(t) + \delta\lambda\rho(t) = q(t) + \delta\lambda \sum_{j=1}^{k} \rho_j t^j, \quad \text{with} \quad \rho_j = \sum_{\nu=0}^{j-1} q_\nu A_{j-\nu}.$$

We obtain $r^*(t)$ and $\rho^*(t)$ by applying the involution to the coefficients. With these data we have

$$A(t) = \frac{r(t)}{q(t)}, \qquad A^*(t) = \frac{r^*(t)}{q^*(t)} \tag{4.6}$$

and, of course, still (4.4) and its conjugate. The identities of power series

$$a(t) = \frac{q^*(t)}{q(t)} \cdot \frac{1-\mu\delta t - t^2}{1-t^2}, \qquad k \quad \text{odd}$$

$$a(t) = \frac{q^*(t)}{q(t)} \cdot \frac{1-\mu^2 t^2}{1-t^2}, \qquad k \quad \text{even}$$

are equivalent to

$$S(t) := \delta\lambda(\rho(t) - s(t)) + q(t) - q^*(t) = 0$$

with

$$s(t) = t^2 \frac{q(t) - q_k q^*(t)}{1-t^2} \qquad k \quad \text{even}$$

$$s(t) = \frac{tq(t) - q_k q^*(t)}{1-t^2} \qquad k \quad \text{odd.}$$

(Note that the fractions which appear in $s(t)$ are polynomials.) For a proof, multiply the power series relations by $q(t)$ and rewrite the result. A small computation shows that if we write $S(t) = \sum_{j=1}^{k-1} S_j t^j$, then the S_j for $1 \le j \le n$ are the polynomials (4.1).

The $S(t)$ have a symmetry property. In its proof we use the following identities.

$$t^k q(t^{-1}) = q_k q^*(t). \tag{4.7}$$

$$t^k s(t^{-1}) = \begin{cases} t^{-2} s(t) & k \quad \text{even} \\ t^{-1} s(t) & k \quad \text{odd.} \end{cases} \tag{4.8}$$

$$s^*(t) = \begin{cases} -q_k^{-1} s(t) & k \quad \text{even} \\ -q_k^{-1} t^{-1} s(t) - q^*(t) & k \quad \text{odd.} \end{cases} \tag{4.9}$$

The first one is an immediate consequence of the definition and the other two are verified by using (4.7) in the definition of $s(t)$. The following identity is a direct consequence of the definition and (2.3).

$$\delta\lambda\rho(t) = -\delta\mu t^k \rho^*(t^{-1}) - \delta\mu A_0 t^k q^*(t^{-1}). \tag{4.10}$$

Here is the symmetry of the $S(t)$:

(4.11) Proposition. *Assume $R_0 = 0$ and $S_0 = 0$. Then*

$$t^k S(t^{-1}) = \lambda\mu S^*(t).$$

PROOF. One uses the assumption on A_0, the quadratic equation and (4.7) – (4.10) in order to verify the identity in question. □

The symmetry property is now used to show that $R_j = 0$ and $S_j = 0$ for $1 \leq j \leq n$ imply $S_j^* = 0$ in the same range. Proof:

(4.12) The basic module. From the last proof we also see that, under the basic relations, the matrices X and Y just defined satisfy $YXYX = 1$ and $X - X^{-1} = \delta(1 - E)$, in both cases. We can therefore define a right module of the algebra with $\mathfrak{K}$-basis $ey^{-n}, \dots, ey^n$ for $k = 2n+1$ and $ey^{1-n}, \dots, ey^n$ for $k = 2n$ such that the right multiplication by X and Y is given by the matrices with the same name of the previous proof. The action of E should be clear. The assignment $ey^j \mapsto EY^j$ is now a $\mathfrak{Z}(k)$-linear map. ♡

We give a heuristic argument for the general form of the relations. We have

$$a(t) = \frac{r(t)}{q(t)} - \frac{\delta\lambda t^2}{1-t^2}$$

and similarly for $a^*(t)$. The basic equation (4.5) thus leads to

$$(1-q^{-2}t^2)(1-q^2t^2)q(t)q^*(t) = (r(t)(1-t^2) - \delta\lambda t^2 q(t)(r^*(t)(1-t^2) + \delta\lambda^{-1}t^2q^*(t)).$$

In a sufficiently generic situation, $q(t)$ and $q^*(t)$ are different irreducible polynomials. So $q^*(t)$ must divide one of the factors on the right. It cannot divide the second factor, because then it would divide $r^*(t)$, and this would imply that $A^*(t)$ is a polynomial. Therefore

$$a(t) = \frac{q^*(t)}{q(t)} \cdot \frac{b(t)}{1-t^2}$$

where $b(t)$ is a quadratic factor of $(1 - q^{-2}t^2)(1 - q^2t^2)$.

We finally show that the relations above imply $R_t = 0$ for all $t \in \mathbb{Z}$. Proof:

5. The splitting

In this section, we discuss conditions under which the ideal $\langle E \rangle$ splits off as an algebra. This amounts to finding a central idempotent which is the unit element of $\langle E \rangle$. We consider the case $k < \infty$ and use the $k \times k$-matrix $A = (A_{i+j} \mid 0 \leq i, j \leq k-1)$.

(5.1) Proposition. *Suppose A is invertible and $B = (B_{ij})$ an inverse. Then $\varepsilon = \sum_{i,j} B_{ij} Y^i E Y^j$ is a unit element of $\langle E \rangle$. If $(Y^i E Y^j \mid 0 \leq i, j \leq k-1)$ is a $\mathfrak{K}$-basis of $\langle E \rangle$ and $\langle E \rangle$ has a unit element, then A is invertible and ε is the unit element.*

PROOF. Let $\varepsilon = \sum_{i,j} B_{ij} Y^i E Y^j$ be an arbitrary element of $\langle E \rangle$. Then

$$\varepsilon Y^a E Y^b = \sum_{i,j} B_{ij} A_{j+a} Y^i E Y^b.$$

If ε is a unit element and (Y^aEY^b) a basis, then $\sum_j B_{ij}A_{j+a} = \delta_{ia}$, hence B is an inverse of A. The same computation shows that ε, constructed from an inverse B of A, is a unit element. □

Suppose from now on that the basic relations are satisfied. Then we have the fundamental right $\mathfrak{Z}$-module M at our disposal (4.12). We have an antihomomorphism of algebras

$$r\colon \langle\, E \,\rangle \to \mathrm{Hom}_{\mathfrak{K}}(M, M)$$

which maps x to the right multiplication r_x by x.

(5.2) Proposition. *Suppose A is invertible. Then r is an isomorphism and $(Y^iEY^j \mid 0 \le i, j \le k-1)$ is a basis of $\langle\, E \,\rangle$.*

Proof. Let $r_x = 0$. The right ideal $[E\,\rangle$ is a surjective image of M. Hence r_x is zero in $[E\,\rangle$. The right module $\langle\, E \,\rangle$ is the sum of the isomorphic right ideals $Y^i[E\,\rangle$. Hence right multiplication by x is zero in $\langle\, E \,\rangle$. Since this algebra has a unit, $x = 0$, and r is injective.

Let $x = \sum_{i,j} C_{ij}Y^iEY^j$. Then r_x has, with respect to ey^j, the matrix AC. Hence, if A is invertible, we can realize any matrix by some r_x. Therefore r is surjective too. This also shows the linear independence of the Y^iEY^j. □

If we work over a field $\mathfrak{K}$, the preceding results yield:

(5.3) Proposition. *The algebra $\mathfrak{Z}(k)$ is semi-simple if and only if $\mathfrak{Z}/\langle\, E \,\rangle$ is semi-simple and A invertible.* □

(5.4) Example. In the case $k = 2$, we have

$$\begin{vmatrix} A_0 & A_1 \\ A_1 & A_2 \end{vmatrix} = -(p_1A_0 + A_1)(p_2A_0 + A_1).$$

In the case $\delta = 0$, $Y^2 = 1$, this is non-zero iff $A_0 \ne \pm A_1$. If δ is invertible this is non-zero iff

$$(q - q^{-1}) - (\lambda - \lambda^{-1}) \ne \pm(p_1 - p_1^{-1} + p_2 - p_2^{-1}).$$

(By the way, the algebras should always be free as a $\mathfrak{K}$-module, if we only assume the basic equations.) ♡

6. Quantum algebra specializations

The generic ring for the definition of the algebra $\mathfrak{Z}(k)$ is obtained by treating the parameters as indeterminates and adding the relations $R_0, \dots, R_n, S_0, \dots, S_n$. This ring is not very tractable. Instead, one looks for specializations into rings which should fulfill the following requirements:

(1) Integral domain
(2) Allows the specialization $q = 1$
(3) Allows specialization to the Brauer algebras defined below

(4) Allows enough specializations so as to capture all information of the generic case
(5) Works for all representations via quantum algebra, e. g. R-matrices and four braid relations.

We suggest that $\mathfrak{R} = \mathbb{Z}[q^{\pm}, \mu_1^{\pm}, \dots, \mu_n^{\pm}]$ is a suitable ring ($k = 2n, 2n+1$). This is based on the following choice of parameters:

Case $k = 2n+1$.

$$\lambda = q^{-t}, \qquad q_k = -q^t$$

$$p_{2j-1} = \mu_j q^{r(j)}, \quad p_{2j} = \mu_j^{-1} q^{r(j)}, \qquad 1 \le j \le n,$$

$$p_{2n+1} = q^{r(2n+1)}$$

with $t = r(2n+1) + 2\sum_{j=1}^n r(j)$. Under these conditions, the coefficients of the polynomial $q(t) - q^*(t)$ are formally divisible by δ. The relations S_j therefore allow to express the a_j as elements of $\mathfrak{R}$, because $q_j^* - q_j$ is divisible by δ. The specialization to the Brauer algebra is obtained by $q = 1$ and $\{\mu_j, \mu_j^{-1} \mid 1 \le j \le n\}$ are the n-th roots of unity different from 1.

Case $k = 2n+2$.

$$\lambda = q^{-t-1}, \qquad q_k = -q^t$$

$$p_{2j-1} = \mu_j q^{r(j)}, \quad p_{2j} = \mu_j^{-1} q^{r(j)}, \qquad 1 \le j \le n,$$

$$p_{2n+1} = q^{r(2n+1)}, \qquad p_{2n+2} = -q^{r(2n+2)}$$

with $t = r(2n+1) + r(2n+2) + \sum_{j=1}^n r(j)$. The specialization to the Brauer algebra is obtained by $q = 1$ and $\{\mu_j, \mu_j^{-1} \mid 1 \le j \le n\}$ are the $1+n$-th roots of unity different from ± 1.

Since the exponents $r(j)$ can be chosen arbitrarily (subject to the requirement about the value of λq_k), we have for each of the parameters p_j an infinite number of specializations.

7. Coloured Brauer graphs

Let $k, l \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ be natural numbers. A (k,l)-*Brauer-graph* is a free involution σ on the set

$$P(k,l) = \{1, \dots, k\} \times 0 \cup \{1, \dots, l\} \times 1.$$

Our aim is to construct certain categories based on Brauer graphs. For this purpose we fix the following data:

(1) A commutative ring $\mathfrak{K}$.
(2) A group G.
(3) A function $z\colon G \to \mathfrak{K}$ with the properties $z(g) = z(g^{-1})$ and $z(ghg^{-1}) = z(h)$ for $g, h \in G$.

We call z the *cycle character*. (The property $z(g) = z(g^{-1})$ holds for characters of selfdual finite dimensional representations of G.)

A *G-coloured* (k,l)-Brauer-graph is a pair (σ, v) which consists of a (k,l)-Brauer-graph σ and a mapping $v\colon P(k,l) \to G$ satisfying the *balance relation* $v(\sigma x) = v(x)^{-1}$.

(7.1) Example. let M be an n-manifold and $D \subset M$ an n-disk. Then a path $w\colon I \to M$ with $w(0), w(1) \in D$ defines uniquely an element of the fundamental group $\pi = \pi_1(M)$ with base point in D. Choose a set $B_k \subset D$ with k points. A set of paths $[0,1] \to M \times [0,1]$ with total boundary point set $B_k \times 0 \cup B_l \times 1$ defines a π-coloured (k,l)-Brauer graph σ as the involution which maps the first point of a path to its last point. Its colouring is given by sending a path to the element of the fundamental group it defines. ♡

We construct a $\mathfrak{K}$-category $T(G,z)$. The objects are the natural numbers $n \in \mathbb{N}_0$. The morphism $\mathfrak{K}$-module $\mathrm{Mor}_{T(G,z)}(k,l)$ is the free $\mathfrak{K}$-module on the set of G-coloured (k,l)-graphs.

By definition of a $\mathfrak{K}$-category, the composition

$$\mathrm{Mor}(l,m) \times \mathrm{Mor}(k,l) \to \mathrm{Mor}(k,m), \quad (\beta,\alpha) \mapsto \beta \circ \alpha$$

has to be a $\mathfrak{K}$-bilinear map. Thus it suffices to specify the composition

$$(\tau, w) \circ (\sigma, v)$$

of two coloured Brauer graphs. The definition of the composition uses the cycle character z.

Let $|\sigma, v|$ denote the one-dimensional CW-complex with set of zero-cells $P(k,l)$ and a one-cell for each orbit $\{x, \sigma x\}$ of σ with boundary set $\{x, \sigma x\}$. Picking x in the orbit $\{x, \sigma x\}$ of σ amounts to the choice of an orientation of the one-cell, directed from x to σx. We define an oriented 1-cochain by mapping the oriented 1-cell $\{x, \sigma x\}$ to $v(x)$. This assigment is called the colouring of $|\sigma, v|$.

Let (τ, w) be an (l,m)-graph. We define a one-complex

$$|\tau, w| \cup |\sigma, v|$$

by identifying the 0-cell $(a,1) \in |\sigma, v|$ with the 0-cell $(a,0) \in |\tau, w|$ for $1 \leq a \leq l$. The path components of this complex are either homeomorphic to the unit interval I or to the circle S^1. We orient the path components of this complex. Each oriented 1-cell of a path component is then a 1-cell of $|\sigma, v|$ or of $|\tau, w|$ and has an assigned colour in G. We assign to an interval component the value $g_r \cdots g_2 g_1$ if $g_1, \ldots, g_r$ are the colours in consecutive order with respect to the chosen orientation. Similarly, to a circle component we assign the colour $g_r \cdots g_1$ by starting at some vertex and then running around cyclically according to the chosen orientation. The colour of an oriented circle component is in general not well defined; but the conjugacy class of the colour is independent of the choice of the initial vertex.

The interval components of $|\tau, w| \cup |\sigma, v|$ with the colours just defined form a G-coloured (k, m)-graph

$$|\tau, w| \wedge |\sigma, v|.$$

By assumption about z, the value $z(g) = z(S)$ of a g-valued oriented circle component S does not depend on the orientation and the choice of the initial vertex. We now define the composition in our category by

$$(\tau, w) \circ (\sigma, v) = \prod_{S\, circle} z(S)\, |\tau, w| \wedge |\sigma, v|.$$

The product is taken over the circle components. It is immediate from the construction, that composition is associative. The identity of the object k is given by the involution $\sigma(a, 0) = (a, 1)$ and colour the neutral element $e \in G$ on each orbit of σ. If $k = 0$, we have to consider the free involution of the empty set; there are no orbits, hence no colours.

Remark. The category $T(G, z)$ may be called the semi-oriented or balanced category. There are analogous unoriented categories and categories of oriented Brauer graphs. In the latter case, the objects are functions $\{1, 2, \ldots, k\} \to \{\pm 1\}$. Can one use character theory to describe the dependence of $T(G, z)$ on z? ♡

8. An example

In order to get a feeling for the definition of $T(G, z)$, we describe the special case of the endomorphism algebra of the object 2. Suppose $k + l = 2n$. There are $1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2n - 1 = (2n - 1)!!$ (k, l)-Brauer-graphs. Therefore, if G is finite, $\mathrm{Mor}(k, l)$ is a free $\mathfrak{K}$-module of rank $|G|^n (2n - 1)!$. In particular, this is the rank of the endomorphism algebra of the object n in $T(G, z)$. As usual, we visualize a (k, l)-Brauer-graph σ by drawing an arc from x to σx in the strip $\mathbb{R} \times [0, 1]$. If the graph is coloured, we orient the arcs and write the corresponding colour next to it.

The endomorphism algebra T_2 of the object 2 thus has the following $\mathfrak{K}$-basis.

Figure

The composition of these morphisms is given by

$$\begin{aligned}
I(a,b)I(c,d) &= I(ac,bd) \\
I(a,b)X(c,d) &= X(ac,bd) \\
X(a,b)I(c,d) &= X(ad,bc) \\
X(a,b)X(c,d) &= I(ad,bc) \\
I(a,b)E(c,d) &= E(c,bda^{-1}) \\
E(a,b)I(c,d) &= E(d^{-1}ac,b) \\
X(a,b)E(c,d) &= E(c,bd^{-1}a^{-1}) \\
E(a,b)X(c,d) &= E(c^{-1}a^{-1}d,b) \\
E(a,b)E(c,d) &= z(a^{-1}d)E(c,b).
\end{aligned}$$

We describe this algebra by generators and relations.

(8.1) Definition. Let U_2 denote the associative unital $\mathfrak{K}$-algebra gerated by X, E, $Y \in G$ with relations

(1) $X^2 = 1$
(2) $EX = XE = 1$
(3) The elements of G multiply according to the group multiplication
(4) $YXZX = XZXY$, for $Y, Z \in G$
(5) $EYXZX = EZ^{-1}Y$, for $Y, Z \in G$
(6) $XZXYE = YZ^{-1}E$, for $Y, Z \in G$
(7) $EYE = z(Y)E$, for $Y \in G$.

(The relation (4) is a generalized four braid relation.) ♡

(8.2) Proposition. *The assignment* $X \mapsto X(1,1)$, $Y \mapsto I(Y,1)$, $E \mapsto E(1,1)$ *yields an isomorphism* $\psi \colon U_2 \to T_2$.

Proof. It is easily verified that the assignment is compatible with the relations of U_2 and therefore yields a well-defined homomorphism ψ of algebras.

The relations

$$\begin{aligned} I(Y,1)X(1,1)I(Z,1) &= X(Y,Z) \\ I(Y,1)X(1,1)I(Z,1)X(1,1) &= I(Y,Z) \\ I(Z^{-1},1)E(1,1)I(Y,1) &= E(Y,Z) \end{aligned}$$

show that ψ is surjective.

Injectivity follows if we show that U_2 is generated as a $\mathfrak{K}$-module by elements of the form $YXZ, YXZX, YEZ$ for $Y, Z \in G$. One first verifies from the relations that the elements of the form YEZ span a twosided ideal $\langle E \rangle$. We factor out this ideal. The resulting algebra has generators $X, Y \in G$ with relations $X^2 = 1$, $YXZX = XZXY$. It is easily seen that this algebra is spanned by elements of the form YXZ and $YXZX$. It is also the group algebra $\mathfrak{K}\Gamma$ of the semi-direct product

$$1 \to G \times G \to \Gamma \to \mathbb{Z}/2 \to 1$$

where $\mathbb{Z}/2$ is generated by X and interchanges the factors of G. □

(8.3) Remark. If the group G is given by a presentation, then we can use this presentation in the presentation of the algebra U_2. Thus, if $G = \langle Y \mid Y^k = 1 \rangle$, then U_2 is generated by X, Y, E with relations $X^2 = 1$, $Y^k = 1$, $XYXY = YXYX$, $EX = E = XE$, $EY^iE = z(Y^i)E$, $EXYXY = E = XYXYE$. This is a special case of our algebras $\mathfrak{Z}(k)$, namely for the parameters $\lambda = q = 1$, $z(Y^i) = A_i$. ♡