Quantum groups and cylinder braiding

Tammo tom Dieck and Reinhard Häring-Oldenburg

Abstract. The purpose of this paper is to introduce a new structure into the representation theory of quantum groups. The structure is motivated by braid and knot theory. Representations of quantum groups associated to classical Lie algebras have an additional symmetry which cannot be seen in the classical limit. We first explain the general formalism of these symmetries (called *cylinder forms*) in the context of comodules. Basic ingredients are tensor representations of braid groups of type B derived from standard R-matrices associated to so-called four braid pairs. These are applied to the Faddeev-Reshetikhin-Takhtadjian construction of bialgebras from R-matrices. As a consequence one obtains four braid pairs on all representations of the quantum group. In the second part of the paper we study in detail the dual situation of modules over the quantum enveloping algebra $U_q(sl_2)$. The main result here is the computation of the *universal cylinder twist*.

1. Cylinder forms

Let $A = (A, m, e, \mu, \varepsilon)$ be a bialgebra[1] (over the commutative ring $\mathfrak{K}$) with multiplication m, unit e, comultiplication μ, and counit ε. Let $r: A \otimes A \to \mathfrak{K}$ be a linear form. We associate to left A-comodules M, N a $\mathfrak{K}$-linear map[2]

$$z_{M,N}: M \otimes N \to N \otimes M, \quad x \otimes y \mapsto \sum r(y^1 \otimes x^1) y^2 \otimes x^2,$$

where we have used the formal notation[3] $x \mapsto \sum x^1 \otimes x^2$ for a left A-comodule structure $\mu_M: M \to A \otimes M$ on M. (See [7, p. 186] formula (5.9) for our map $z_{M,N}$ and also formula (5.8) for a categorical definition.) We call r a *braid form* on A, if the $z_{M,N}$ yield a braiding on the tensor category A-COM of left A-comodules.

[1] We prefer Latin-Greek duality; thus the comultiplication is not Δ.

[2] A notation like $f: A \to B$, $a \mapsto f(a)$ is used in two different ways. Either the symbols $a \mapsto f(a)$ *define* $f: A \to B$, or they specify a *notation* for an already defined morphism. For typographical reasons it often seems better not to obscure and interrupt formulas by inserting phrases like 'defined by'.

[3] In the literature, notations of this type are attributed to Heyneman and Sweedler. Also the name *sigma notation* is used. We refer to this notation as the *μ-convention.* In this paper, we use superscripts for comodule actions and supscripts for comultiplications. The reader may notice that by proper use of the μ-convention summation indices are redundant. Strings of super- or supscripts have to be lexicographic sequences without gaps, and associativity amounts to the rule that any such string can be replaced by another one of the same length.

We refer to [7, Def. VIII.5.1 on p. 184] for the properties of r which make it into a braid form and (A, r) into a cobraided bialgebra. (What we call braid form is called universal R-form in [7].)

Let (C, μ, ε) be a coalgebra. Examples of our μ-convention for coalgebras are $\mu(a) = \sum a_1 \otimes a_2$ and $(\mu \otimes 1)\mu(a) = \sum a_{11} \otimes a_{12} \otimes a_2$; if we set $\mu_2(a) = (\mu \otimes 1)\mu$, then we write $\mu_2(a) = \sum a_1 \otimes a_2 \otimes a_3$. The counit axiom reads in this notation $\sum \varepsilon(a_1)a_2 = a = \sum \varepsilon(a_2)a_1$. The multiplication in the dual algebra C^* is denoted by $*$ and called convolution: If $f, g \in C^*$ are $\mathfrak{K}$-linear forms on C, then the convolution product $f * g$ is the element of C^* defined by $a \mapsto \sum f(a_1)g(a_2)$. The unit element of the algebra C^* is ε. Therefore g is a (convolution) inverse of f, if $f * g = g * f = \varepsilon$. We apply this formalism to the coalgebras A and $A \otimes A$. If f and g are linear forms on A, we denote their exterior tensor product by $f \hat{\otimes} g$; it is the linear form on $A \otimes A$ defined by $a \otimes b \mapsto f(a)g(b)$. The twist on $A \otimes A$ is $\tau(a \otimes b) = b \otimes a$.

Here is the main definition of this paper. Let (A, r) be a cobraided bialgebra with braid form r. A linear form $f\colon A \to \mathfrak{K}$ is called a *cylinder form* for (A, r), if it is convolution invertible and satisfies

(1.1) $$f \circ m = (f \hat{\otimes} \varepsilon) * r\tau * (\varepsilon \hat{\otimes} f) * r = r\tau * (\varepsilon \hat{\otimes} f) * r * (f \hat{\otimes} \varepsilon).$$

In terms of elements and the μ-convention, (1.1) assumes the following form:

(1.2) Proposition. *For any two elements $a, b \in A$ the identities*

$$f(ab) = \sum f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3) = \sum r(b_1 \otimes a_1) f(b_2) r(a_2 \otimes b_3) f(a_3)$$

hold.

PROOF. Note that a four-fold convolution product is computed by the formula

$$(f_1 * f_2 * f_3 * f_4)(x) = \sum f_1(x_1) f_2(x_2) f_3(x_3) f_4(x_4).$$

We apply this to the second term in (1.1). The value on $a \otimes b$ is then

$$(f(a_1) \cdot \varepsilon(b_1)) \cdot r(b_2 \otimes a_2) \cdot (\varepsilon(a_3) \cdot f(b_3)) \cdot r(a_4 \otimes b_4).$$

By the counit axiom, we can replace $\sum \varepsilon(b_1) \cdot b_2 \otimes b_3 \otimes b_4$ by $\sum b_1 \otimes b_2 \otimes b_3$ (an exercise in the μ-convention), and $\sum a_1 \otimes a_2 \otimes \varepsilon(a_3) \cdot a_4$ can be replaced by $\sum a_1 \otimes a_2 \otimes a_3$. This replacement yields the second expression in (1.2). The third expression is verified in a similar manner. The first value is obtained from the definition of $f \circ m$. $\square$

A cylinder form f (in fact any linear form) yields for each left A-comodule M a $\mathfrak{K}$-linear endomorphism

$$t_M\colon M \to M, \quad x \mapsto \sum f(x^1)x^2.$$

If $\varphi\colon M \to N$ is a morphism of comodules, then $\varphi \circ t_M = t_N \circ \varphi$. Since t_M is in general not a morphism of comodules we express this fact by saying: The t_M

constitute a *weak endomorphism of the identity functor* of A-COM. We call t_M the *cylinder twist* on M. The axiom (1.1) for a cylinder form has the following consequence.

(1.3) Proposition. *The linear map t_M is invertible. For any two comodules M, N the identities*

$$t_{M\otimes N} = z_{N,M}(t_N \otimes 1_M)z_{M,N}(t_M \otimes 1_N) = (t_M \otimes 1_N)z_{N,M}(t_N \otimes 1_M)z_{M,N}$$

hold.

PROOF. Let g be a convolution inverse of f. Define the endomorphism $s_M\colon M \to M$ via $x \mapsto \sum g(x^1)x^2$. Then

$$s_M t_M(x) = \sum f(x^1)g(x^{21})x^{22} = \sum \varepsilon(x^1)x^2 = x,$$

by the definition of a convolution inverse and the counit axiom. Hence s_M is inverse to t_M.

In order to verify the second equality, we insert the definitions and see that the second map is

$$x \otimes y \mapsto \sum f(x^1)r(y^1 \otimes x^{21})f(y^{21})r(y^{221} \otimes x^{221})y^{222} \otimes x^{222}$$

while the third map is

$$x \otimes y \mapsto \sum r(y^1 \otimes x^1)f(y^{21})r(y^{21} \otimes x^{221})f(x^{221})y^{222} \otimes x^{222}.$$

The coassociativity of the comodule structure yields a rewriting of the form

$$\sum y^1 \otimes y^{21} \otimes y^{221} \otimes y^{222} = \sum (y^1)_1 \otimes (y^1)_2 \otimes (y^1)_3 \otimes y^2$$

and one has a similar formula for x. We now apply (1.2) in the case where $(a, b) = (x^1, y^1)$.

By definition of the comodule structure of $M \otimes N$, the map $t_{M\otimes N}$ has the form $x \otimes y \mapsto \sum f(x^1y^1)x^2 \otimes y^2$. Again we use (1.2) in the case where $(a, b) = (x^1, y^1)$ and obtain the first equality of (1.3). □

We also mention dual notions. Let A be a bialgebra with a universal R-matrix $R \in A \otimes A$. An element $v \in A$ is called a (universal) *cylinder twist* for (A, R), if it is invertible and satisfies

(1.4) $$\mu(v) = (v \otimes 1) \cdot \tau R \cdot (1 \otimes v) \cdot R = \tau R \cdot (1 \otimes v) \cdot R \cdot (v \otimes 1).$$

The R-matrix $R = \sum a_r \otimes b_r$ induces the braiding

$$z_{M,N}\colon M \otimes N \to N \otimes M,\ x \otimes y \mapsto \sum b_r y \otimes a_r x.$$

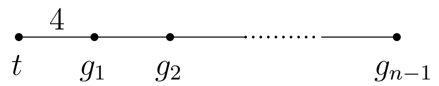
Let $t_M\colon M \to M$ be the induced cylinder twist defined by $x \mapsto vx$. Again the t_M form a weak endomorphism of the identity functor. If v is not central in A, then the t_M are not in general A-module morphisms. The relations (1.3) also holds in this context.

In practice one has to consider variants of this definition. The universal R-matrix for the classical quantum groups A is not contained in the algebra $A \otimes A$, but rather is an operator on suitable modules. The same phenomenon will occur for the cylinder twist. Will see an example of this situation in Section 8.

If a ribbon algebra is defined as in [7, p. 361], then the element θ^{-1}, *loc. cit.*, is a cylinder twist in the sense above.

2. Tensor representations of braid groups

The braid group ZB_n associated to the Coxeter graph B_n



B_n

with n vertices has generators $t, g_1, \ldots, g_{n-1}$ and relations (2.1).

(2.1)
$$\begin{array}{rcll} tg_1tg_1 & = & g_1tg_1t & \\ tg_i & = & g_it & \text{for} \quad i > 1 \\ g_ig_j & = & g_jg_i & \text{for} \quad |i-j| \geq 2 \\ g_ig_jg_i & = & g_jg_ig_j & \text{for} \quad |i-j| = 1 \end{array}$$

We recall: The group ZB_n is the group of braids with n strings in the cylinder $(\mathbb{C} \setminus 0) \times [0,1]$ from $\{1, \ldots, n\} \times 0$ to $\{1, \ldots, n\} \times 1$. This topological interpretation is the reason for using the cylinder terminology. For the relation between the root system B_n and ZB_n see [2].

Let V be a $\mathfrak{K}$-module. Suppose $X\colon V \otimes V \to V \otimes V$ and $F\colon V \to V$ are $\mathfrak{K}$-linear automorphisms with the following properties:

(1) X is a Yang-Baxter operator, i. e., X satisfies the equation

$$(X \otimes 1)(1 \otimes X)(X \otimes 1) = (1 \otimes X)(X \otimes 1)(1 \otimes X)$$

on $V \otimes V \otimes V$.

(2) With $Y = F \otimes 1_V$, the *four braid relation* $YXYX = XYXY$ is satisfied.

If (1) and (2) hold, we call (X, F) a *four braid pair.* For the construction of four braid pairs associated to standard R-matrices see [4]. For a geometric interpretation of (2) in terms of symmetric braids with 4 strings see [3].

Given a four braid pair (X, F), we obtain a tensor representation of ZB_n on the n-fold tensor power $V^{\otimes n}$ of V by the following assignment:

(2.2)
$$\begin{array}{rcl} t & \mapsto & F \otimes 1 \otimes \cdots \otimes 1 \\ g_i & \mapsto & X_i = 1 \otimes \cdots \otimes X \otimes \cdots \otimes 1. \end{array}$$

The X in X_i acts on factors i and $i+1$.

These representations give raise to further operators if we apply them to special elements in the braid groups. We set

$$t(1) = t, \quad t(j) = g_{j-1}g_{j-2}\cdots g_1 t g_1 g_2 \cdots g_{j-1}, \quad t_n = t(1)t(2)\cdots t(n),$$

$$g(j) = g_j g_{j+1}\cdots g_{j+n-1}, \quad x_{m,n} = g(m)g(m-1)\cdots g(1).$$

The elements $t(j)$ pairwise commute. We denote by $T_n\colon V^{\otimes n} \to V^{\otimes n}$ and by $X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$, respectively, the operators induced by t_n and by $x_{m,n}$.

(2.3) Proposition. *The following identities hold*

$$T_{m+n} = X_{n,m}(T_n \otimes 1)X_{m,n}(T_m \otimes 1) = (T_m \otimes 1)X_{n,m}(T_n \otimes 1)X_{m,n}.$$

Proof. We use some facts about Coxeter groups [1, CH. IV, §1]. If we adjoin the relations $t^2 = 1$ and $g_j^2 = 1$ to (2.1) we obtain the Coxeter group CB_n. The element t_n is given as a product of n^2 generators t, g_j. The uniquely determined element of CB_n has length n^2 and is equal to t_n. The element $x_{n,m}t_n x_{m,n} t_m$ of CB_{m+n} has length $(m+n)^2$ and therefore equals t_{m+n} in CB_{m+n}. By a fundamental fact about braid groups [1, CH. IV, §1.5, Prop. 5], the corresponding elements in the braid group are equal. We now apply the tensor representation and obtain the first equality of (2.3). □

For later use we record:

(2.4) Proposition. *The element t_n is contained in the center of ZB_n.* □

3. Cylinder forms from four braid pairs

Let V be a free $\mathfrak{K}$-module with basis $\{v_1, \ldots, v_n\}$. Associated to a Yang-Baxter operator $X\colon V \otimes V \to V \otimes V$ is a bialgebra $A = A(V, X)$ with braid form r, obtained via the FRT-construction (see [7, VIII.6 for the construction of A and r). We show that a four braid pair (X, F) induces a canonical cylinder form on (A, r).

Recall that A is a quotient of a free algebra $\tilde{A}$. We use the model

$$\tilde{A} := \bigoplus_{n=0}^{\infty} \mathrm{Hom}(V^{\otimes n}, V^{\otimes n}).$$

The multiplication of $\tilde{A}$ is given by the canonical identification $E_k \otimes E_l \cong E_{k+l}, furnished by f \otimes g \mapsto f \otimes g$ where $E_k = \mathrm{Hom}(V^{\otimes k}, V^{\otimes k})$. The canonical basis of E, given by $T_i^j\colon v_k \mapsto \delta_{i,k} v_j$ of E_1, induces the basis

$$T_i^j = T_{i_1}^{j_1} \otimes \cdots \otimes T_{i_k}^{j_k}$$

of E_k, where, in multi-index notation, $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_k)$. The comultiplication of $\tilde{A}$ is given by $\mu(T_i^j) = \sum_k T_i^k \otimes T_k^j$ while the counit of $\tilde{A}$ is given by $\varepsilon(T_i^j) = \delta_i^j$.

In Section 2 we defined an operator $T_k \in E_k$ from a given four braid pair (X, F). We express T_k in terms of our basis

$$T_k(v_i) = \sum_j F_i^j v_j$$

again using the multi-index notation $v_i = v_{i_1} \otimes \cdots \otimes v_{i_k}$ when $i = (i_1, \ldots, i_k)$. We use these data in order to define a linear form

$$\tilde{f}\colon \tilde{A} \to \mathfrak{K}, \quad T_i^j \mapsto F_i^j.$$

(3.1) Theorem. *The linear form $\tilde{f}$ factors through the quotient map $\tilde{A} \to A$ and induces a cylinder form f for (A, r).*

Proof. Suppose the operator $X = X_{m,n}\colon V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ has the form $X(v_i \otimes v_j) = \sum_{ab} X_{ij}^{ab} v_a \otimes v_b$. We define a form $\tilde{r}\colon \tilde{A} \otimes \tilde{A} \to \mathfrak{K}$ by defining

$$\tilde{r}\colon E_k \otimes E_l \to \mathfrak{K}, \quad T_i^a \otimes T_j^b \mapsto X_{ji}^{ab}.$$

The form $\tilde{r}$ factors through the quotient $A \otimes A$ and induces r.

Claim: The forms $\tilde{r}$ and $\tilde{f}$ satisfy (1.1) and (1.2). *Proof of the Claim:* In the proof we use the following summation convention: Summation occurs over an upper-lower index. We can then write $\mu_2(T_i^c) = T_i^k \otimes T_k^a \otimes T_a^c$ and $\mu_2(T_j^d) = T_j^l \otimes T_l^b \otimes T_b^d$. The equality (1.2) amounts to

$$F_{ij}^{cd} = F_i^k X_{kj}^{la} F_l^b X_{ba}^{cd} = X_{ij}^{lk} F_l^b X_{bk}^{ad} F_a^c.$$

These equations are also a translation of (2.3) into matrix form. This completes the proof of the claim.

We have to show that $\tilde{f}$ maps the kernel I of the projection $\tilde{A} \to A$ to zero. But this is a consequence of (1.2), applied in the case $b = 1$, since one of the terms a_1, a_2, a_3 is contained in I and $\tilde{r}$ is the zero map on $I \otimes \tilde{A}$ and $\tilde{A} \otimes I$.

It remains to show that f is convolution invertible. The pair (X^{-1}, F^{-1}) is a four braid pair. Let $\bar{r}$ and $\bar{f}$ be the induced operators on $\tilde{A}$. Then $\tilde{f} * \bar{f} = \varepsilon = \bar{f} * \tilde{f}$ on $\tilde{A}$, and (1.2) holds for $(\bar{f}, \bar{r})$ in place of (f, r). The Yang-Baxter operator X^{-1} defines the same quotient A of $\tilde{A}$ as X. Hence the kernel ideal obtained from X^{-1} equals I; therefore $\bar{f}(I) = 0$. □

We have the comodule structure map $V \to A \otimes V$ defined via $v_i \mapsto \sum_j T_i^j \otimes v_j$. One has a similar formula for $V^{\otimes k}$ using multi-index notation. By construction we have:

(3.2) Proposition. *The cylinder form f induces on $V^{\otimes k}$ the cylinder twist $t_{V^{\otimes k}} = T_k$.* □

4. Tensor categories with cylinder braiding

We may summarize the results of the previous section from the following categorical perspective:

(1) $\mathcal{B}$ is a category;

(2) $\mathcal{A}$ is a subcategory with the same objects;

(3) The category $\mathcal{A}$ carries the structure of a tensor category with a braiding $z_{M,N}$;

(4) $\mathcal{B}$ is a right tensor module category over $\mathcal{A}$;

(5) For each object V an automorphism $t_V\colon V \to V$ in $\mathcal{B}$ is given. The t_V constitute a *weak endomorphism of the identity functor* of $\mathcal{A}$.

(6) For each pair M, N of objects the identities (1.3) hold.

The meaning of (4) is the following: There is given a functor $\otimes\colon \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ and a natural associativity isomorphism $a\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ of functors $\mathcal{B} \times \mathcal{A} \times \mathcal{A} \to \mathcal{B}$. The pentagon axiom of tensor category theory (which still makes sense in this context) is also assumed. The tensor product functor and the associativity a restrict to the given tensor product and associativity in the tensor category $\mathcal{A}$. The unit object of $\mathcal{A}$ is a left and right unit for $\otimes\colon \mathcal{B} \times \mathcal{A} \to \mathcal{B}$ and the triangle axiom holds. An example of this type of module category arises from a tensor category $\mathcal{B}$ and a tensor subcategory $\mathcal{A}$. (See [7, XI.2] for such notions.)

We considered the case where $\mathcal{B}$ was the category of A-comodules and $\mathfrak{K}$-linear maps and $\mathcal{A}$ the category of A-comodules and A-linear maps. (5) is induced by a cylinder form.

The prototype is given by the braid categories themselves. The objects are the natural numbers $n \geq 0$. The morphisms in $\mathcal{B}$ from n to n are the elements in ZB_n with composition the group multiplication. There are no morphisms from m to n for $m \neq n$. The morphisms in $\mathcal{A}$ from n to n are the elements of the Artin braid group ZA_{n-1}, the subgroup of ZB_n generated by $g_1, \ldots, g_{n-1}$. The tensor product is given on objects as $m \otimes n = m + n$ and on morphisms as the following homomorphism $ZB_m \times ZB_n \to ZB_{m+n}$

$$t, g_1, \ldots, g_{m-1} \in ZB_m \mapsto t, g_1, \ldots, g_{m-1} \in ZB_{m+n}$$

$$t, g_1, \ldots, g_{n-1} \in ZB_n \mapsto g_m g_{m-1} \ldots g_1 t g_1 g_2 \ldots g_m, g_{m+1}, \ldots, g_{m+n-1} \in ZB_{m+n}.$$

The braiding is given by the morphisms $x_{m,n}$ of section 2 and the morphisms t_n are also specified in that section. By (2.4), the t_n constitute an endomorphism of the identity of $\mathcal{B}$.

There is a natural quotient category of this braid category (when $\mathfrak{K}$-linearized), namely the Temperley-Lieb category of type B via the Kauffman functor (see [3]).

For an elaboration of the categorical viewpoint and applications to knot theory along the lines of [9] see [5] and [6].

5. The example $SL_q(2)$

We illustrate the theory with the quantum group associated to SL_2. For simplicity we work over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$.

Let V be a two-dimensional $\mathfrak{K}$-module with basis $\{v_1, v_2\}$. In terms of the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$ the matrix

(5.1)
$$X = q^{-1/2} \begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q \end{pmatrix}$$

defines a Yang-Baxter operator. The FRT-construction associates to X the algebra A generated by a, b, c, and d (corresponding, respectively, to T_1^1, T_1^2, T_2^1, and T_2^2 in the general setting) with relations

$$\begin{aligned} ab &= qba & bd &= qdb \\ ac &= qca & cd &= qdc \end{aligned}$$
$$\begin{aligned} bc &= cb \\ ad - da &= (q-q^{-1})bc. \end{aligned}$$

The matrix

(5.2)
$$F = \begin{pmatrix} 0 & \beta \\ \alpha & \theta \end{pmatrix}$$

yields a four braid pair (X, F) for arbitrary parameters with invertible $\alpha\beta$. (See [4], also for an n-dimensional generalization.) The quantum plane $P = \mathfrak{K}\{x, y\}/(xy - qyx)$ is a left A-comodule via the map $\mu_P \colon P \to A \otimes P$ given by

$$\mu_P(x^i y^j) = \sum_{r=0}^{i} \sum_{s=0}^{j} q^{-s(i+j-r-s)-r(i-r)} \begin{bmatrix} i \\ r \end{bmatrix} \begin{bmatrix} j \\ s \end{bmatrix} a^r b^{i-r} c^s d^{j-s} \otimes x^{r+s} y^{i+j-r-s}$$

where $\begin{bmatrix} i \\ r \end{bmatrix}$ is a q-binomial coefficient

$$\begin{bmatrix} i \\ r \end{bmatrix} = \frac{[i]!}{[r]![i-r]!}, \quad [i]! = [1][2]\cdots[i], \quad [i] = \frac{q^i - q^{-i}}{q - q^{-1}}.$$

(Compare with [7, IV], where different conventions are used.) The operator $T_2 = (F \otimes 1)X(F \otimes 1)X$ on $V \otimes V$ has the matrix (with $\delta = q - q^{-1}$)

$$\begin{pmatrix} 0 & 0 & 0 & \beta^2 \\ 0 & \alpha\beta\delta & \alpha\beta & q\beta\theta \\ 0 & \alpha\beta & 0 & \beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix} = \begin{pmatrix} F_{11}^{11} & F_{12}^{11} & F_{21}^{11} & F_{22}^{11} \\ F_{11}^{12} & F_{12}^{12} & F_{21}^{12} & F_{22}^{12} \\ F_{11}^{21} & F_{12}^{21} & F_{21}^{21} & F_{22}^{21} \\ F_{11}^{22} & F_{12}^{22} & F_{21}^{22} & F_{22}^{22} \end{pmatrix}$$

with respect to the basis $\{v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2\}$. This is also the matrix of values of the cylinder form f

$$f\begin{pmatrix} aa & ac & ca & cc \\ ab & ad & cb & cd \\ ba & bc & da & dc \\ bb & bd & db & dd \end{pmatrix}.$$

(The notation means: If we apply f to entries of the matrix we obtain the matrix for T_2 diplayed above.) Let $\det_q = ad - qbc$ be the quantum determinant. It is a group-like central element of A. The quotient of A by the ideal generated by $\det_q$ is the Hopf algebra $SL_q(2)$.

(5.3) Proposition. *The form f has the value $-q^{-1}\alpha\beta$ on $\det_q$. If $-q^{-1}\alpha\beta = 1$, then f factors over $SL_q(2)$.*

Proof. The stated value of $f(\det_q)$ is computed from the data above. We use the fact that

$$r(x \otimes \det_q) = r(\det_q \otimes x) = \varepsilon(x).$$

(See [7, p. 195].) From (1.2) we obtain, for $a \in A$ and $b = \det_q$, that

$$\begin{aligned} f(ab) &= \sum f(a_1) r(b_1 \otimes a_2) f(b_2) r(a_3 \otimes b_3) \\ &= \sum f(a_1)\varepsilon(a_2) f(\det_q)\varepsilon(a_3) \\ &= f(a), \end{aligned}$$

by using the assumption that $f(\det_q) = 1$ together with the counit axiom. $\square$

We consider the subspace $W = V_2$ of the quantum plane generated by x^2, xy, and y^2. We have

$$\begin{aligned} \mu_P(x^2) &= b^2 \otimes y^2 + (1 + q^{-2})ab \otimes xy + a^2 \otimes x^2 \\ \mu_P(xy) &= bd \otimes y^2 + (ad + q^{-1}bc) \otimes xy + ac \otimes x^2 \\ \mu_P(y^2) &= d^2 \otimes y^2 + (1 + q^{-2})cd \otimes xy + c^2 \otimes x^2. \end{aligned}$$

This yields the following matrix for t_W with respect to the basis $\{x^2, xy, y^2\}$:

$$\begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & (q + q^{-1})\beta\theta \\ \alpha^2 & q\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In the Clebsch-Gordan decomposition $V \otimes V = V_2 \oplus V_0$ the subspace V_0 (the trivial irreducible module) is spanned by $u = v_2 \otimes v_1 - q^{-1} v_1 \otimes v_2$. This is the eigenvector of X with eigenvalue $-q^{-3/2}$. It is mapped by T_2 to $-q^{-1}\alpha\beta u$. If we require this to be the identity we must have $\alpha\beta = -q$. We already obtained this condition by considering the quantum determinant.

The matrix of t_W with respect to the basis $\{w_1 = x^2, w_2 = \sqrt{1 + q^{-2}}xy, w_3 = y^2\}$ is

(5.4)
$$F_2 = \begin{pmatrix} 0 & 0 & \beta^2 \\ 0 & q\alpha\beta & \sqrt{1+q^2}\beta\theta \\ \alpha^2 & \sqrt{1+q^2}\alpha\theta & \alpha\beta\delta + q\theta^2 \end{pmatrix}.$$

In case $\alpha = \beta$ this matrix is symmetric.

The R-matrix X on $W \otimes W$ with respect to the *lexicographic basis* consisting of elements $w_i \otimes w_j$ with $w_1 = x^2, w_2 = \sqrt{1+q^{-2}}xy$, and $w_3 = y^2$ has the form

(5.5)
$$X_2 = \left|\begin{array}{ccc|ccc|ccc} q^2 & & & & & & & & \\ & \delta^* & & 1 & & & & & \\ & & \mu & & \lambda & & q^{-2} & & \\ \hline & 1 & & 0 & & & & & \\ & & \lambda & & 1 & & & & \\ & & & & & \delta^* & & 1 & \\ \hline & & q^{-2} & & & & 0 & & \\ & & & & & 1 & & 0 & \\ & & & & & & & & q^2 \end{array}\right|$$

It makes use of the identities $\delta^* = q^2 - q^{-2}$, $\mu = \delta^*(1 - q^{-2})$, and $\lambda = q^{-1}\delta^*$. By construction, (X_2, F_2) is a four braid pair.

One has the the problem of computing t_W on irreducible comodules W. We treat instead the more familiar dual situation of modules over the quantized universal enveloping algebra.

6. The cylinder braiding for U-modules

The construction of the cylinder form is the simplest method to produce a universal operator for the cylinder twist. In order to compute the cylinder twist explicitly we pass to the dual situation of the quantized universal enveloping algebra U. One can formally dualize comodules to modules and thus obtain a cylinder braiding for suitable classes of U-modules from the results of the previous sections. But we rather start from scratch.

We work with the Hopf algebra $U = U_q(sl_2)$ as in [8]. As an algebra, it is the the associative algebra over the function field $\mathbb{Q}(q^{1/2}) = \mathfrak{K}$ generated[4] by K, K^{-1}, E, and F subject to the relations $KK^{-1} = K^{-1}K = 1$, $KE = q^2EK$, $KF = q^{-2}FK$, and $EF - FE = (K - K^{-1})/(q - q^{-1})$. Its coalgebra structure is defined by setting $\mu(K) = K \otimes K$, $\mu(E) = E \otimes 1 + K \otimes E$, $\mu(F) = F \otimes K^{-1} + 1 \otimes F$, $\varepsilon(K) = 1$, and $\varepsilon(E) = \varepsilon(F) = 0$. A left U-module M is called *integrable* if the following condition holds:

(1) $M = \bigoplus M^n$ is the direct sum of weight spaces M^n on which K acts as multiplication by q^n for $n \in \mathbb{Z}$.

(2) E and F are locally nilpotent on M.

Let U-INT denote the *category of integrable U-modules* and U-linear maps. (It would be sufficient to consider only finite dimensional such modules.) An

[4] There is another use of the letter F. It has nothing to do with the 2×2-matrix F in (5.2).

integrable U-module M is semi-simple: It has a unique isotypic decomposition $M = \bigoplus_{n\ge 0} M(n)$ with $M(n)$ isomorphic to a direct sum of copies of the irreducible module V_n. The module V_n has a $\mathfrak{K}$-basis $x_0, x_1, \dots, x_n$ with $F(x_i) = [i+1]x_{i+1}$, $E(x_i) = [n-i+1]x_{i-1}$, $x_{-1} = 0$, $x_{n+1} = 0$; moreover, $x_i \in V_n^{n-2i}$. The category of integrable U-modules is *braided*. The braiding is induced by the universal R-matrix $R = \kappa \circ \Psi$ with

(6.1)
$$\Psi = \sum_{n\ge 0} q^{n(n-1)/2} \frac{(q-q^{-1})^n}{[n]!} F^n \otimes E^n$$

and $\kappa = q^{H\otimes H/2}$. Note that Ψ is a well-defined operator on integrable U-modules. (This operator is called $\bar\Theta$ in [8, section 4.1] and L_i' in [8, p. 46].) The operator κ acts on $M^m \otimes N^n$ as multiplication by $q^{mn/2}$. If we view H as the operator $H\colon M^m \to M^m$ given by $x \mapsto mx$, then $q^{H\otimes H/2}$ is a suggestive notation for κ. The braiding $z_{M,N}\colon M\otimes N \to N \otimes M$ is $\tau \circ R$, i. e., the action of R followed by the interchange operator $\tau\colon x\otimes y \mapsto y \otimes x$.

A four braid pair (X, F) on the vector space V yields a tensor representation of ZB_n on $V^{\otimes n}$. We start with the standard four braid pair determined by (5.1) and (5.2) on the two-dimensional U-module $V = V_1$. Let $T_n\colon V^{\otimes n} \to V^{\otimes n}$ be the associated cylinder twist as defined in Section 2. By the Clebsch-Gordan decomposition, V_n is contained in $V^{\otimes n}$ with multiplicity 1. Similarly, $V_{m+n} \subset V_m \otimes V_n$ with multiplicity one [7, VII.7].

(6.2) Lemma. *There exists a projection operator $e_n\colon V^{\otimes n} \to V^{\otimes n}$ whose image, V_n, commutes with T_n.*

Proof. Let H_n be the Hecke algebra over $\mathfrak{K}$ generated by $x_1, \dots, x_{n-1}$ with braid relations $x_i x_j x_i = x_j x_i x_j$ for $|i-j| = 1$ and $x_j x_i = x_i x_j$ for $|i-j| > 1$ and quadratic relations $(x_i+1)(x_i-q^2) = 0$. Since X satisfies $(X-q^{1/2})(X+q^{-3/2}) = 0$, we obtain an action of H_n from the action of $ZA_{n-1} \subset ZB_n$ on $V^{\otimes n}$ if we let x_i act as $q^{3/2}g_i$. Since T_n comes from a central element of ZB_n as noted in (2.4), the H_n-action commutes with T_n. It is well known that there exists an idempotent $e_n \in H_n$ for which $e_n V^{\otimes n} = V_n$. (This is quantized Schur-Weyl duality.) This fact implies the assertion of the Lemma. □

(6.3) Corollary. *The subspace $V_n \subset V^{\otimes n}$ is T_n-stable.* □

A similar proof shows that all summands in the isotypic decomposition of $V^{\otimes n}$ are T_n-stable.

We denote by τ_n the restriction of T_n to V_n; and we denote by $\tau_{m,n} = z_{n,m}(\tau_n \otimes 1)z_{m,n}(\tau_m \otimes 1)$ the induced operator on $V_m \otimes V_n$ where $z_{m,n}$ denotes the braiding on $V_m \otimes V_n$.

(6.4) Lemma. *The subspace $V_{m+n} \subset V_m \otimes V_n$ is $\tau_{m,n}$-stable. The induced morphism equals τ_{m+n}.*

Proof. Consider $V_m \otimes V_n \subset V^{\otimes m} \otimes V^{\otimes n} = V^{\otimes(m+n)}$. The projection operator $e_m \otimes e_n$ is again obtained from the action of a certain element of the Hecke algebra H_{m+n}. Hence $V_m \otimes V_n$ is T_{m+n}-stable and the action on the subspace V_{m+n} is τ_{m+n}. We now use the equality (2.3)

$$T_{m+n} = X_{n,m}(T_n \otimes 1)X_{m,n}(T_m \otimes 1).$$

The essential fact is that $X_{m,n}$ is the braiding on $V^{\otimes m} \otimes V^{\otimes n}$. It induces, by naturality of the braiding, the braiding $z_{m,n}$ on $V_m \otimes V_n$. □

Let $A(n) = (\alpha_i^j(n))$ be the matrix of τ_n with respect to $x_0, \ldots, x_n$. In the next theorem we derive a recursive description of $A(n)$. We need more notation to state it. Define inductively polynomials γ_k by $\gamma_{-1} = 0$, $\gamma_0 = 1$ and, for $k > 0$,

(6.5)
$$\alpha\gamma_{k+1} = q^k\theta\gamma_k + \beta q^{k-1}\delta[k]\gamma_{k-1}.$$

Here $\delta = q - q^{-1}$, and $\gamma_k = \gamma_k(\theta, q, \alpha, \beta)$ is a polynomial in θ with coefficients in $\mathbb{Z}[q, q^{-1}, \alpha^{-1}, \beta]$. Let $D(n)$ denote the codiagonal matrix with $\alpha^k\beta^{n-k}q^{k(n-k)}$ in the k-th row and $(n-k)$-th column and zeros otherwise. (We enumerate rows and columns from 0 to n.) Let $B(n)$ be the upper triangular matrix

(6.6)
$$B(n) = \begin{pmatrix} \gamma_0 & \begin{bmatrix} n \\ 1 \end{bmatrix}\gamma_1 & \begin{bmatrix} n \\ 2 \end{bmatrix}\gamma_2 & \cdots & \gamma_n \\ & \gamma_0 & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}\gamma_1 & \cdots & \gamma_{n-1} \\ & & \cdots & \cdots & \cdots \\ & & & \gamma_0 & \gamma_1 \\ & & & & \gamma_0 \end{pmatrix}.$$

Thus the $(n-k)$-th row of $B(n)$ is

$$0, \ldots, 0, \begin{bmatrix} k \\ 0 \end{bmatrix}\gamma_0, \begin{bmatrix} k \\ 1 \end{bmatrix}\gamma_1, \begin{bmatrix} k \\ 2 \end{bmatrix}\gamma_2, \ldots, \begin{bmatrix} k \\ k-1 \end{bmatrix}\gamma_{k-1}, \begin{bmatrix} k \\ k \end{bmatrix}\gamma_k.$$

(6.7) Theorem. *The matrix $A(n)$ is equal to the product $D(n)B(n)$.*

Proof. The proof is by induction on n. We first compute the matrix of $\tau_{n,1}$ on $V_n \otimes V_1$ and then restrict to V_{n+1}. In order to display the matrix of $\tau_{n,1}$ we use the basis

$$x_0 \otimes x_0, \ldots, x_n \otimes x_0, x_0 \otimes x_1, \ldots, x_n \otimes x_1.$$

The matrix of $\tau_{n,1}$ has the block form

$$\begin{pmatrix} 0 & \beta A(n) \\ \alpha A(n) & A'(n) \end{pmatrix}.$$

The matrix $A'(n)$ is obtained from $A(n)$ in the following manner: Let $\alpha_0, \ldots, \alpha_n$ denote the columns of $A(n)$ and $\beta_0, \ldots, \beta_n$ the columns of $A'(n)$. We claim that

$$\beta_i = \alpha q^{2i-n}\theta\alpha_i + \beta q^{2i-n-1}\delta[n-i+1]\alpha_{i-1} + \alpha\delta[i+1]\alpha_{i+1},$$

with $\alpha_{-1} = \alpha_{n+1} = 0$.

Recall that $\tau_{n,1} = (\tau_n \otimes 1)z_{1,n}(\tau_1 \otimes 1)z_{n,1}$. In our case the universal R-matrix has the simple form

$$R = \kappa \circ (1 + (q - q^{-1})F \otimes E).$$

For the convenience of the reader we display the four steps in the calculation of $\tau_{n,1}$, separately for $x_i \otimes x_0$ and $x_i \otimes x_1$.

$$\begin{aligned} x_i \otimes x_0 &\mapsto q^{(n-2i)/2} x_0 \otimes x_i \\ &\mapsto \alpha q^{(n-2i)/2} x_1 \otimes x_i \\ &\mapsto \alpha x_i \otimes x_1 \\ &\mapsto \sum_j \alpha \alpha_i^j x_j \otimes x_0. \end{aligned}$$

$$\begin{aligned} x_i \otimes x_1 &\mapsto q^{-(n-2i)/2} x_1 \otimes x_i + \delta[i+1] q^{(n-2i-2)/2} x_0 \otimes x_{i+1} \\ &\mapsto q^{-(n-2i)/2}(\beta x_0 + \theta x_1) \otimes x_i + \alpha\delta[i+1] q^{(n-2i-2)/2} x_1 \otimes x_{i+1} \\ &\mapsto \beta x_i \otimes x_0 + \beta q^{-n+2i-1} \delta[n-i+1] x_{i-1} \otimes x_1 \\ &\quad + q^{2i-n}\theta x_i \otimes x_1 + \alpha\delta[i+1] x_{i+1} \otimes x_1 \\ &\mapsto \sum_j \alpha_i^j x_i \otimes x_0 + \sum_j \beta q^{2i-n+1} \delta[n-i+1] \alpha_{i-1}^j x_j \otimes x_1 \\ &\quad + \sum_j q^{2i-n} \theta \alpha_i^j x_j \otimes x_1 + \sum_j \alpha\delta[i+1] \alpha_{i+1}^j x_j \otimes x_1. \end{aligned}$$

This proves the claim about the matrix for $\tau_{n,1}$.

We now use the following fact about the Clebsch-Gordan decomposition (it is easily verified in our case, but see e. g. [7, VII.7] for more general results): In the Clebsch-Gordan decomposition $V_n \otimes V_1 = V_{n+1} \oplus V_{n-1}$ a basis of V_{n+1} is given by

$$y_j = \frac{F^j}{[j]!}(x_0 \otimes x_0) = q^{-j} x_j \otimes x_0 + x_{j-1} \otimes x_1.$$

We apply $\tau_{n,1}$ to the y_j. Since there are no overlaps between the coordinates of the y_j, we can directly write $\tau_{n,1}(y_j)$ as a linear combination of the y_k.

We assume inductively that $A(n)$ has *bottom-right triangular form*, i. e., zero entries above the codiagonal, with codiagonal as specified by $D(n)$. Then $A'(n)$ has a nonzero line one step above the codiagonal and is bottom-right triangular otherwise. From the results so far we see that the columns of $A(n+1)$, enumerated from 0 to $n+1$, are obtained inductively as follows: The 0-th row is $(0, \ldots, 0, \beta^{n+1})$. Below this 0-th row the j-th column, for $0 \leq j \leq n+1$, has the form

(6.8) $$\alpha q^j \alpha_j + q^{2j-n-2}\theta\alpha_{j-1} + \beta q^{2j-n-3}\delta[n-j+2]\alpha_{j-2}.$$

From this recursive formula one derives immediately that the codiagonal of $A(n)$ is given by $D(n)$.

Finally, we prove by induction that $A(n)$ is as claimed. The element in row k and column $n-k+j$ equals

$$\alpha^k \beta^{n-k} q^{k(n-k)} \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j .$$

For $n = 1$, we have defined τ_1 as $A(1)$. For the inductive step we use (6.8) in order to determine the element of $A(n)$ in column $n-k+j$ and row $k+1$. The assertion is then equivalent to the following identity:

$$\begin{aligned} & \alpha^k \beta^{n-k} q^{k(n-k)} \left(\alpha \begin{bmatrix} k \\ j \end{bmatrix} \gamma_j + q^{n-2k+2j-2} \theta \begin{bmatrix} k \\ j-1 \end{bmatrix} \gamma_{j-1} \right. \\ & \left. + \beta q^{n-2k+2j-3} \delta[k-j+2] \begin{bmatrix} k \\ j-2 \end{bmatrix} \gamma_{j-2} \right) \\ = \; & \alpha^{k+1} \beta^{n-k} q^{(n-k)(k+1)} \begin{bmatrix} k+1 \\ j \end{bmatrix} \gamma_j . \end{aligned}$$

We cancel α-, β-, and q-factors, use the Pascal formula

(6.9)
$$\begin{bmatrix} a+1 \\ b \end{bmatrix} = q^b \begin{bmatrix} a \\ b \end{bmatrix} + q^{-a+b-1} \begin{bmatrix} a \\ b-1 \end{bmatrix}$$

and the identity

$$\delta[k-j+2] \begin{bmatrix} k \\ j-2 \end{bmatrix} = \begin{bmatrix} k \\ j-1 \end{bmatrix} [j-1]$$

and see that the identity in question is equivalent to the recursion formula (6.5) defining the γ-polynomials. This completes the proof. □

We now formulate the main result of this section in a different way. First, we note that it was not essential to work with the function field $\mathfrak{K}$. In fact, $\mathfrak{K}$ could have been any commutative ring and q, α, and β could have been any suitable parameters in it. We think of θ as being an indeterminate.

Let $L(\alpha, \beta)$ be the operator on integrable U-modules which acts on V_n via

$$x_j \mapsto \alpha^{n-j} \beta^j q^{j(n-j)} x_{n-j} .$$

Let

(6.10)
$$T(\alpha, \beta) = \sum_{k=0}^{\infty} \gamma_k \frac{E^k}{[k]!};$$

$T(\alpha, \beta)$ is well-defined as an operator on integrable U-modules. Then (6.7) can be expressed as follows:

(6.11) Theorem. *The operator $t(\alpha, \beta)$ defined by setting $t(\alpha, \beta) = L(\alpha, \beta) \circ T(\alpha, \beta)$ acts on V_n as τ_n.* □

In Section 8 we give another derivation of this operator from the universal point of view.

One can develop a parallel theory by starting with the four braid pair (X^{-1}, F^{-1}). This leads to matrices which are *top-left triangular*, i. e., zero entries below the codiagonal. By computing the inverse of (5.1) and of (5.2) we see that, in the case $(\alpha, \beta) = (1, 1)$, we have to replace (q, θ) by $(q^{-1}, -\theta)$.

The following proposition may occasionally be useful. Introduce a new basis $u_0, \dots, u_n$ in V_n by

$$x_i = q^{-i(n-i)/2} \sqrt{\begin{bmatrix} n \\ i \end{bmatrix}} u_i.$$

Then a little computation shows:

(6.12) Proposition. *Suppose $\alpha = \beta$. With respect to the basis (u_i) the R-matrix and the matrix for τ_n are symmetric.* □

7. The γ-polynomials

For later use we derive some identities for the γ-polynomials of the previous section. A basic one, (7.1), comes from the compatibility of the cylinder twist with tensor products. Again we use $\delta = q - q^{-1}$. We give two proofs of (7.1).

(7.1) Theorem. *The γ-polynomials satisfy the product formula*

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} \alpha^{-k} \beta^k q^{mn-k(k+1)/2} \delta^k [k]! \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix} \gamma_{m-k} \gamma_{n-k}.$$

First proof of (7.1). The first proof is via representation theory. We have a unique U-submodule of $V_m \otimes V_n$ which is isomorphic to V_{m+n} (Clebsch-Gordan decomposition). We use the symbol V_{m+n} also for this module. The vector $x_m \otimes x_n$ is contained in this module and satisfies $F(m_m \otimes x_n) = 0$. The latter property characterizes $x_m \otimes x_n$ inside V_{m+n} up to a scalar (lowest weight vector, F-primitive vector).

We consider

$$\tau_{m,n} = (\tau_m \otimes 1) z_{n,m} (\tau_n \otimes 1) z_{m,n}$$

on $V_{m+n} \subset V_m \otimes V_n$ where it equals τ_{m+n}. We first express this equality formally in terms of matrices and then evaluate the formal equation by a small computation. We already have intruced the matrices for τ_m in Section 6

$$\tau_m(x_j) = \sum_k \alpha_j^k(m) x_k.$$

We write

$$z_{n,m}(x_j \otimes x_m) = \sum_{u,v} r_{jm}^{uv} x_u \otimes x_v.$$

From the form of the action of E and F on the modules V_t and from the form of the universal R-matrix we see that the sum is over (u,v) with $u+v=j+m$. Since $Fx_m = 0$ for $x_m \in V_m$, we also observe directly $z_{m,n}(x_m \otimes x_n) = q^{mn/2} x_n \otimes x_m$. The two expressions for τ_{m+n}, applied to $x_m \otimes x_n \in V_{m+n} \subset V_m \otimes V_n$, now yield the formal identity

$$\alpha^{m+n}_{m+n}(m+n) = \sum_{k\geq 0} q^{mn/2} \alpha^{n-k}_n(n) r^{m-k,n}_{n-k,m} \alpha^m_{m-n}(n).$$

By (6.7) we have

$$\begin{aligned} \alpha^{m+n}_{m+n}(m+n) &= \alpha^{m+n}\gamma_{m+n}, \\ \alpha^{n-k}_n(n) &= \alpha^{n-k}\beta^k q^{(n-k)k}\gamma_{n-k}, \quad \text{and} \\ \alpha^m_{m-k}(m) &= \alpha^m \begin{bmatrix} m \\ k \end{bmatrix} \gamma_{m-k}. \end{aligned}$$

This already yields a relation of type (7.1). It remains to compute the coefficient $r^{m-k,n}_{n-k,m}$. For this purpose we use the definition $z_{n,m} = \tau \circ \kappa \circ \Psi$ of the braiding, the action of E and F on vectors x_j, and the explicit form (6.1) of the operator Ψ. Put together, this yields

$$z_{n,m}(x_j \otimes x_m) = \sum_{k\geq 0} v^{\bullet(k)} \delta^k [j+1]\cdots[j+k] x_{m-k} \otimes x_{j+k}$$

with

$$\bullet(k) = k(k-1)/2 + (n-2j-2k)(2k-m)/2.$$

We now have enough data to rewrite the formal identity above and give it the form (7.1). $\square$

The dependence of γ_k on the parameters α and β is not essential. Define, inductively, polynomials γ'_k in θ over $\mathbb{Z}[q,q^{-1}]$ by setting $\gamma'_{-1} = 0, \gamma'_0 = 1$ and, for $k \geq 0$,

$$\gamma'_{k+1} = q^k \theta \gamma'_k + q^{k-1}\delta[k]\gamma'_{k-1},$$

i. e., by setting $\gamma'_k(\theta,q) = \gamma_k(\theta,q,1,1)$. A simple rewriting of the recursion formula then yields the identity

(7.2) $$\gamma_k(\theta,q,\alpha,\beta) = \gamma'_k\left(\frac{\theta}{\sqrt{\alpha\beta}}, q\right)\left(\frac{\beta}{\alpha}\right)^{k/2}.$$

Note that γ'_k contains only powers θ^l with $l \equiv k \bmod 2$.

Normalize the γ' to obtain monic polynomials $\beta_k(\theta) = q^{-k(k-1)/2}\gamma'_k(\theta)$. The new polynomials are determined by the recursion relation

(7.3) $\beta_{-1} = 0, \quad \beta_0 = 1, \quad$ and $\beta_{k+1} = \theta\beta_k + (1-q^{-2k})\beta_{k-1}$ for $k \geq 0$.

In order to find an explicit expression for the β_k, we introduce a new variable ρ

via the quadratic relation $\theta = \rho - \rho^{-1}$. We then consider the recursion formally over the ring $\mathbb{Z}[q, q^{-1}, \rho, \rho^{-1}]$. Let us set

$$B_n(\rho) = \sum_{j=0}^{n} (-1)^j q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} \rho^{n-2j}.$$

(7.4) Proposition. *The polynomials β satisfy the identity*

$$\beta_k(\rho - \rho^{-1}) = B_k(\rho).$$

Proof. We verify the recursion (7.3) with θ replaced by $\rho - \rho^{-1}$ and β_k replaced by B_k. We use the definition of the B_k in the right hand side of (7.3). Then the coefficient of ρ^{k+1-2j}, for $1 \leq j \leq k$, turns out to be

$$(-1)^j q^{-j(k-j} \left(\begin{bmatrix} k \\ j \end{bmatrix} + q^{k-2j+1} \begin{bmatrix} k \\ j-1 \end{bmatrix} - q^{-j} \delta [k] \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} \right).$$

We use the identity

$$[k] \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} = [k-j+1] \begin{bmatrix} k \\ j-1 \end{bmatrix}$$

and arrive at

$$(-1)^j q^{-j(k-j} \left(\begin{bmatrix} k \\ j \end{bmatrix} + q^{-k-1} \begin{bmatrix} k \\ j-1 \end{bmatrix} \right).$$

The Pascal formula (6.9) now shows that this is the coefficient of ρ^{k+1-2j} in B_{k+1}. It is easy to check that the coefficients of $\rho^{\pm(k+1)}$ on both sides coincide. $\square$

We can write $\rho^k + (-1)^k \rho^{-k}$ as an integral polynomial P_k in θ where $\theta = \rho - \rho^{-1}$. That polynomial satisfies the recursion relation

$$\theta P_k = P_{k+1} - P_{k-1}.$$

It is possible to write P_k in terms of Tschebischev- or Jacobi-polynomials. The last proposition says that

$$\beta_n(\theta) = \sum_{j=0}^{[n/2]} (-1)^j q^{-j(n-j)} \begin{bmatrix} n \\ j \end{bmatrix} P_{n-2j}(\theta).$$

The product formula (7.1) was a consequence of representation theory. In view of the applications to be made in Section 8 it is desirable to have a proof which uses only the recursive definition of the γ-polynomials. We now give such a proof. By (7.2), it suffices to consider the case $\alpha = \beta = 1$.

Second proof of (7.1). We write

$$C_k^{m,n} = q^{mn-k(k+1)/2} \delta^k [k]! \begin{bmatrix} m \\ k \end{bmatrix} \begin{bmatrix} n \\ k \end{bmatrix},$$

and want to show that

$$\gamma_{m+n} = \sum_{k=0}^{\min(m,n)} C_k^{m,n} \gamma_{m-k}\gamma_{n-k}.$$

Denote the right hand side by $\gamma(m,n)$. Then $\gamma(m,n) = \gamma(n,m)$. We will use the recursion (6.5) and the Pascal formula (6.9), with q replaced by q^{-1}, to show $\gamma(m+1,n) = \gamma(m,n+1)$. Since $\gamma(m+n,0) = \gamma_{m+n}$ the proof will then be complete. Set $\gamma_k = 0$ for $k < 0$. We can then sum just over $k \geq 0$. The C-coefficients satisfy the following Pascal type relation

(7.5) $$C_k^{m+1,n} = q^{n-k} C_k^{m,n} + \delta q^{n-k+1} q^{m-k} [n-k+1] C_{k-1}^{m,n}.$$

The verification that this is so uses the Pascal formula for $\left[\begin{smallmatrix} m+1 \\ k \end{smallmatrix}\right]$ and a little rewriting. Next we apply this relation in the sum $\gamma(m+1,n)$ and obtain (with an index shift $k \to k+1$ in the second summand) the identity

$$\begin{aligned}\gamma(m+1,n) &= \sum_k q^{n-k} C_k^{m,n} \gamma_{m-k+1}\gamma_{n-k} \\ &\quad + \sum_k \left(\delta [n-k] q^{n-k-1} \gamma_{n-k-1}\right) q^{m-k} C_k^{m,n} \gamma_{m-k}.\end{aligned}$$

In the second sum apply the recursion to the factor in parentheses to obtain the identity

$$\gamma(m+1,n) = \sum_k C_k^{m,n} (q^{n-k}\gamma_{m-k+1}\gamma_{n-k} + q^{m-k}\gamma_{m-k}\gamma_{n-k+1} - q^{n+m-2k}\theta\gamma_{m-k}\gamma_{n-k}).$$

Since $\gamma(m,n) = \gamma(n,m)$, we obtain $\gamma(m,n+1)$ upon interchanging m and n in the foregoing identity: That interchanges the first two summands in the parentheses and leaves the third fixed. □

8. The universal cylinder twist

In this section we work with operators on integrable U-modules. These are $\mathfrak{K}$-linear weak endomorphisms of the category U-INT. Left multiplication by $x \in U$ is such an operator; it will be denoted by x or by l_x. If t is an operator, then $\mu(t)$ is the operator on U-INT $\times$ U-INT which is given by the action of t on tensor products of modules. If τ denotes the twist operator, then we define $\tau(t) = \tau \circ t \circ \tau$. We have the compatibilities $\mu(l_x) = l_{\mu(x)}$ and $\tau\mu(l_x) = l_{\tau\mu(x)}$. The operators $\mu(t)$ and $\tau(t)$ are again weak endomorphisms of the categories involved.

Typical examples of such operators which are not themselves elements of U are the universal R-matrix R and its factors κ and Ψ, (See (6.1).) as are the operators $L = T'_{i,1}$ and $L^{\#} = T''_{i,1}$ of Lusztig [8, p. 42].

Since R acts by U-linear maps each operator t satisfies the standard relation

(8.1) $$R \circ \mu(t) = \tau\mu(t) \circ R$$

of a braiding.

An operator t is called a *universal cylinder twist* on U-INT if it is invertible and satisfies the analogue of (1.4), namely,

$$\textbf{(8.2)} \qquad \mu(t) = \tau R(1 \otimes t) R(t \otimes 1) \qquad \text{and}$$

$$\textbf{(8.3)} \qquad \tau R(1 \otimes t) R(t \otimes 1) = (t \otimes 1) \tau R(1 \otimes t) R.$$

We denote by t_V the action of t on the module V. Then (1.3) holds if we use R to define the braiding. Recall the operator $t(\alpha, \beta)$ defined at the end of Section 6. Here is the main result, proved following (8.6).

(8.4) Theorem. *Suppose $\alpha\beta = -q$. Then $t(\alpha, \beta)$ is a universal cylinder twist.*

We treat the case $(\alpha, \beta) = (-q, 1)$ in detail and reduce the general case formally to this one. We skip the notation α, β and work with $t = LT$. Note that L is Lusztig's operator referred to above. We collect a few properties of L in the next lemma.

(8.5) Lemma. *The operator L satisfies the following identities:*

(1) $LEL^{-1} = -KF$, $LFL^{-1} = -EK^{-1}$, $LKL^{-1} = K^{-1}$.
(2) $\mu(L) = (L \otimes L)\Psi = \tau R(L \otimes L)\kappa^{-1}$.
(3) $\kappa(L \otimes 1) = (L \otimes 1)\kappa^{-1}$, $\kappa(1 \otimes L) = (1 \otimes L)\kappa^{-1}$.
(4) $(L \otimes L)\Psi(L \otimes L)^{-1} = \kappa \circ \tau\Psi \circ \kappa^{-1}$.

Proof. For (1), in the case $L^{\#}$, see [8, Proposition 5.2.4.]. A simple computation from the definitions yields (3) and (4). For the first equality in (2) see [8, Proposition 5.3.4]; the second one follows by using (3) and (4). □

In the universal case one of the axioms for a cylinder twist is redundant, namely:

(8.6) Proposition. *If the operator t satisfies* (8.2), *then it also satisfies* (8.3).

Proof. Apply τ to (8.2) and use (8.1). □

Proof of theorem (8.4). The operator L is invertible. The operator T is invertible since its constant term is 1. Thus it remains to verify (8.2). We show that (8.2) is equivalent to

$$\textbf{(8.7)} \qquad \mu(T) = \kappa(1 \otimes T)\kappa^{-1} \circ (L^{-1} \otimes 1)\Psi(L \otimes 1) \circ (T \otimes 1),$$

given the relations of Lemma (8.5). Given (8.2), we have

$$\mu(T) = \mu(L^{-1})\tau(R)(1 \otimes LT)\kappa\Psi(LT \otimes 1).$$

We use (8.5.2) for $\mu(L^{-1})$, cancel $\tau(R)$ and its inverse, and then use (8.5.3); (8.7) drops out. In like manner, (8.2) follows from (8.1).

In order to prove (8.7), one verifies the following identities from the definitions:

$$\kappa(1 \otimes T)\kappa^{-1} = \sum_{k=0}^{\infty} \frac{\gamma_k}{[k]!}(K^k \otimes E^k) \qquad \text{and}$$

$$(L^{-1} \otimes 1)\Psi(L \otimes 1) = \sum_{k=0}^{\infty} (-1)^k q^{-k(k-1)/2} \frac{\delta^k}{[k]!} K^k E^k \otimes E^k.$$

Using this information, we compute the coefficient of $K^r E^s \otimes E^r$ on the right hand side of (8.7) to be

$$\sum_{n=0}^{\min(r,s)} (-1)^n q^{-n(n-1)/2} \frac{\delta^n}{[n]![s-n]![r-n]!} \gamma_{s-n}\gamma_{r-n}.$$

The coefficient of the same element in $\mu(T)$ is, by the q-binomial formula, equal to

$$q^{-rs} \frac{1}{[s]![r]!} \gamma_{r+s}.$$

Equality of these coefficients is exactly the product formula (7.1) in the case where $(\alpha, \beta) = (-q, 1)$. This finishes the proof of the theorem in this special case.

A similar proof works in the general case. Specifically, a formal reduction to the special case uses the following observation. Write $\alpha = q^{\zeta}$. Then, formally, $L(\alpha, \beta) = K^{\zeta} L$ in case $\alpha\beta = -q$. This fact is used to deduce similar properties for $L_{\#} = L(\alpha, \beta)$ from lemma (8.5), in particular

$$L_{\#}^{-1} F L_{\#} = \alpha^{-1} \beta q K E.$$

The final identity leads to (7.1) in the general case. □

We point out that the main identity in the construction of the universal twist involves only the Borel subalgebra of U generated by E and K. Of course, there is a similar theory based on F and K and another braiding. The constructions of section 6 show that the universal twist is determined by its action on the 2-dimensional module V_1. Hence our main theorem gives all possible universal cylinder twists associated to the given *braided* category U-Int.

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