REPRESENTATION THEORY

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Chapter 1

Representations

1.1 Basic Definitions

Groups are intended to describe symmetries of geometric and other mathematical objects. Representations are symmetries of some of the most basic objects in geometry and algebra, namely vector spaces.

Representations have three different aspects — geometric, numerical and algebraic — and manifest themselves in corresponding form. We begin with the numerical form.

In a general context we write groups G in multiplicative form. The group structure (multiplication) is then a map $G \times G \to G, (g, h) \mapsto g \cdot h = gh$, the unit element is e or 1, and g^{-1} is the inverse of g.

An n-dimensional **matrix representation** of the group G over the field K is a homomorphism $\varphi\colon G \to GL_n(K)$ into the general linear group $GL_n(K)$ of invertible (n, n)-matrices with entries in K. Two such representations φ, ψ are said to be **conjugate** if there exists a matrix $A \in GL_n(K)$ such that the relation $A\varphi(g)A^{-1} = \psi(g)$ holds for all $g \in G$. The representation φ is called **faithful** if φ is injective. If $K = \mathbb{C}, \mathbb{R}, \mathbb{Q}$, we talk about complex, real, and rational representations.

The group $GL_1(K)$ will be identified with the multiplicative group of non-zero field elements $K^* = K \smallsetminus \{0\}$. In this case we are just considering homomorphisms $G \to K^*$.

Next we come to the geometric form of a representation as a symmetry group of a vector space. The field K will be fixed.

A **representation** of G on the K-vector space V, a KG-representation for short, is a map

$$\rho\colon G \times V \to V, \quad (g, v) \mapsto \rho(g, v) = g \cdot v = gv$$

with the properties:

(1) $g(hv) = (gh)v, ev = v$ for all $g, h \in G$ and $v \in V$.
(2) The **left translation** $l_g : V \to V, v \mapsto gv$ is a K-linear map for each $g \in G$.

We call V the **representation space**. Its dimension as a vector space is the **dimension** $\dim V$ of the representation (sometimes called the **degree** of the representation). The rules (1) are equivalent to $l_g \circ l_h = l_{gh}$ and $l_e = \mathrm{id}_V$. They express the fact that ρ is a group action – see the next section. From $l_g l_{g^{-1}} = l_{gg^{-1}} = l_e = \mathrm{id}$ we see that l_g is a linear isomorphism with inverse $l_{g^{-1}}$.

Occasionally it will be convenient to define a representation as a map

$$V \times G \to V, \quad (v, g) \mapsto vg$$

with the properties $v(hg) = (vh)g$ and $ve = v$, and K-linear right translations $r_g : v \mapsto vg$. These will be called **right** representations as opposed to **left** representations defined above. The map $r_g : v \mapsto vg$ is then the **right translation** by g. Note that now $r_g \circ r_h = r_{hg}$ (contravariance). If V is a right representation, then $(g, v) \mapsto vg^{-1}$ defines a left representation. We work with left representations if nothing else is specified.

One can also use both notions simultaneously. A **(G, H)-representation** is a vector space V with the structure of a left G-representation and a right H-representation, and these structures are assumed to commute $(gv)h = g(vh)$.

A **morphism** $f : V \to W$ between KG-representations is a K-linear map f which is **G-equivariant**, i.e., which satisfies $f(gv) = gf(v)$ for $g \in G$ and $v \in V$. Morphisms are also called **intertwining operators**. A bijective morphism is an **isomorphism**. The vector space of all morphisms $V \to W$ is denoted $\mathrm{Hom}_G(V, W) = \mathrm{Hom}_{KG}(V, W)$. Finite-dimensional KG-representations and their morphisms form a category KG-Rep.

Let V be an n-dimensional representation of G over K. Let B be a basis of V and denote by $\varphi^B(g) \in GL_n(K)$ the matrix of l_g with respect to B. Then $g \mapsto \varphi^B(g)$ is a matrix representation of G. Conversely, from a matrix representation we get in this manner a representation.

(1.1.1) Proposition. *Let V, W be representations of G, and B, C bases of V, W. Then V, W are isomorphic if and only if the corresponding matrix representations φ^B, φ^C are conjugate.*

Proof. Let $f : V \to W$ be an isomorphism and A its matrix with respect to B, C. The equivariance $f \circ l_g = l_g \circ f$ then translates into $A\varphi^B(g) = \varphi^C(g)A$; and conversely. □

Conjugate 1-dimensional representations are equal. Therefore the isomorphism classes of 1-dimensional representations correspond bijectively to homomorphisms $G \to K^*$. The aim of representation theory is not to determine matrix representations. But certain concepts are easier to explain with the help of matrices.

Let V be a representation of G. A **sub-representation** of V is a subspace U which is G-invariant, i.e., $gu \in U$ for $g \in G$ and $u \in U$. A non-zero representation V is called **irreducible** if it has no sub-representations other than $\{0\}$ and V. A representation which is not irreducible is called **reducible**.

(1.1.2) Schur's Lemma. *Let V and W be irreducible representations of G.*

(1) *A morphism $f \colon V \to W$ is either zero or an isomorphism.*

(2) *If K is algebraically closed then a morphism $f \colon V \to V$ is a scalar multiple of the identity, $f = \lambda \cdot \mathrm{id}$.*

Proof. (1) Kernel and image of f are sub-representations. If $f \neq 0$, then the kernel is different from V hence equal to $\{0\}$ and the image is different from $\{0\}$ hence equal to W.

(2) Algebraically closed means: Non-constant polynomials have a root. Therefore f has an eigenvalue $\lambda \in K$ (root of the characteristic polynomial). Let $V(\lambda)$ be the eigenspace and $v \in V(\lambda)$. Then $f(gv) = gf(v) = g(\lambda v) = \lambda gv$. Therefore $gv \in V(\lambda)$, and $V(\lambda)$ is a sub-representation. By irreducibility, $V = V(\lambda)$. □

(1.1.3) Proposition. *An irreducible representation of an abelian group G over an algebraically closed field is one-dimensional.*

Proof. Since G is abelian, the l_g are morphisms and, by 1.1.2, multiples of the identity. Hence each subspace is a sub-representation. □

(1.1.4) Example. Let S_n be the symmetric group of permutations of $\{1, \ldots, n\}$. We obtain a right(!) representation of S_n on K^n by permutation of coordinates

$$K^n \times S_n \to K^n, \quad ((x_1, \ldots, x_n), \sigma) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

This representation is not irreducible if $n > 1$. It has the sub-representations $T_n = \{(x_i) \mid \sum_{i=1}^n x_i = 0\}$ and $D = \{(x, \ldots, x) \mid x \in K\}$. ◇

Schur's lemma can be expressed in a different way. Recall that an **algebra** A over K consists of a K-vector space together with a K-bilinear map $A \times A \to A, (a,b) \mapsto ab$ (the multiplication of the algebra). The algebra is called associative (commutative), if the multiplication is associative (commutative). An associative algebra with unit element is therefore a ring with the additional property that the multiplication is bilinear with respect to the scalar multiplication in the vector space. In a **division algebra** (also called **skew field**) any non-zero element has a multiplicative inverse. A typical example of an associative algebra is the endomorphism algebra $\mathrm{Hom}_G(V,V)$ of a representation V; multiplication is the composition of endomorphisms. Other examples are the algebra $M_n(K)$ of (n,n)-matrices with entries in K and the polynomial algebra $K[x]$. The next proposition is a reformulation of Schur's lemma.

(1.1.5) Proposition. *Let V be an irreducible G-representation. Then the endomorphism algebra $A = \mathrm{Hom}_G(V,V)$ is a division algebra. If K is algebraically closed, then $K \to A$, $\lambda \mapsto \lambda \cdot \mathrm{id}$ is an isomorphism of K-algebras.* □

Let V be an irreducible G-representation over $\mathbb{R}$. A finite-dimensional division algebra over $\mathbb{R}$ is one of the algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$. We call V of **real, complex, quaternionic type** according to the type of its endomorphism algebra.

The third form of a representation — namely a module over the group algebra — will be introduced later.

(1.1.6) Cyclic groups. The cyclic group of order n is the additive group $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n$ of integers modulo n. We also use a formal multiplicative notation for this group $C_n = \langle\, a \mid a^n = 1 \,\rangle$; this means: a is a generator and the n-th power is the unit element.

Homomorphisms $\alpha\colon C_n \to H$ into another group H correspond bijectively to elements $h \in H$ such that $h^n = 1$, via $a \mapsto \alpha(a)$. Hence there are n different 1-dimensional representations over the complex numbers $\mathbb{C}$, given by $a \mapsto \exp(2\pi i t/n)$, $0 \le t < n$.

The rotation matrices $D(\alpha)$

$$D(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

satisfy $D(\alpha)D(\beta) = D(\alpha+\beta)$ and $BD(\alpha)B^{-1} = D(-\alpha)$. We obtain a 2-dimensional real representation $\varphi_t\colon a \mapsto D(2\pi t/n)$. The representations φ_t and $\varphi_{-t} = \varphi_{n-t}$ are conjugate. ◇

(1.1.7) Dihedral groups. Groups can be presented in terms of generators and relations. We do not enter the theory of such presentations but consider an example. Let

$$D_{2n} = \langle\, a, b \mid a^n = 1 = b^2,\ bab^{-1} = a^{-1} \,\rangle.$$

This means: The group is generated by two elements a and b, and these generators satisfy the specified relations. The **universal property** of this presentation is: The homomorphisms $\alpha\colon D_{2n} \to H$ into any other group H correspond bijectively to pairs $(A = \alpha(a), B = \alpha(b))$ in H such that $A^n = 1 = B^2, BAB^{-1} = A^{-1}$.

Thus 1-dimensional representations over $\mathbb{C}$ correspond to complex numbers A, B such that $A^n = B^2 = A^2 = 1$. If n is odd there are two pairs $(1, \pm 1)$; if n is even there are four pairs $(\pm 1, \pm 1)$.

A 2-dimensional representation on the $\mathbb{R}$-vector space $\mathbb{C}$ is specified by $a \cdot z = \lambda z$, $b \cdot z = \overline{z}$ where $\lambda^n = 1$. Complex conjugation shows that the representations which correspond to λ and $\overline{\lambda}$ are isomorphic. Denote the representation obtained from $\lambda = \exp(2\pi i t/n)$ by V_t.

The group D_{2n} has order $2n$ and is called the **dihedral group** of this order. From a geometric viewpoint, D_{2n} is the orthogonal symmetry group of the regular n-gon in the plane. A faithful matrix representation $D_{2n} \to O(2)$ is obtained by choosing $\lambda = \exp(2\pi i/n)$. The powers of a correspond to rotations, the elements $a^t b$ to reflections. ◇

(1.1.8) Example. The real representations $\varphi_t, 1 \le t < n/2$, of C_n in 1.1.6 are irreducible. A nontrivial sub-representation would be one-dimensional and spanned by an eigenvector of $\varphi_t(a)$.

If we consider φ_t as a complex representation, then it is no longer irreducible, since eigenvectors exist. In terms of matrices

$$PD(\alpha)P^{-1} = \begin{pmatrix} \exp(i\alpha) & 0 \\ 0 & \exp(-i\alpha) \end{pmatrix}, \qquad P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

The representations $V_t, 1 \le t < n/2$ of D_{2n} in 1.1.7 are irreducible, since they are already irreducible as representations of C_n. But this time they remain irreducible when considered as complex representations. The reason is, that PBP^{-1} does not preserve the eigenspaces. ◇

Problems

1. The dihedral group D_{2n} has the presentation $\langle\, s,t \mid s^2 = t^2 = (st)^n = 1 \,\rangle$.
2. Recall the notion of a semi-direct product of groups and show that D_{2n} is the semi-direct product of C_n by C_2.
3. Let $Q_{4n} = \langle\, a,b \mid a^n = b^2, bab^{-1} = a^{-1} \,\rangle$, $n \ge 2$. Deduce from the relation $b^4 = a^{2n} = 1$. Show that Q_{4n} is a group of order $4n$. Show that $a \mapsto \exp(\pi i/n), b \mapsto j$ induces an isomorphism of Q_{4n} with a subgroup of the multiplicative group of the quaternions. The group Q_{4n} is called a **quaternion group**. Show that Q_{4n} has also the presentation $\langle\, s,t \mid s^2 = t^2 = (st)^n \,\rangle$. Construct a two-dimensional faithful irreducible (matrix) representation over $\mathbb{C}$.
4. Let A_4 be the alternating group of order 12 (even permutations in S_4). Show: A_4 has 3 elements of order 2, 8 elements of order 3. Show that A_4 is the semi-direct product of $C_2 \times C_2$ by C_3. Show $A_4 = \langle\, a,b \mid a^3 = b^3 = (ab)^2 = 1 \,\rangle$; show that $s = ab$ and $t = ba$ are commuting elements of order 2 which generate a normal subgroup. Show that the matrices

$$a = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

define a representation on $\mathbb{R}^3$ as a symmetry group of a regular tetrahedron.
5. A representation of C_2 on V amounts to specifying an involution $T\colon V \to V$, i.e., a linear map T with $T^2 = \mathrm{id}$. If the field K has characteristic different from 2 show that V is the direct sum of the ± 1-eigenspaces of T. (Consider the operators $\frac{1}{2}(1+T)$ and $\frac{1}{2}(1-T)$.)

1.2 Group Actions and Permutation Representations

In this section we collect basic terminology about group actions. We use group actions to construct the important class of permutation representations.

Let G be a multiplicative group with unit element e. A **left action** of a group G on a set X is a map

$$\rho\colon G \times X \to X, \quad (g,x) \mapsto \rho(g,x) = g \cdot x = gx$$

with the properties $g(hx) = (gh)x$ and $ex = x$ for $g, h \in G$ and $x \in X$. The pair (X, ρ) is called a **(left)** G**-set**. Each $g \in G$ yields the **left translation** $l_g\colon X \to X$, $x \mapsto gx$ by g. It is a bijection with inverse the left translation by g^{-1}. An action is called **effective**, if l_g for $g \neq e$ is never the identity. We also use **(right) actions** $X \times G \to X$, $(x,g) \mapsto xg$. They satisfy $(xh)g = x(hg)$ and $xe = x$. Usually we work with left G-actions.

A subset A of a G-set X is called G**-stable** or G**-invariant**, if $g \in G$ and $a \in A$ implies $ga \in A$.

Recall that we defined a representation as a group action on a vector space with the additional property that the left translations are linear maps. We now use group actions to construct representations.

Let S be a finite (left) G-set and denote by KS the vector space with K-basis S. Thus elements in KS are linear combinations $\sum_{s \in S} \lambda_s s$ with $\lambda_s \in K$. The left action of G on S is extended linearly to KS

$$g \cdot (\textstyle\sum_{s \in S} \lambda_s s) = \sum_{s \in S} \lambda_s (g \cdot s) = \sum_{x \in S} \lambda_{g^{-1}x} x.$$

The resulting representation is called the **permutation representation** of S. An important example is obtained from the group $G = S$ with left action by group multiplication. The associated permutation representation is the **left regular representation** of the finite group G. Right multiplication leads to the right regular representation.

Let X be a G-set. Then $R = \{(x, gx) \mid x \in X, g \in G\}$ is an equivalence relation on X. Let X/G denote the set of equivalence classes. The class of x is $Gx = \{gx \mid g \in G\}$ and called the **orbit** through x. We call X/G the **orbit set** or (**orbit space**) of the G-set X. An action is called **transitive**, if it consists of a single orbit. For systematic reasons it would be better to denote the orbit set of a left action by $G\backslash X$. If right and left actions occur, we use both notations.

A group acts on itself by conjugation $G \times G \to G$, $(g,h) \mapsto ghg^{-1}$. Elements are **conjugate** if they are in the same orbit. The orbits are called **conjugation classes**. A function on G is called a **class function**, if it is constant on conjugacy classes.

Let H be a subgroup of G. We have the set G/H of right cosets gH with left G-action by left translation

$$G \times G/H \to G/H, \quad (k, gH) \mapsto kgH.$$

G-sets of this form are called **homogeneous** G-sets. Similarly, we have the set $H\backslash G$ of left cosets Hg with an action by right translation.

We write $H \le G$, if H is a subgroup of G, and $H < G$, if it is a proper subgroup. On the set $\mathrm{Sub}(G)$ of subgroups of G the relation $\le$ is a partial order.

The group G acts on $\mathrm{Sub}(G)$ by conjugation $(g,H) \mapsto gHg^{-1} = {}^gH$. The orbit through H consists of the subgroups H of G which are **conjugate** to H. We write $K \sim L$ or $K \sim_G L$, if there exists $g \in G$ such that $gKg^{-1} = L$. We denote by (H) the conjugacy class of H. Let $\mathrm{Con}(G)$ be the set of conjugacy classes of subgroups of G. We say H is **subconjugate** to K in G, if H is conjugate in G to a subgroup of K. We denote this fact by $(H) \le (K)$; and by $(H) < (K)$, if equality is excluded.

The **stabilizer** or **isotropy group** of $x \in X$ is the subgroup $G_x = \{g \in G \mid gx = x\}$. We have $G_{gx} = gG_xg^{-1}$. An action is called **free**, if all isotropy groups are trivial. The set of isotropy groups of X is denoted $\mathrm{Iso}(X)$.

A **family $\mathcal{F}$ of subgroups** is a subset of $\mathrm{Sub}(G)$ which consists of complete conjugacy classes. If $\mathcal{F}$ and $\mathcal{G}$ are families, we write $\mathcal{F} \circ \mathcal{G}$ for the family of intersections $\{K \cap L \mid K \in \mathcal{F}, L \in \mathcal{G}\}$. We call $\mathcal{F}$ **multiplicative**, if $\mathcal{F} \circ \mathcal{F} = \mathcal{F}$, and $\mathcal{G}$ is called $\mathcal{F}$**-modular**, if $\mathcal{F} \circ \mathcal{G} \subset \mathcal{G}$. A family is called **closed**, if it contains with a group all supergroups, and it is called **open**, if it contains with a group all subgroups. Let $(\mathcal{F})$ denote the set of conjugacy classes of $\mathcal{F}$. Suppose $\mathrm{Iso}(X) \subset \mathcal{F}$, then we call X an $\mathcal{F}$-set. We denote by $X(\mathcal{F})$ the subset of points in X with isotropy groups in $\mathcal{F}$.

A G**-map** $f \colon X \to Y$ between G-sets, also called a G**-equivariant map**, is a map which satisfies $f(gx) = gf(x)$ for all $g \in G$ and $x \in X$. Left G-sets and G-equivariant maps form the category G-SET. By passage to orbits, a G-map $f \colon X \to Y$ induces $f/G \colon X/G \to Y/G$. The category G-SET of G-sets and G-maps has products: If $(X_j \mid j \in J)$ is a family of G-sets, then the Cartesian product $\prod_{j \in J} X_j$ with so-called **diagonal action** $g(x_j) = (gx_j)$ is a product in this category.

(1.2.1) Proposition. *Let C be a transitive G-set and $c \in C$. Then $G/G_c \to C$, $gG_c \mapsto gc$ is a well-defined isomorphism of G-sets (a simple algebraic verification). The orbits of a G-set are transitive. Therefore each G-set is isomorphic to a disjoint sum of homogeneous G-sets.* □

For a G-set X and a subgroup H of G we use the following notations

$$\begin{aligned}
X_H &= \{x \in X \mid G_x = H\}, \\
X_{(H)} &= \{x \in X \mid (G_x) = (H)\} \\
X^H &= \{x \in X \mid hx = x,\, h \in H\}, \\
X^{>H} &= X^H \smallsetminus X_H \\
X^{(H)} &= GX^H = \{x \in X \mid (H) \le (G_x)\}, \\
X^{>(H)} &= X^{(H)} \smallsetminus X_{(H)}.
\end{aligned}$$

We call X^H the **H-fixed point set** of X. If $f\colon X \to Y$ is a G-map, then $f(X^H) \subset Y^H$. The left translation $l_g\colon X \to X$ induces a bijection $X^H \to X^K$, $K = gHg^{-1}$. The subset $X_{(H)}$ is G-stable; it is called the **(H)-orbit bundle** of X.

(1.2.2) Example. The permutation representation KS is always reducible ($|S| > 1$). The fixed point set is a non-zero proper sub-representation. The dimension of $(KS)^G$ is $|S/G|$; a basis of $(KS)^G$ consists of the $x_C = \sum_{s\in C} s$ where C runs through the orbits of S. ◇

Suppose X is a right and Y a left H-set. Then $X \times_H Y$ denotes the quotient of $X \times Y$ with respect to the equivalence relation $(xh, y) \sim (x, hy)$, $h \in H$. This is the orbit set of the action $(h, (x, y)) \mapsto (xh^{-1}, hy)$ of H on $X \times Y$.

Let G and H be groups. A **(G, H)-set** X is a set X together with a left G-action and a right H-action which commute $(gx)h = g(xh)$. If we form $X \times_H Y$, then this set carries an induced G-action $g \cdot (x, y) = (gx, y)$. If $f\colon Y_1 \to Y_2$ is an H-map, then we obtain an induced G-map $X \times_H f\colon X \times_H Y_1 \to X \times_H Y_2$. This construction yields a functor $\rho(X)\colon H\text{-}\mathrm{SET} \to G\text{-}\mathrm{SET}$.

We apply this construction to the (G, H)-set $G = X$ with action by left G-translation and right H-translation for $H \le G$. The resulting functor is called **induction functor**

$$\mathrm{ind}_H^G\colon H\text{-}\mathrm{SET} \to G\text{-}\mathrm{SET}\,.$$

It is left adjoint to the **restriction functor**

$$\mathrm{res}_H^G\colon G\text{-}\mathrm{SET} \to H\text{-}\mathrm{SET},$$

given by considering a G-set as an H-set. The adjointness means that there is a natural bijection

$$\mathrm{Hom}_G(\mathrm{ind}_H^G X, Y) \cong \mathrm{Hom}_H(X, \mathrm{res}_H^G Y).$$

It assigns to an H-map $f\colon X \to Y$ the G-map $G \times_H X \to Y$, $(g,x) \mapsto gf(x)$.

One can obtain interesting group theoretic results by counting orbits and fixed points. We give some examples.

(1.2.3) Proposition. *Let P be a p-group and X a finite P-set; then $|X| \equiv |X^P| \bmod p$. Let C be cyclic of order p^t and $D \leq C$ the unique subgroup of order p; then $|X| \equiv |X^D| \bmod p^t$.*

Proof. Each orbit in $X \smallsetminus X^P$ has cardinality divisible by p. Each orbit in $X \smallsetminus X^D$ has cardinality p^t. □

(1.2.4) Proposition. *Let $P \neq 1$ be a p-group. Then P has a non-trivial center $Z(P) = \{x \in P \mid \forall y \in P,\, xy = yx\}$.*

Proof. Let P act on itself by conjugation $(x,y) \mapsto xyx^{-1}$. The fixed point set is the center. Since $1 \in Z(P)$, we see from 1.2.3 that $|Z(P)|$ is non-zero and divisible by p. □

(1.2.5) Proposition. *Let P be a p-group. There exists a chain of normal subgroups $P_i \lhd P$*

$$1 = P_0 \lhd P_1 \lhd \ldots \lhd P_r = P$$

such that $|P_i/P_{i-1}| = p$.

Proof. Induct on $|P|$. Since subgroups of the center are normal, there exists by 1.2.4 a normal subgroup P_1 of order p. Apply the induction hypothesis to the factor group P/P_1 and lift a normal series to P. □

(1.2.6) Proposition. *Let K be a field of characteristic p and V a KP-representation for a p-group P. Then $V^P \neq \{0\}$.*

Proof. Let P have order p with generator x. Then $l_x\colon V \to V$ has eigenvalues a root of $X^p - 1 = (X-1)^p$. Hence 1 is the only eigenvalue. In the general case choose a normal subgroup $Q \lhd P$ and observe that V^Q is a $K(P/Q)$-representation. □

A group A is called **elementary abelian of rank** n if it is isomorphic to the n-fold product $(C_p)^n$ of cyclic groups C_p of prime order p. We can view this group as n-dimensional vector space over the prime field $\mathbb{F}_p$.

(1.2.7) Proposition. *Let the p-group P act on the elementary abelian p-group A of rank n by automorphisms. Then there exists a chain $1 = A_0 < \ldots < A_n = A$ of subgroups which are P-invariant and $|A_i/A_{i-1}| = p$.* □

(1.2.8) Counting lemma. *Let G be a finite group, M a finite G-set, and $\langle g \rangle$ the cyclic subgroup generated by $g \in G$. Then $|G| \cdot |M/G| = \sum_{g\in G} |M^{\langle g \rangle}|$.*

Proof. Let $X = \{(g,x) \in G \times M \mid gx = x\}$. Consider the maps

$$p\colon X \to G,\ (g,x) \mapsto g, \qquad q\colon X \to M/G,\ (g,x) \mapsto Gx.$$

Since $p^{-1}(g) = |\{g\} \times M^{\langle g \rangle}|$, the right hand side is the sum of the cardinalities of the fibres of p. Since $|q^{-1}(Gx)| = |G_x||Gx| = |G|$, the left hand side is the sum of the cardinalities of the fibres of q. □

Problems

1. Let X and Y be G-sets. The set $\mathrm{Hom}(X,Y)$ of all maps $X \to Y$ carries a left G-action $(g \cdot f)(x) = gf(g^{-1}x)$. The G-fixed point set $\mathrm{Hom}(X,Y)^G$ is the subset $\mathrm{Hom}_G(X,Y)$ of G-maps $X \to Y$.
2. Let X be a G-set and $H \leq G$. Then $G \times_H X \to G/H \times X$, $(g,x) \mapsto (gH, gx)$ is a bijection of G-sets. If Y is a further H-set, then we have an isomorphism of G-sets $G \times_H (X \times Y) \cong X \times (G \times_H Y)$.
3. Determine the conjugacy classes of D_{2n} and Q_{4n}.
4. The orbits of $G/K \times G/L$ correspond bijectively to the double cosets $K\backslash G/L$; the maps

$$\begin{aligned} &G\backslash(G/K \times G/L) \to K\backslash G/L, && G \cdot (uK, vL) \mapsto Ku^{-1}vL \\ &K\backslash G/L \to G\backslash(G/K \times G/L), && v \mapsto G \cdot (eK, vL) \end{aligned}$$

are inverse bijections.

1.3 The Orbit Category

The full subcategory of G-SET with object the homogeneous G-sets is called the **orbit category** $\mathrm{Or}(G)$ of G.

(1.3.1) Proposition. *Let H and K be subgroups of G.*
(1) *There exists a G-map $G/H \to G/K$ if and only if (H) is subconjugate to (K).*
(2) *Each G-map $G/H \to G/K$ has the form $R_a\colon gH \mapsto gaK$ for an $a \in G$ such that $a^{-1}Ha \subset K$.*
(3) *$R_a = R_b$ if and only if $a^{-1}b \in K$.*
(4) *G/H and G/K are G-isomorphic if and only if H and K are conjugate in G.*

Proof. Let $f\colon G/H \to G/K$ be equivariant and suppose $f(eH) = aK$. By equivariance, we have for all $h \in H$ the equalities $aK = f(eH) = f(hH) = hf(eH) = haK$ and hence $a^{-1}Ha \subset K$. The other assertions are easily verified. □

We denote by $N_G H = NH = \{n \in G \mid nHn^{-1} = H\}$ the **normalizer** of H in G and by $W_G(H) = WH$ the associated quotient group NH/H (**Weyl-group**). Suppose G is finite. Then $n^{-1}Hn \subset H$ implies $n^{-1}Hn = H$. Hence each endomorphism of G/H is an automorphism. A G-map $f\colon G/H \to G/H$ has the form $gH \mapsto gn_f H$ for a uniquely determined coset $n_f \in WH$. The assignment $f \mapsto n_f^{-1}$ is an isomorphism $\mathrm{Aut}_G(G/H) \cong WH$.

(1.3.2) Proposition. *The right action of the automorphism group*

$$G/H \times WH \to G/H, \quad (gH, nH) \mapsto gnH$$

is free. Hence for each $K \le G$ the set G/H^K carries a free WH-action and the cardinality $|G/H^K|$ is divisible by $|WH|$. We have $G/H^H = WH$. □

(1.3.3) Example. The assignment

$$\Psi_L\colon \mathrm{Hom}_G(G/L, X) \to X^L, \quad \alpha \mapsto \alpha(eL)$$

is a bijection. The inverse sends $x \in X^L$ to $gL \mapsto gx$. We have

$$G/L^K = \{sL \mid s^{-1}Ks \le L\}.$$

Given $sL \in G/L^K$ then $R_s\colon G/K \to G/L, gK \mapsto gsL$ is the associated morphism. The diagram

$$\begin{array}{ccc} \mathrm{Hom}_G(G/L,X) & \xrightarrow{\Psi_L} & X^L \\ \downarrow{\scriptstyle R_s^*} & & \downarrow{\scriptstyle l_s} \\ \mathrm{Hom}_G(G/K,X) & \xrightarrow{\Psi_K} & X^K \end{array}$$

is commutative. We view the Ψ_L as a natural isomorphism from the Hom-functor $\mathrm{Hom}_G(-,X)$ to the **fixed point functor**. The left translation by $n \in NK$ maps X^K into itself. In this way, X^K becomes a WK-set. ◇

Let G be a finite group. The fixed point set G/L^K is the set $\{sL \mid s^{-1}Ks \le L\}$. Let $A \le L$ be G-conjugate to K. Consider the subset

$$G/L^K(A) = \{tL \mid t^{-1}Kt \sim_L A\}.$$

The set G/L^K has a left $N_G K$-action $(n, sL) \mapsto nsL$. The subsets $G/L^K(A)$ are $N_G K$-invariant.

(1.3.4) Proposition. *Suppose $s^{-1}Ks = A$. The assignment*

$$N_G(A)/N_L(A) \to G/L^K(A), \quad nN_L(A) \mapsto snL$$

is a bijection.

Proof. Since $n^{-1}s^{-1}Ksn = n^{-1}An = A \leq L$ the element snL is contained in G/L^K. The map is well-defined, because $N_L(A) \leq A$. If $snL = smL$, then $m^{-1}n \in L \cap N_G(A) = N_L(A)$, and we see that the map is injective.

Suppose $A \sim_L t^{-1}Kt \leq L$. Then there exists $l \in L$ such that $t^{-1}Kt = l^{-1}Al$, hence $s^{-1}tl^{-1} \in N_G(A)$, and $tL = snL$. We see that the map is surjective. □

We can rewrite this result in terms of the $N_G(K)$-action on G/L^K. The subset $G/L^K(A)$ is an $N_G(K)$-orbit; and the isotropy group at sL is $sN_L(A)s^{-1}$. The fixed point set G/L^K is the disjoint union of the $G/L^K(A)$ where (A) runs over the L-conjugacy classes of the subgroups $A \leq L$ which are G-conjugate to K.

Since the homogeneous G-sets correspond to the subgroups of G we can consider a modified orbit category: The objects are the subgroups of G and the morphisms $K \to L$ are the G-maps $G/K \to G/L$. Since we are working with left actions we denote this category by ${}_\bullet\mathrm{Or}(G)$. There is a similar category $\mathrm{Or}_\bullet(G)$ where the morphisms $K \to L$ are the G-maps $K\backslash G \to L\backslash G$. If we assign to $R_s\colon G/K \to G/L$ the map $L_{s^{-1}}\colon K\backslash G \to L\backslash G, Kg \mapsto Ls^{-1}g$, then we obtain an isomorphism ${}_\bullet\mathrm{Or}(G) \to \mathrm{Or}_\bullet(G)$.

The **transport category** $\mathrm{Tra}(G)$ of G has as object set the subgroups of G, and the morphism set $\mathrm{Tra}(K, L)$ consists of the triples (K, L, s) with $s \in G$ and $sKs^{-1} \subset L$. We denote $\mathrm{Tra}(K, L)$ also as $\{s \in G \mid sKs^{-1} \leq L\}$ and pretend that the morphism sets are disjoint. Composition is defined by multiplication of group elements. In this context we work with the orbit category $\mathrm{Or}_\bullet(G)$ of right homogeneous G-sets. We have a functor

$$q\colon \mathrm{Tra}(G) \to \mathrm{Or}_\bullet(G).$$

It is the identity on objects and sends (K, L, s) to $l_s\colon K\backslash G \to L\backslash G,\ Kg \mapsto Lsg$. The endomorphism sets in both categories are groups

$$\mathrm{Tra}(K, K) = NK, \qquad \mathrm{Or}_\bullet(K, K) = WK.$$

Via composition, $\mathrm{Tra}(K, L)$ carries a left action of $NL = \mathrm{Tra}(L, L)$ and a right action of $NK = \mathrm{Tra}(K, K)$. These actions commute. Similarly for the category $\mathrm{Or}_\bullet(G)$. The functor q is surjective on Hom-sets and induces a bijection

$$L\backslash\mathrm{Tra}(K, L) \cong \mathrm{Or}_\bullet(K, L).$$

Let

$$(K, L)_* = \{A \mid A \leq L, A \sim_G K\} \tag{1.1}$$
$$(K, L)^* = \{B \mid K \leq B, B \sim_G L\}. \tag{1.2}$$

(These sets can be empty.) We have bijections

$$\mathrm{Tra}(K, L)/NK \cong (K, L)_*, \quad s \cdot NK \mapsto sKs^{-1},$$

$$NL\backslash\mathrm{Tra}(K,L) \cong (K,L)^*, \quad NL\cdot s \mapsto s^{-1}Ls.$$

They imply the counting identities

$$|(K,L)_*|\cdot|NK| = |(K,L)^*|\cdot|NL| = |G/L^K|\cdot|L|. \tag{1.3}$$

The integers $\zeta^*(K,L) = |(K,L)^*|$ and $\zeta_*(K,L) = |(K,L)_*|$ depend only on the conjugacy classes of K and L. The $\mathrm{Con}(G)\times\mathrm{Con}(G)$-matrices ζ^* and ζ_* have the property that their entries at $(K),(L)$ are zero if $(K)\not\leq(L)$, and the diagonal entries are 1. They are therefore invertible over $\mathbb{Z}$, and their respective inverses μ^* and μ_* have similar properties. In order to see this, ones solves the equation

$$\textstyle\sum_{(A)}\zeta^*(K,A)\mu^*(A,L) = \delta_{(K),(L)} \tag{1.4}$$

inductively for $\mu^*(K,L)$; the induction is over $|\{(A)\mid(K)\leq(A)\leq(L)\}|$. The matrices μ^* and μ_* are called the **Möbius-matrices** of $\mathrm{Con}(G)$. Let N denote the diagonal matrix with entry $|NK|$ at $(K),(K)$. Then (1.3) says

$$\zeta_*N = N\zeta^*, \qquad N\mu_* = \mu^*N.$$

Let G be abelian. Then

$$\zeta^*(K,L) = \zeta_*(K,L) = \zeta(K,L) = 1, \quad \text{for } (K)\leq(L).$$

Hence also $\mu^* = \mu_* = \mu$ in this case.

(1.3.5) Proposition. *Let $G\cong(\mathbb{Z}/p)^d$ be elementary abelian. Then $\mu(1,G) = (-1)^d p^{d(d-1)/2}$.*

Proof. The direct proof from the definition is a classical q-identity. For an indeterminate q we define the quantum number

$$[n]_q = \frac{q^n-1}{q-1} = 1+q+q^2+\cdots+q^{n-1},$$

and the q-binomial coefficient

$$[n]_q! = [1]_q[2]_q\cdots[n_q], \quad \binom{n}{a}_q = \frac{[n]_q!}{[a]_q![n-a]_q!}.$$

With a further indeterminate z the following generalized binomial identity holds

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}_q q^{j(j-1)/2}z^j = \prod_{k=0}^{n-1}(1-q^kz). \tag{1.5}$$

A proof can be given by induction over n, as in the case of the classical binomial identity. If q is a prime, then $\binom{n}{j}_q$ is the number of j-dimensional subspaces of $\mathbb{F}_q^n$. For $z=1$ the identity (1.5) yields inductively the values of the μ-function as claimed. □

There are two more quotient categories of the transport category. The **homomorphism category** $Sc(G)$ has as morphism set the homomorphisms $K \to L$ which are of the form $k \mapsto gkg^{-1}$ for some $g \in G$. Two elements of G define the same homomorphism if they differ by an element of the centralizer ZK of K in G. We can therefore identify the morphism set $\mathrm{Sc}(K, L)$ with $\mathrm{Tra}(K, L)/ZK$.

Finally we can combine the orbit category and the homomorphism category. In the category $\mathrm{Sci}(G)$ we consider homomorphisms $K \to L$ up to inner automorphisms of L; thus the morphism set $\mathrm{Sci}(K, L)$ can be identified with the double coset $L\backslash\mathrm{Tra}(K, L)/ZK$.

Problems

1. S be a G-set and $K \leq G$. We have a free left WK-action on S_K via left translation. The inclusion $S_K \subset S_{(K)}$ induces a bijection $S_K/WK \cong S_{(K)}/G$. The map

$$G/K \times_{WK} S_K \to S_{(K)}, \quad (gK, x) \mapsto gx$$

is a bijection of G-sets.

2. Suppose $D \leq G$ is cyclic. Then $|D||G/D^A| = |N_G A|$ if $(A) \leq (D)$. Hence $\zeta_*(A, D) = 1$ for $(A) \leq (D)$. If $\mu\colon \mathbb{N} \to \mathbb{Z}$ denotes the classical Möbius-function, then $\mu_*(A, D) = \mu(|D/A|)$ and $\mu^*(A, D) = NA/ND\mu(|D/A|)$. (The function μ is defined inductively by $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ in the case that $n > 1$.)

1.4 Möbius Inversion

We discuss in this section the Möbius matrices from a combinatorial view point. Let $(S, \leq)$ be a finite partially ordered set (= poset). The Möbius-function of this poset is the function $\mu\colon S \times S \to \mathbb{Z}$ with the properties

$$\mu(x, x) = 1, \quad \textstyle\sum_{y, x \leq y \leq z} \mu(x, y) = 0 \quad \text{for } x < z, \quad \mu(x, y) = 0 \quad \text{for } x \not\leq y.$$

These properties allow for an inductive computation of μ. We use the Möbius-function for the Möbius-inversion: Let $f, g\colon S \to \mathbb{Z}$ be functions such that

$$g(x) = \textstyle\sum_{y, x \leq y} f(y). \tag{1.6}$$

Then

$$f(x) = \textstyle\sum_{y, x \leq y} \mu(x, y) g(y). \tag{1.7}$$

A more general combinatorial formalism uses the associative incidence algebra with unit $I(S, \leq)$ of a poset. It consists of all functions $f\colon S \times S \to \mathbb{Z}$ such that $f(x,y) = 0$ if $x \not\leq y$ with pointwise addition and multiplication

$$(f * g)(x,y) = \textstyle\sum_{z, x \leq z \leq y} f(x,z)g(z,y).$$

(One can, more generally, define a similar algebra for functions into a commutative ring R.) The unit element of this algebra is the Kronecker-delta

$$\delta(x,y) = 1, \text{ for } x = y, \quad \delta(x,y) = 0, \text{ otherwise.}$$

If we define the function ζ by $\zeta(x,y) = 1$ for $x \leq y$, then the Möbius-function is the inverse of ζ in this algebra $\mu = \zeta^{-1}$. The group of functions $\alpha\colon S \to \mathbb{Z}$ becomes a left module over the incidence algebra via $(f * \alpha)(x) = \sum_y f(x,y)\alpha(y)$. We can now write (1.6) and (1.7) in the form $g = \zeta * f$, $f = \zeta^{-1} * g = \mu * g$.

Let G be a finite group. We apply this to the poset $(\mathrm{Sub}(G), \leq)$ and write $\mu(1,H) = \mu(H)$, with the trivial group 1. Conjugation of subgroups yields an action of G on this poset by poset automorphisms. Let $I^{Con}(G)$ denote the subalgebra of $I(\mathrm{Sub}(G), \leq)$ of invariant functions $f(K,L) = f(gKg^{-1}, gLg^{-1})$. We also have the poset $(\mathrm{Con}(G), \leq)$ of conjugacy classes. For $f \in I^{Con}(G)$ we define $f_*(K,L) = \sum\{f(A,L) \mid A \in (K,L)_*\}$ and $f^*(K,L) = \sum\{f(K,B) \mid B \in (K,L)^*\}$; see (1.1) and (1.2) for the notation. One verifies that $f_*(K,L)$ and $f^*(K,L)$ only depend on the conjugacy classes of K and L. Moreover:

(1.4.1) Proposition. *The assignments*

$$c_*\colon I^{Con}(G) \to I(\mathrm{Con}(G)),\ f \mapsto f_*, \quad c^*\colon I^{Con}(G) \to I(\mathrm{Con}(G)),\ f \mapsto f^*$$

are unital algebra homomorphisms. □

With these notations $c_*(\zeta) = \zeta_*$, $c^*(\zeta) = \zeta^*$ In particular, since $\zeta \in I^{Con}$, we have in $I(\mathrm{Con}(G))$ the inverses μ^* of ζ^* and μ_* of ζ_*. Recall that $\zeta^*(K,L) = |(K,L)^*|$ and $\zeta_*(K,L) = |(K,L)_*|$.

Let S be a finite G-set. Then we have $S^H = \coprod_{H \leq K} S_K$ and hence

$$|S_H| = \textstyle\sum_{K, H \leq K} \mu(H,K)|S^K|.$$

Since $S_{(H)} \cong G/H \times_{WH} S_H$, we see that the number $m_H(S)$ of orbits of type H in S is given by

$$m_H(S) = \tfrac{1}{|WH|} \textstyle\sum_{K, H \leq K} \mu(H,K)|S^K|;$$

note that $|S_{(H)}/G| = |S_H/WH|$, and WH acts freely on S_H. We rewrite this in terms of conjugacy classes:

$$m_H(S) = \tfrac{1}{|WH|} \textstyle\sum_{(K), (H) \leq (K)} \mu^*(H,K)|S^K|.$$

1.5 The Möbius Function

In this section we investigate the Möbius-function by combinatorial methods.

(1.5.1) Lemma. *Let $1 \neq N \lhd G$ and $N \leq K \leq G$. Then $\sum_{X, XN=K} \mu(X) = 0$.*

Proof. By definition of μ, this holds for $K = N$. We assume inductively, that the assertion holds for all proper subgroups Y of K which do not contain N. The computation

$$0 = \sum_{X \leq K} \mu(K) = \sum_{X, XN=K} \mu(X) + \sum_{N \leq Y < K} \left(\sum_{XN=Y} \mu(X) \right) = \sum_{X, XN=K} \mu(X)$$

yields the claim. □

(1.5.2) Proposition. *Let $N \lhd G$ and let $Co(G,N) = \{K \leq G \mid KN = G, K \cap N = 1\}$ be the set of complements of N in G. Then*

$$\mu(G) = \mu(G/N) \cdot \sum_{K \in Co(G,N)} \mu(K,G).$$

Proof. (An empty sum yields zero.) The assertion is trivial in the case that $N = 1$; hence assume $N > 1$. By 1.5.1,

$$\mu(G) = -\sum_{X < G, XN=G} \mu(X).$$

We use induction over the order of G. This yields for the summand $\mu(X)$

$$\mu(X) = \mu(X/X \cap N) \cdot \sum_{K \in Co(X, X \cap N)} \mu(K,X).$$

Since $XN = G$ we have $G/N \cong X/X \cap N$. Therefore $\mu(G)$ equals

$$-\mu(G/N) \cdot \textstyle\sum_{X<G, XN=G} \left(\displaystyle\sum_{K \in Co(X, X \cap N)} \mu(K,X) \right).$$

One verifies that the following conditions (1) and (2) on X, K are equivalent:

(1) $X < G$, $XN = G$, $K \in Co(X, X \cap N)$

(2) $K \leq X < G$, $K \in Co(G,N)$.

For (1) says $K \leq X < G, K \cap N = 1; XN = G, K \cdot (X \cap N) = X$, and (2) says $K \leq X < G, K \cap N = 1; K \cdot N = G$. In order to prove (1) $\Rightarrow$ (2) we multiply the last equation in (1) with N. In order to prove (2) $\Rightarrow$ (1) we use the modular property of the subgroup lattice which says in general terms: For $A, B, C \leq G$ and $A \leq C$ the equality $AB \cap C = A(B \cap C)$ holds. By definition of μ we know

$$\sum_{X, K \leq X < G} \mu(K,X) = -\mu(K,G).$$

Now we put everything together and obtain the claim. □

(1.5.3) Proposition. *Let G be a group with $\mu(G) \neq 0$. Let $N \leq M \lhd G$ and $N \lhd G$. Then M/N has a complement in G/N.*

Proof. By the previous note, $\mu(G/N) \neq 0$. Therefore it suffices to treat the case $N = 1$. But then, again by 1.5.2, $Co(G, M) \neq \emptyset$. □

The **Frattini-subgroup** $\Phi(G)$ of G is the intersection of its maximal subgroups. The first assertion of the next note follows immediately from the definition of $\Phi(G)$. For the second one we use the fact that a maximal subgroup of a p-group is a normal subgroup of index p. See [**?**, III.3.2 and III.3.14].

(1.5.4) Proposition. (1) *Let $N \lhd G$. Then there exists $H < G$ with $G = NH$ if and only if N is not contained in $\Phi(G)$.*
(2) *Let G be a p-group. Then $G/\Phi(G)$ is elementary abelian, and $\Phi(G)$ is the smallest normal subgroup N such that G/N is elementary abelian.* □

(1.5.5) Corollary. 1.5.3 *and* 1.5.4 *imply:*

(1) *Let $\mu(G) \neq 0$. Then $\Phi(G) = 1$.*
(2) *If G is a p-group and $\mu(G) \neq 0$, then G is elementary abelian.*

Proof. (1) Suppose $\mu(G) \neq 0$. Then we know from 1.5.3 that $\Phi(G)$ has a complement in G, and this is impossible, by 1.5.4(1), if $\Phi(G) \neq 1$.

(2) If G is not elementary abelian, then $\Phi(G) \neq 1$, by 1.5.4(2), and therefore $\mu(G) = 0$. □

We now reprove 1.3.5.

(1.5.6) Proposition. *Let P be a p-group. Then $\mu^*(1, H) \neq 0$ if and only if $H \leq P$ is elementary abelian. If H is elementary abelian of order $|H| = p^d$, then $\mu^*(1, H) = (-1)^d p^{d(d-1)/2} |P/NH|$.*

Proof. We have just seen a proof of the first assertion. It remains to determine $\mu(G)$ for elementary abelian G. We induct over $|G|$. Suppose $A \leq G$, $|A| = p$, and $|G| = p^d$. Among the $(p^d - 1)/(p - 1) = b$ maximal subgroups exactly $(p^{d-1} - 1)/(p - 1) = a$ contain the subgroup A. Therefore A has $b - a = p^{d-1}$ complements. By 1.5.2, $\mu(G) = -p^{d-1}\mu(G/A)$, since $\mu(K, G) = -1$ for a maximal subgroup K. □

Problems

1. The function $H \mapsto \mu(H, A_5)$ is displayed in the next table.

1	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	D_2	D_3	D_5	A_4	A_5
-60	4	2	0	0	-1	-1	-1	1

1.6 One-dimensional Representations

We study in some detail the simplest type of representations, namely one-dimensional representations of finite groups G over the complex numbers; these are just the homomorphisms $G \to \mathbb{C}^*$. These representations do not need much theory, and they will be used at various occasions, e.g., as input for the construction of more complicated representations (later called induced representations). The set $X(G) = G^*$ of these homomorphisms becomes an abelian group with product $(\alpha \cdot \beta)(g) = \alpha(g)\beta(g)$. A homomorphism $\varphi\colon A \to B$ induces a homomorphism $X(\varphi)\colon X(B) \to X(A)$, $\beta \mapsto \beta \circ \varphi$. In this manner X yields a contravariant functor from finite groups to finite abelian groups. The group $X(G)$ will be called the **character group** of G, and $\alpha \in X(G)$ is a (linear) **character** of G.

Since $\mathbb{C}^*$ is abelian, a homomorphism $\alpha\colon G \to \mathbb{C}^*$ maps the **commutator subgroup** $[G,G] = G'$, generated by the commutators $uvu^{-1}v^{-1}$, to 1 and induces $\overline{\alpha}\colon G/[G,G] \to \mathbb{C}^*$. The factor group $G/[G,G]$ is abelian and it is called the **abelianized quotient** G^{ab} of G.

For the cyclic group $C_m = \langle\, c \mid c^m = 1 \,\rangle$ the character group $X(C_m)$ is the cyclic group of order m generated by $\rho\colon c \mapsto \exp(2\pi i/m)$. Let G and H be groups. Let $\alpha\colon G \to \mathbb{C}^*$ and $\beta\colon H \to \mathbb{C}^*$ be homomorphisms. Then $\alpha \odot \beta\colon G \times H \to \mathbb{C}^*$, $(g,h) \mapsto \alpha(g)\beta(h)$ is again a homomorphism, and

$$\odot\colon G^* \times H^* \longrightarrow (G \times H)^*, \quad (\alpha,\beta) \longmapsto \alpha \odot \beta$$

is a homomorphism between character groups. One verifies that $\odot$ is an isomorphism[1].

For a finite abelian group A the group A^* is isomorphic to A. This follows from the previous remarks and the structure theorem about finite abelian groups which says that each such group is isomorphic to a product of cyclic groups.

Let $H \lhd G$ be a normal subgroup of G. Then the group G acts as a group of automorphisms on $X(H)$ by $(g \cdot \gamma)(h) = \gamma(ghg^{-1})$. If γ is the restriction of a homomorphism $\alpha \in X(G)$, then $g \cdot \gamma = \gamma$. Therefore the restriction homomorphism $X(G) \to X(H)$ has an image in the fixed point group $X(H)^G$; its elements are called G-invariant. Note that the G-action on $X(H)$ factors over G/H.

Let G be the semi-direct product of the normal subgroup A and a group P, i.e. $G = AP$ and $A \cap P = 1$. Let

$$\Gamma\colon X(G) \to X(A)^P \times X(P)$$

be the product of the restriction homomorphisms.

[1] Recall the categorical notion: sum in the category of abelian groups.

(1.6.1) Proposition. Γ *is an isomorphism.*

Proof. Since $\gamma \in X(G)$ is determined by the restrictions to A and P, the map Γ is injective. Given $\alpha \in X(A)^P$, i.e. $\alpha(a) = \alpha(xax^{-1})$ for $a \in A, x \in P$, and $\beta \in X(P)$. Define a map $\gamma\colon G \to \mathbb{C}^*$ by $\gamma(ax) = \alpha(a)\beta(x)$. We verify that γ is a homomorphism

$$\begin{aligned}\gamma(axa_1x_1) &= \gamma(axa_1x^{-1}xx_1)\\ &= \alpha(axa_1x^{-1})\beta(xx_1)\\ &= \alpha(a)\alpha(xa_1x^{-1})\beta(x)\beta(x_1)\\ &= \alpha(a)\beta(x)\alpha(a_1)\beta(x_1)\\ &= \gamma(ax)\gamma(a_1x_1).\end{aligned}$$

By construction, $\Gamma(\gamma) = (\alpha, \beta)$. □

A homomorphism $\alpha \in X(A)$ is P-invariant if and only if it vanishes on the normal subgroup A_P generated by the elements $axa^{-1}x^{-1}$ for $a \in A$ and $x \in P$. Thus the quotient map $\pi\colon A \to A/A_P$ induces an isomorphism $X(A/A_P) \to X(A)^P$.

(1.6.2) Proposition. *Let the group P act on the abelian group A by automorphisms $(x, a) \mapsto x \diamond a$. Suppose $(|A|, |P|) = 1$. Let $A_P \le A$ denote the subgroup generated by the elements $a \cdot (x \diamond a)^{-1}$. Then the inclusion $\iota\colon A^P \mapsto A/A_P$ is an isomorphism.*

Proof. For $a \in A$ set $\mu(a) = \prod_{x \in P}(x \diamond a)$. Then $\mu(a) \in A^P$, and for $a \in A^P$ we have $\mu(a) = a^{|P|}$. Since $|P|$ is prime to the order of A, the map $a \mapsto a^{|P|}$ is an automorphism of A and A^P. The group A_P is contained in the kernel of μ, since, by construction, $\mu(y \diamond a) = \mu(a)$ for $a \in A$ and $x \in P$. Hence we obtain an induced map $\nu\colon A/A_P \to A^P$, and $\nu \circ \iota$ is an isomorphism. On the other hand $\iota\nu(a) = a^{|P|} \prod((x \diamond a)a^{-1})$, and this shows that $\iota \circ \nu$ is an isomorphism too. □

As a consequence of 1.6.1 and 1.6.2 we obtain the next result which will later be used in the proof of the Brauer induction theorem 4.6.5.

(1.6.3) Proposition. *Let $G = AP$ be the semi-direct product of the abelian subgroup A by P. Suppose $(|A|, |P|) = 1$. Then the restriction $X(G) \to X(A^P \times P)$ is an isomorphism.* □

We continue the study of one-dimensional representations and demonstrate their use in group theory. Let $H \le G$ and $\alpha \in X(H)$. We associate to α an element $m_H^G \alpha \in X(G)$. For this purpose we choose a representative system

$g_1, \dots, g_r$ of G/H. For each $g \in G$ we have $gg_i = g_{\sigma(i)} h_i$ with a permutation $\sigma \in S_r$ and certain $h_i \in H$. We set

$$(m_H^G \alpha)(g) = \prod_{i=1}^r \alpha(g_{\sigma(i)}^{-1} g g_i) = \prod_{i=1}^r \alpha(h_i).$$

One verifies that $m_H^G \alpha$ is a well-defined homomorphism and that, moreover, $m_H^G \colon X(H) \to X(G)$ is a homomorphism. We call it **multiplicative induction**.

We use this construction to deal with the question: Given $\alpha \in X(H)$, when does there exist an extension $\beta \in X(G)$ such that $\beta|H = \alpha$? Suppose it exists. Then for $h \in H$, $u \in G$ and $g = uhu^{-1}$ we have $\beta(g) = \beta(h)$. Thus a necessary condition for the existence of β is that α is trivial on the subgroup H_0 generated by $\{xy^{-1} \mid x \sim_G y, x, y \in H\}$.

(1.6.4) Proposition. *Suppose H and G/H have coprime order. Then an extension β exists if and only if α vanishes on H_0.*

Proof. Let α have the stated property. We compute $m_H^G \alpha(h)$ for $h \in H$. For this purpose we make a special choice of the coset representatives: The cyclic group $\langle h \rangle$ acts on G/H; let $gH, hgH, \dots, h^{t-1}gH$ be an orbit, and suppose $h^t g = g\tilde{h}$. Then this orbit contributes $\alpha(\tilde{h}) = \alpha(g^{-1}h^t g) = \alpha(h)^t$ to the product in the definition of $m_H^G \alpha$. Altogether we obtain $\mathrm{res}_H^G \, m_H^G \alpha = \alpha^{|G/H|}$. Hence if H and G/H have coprime order, then $\mathrm{res}_H^G \, m_H^G$ is an automorphism because $|X(H)|$ is coprime to $|G/H|$. Therefore there exists an extension. □

(1.6.5) Proposition. *Let H be a Sylow p-subgroup of G. Then $H_0 = H \cap G'$.*

Proof. If $y = g^{-1}xg$, then $xy^{-1} = xg^{-1}x^{-1}g$, so that $P' \le P_0 \le P \cap G'$. It remains to show $P \cap G' \le P_0$. Given $x \in P \smallsetminus P_0$, there exists $\lambda \in X(P)$ such that $\lambda(x) \neq 1$ with trivial $\lambda|P_0$. By the previous proposition, λ has an extension $\theta \colon G \to \mathbb{C}^*$. Since $\theta(x) \neq 1$, we see that $x \notin P \cap G'$. □

The previous considerations lead to a simple proof of the so-called **normal complement theorem**.

(1.6.6) Proposition. *Let $G(p)$ be an abelian Sylow p-group of G and assume that $NG(p) = G(p)$. Then there exists a normal subgroup $H \lhd G$ such that $N \cap G(p) = 1$.*

Proof. The quotient G/G' has Sylow group $G(p)$ if $G' \cap P = P_0 = 1$. This means: Suppose $x, y \in P$ are conjugate in G, then $x = y$. This is a consequence of the next lemma. Since G/G' is abelian, there exists a complement of $G(p)$, and the pre-image in G is the required complement. □

(1.6.7) Lemma. *Let G have abelian Sylow p-group P. Suppose $x, y \in P$ are conjugate in G. Then they are conjugate in $N_G P$.*

Proof. Let $y = gxg^{-1}$. Since P is abelian, P is a subgroup of the centralizer $C_G(y)$ of y in G, moreover $g(C_G(x))g^{-1} = C_G(y)$. Hence gPg^{-1} and P are Sylow groups of $C_G(y)$. Therefore there exists $n \in C_G(y)$ such that $ngPg^{-1}n^{-1} = P$. Hence $ng \in NG(P)$ and $y = ny^{-1} = ngxg^{-1}n^{-1}$. □

Problems

1. Let B be a subgroup of the finite abelian group A. Show that for each $a \in A \smallsetminus B$ there exists $\alpha \in X(A)$ with $\alpha(a) \neq 1$.
2. The isomorphism between G and G^* is not natural, but there exists a canonical and natural isomorphism $G \to X(X(G))$, G abelian. (This is analogous to the double dual of finite dimensional vector spaces.)
3. Let $1 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 1$ be an exact sequence of finite abelian groups. Then the functor X transforms it into an exact sequence. Exactness at $X(B)$ and $X(C)$ is formal; for the exactness at $X(A)$ one can use the knowledge of the order of this group.

1.7 Representations as Modules

The vector space KG has more structure than just carrying the left and right regular representation.

There is a bilinear map $KG \times KG \to KG$ which extends the group multiplication $(g, h) \mapsto gh$ of the basis elements. This bilinear map defines on KG the structure of an associative algebra with unit. This algebra is called the **group algebra** KG of G over K. The multiplication in the group algebra is therefore defined by the formula

$$(\textstyle\sum_{g\in G} \lambda(g)g) \cdot (\sum_{h\in G} \mu(h)h) = \sum_{g,h} \lambda(g)\mu(h)gh = \sum_{u\in G} \gamma(u)u$$

with $\gamma(u) = \sum_{g\in G} \lambda(g)\mu(g^{-1}u)$. Another model for the group algebra is the vector space $C(G, K)$ of functions $G \to K$ with **convolution product**

$$(\alpha * \beta)(u) = \textstyle\sum_{g\in G} \alpha(g^{-1})\beta(u^{-1}g).$$

The assignment $C(G, K) \to KG, \varphi \mapsto \sum_g \varphi(g^{-1})g$ is an isomorphism of algebras. Under this isomorphism the natural left-right action on $C(G, K)$, given by

$$(g \cdot \varphi \cdot h)(x) = \varphi(hxg),$$

corresponds to the left-right action on KG.

(1.7.1) Example. The group algebra of the cyclic group $C_n = \langle\, x \mid x^n = 1 \,\rangle$ is the quotient $K[x]/(x^n - 1)$ of the polynomial algebra $K[x]$ by the principal ideal $(x^n - 1)$. $\diamond$

We now come to the third form of a representation, that of a module over the group algebra. Let V be a KG-representation. The bilinear map

$$KG \times V \to V, \quad (\textstyle\sum_g \lambda(g)g, v) \mapsto \sum_g \lambda(g)(g \cdot v)$$

is the structure of a unital KG-module on the vector space V. The element $\sum_g \lambda(g)g \in KG$ acts on V as the linear combination $\sum_g \lambda(g)g$. A morphism $V \to W$ of representations becomes a KG-linear map. Conversely, given a KG-module M we obtain a representation on M by defining l_g as the scalar multiplication by $g \in KG$ in the module. In this manner, the category KG-Rep of finite-dimensional KG-representations becomes the category KG-Mod of left KG-modules which are finite-dimensional as vector spaces. Direct sums correspond in these categories. A module M over an algebra A is **irreducible**, if it has no submodules different from 0 and M.

The view point of modules allows for an algebraic construction of representations. Consider KG as a left module over itself. Then a left ideal is a representation. A non-zero left ideal yields an irreducible module, if it is a minimal left ideal with respect to inclusion. Let M be an irreducible KG-module and $0 \neq x \in M$. Then $KG \to M, \lambda \mapsto \lambda x$ is KG-linear; its kernel I is a left ideal and the induced map $KG/I \to M$ an isomorphism, since M is irreducible; the ideal I is then a maximal ideal.

(1.7.2) Example. The maximal ideals in the group algebra $KC_n = K[x]/(x^n - 1)$ correspond to principal ideals $(q) \subset K[x]$ where q is an irreducible factor of $x^n - 1$. If K is a splitting field for $x^n - 1$, then the irreducible factors are linear, and irreducible representations one-dimensional. Over $\mathbb{Q}$, the polynomial is the product $\prod_{d|n} \Phi_d(x)$ of the irreducible cyclotomic polynomials Φ_d. The complex roots of Φ_d are the primitive d-th roots of unity. As an example

$$x^6 - 1 = (x-1)(x+1)(x^2+x+1)(x^2-x+1).$$

The representation on $\mathbb{Q}[x]/(x^2 - x + 1)$ is given in the basis $1, x$ by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Thus we know that this matrix has order 6; this can, of course, be checked by a calculation. One the other hand, it is a nontrivial task to find matrices in $SL_2(\mathbb{Z})$ of order 6. $\diamond$

1.8 Linear Algebra of Representations

Standard constructions of linear algebra may be used to obtain new representations from old ones. We begin with direct sums.

Let $V_1, \dots V_r$ be vector spaces over K. Their (external) direct sum $V_1 \oplus \dots \oplus V_r$ consists of all r-tuples $(v_1, \dots, v_r)$, $v_j \in V_j$ with component-wise addition and scalar multiplication. If the V_j are subspaces of a vector space V, we say, V is the (internal) direct sum of these subspaces, if each $v \in V$ has a unique presentation of the form $v = \sum_{j=1}^{r} v_j$ with $v_j \in V_j$. We also use the notation $V = \oplus_{j=1}^{r} V_j$, because V is canonically isomorphic to the external direct sum of the V_j. A subspace U of W is a direct summand, if there exists a complementary subspace V, i.e., a subspace V such that $U \oplus V = W$.

Let $(V_j \mid j \in J)$ be a family of subspaces of V. The sum $\sum_{j \in J} V_j$ is the subspace of V generated by the V_j. It is the smallest subspace containing the V_j and consists of the elements which are sums of elements in the various V_j. We use the following fact from linear algebra.

(1.8.1) Proposition. *Let $V_1, \dots, V_n$ be subspaces of V. The following are equivalent:*

(1) *V is the internal direct sum of the V_j.*

(2) *V is the sum of the V_j, and $V_j \cap \sum_{i \neq j} V_i = \{0\}$ for all j.* □

We now apply these concepts to representations. We use two simple observations. If $(V_j \mid j \in J)$ are sub-representations of V, then their sum is again a sub-representation. The direct sum $U \oplus V$ of representations becomes a representation with respect to the component-wise group action $g \cdot (u, v) = (g \cdot u, g \cdot v)$. Similarly for an arbitrary number of summands. This defines the **direct sum** of representations. If $g \mapsto A(g)$ and $g \mapsto B(g)$ are matrix representations associated to U and V, then the block matrices

$$\begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}$$

yield a matrix representation for $U \oplus V$. A representation is called **indecomposable**, if it is not the direct sum of non-zero sub-representations. An irreducible representation is clearly indecomposable, but the converse does not hold in general.

(1.8.2) Example. In 1.1.4 we defined two sub-representations T_n, D of the permutation representation of S_n on K^n. Given $(x_1, \dots, x_n) \in K^n$ write $x = n^{-1} \sum_j x_j$. Then $(x_1 - x, \dots, x_n - x) \in T_n$ and $(x, \dots, x) \in D$. Hence $T_n + D = K^n$. The intersection $T_n \cap D$ consists of the $(y, \dots, y)$ with $ny = 0$. This implies $y = 0$. Hence $T_n \oplus D = K^n$. But note: This argument requires that n^{-1} makes sense in K, i.e., the characteristic of K does not divide n.

If $n = 2$ and $K = \mathbb{F}_2$ is the field with two elements, then $V = W$! The regular representation is not irreducible, because it has a one-dimensional fixed point set. If this fixed point set had a complement it would be a one-dimensional representation, hence a trivial representation. Therefore the regular representation is indecomposable. ◇

Here is another result from linear algebra.

(1.8.3) Proposition. *A sub-representation W of V is a direct factor if and only if there exists a projection morphism $q\colon V \to V$ with image W. A projection is a morphism q such that $q \circ q = q$. If q is a projection, then V is the direct sum of the image and the kernel of q.* □

Let V and W be representations of G. The **tensor product** representation $V \otimes_K W$ has the action $g(v \otimes w) = gv \otimes gw$. If $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W, then the $v_i \otimes w_k$ form a basis of $V \otimes W$. The map $V \times W \to V \otimes W, (v, w) \mapsto v \otimes w$ is bilinear. If g acts on V and W via matrices (r_{ij}) and (s_{kl}), then g acts on $V \otimes W$ via the matrix $(r_{ij} s_{kl})$ whose entry in the (i, k)-th row and (j, l)-th column is $r_{ij} s_{kl}$. More explicitely, if $gv_j = \sum_i r_{ij} v_i$ and $gw_l = \sum_k s_{kl} w_k$, then

$$g(v_j \otimes w_l) = \textstyle\sum_{i,k} r_{ij} s_{kl} v_i \otimes w_k.$$

If V is one-dimensional and given by a homomorphism $\alpha\colon G \to K^*$, then we simply multiply the matrix (s_{kl}) with $\alpha(g)$ in order to obtain the tensor product.

Let V and W be G-representations. We have a G-action on the vector space $\mathrm{Hom}(V, W)$ of K-linear maps, given by $(g \cdot \varphi)(v) = g\varphi(g^{-1}v)$. The fixed point set is $\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W)$. When $W = K$ is the trivial representation we obtain the **dual representation** $V^* = \mathrm{Hom}(V, K)$ of V. If $g \mapsto A(g)$ is the matrix representation of V with respect to a basis, then $g \mapsto {}^tA(g)^{-1}$ (inverse of the transpose) is the matrix representation of V^* with respect to the dual basis.

(1.8.4) Note. *There is a canonical isomorphism*

$$V^* \otimes W \xrightarrow{\cong} \mathrm{Hom}(V, W), \quad \varphi \otimes w \mapsto (u \mapsto \varphi(u)w).$$

One verifies that it is a morphism of G-representations. □

In some of the constructions one can also use representations for different groups. Let V be a G-representation and W an H-representation. Then $V \otimes W$ becomes a $G \times H$-representation via $(g, h)(v \otimes w) = gv \otimes hw$. Similarly, we have a $G \times H$-action on $\mathrm{Hom}(V, W)$ defined as $((g, h) \cdot \psi)(v) = h\psi(g^{-1}v)$. With these actions, 1.8.4 is an isomorphism of $G \times H$-representations.

(1.8.5) Example. Let S and T be finite G-sets. There are canonical isomorphisms

$$K(S \amalg T) \cong K(S) \oplus K(T), \quad K(S \times T) \cong K(S) \otimes K(T), \quad K(S)^* \cong K(S).$$

They are induced by a G-equivariant bijection of the canonical bases. We combine with 1.8.4 and obtain

$$\mathrm{Hom}(K(S), K(T)) \cong K(S)^* \otimes K(T) \cong K(S) \otimes K(T) \cong K(S \times T).$$

Together with 1.2.2 we get

$$\dim_K \mathrm{Hom}_G(K(S), K(T)) = |(S \times T)/G|.$$

The representation 1.1.4 of S_n on K^n by permutation of coordinates (made into a left representation by inversion) is isomorphic to $K(S_n/S_{n-1})$ where S_{n-1} is the subgroup of S_n which fixes $1 \in \{1, \dots, n\}$. The action of S_{n-1} on S_n/S_{n-1} has two orbits, of length 1 and $n-1$. Hence $\mathrm{Hom}_{S_n}(K^n, K^n)$ is two-dimensional. ◇

1.9 Semi-simple Representations

The topic of this section is the decomposition of a representation into a direct sum of sub-representations.

We begin with a simple and typical example. Let $\alpha\colon G \to K^*$ be a homomorphism. Consider $x_\alpha = \sum_{g \in G} \alpha(g^{-1})g \in KG$. The computation

$$h \cdot x_\alpha = \textstyle\sum_g \alpha(g^{-1})hg = \sum_g \alpha(h)\alpha(g^{-1}h^{-1})hg = \alpha(h)x_\alpha$$

shows that x_α spans a one-dimensional sub-representation $V(\alpha)$ of the regular representation. Let $K = \mathbb{C}$ and $G = C_n = \langle\, a \mid a^n = 1 \,\rangle$ the cyclic group. There are n different homomorphisms $\alpha(j)\colon C_n \to \mathbb{C}^*, 1 \le j \le n$. The vectors $x_{\alpha(j)}$ are different eigenvectors of l_a. Therefore we have a decomposition $\mathbb{C}C_n = \oplus_j V(\alpha(j))$ into one-dimensional representations. A similar decomposition exists for finite abelian groups G, since we still have $|G|$ homomorphisms $G \to \mathbb{C}^*$. Our aim is to find analogous decompositions for general finite groups.

(1.9.1) Theorem. *Let V be the sum of irreducible representations $(U_j \mid j \in J)$ and let U be a sub-representation. Then there exist a finite subset $E \subset J$ such that V is the direct sum of U and the $U_j, j \in E$.*

Proof. If $W \neq V$ is any sub-representation, then there exists $k \in J$ such that $V_k \not\subset W$, since V is the sum of the V_j. Then $V_k \cap W = 0$, and $W + V_k = W \oplus V_k$. If now $E \subset J$ is a maximal subset such that the sum W of U and the $V_j, j \in E$ is direct, then necessarily $W = V$. □

(1.9.2) Theorem. *The following assertions about a representation M are equivalent:*

(1) *M is a direct sum of irreducible sub-representations.*
(2) *M is a sum of irreducible sub-representations.*
(3) *Each sub-representation is a direct summand.*

Proof. (1) $\Rightarrow$ (2) as special case; and (2) $\Rightarrow$ (3) is a special case of 1.9.1.

(3) $\Rightarrow$ (1). Let $\{M_1, \dots, M_n\}$ be a set of irreducible sub-representations such that their sum N is the direct sum of the M_j. If $N \neq M$ then, by hypothesis, there exists a sub-representation L such that $M = N \oplus L$. Each sub-representation contains an irreducible one. If $M_{n+1} \subset L$ is irreducible, then the sum of the $\{M_1, \dots, M_{n+1}\}$ is direct. □

A representation is called **semi-simple** or **completely reducible** if it has one of the properties (1)-(3) in 1.9.2.

(1.9.3) Proposition. *Sub-representations and quotient representations of semi-simple representations are semi-simple.*

Proof. Let M be semi-simple and $F \subset N \subset M$ sub-representations. A projection $M \to M$ with image F restricts to a projection $N \to N$ with image F. Hence F is a direct summand in N.

Suppose $N \oplus P = M$; then the quotient $M/N \cong P$ is semi-simple. □

(1.9.4) Proposition. *Let V be the sum of irreducible sub-representations $(V_j \mid J)$. Then each irreducible sub-representation W is isomorphic to some V_j.*

Proof. There exists a surjective homomorphism $\beta \colon V \to W$, by 1.8.3 and 1.9.2. If W were not isomorphic to some V_j, then the restriction of β to each V_j would be zero, by Schur's lemma, hence β would be the zero morphism. □

We write

$$\langle U, V \rangle = \dim_K \operatorname{Hom}_G(U, V)$$

for G-representations U and V. This integer depends only on the isomorphism classes of U and V. Note the additivity $\langle U_1 \oplus U_2, V \rangle = \langle U_1, V \rangle + \langle U_2, V \rangle$, and similarly for the second argument.

(1.9.5) Proposition. *Suppose $V = V_1 \oplus \cdots \oplus V_r$ is a direct sum of irreducible representations V_j. Let W be any irreducible representation and denote by $n(W, V)$ the number of V_j which are isomorphic to W. Then*

$$\langle W, W \rangle n(W, V) = \langle W, V \rangle = \langle V, W \rangle.$$

Therefore $n(W, V)$ is independent of the decomposition of V into irreducibles.

Proof. For a direct sum as above, we have a canonical isomorphism

$$\mathrm{Hom}_G(W,V) \cong \prod_{j=1}^r \mathrm{Hom}_G(W,V_j).$$

This expresses the fact that a morphism $W \to V$ is nothing else but an r-tuple of morphisms $W \to V_j$. The assertion is now a direct consequence of Schur's lemma. For the second assertion we use the canonical isomorphism $\mathrm{Hom}_G(V,W) \cong \prod_j \mathrm{Hom}_G(V_j,W)$. (In conceptual terms: We are using the fact that $\oplus_j V_j$ is the sum and the product of the V_j in the category of representations.) □

We call the integer $n(W,V)$ in 1.9.5 the **multiplicity** of the irreducible representation W in the semi-simple representation V. We say W **occurs** in V or is **contained** in V if $n(W,V) \neq 0$. In fact, if $n(W,V) \neq 0$, then V has a sub-representation which is isomorphic to W: take a non-zero morphism $W \to V$ and apply Schur's lemma. The irreducible representation W appears in V if and only if $\mathrm{Hom}_G(W,V)$ or $\mathrm{Hom}_G(V,W)$ is non-zero.

Let W be irreducible and denote by $V(W)$ the sum of the irreducible sub-representations of V which are isomorphic to W. We call $V(W)$ the W-**isotypical** part of V, if $V(W) \neq 0$, and the decomposition in 1.9.6 is the **isotypical decomposition** of V. Let $I = \mathrm{Irr}(G;K)$ denote a complete set of pairwise non-isomorphic irreducible representations of G over K.

(1.9.6) Theorem. *A semi-simple representation V is the direct sum of its isotypical parts.*

Proof. Since V is semi-simple it is the direct sum of irreducible sub-representations and therefore the sum of its isotypical parts. Let $A \in I$ and let Z be the sum of the $V(B), B \in I, B \neq A$. We refer to 1.8.1 and have to show: $V(A) \cap Z = 0$. Suppose this is not the case. Then the intersection would contain an irreducible sub-representation, and by 1.9.4 it would be isomorphic to A and to some $B \neq A$. Contradiction. □

1.10 The Regular Representation

We now consider KG as left and right regular representation. For each representation U the vector space $\mathrm{Hom}_G(KG,U)$ becomes a left G-representation via $(g \cdot \varphi)(x) = \varphi(x \cdot g)$.

(1.10.1) Lemma. *The evaluation $\mathrm{Hom}_G(KG,U) \to U$, $\varphi \mapsto \varphi(e)$ is an isomorphism of representations.*

Proof. We use the fact that KG is a free KG-module with basis e. It is verified from the definitions that the evaluation is a morphism. Clearly, a morphism $KG \to U$ is determined by its value at e, and this value can be any prescribed element of U. □

(1.10.2) Theorem. *Suppose the left regular representation is semi-simple. Then each irreducible representation U appears in KG with multiplicity $n_U = \langle U, U \rangle^{-1} \dim_K U$.*

Proof. Since $U \cong \mathrm{Hom}_G(KG, U)$ is non-zero, each irreducible representation U appears in KG, see the remarks after 1.9.5. Suppose $KG \cong \bigoplus_{W \in I} n_W W$ where $n_W W$ denotes the direct sum of n_W copies of W. Then

$$\dim U = \langle KG, U \rangle = \textstyle\sum_{W \in I} n_W \langle W, U \rangle = n_U \langle U, U \rangle,$$

the latter by Schur's lemma. □

(1.10.3) Proposition. *Suppose the left regular representation is semi-simple.*

(1) *The number of isomorphism classes of irreducible representations is finite.*
(2) $|G| = \sum_{V \in I} \langle V, V \rangle^{-1} (\dim V)^2$.
(3) *If K is algebraically closed, then $|G| = \sum_{V \in I} (\dim V)^2$.*

Proof. (1) is a corollary of 1.10.2.

(2) Let $KG \cong \bigoplus_{W \in I} n_W W$. We insert the values of n_W obtained in 1.10.2.

(3) If the field K is algebraically closed then, by Schur's lemma, $\langle V, V \rangle = 1$ for an irreducible representation V. □

Part (3) of 1.10.3 gives us a method to decide whether a given set of pairwise non-isomorphic irreducible representations is complete. If G is abelian then irreducible representations over $\mathbb{C}$ are 1-dimensional. By 1.10.3 we see that there are $|G|$ non-isomorphic such representations; we know this, of course, from a direct elementary argument.

(1.10.4) Proposition. *A finite group G is abelian if and only if the irreducible complex representations are one-dimensional.*

Proof. A one-dimensional complex representation is given by a homomorphism $G \to \mathbb{C}^*$. The regular representation is faithful. If the regular representation is a sum of one-dimensional representations, then G has an injective homomorphism into an abelian group. The reversed implication was proved in 1.1.3. □

There remains the question: When is KG semi-simple? Recall some elementary algebra. For $n \in \mathbb{N}$ and $x \in K$, an expression nx stands for an n-fold sum $x + \cdots + x$. A relation $nx = 1$ exists in K if and only if either K has characteristic zero or the characteristic $p > 0$ of K does not divide n. In this case we say, n is invertible in K. We denote this inverse as usual by n^{-1}.

(1.10.5) Proposition. *If KG is semi-simple, then $|G|$ is invertible in K.*

Proof. Suppose KG is semi-simple. Then the fixed set $F = \{\lambda\Sigma \mid \lambda \in K, \Sigma = \sum_{g\in G} g\}$ is a sub-representation. Hence there exists a projection $p\colon KG \to F$. Since p is G-equivariant, it is determined by the value $p(e)$, say $p(e) = \mu\Sigma$. Since p is a projection we obtain $\Sigma = p(\Sigma) = \sum_{g\in G} p(e) = \sum_{g\in G} \mu\Sigma = |G|\mu\Sigma$. Thus we have shown: If KG is semi-simple, then $|G|$ is invertible in K. □

(1.10.6) Theorem (Maschke). *Suppose $|G|$ is invertible in K. Then G-representations are semi-simple.*

Proof. We show that each sub-representation W of a representation V is a direct summand (see 1.9.2). There certainly exists a K-linear projection $p\colon V \to V$ with image W. We make it equivariant by an averaging process. Namely we define

$$q(v) = \tfrac{1}{|G|} \textstyle\sum_{g\in G} g^{-1}p(gv).$$

At this point we use the fact that $|G|^{-1}$ makes sense in K. By construction, q is K-linear as a linear combination of linear maps. For $h \in G$ we compute

$$q(hv) = \tfrac{1}{|G|} \textstyle\sum_{g\in G} g^{-1}p(ghv) = \tfrac{1}{|G|} h \textstyle\sum_{g\in G} h^{-1}g^{-1}p(ghv) = hq(v),$$

and this verifies the equivariance. By hypothesis, $p(w) = w$ for $w \in W$, hence $p(gw) = gw$ and therefore $q(w) = w$. The values $q(v)$ are contained in W, hence $W = q(V)$ and $q^2 = q$. □

(1.10.7) Proposition. *Suppose V is semi-simple. Then $\langle V, V\rangle = 1$ implies that V is irreducible.*

Proof. Decompose into irreducibles $V = \sum n_W W$. Then $1 = \langle V, V\rangle = \sum n_W^2 \langle W, W\rangle$, by Schur's lemma. Hence one of the n_W is 1 and the others are 0. □

Assume that $|G|$ is invertible in K. In order that 1.10.3 holds, it is necessary to assume that $\langle V, V\rangle = 1$ for each $V \in I$. If $K \subset L$ is a field extension, then we can view a K-representation as an L-representation (just take the same matrices). However, an irreducible representation over K may become reducible over a larger field. This already happens for cyclic groups, as we have seen in the first section. If KG is semi-simple, then also LG. If the relation 1.10.3 holds for K-representations, then it also holds for L-representations. The relation 1.10.3 is equivalent to $\langle V, V\rangle = 1$ for all $V \in I$. If this is the case, we call K a **splitting field** for G.

We now present the isotypical decomposition in a more canonical form. Let V be semi-simple. For each $U \in I$ we let $D(U)$ be its endomorphism algebra. Evaluation of endomorphisms makes U into a left $D(U)$-module. The vector space $\mathrm{Hom}_G(U, V)$ becomes a right $D(U)$-module via composition of endomorphisms. The evaluation $\mathrm{Hom}_G(U,V) \otimes U \to V, \varphi \otimes u \mapsto \varphi(u)$ induces a linear map $\iota_U \colon \mathrm{Hom}_G(U,V) \otimes_{D(U)} U \to V$.

(1.10.8) Theorem. *Let* $\iota\colon \bigoplus_{U \in I} \mathrm{Hom}_G(U,V) \otimes_{D(U)} U \to V$ *have components* ι_U*. Then* ι *is an isomorphism. The image of* ι_U *is the* U*-isotypical component of* V*.*

Proof. The maps ι constitute, in the variable V, a natural transformation on the category of semi-simple representations, and they are compatible with directs sums. Thus it suffices to consider irreducible V. In that case, by Schur's lemma, only the summand $\mathrm{Hom}_G(V,V) \otimes_{D(V)} V$ is non-zero, and evaluation is the canonical isomorphism $D(V) \otimes_{D(V)} V \cong V$. By construction, ι_U has an image in the U-isotypical part. □

Problems

1. Let $U^* = \mathrm{Hom}_K(U,K)$ be the dual vector space. This becomes a right G-representation via $(g \cdot \varphi)(u) = \varphi(gu)$. The vector space $\mathrm{Hom}_G(U,KG)$ becomes a right representation via $(\varphi \cdot g)(u) = \varphi(u) \cdot g$. Show: The linear map

$$U^* \to \mathrm{Hom}_G(U,KG), \quad \varphi \mapsto (u \mapsto \textstyle\sum_{g \in G} \varphi(g^{-1}u)g)$$

is an isomorphism of right representations. An inverse morphism assigns to $\alpha \in \mathrm{Hom}_G(U,KG)$ the linear form $U \to K$ which maps u to the coefficient of e in $\alpha(u)$.

Let KG be semi-simple. Then the isotypical decomposition **??** of KG assumes the form

$$\textstyle\bigoplus_{U \in I} U^* \otimes_{D(U)} U \to KG, \quad \varphi \otimes u \mapsto \sum_{g \in G} \varphi(g^{-1}u)g.$$

This is an isomorphism of left and right G-representations.

2. Use 1.10.3 in order to show that we found (in section 1) enough irreducible complex representations of the dihedral group D_{2n}. In the case that n is odd there are 2 one-dimensional and $(n-1)/2$ two-dimensional irreducibles. In the case that n is even there are 4 one-dimensional and $n/2 - 1$ two-dimensional irreducibles.

3. Use 1.10.7 in order to show that the representation of S_n on $T_n = \{(x_i) \in K^n \mid \sum_i x_i = 0\}$ by permutation of coordinates is irreducible (K characteristic zero).

4. Let V be a KG-representation. Let V_G denote the subrepresentation spanned by the vectors $v - gv, v \in V, g \in G$. Consider $\alpha\colon V^G \to V/V_G$ induced by the inclusion $V^G \subset V$. Show that α is an isomorphism if the characteristic of K does not divide $|G|$, and give an example where α is not bijective.

Chapter 2

Characters

2.1 Characters

We assume in this chapter that K has characteristic zero. It is then no essential restriction to assume moreover that $\mathbb{Q}$ is a subfield of K.

(2.1.1) Proposition. *Let U be a G-representation. Then the linear map*

$$p\colon U \to U, \quad u \mapsto |G|^{-1} \textstyle\sum_{g\in G} gu$$

is a G-equivariant projection onto the fixed point space U^G.

Proof. The map p is the identity on U^G, equivariant by construction, and the image is contained in U^G. □

Let V be a G-representation. We denote the trace of $l_g\colon V \to V$ by $\chi_V(g)$. The **character** of V is the function $\chi_V\colon G \to K$, $g \mapsto \chi_V(g)$. The character of an irreducible representation is an **irreducible character**.

The trace of a projection operator is the dimension of its image. Therefore 2.1.1 yields the identity

$$\dim U^G = |G|^{-1} \textstyle\sum_{g\in G} \chi_U(g). \tag{2.1}$$

Recall from linear algebra: The trace of a matrix is the sum of the diagonal elements, and conjugate matrices have the same trace. If we express l_g in matrix form with respect to a basis, then the trace does not depend on the chosen basis. Since conjugate matrices have the same trace, isomorphic representations have the same character 1.1.1. Conjugation invariance also yields:

$$\chi_V(ghg^{-1}) = \chi_V(h), \qquad g,h \in G.$$

Thus characters are class functions. From the matrix form of representations we derive some properties of characters:

$$\chi_{V\oplus W} = \chi_V + \chi_W, \tag{2.2}$$
$$\chi_{V\otimes W} = \chi_V \chi_W, \tag{2.3}$$
$$\chi_{V^*}(g) = \chi_V(g^{-1}). \tag{2.4}$$

Let V and W be G-representations. In section 1.8 we introduced the representation $\mathrm{Hom}_K(V,W)$ with fixed point set $\mathrm{Hom}_{KG}(V,W)$.

(2.1.2) Proposition. *The character of* $\mathrm{Hom}_K(V,W)$ *is* $g \mapsto \chi_V(g^{-1})\chi_W(g)$.

Proof. This is a consequence of 1.8.4, (2.3) and (2.4). We also prove it by a direct calculation with matrices, thus avoiding 1.8.4. We express the necessary data in matrix form. Let $v_1,\dots,v_m$ be a basis of V and $w_1,\dots,w_n$ a basis of W. We set $l_g^{-1}(v_i) = \sum_j a_{ji}v_j$ and $l_g(w_k) = \sum_l b_{lk}w_l$. Then a basis of $\mathrm{Hom}_K(V,W)$ is $e_{rs}\colon v_i \mapsto \delta_{si}w_r$. We compute:

$$\begin{aligned} & (g\cdot e_{rs})(v_i) = ge_{rs}(g^{-1}v_i) = ge_{rs}(\textstyle\sum_j a_{ji}v_j) = g(\sum_j a_{ji}\delta_{sj}w_r) \\ = \ & \textstyle\sum_{j,l}\delta_{sj}a_{ji}b_{lr}w_l = \sum_l a_{si}b_{lr}w_l = \sum_l a_{si}b_{lr}e_{li}(v_i). \end{aligned}$$

The trace is the sum of the diagonal elements $\sum_{r,s} a_{ss}b_{rr} = \chi_V(g^{-1})\chi_W(g)$. □

We now combine 2.1.1 and 2.1.2 and obtain

$$\langle V,W\rangle = \dim_K \mathrm{Hom}_G(V,W) = |G|^{-1}\textstyle\sum_{g\in G}\chi_V(g^{-1})\chi_W(g). \tag{2.5}$$

This formula tells us that we can compute $\langle V,V\rangle$ from the character. The character does not change under field extensions. We know that $\langle V,V\rangle = 1$ implies that V is irreducible; it then remains irreducible under field extensions. If this is the case, we call the representation **absolutely irreducible**. When K is algebraically closed, Schur's lemma says $\langle V,V\rangle = 1$. Therefore V is absolutely irreducible if and only if $\langle V,V\rangle = 1$.

(2.1.3) Theorem. *Two representations of G are isomorphic if and only if they have the same character.*

Proof. Let V and V' have the same character. Then, by (2.5), the values $\langle W,V\rangle$ and $\langle W,V'\rangle$ are equal for all W. From 1.9.5 we now see that the multiplicities of $W\in \mathrm{Irr}(G,K)$ in V and V' are equal. □

The previous theorem has an interesting consequence; it roughly says, that cyclic subgroups detect representations. If V is a G-representation and H a subgroup of G, we can view V as an H-representation by restriction of the group action. Denote it $\mathrm{res}^G_H V$ for emphasis. The character value $\varphi_V(g)$ only depends on the restriction to the cyclic subgroup generated by g. Therefore:

(2.1.4) Theorem. *G-representations V and W are isomorphic if and only if* $\operatorname{res}^G_H V$ *and* $\operatorname{res}^G_H W$ *are isomorphic for each cyclic subgroup H of G.* □

(2.1.5) Proposition. *Let $V = KS$ be the permutation representation of the finite G-set S. Then $\chi_V(g) = |S^g|$. Here $S^g = \{s \in S \mid gs = s\}$.*

Proof. Consider the matrix of l_g with respect to the basis S. A basis element $s \in S$ yields a non-zero entry on the diagonal if and only if $gs = s$, and this entry is 1. □

(2.1.6) Proposition. *Let K be a splitting field for G and H. Then*

$$\operatorname{Irr}(G;K) \times \operatorname{Irr}(H;K) \to \operatorname{Irr}(G \times H;K), \quad (V,W) \mapsto V \otimes W$$

is a well-defined bijection.

Proof. In the statement of the proposition we view $V \otimes W$ as $G \times H$-representation, as explained in section 1.8. From $\chi_{V\otimes W}(g,h) = \chi_V(g)\chi_W(h)$ and (2.5) we obtain

$$\langle V_1 \otimes W_1, V_2 \otimes W_2 \rangle_{G\times H} = \langle V_1, V_2 \rangle_G \langle W_1, W_2 \rangle_H.$$

This shows that $V \otimes W$ is irreducible, if we start with irreducible representations V and W. It also shows that the map in question is injective. We use 1.10.3 and see that we got the right number of irreducible $G \times H$-representations. □

Problems

1. Let $H \lhd G$ and V a G-representation. Then V^H is a G/H-representation. Its character is given by $\chi_{V^H}(gH) = |H|^{-1} \sum_{h\in H} \chi_V(gh)$.

2.2 Orthogonality

We derive orthogonality properties of characters and show that the irreducible characters form an orthonormal basis in the ring of class functions. We assume that K has characteristic zero and is a splitting field for G.

Let $Cl(G,K) = Cl(G)$ be the ring of class functions $G \to K$ (pointwise addition and multiplication). We define on $Cl(G)$ a symmetric bilinear form

$$\langle \alpha, \beta \rangle = \frac{1}{|G|} \sum_{g\in G} \alpha(g^{-1})\beta(g). \tag{2.6}$$

Bilinearity is clear and the reason for symmetry is that we can replace summation over g by summation over g^{-1}. By (2.5), $\langle V, W \rangle = \langle \chi_V, \chi_W \rangle$. This gives us together with Schur's lemma the orthogonality properties of characters:

(2.2.1) Proposition. *The irreducible characters form an orthonormal system with respect to the bilinear form* (2.6). □

The main result 2.2.5 of this section says that the irreducible characters are a basis of the vector space $Cl(G)$. We prepare for the proof.

(2.2.2) Proposition. *The linear map* $q_\alpha = \sum_{g \in G} \alpha(g) l_g \colon V \to V$ *is a morphism for each representation* V *if and only if* $\alpha \colon G \to K$ *is a class function.*

Proof. Let α be a class function. We compute

$$q_\alpha(hv) = \sum \alpha(g) l_g(hv) = \sum \alpha(g) ghv = \sum \alpha(h^{-1}gh) h(h^{-1}gh) = hq_\alpha(v).$$

For the converse we evaluate the equation $q_\alpha(h) = hq_\alpha(e)$ in the regular representation and compare coefficients. □

(2.2.3) Proposition. *Let* α *be a class function. Then* $p_\alpha = \sum_{g \in G} \alpha(g^{-1}) l_g$ *acts on* V *as the multiplication by the scalar* $|G|(\dim V)^{-1} \langle \alpha, \chi_V \rangle$.

Proof. By 2.2.2, p_α is an endomorphism of V, and $\langle V, V \rangle = 1$ tells us that p_α is the multiplication with some scalar λ. The computation (Tr = Trace)

$$\begin{aligned} \lambda \dim V &= \operatorname{Tr}(\lambda \cdot \mathrm{id}) = \operatorname{Tr}\left(\sum \alpha(g^{-1}) l_g\right) \\ &= \sum \alpha(g^{-1}) \operatorname{Tr}(l_g) = \sum \alpha(g^{-1}) \chi_V(g) \\ &= |G| \langle \alpha, \chi_V \rangle. \end{aligned}$$

determines λ. □

(2.2.4) Lemma. *Let* $\alpha \in Cl(G)$ *be orthogonal to the characters of irreducible representations. Then* $\alpha = 0$.

Proof. The hypothesis of the lemma and 2.2.3 imply that p_α acts as zero morphism in each irreducible representation, hence in each representation. In the regular representation we have $0 = p_\alpha(e) = \sum_g \alpha(g^{-1}) g$. Hence $\alpha(g) = 0$ for all $g \in G$. □

(2.2.5) Theorem. *The irreducible characters of* G *are an orthonormal basis of of* $Cl(G)$. *The number of irreducible representations is equal to the number of conjugacy classes of* G.

Proof. Let $U \subset Cl(G)$ be a linear subspace. If $U \neq Cl(G)$ then the orthogonal complement $U^\perp$ with respect to $\langle -, - \rangle$ is different from zero, since $U^\perp$ is the kernel of the linear map $Cl(G) \to \operatorname{Hom}(U, K)$, $x \mapsto (u \mapsto \langle x, u \rangle)$. For the subspace U generated by characters, $U^\perp = 0$, by 2.2.4, hence $U = Cl(G)$. Now recall 2.2.1.

The dimension of $Cl(G)$ is the number of conjugacy classes, because a basis of $Cl(G)$ consists of those functions which have value 1 on one class and value 0 on all the other classes. □

There exist more general orthogonality relations. They are concerned with the entries of matrix representations and are consequences of the next result.

(2.2.6) Proposition. *Let V be irreducible and suppose that $\langle V, V \rangle = 1$. Then for each linear map $f \in \operatorname{Hom}(V,V)$*

$$\frac{1}{|G|}\sum_{g\in G} l_g f l_g^{-1} = \frac{\operatorname{Tr}(f)}{\dim V}\cdot \mathrm{id}\,.$$

Proof. The left hand side is contained in $\operatorname{Hom}_G(V,V)$ and has the form $\lambda \cdot \mathrm{id}$, since $\langle V, V \rangle = 1$. We apply the trace operator

$$\lambda \dim V = \operatorname{Tr}(\lambda\cdot \mathrm{id}) = |G|^{-1}\textstyle\sum_{g\in G}\operatorname{Tr}(l_g f l_g^{-1}) = |G|^{-1}\sum_{g\in G}\operatorname{Tr}(f) = \operatorname{Tr}(f)$$

and determine λ. □

For $v \in V$ and $\varphi \in V^*$ we obtain from 2.2.6

$$\textstyle\sum_{g\in G}\varphi(gf(g^{-1}v)) = (\dim V)^{-1}|G|\operatorname{Tr}(f)\varphi(v).$$

We apply this to the linear map $f\colon v \mapsto \psi(v)w$, $w \in V$, $\psi \in V^*$ with trace $\operatorname{Tr}(f) = \psi(w)$ and obtain

$$|G|^{-1}\textstyle\sum_{g\in G}\psi(g^{-1}v)\varphi(gw) = (\dim V)^{-1}\varphi(v)\psi(w). \tag{2.7}$$

Note that we can use the definition (2.6) of $\langle \alpha, \beta \rangle$ for arbitrary functions $\alpha, \beta\colon G \to K$. This remark can be applied to the left hand side of (2.7). Let $v_1, \dots, v_n$ be a basis of V and $\varphi_1, \dots, \varphi_n$ the dual basis. In a matrix representation $gv_i = \sum_j r^V_{ji}(g)v_j$ we have $\varphi_j(gv_i) = r^V_{ji}(g)$. We apply (2.7) to this situation and arrive at the following:

(2.2.7) Orthogonality for matrix entries. Let V and W be irreducible representations of G. Then

$$\langle r^V_{lk}, r^W_{ji} \rangle = \frac{1}{\dim V}\delta_{li}\delta_{jk}\delta_{VW}. \tag{2.8}$$

We have treated the case $V = W$. If V is not isomorphic to W and $f \in \operatorname{Hom}(V,W)$, then the left hand side of the equality in (2.8) is zero. □

Problems

1. Let $V_1, \dots V_r$ be a complete set of pairwise non-isomorphic irreducible KG-representations. Let (a^j_{rs}) denote a matrix representation of V_j. Then the functions a^j_{rs} are an orthogonal basis of the space of functions $G \to K$ with respect to the form (2.6).

2.3 Complex Representations

We begin with some special and useful properties of representations over the complex numbers.

An (Hermitian) inner product $V \times V \to \mathbb{C}$, $(u,v) \mapsto \langle u, v \rangle$ on a G-representation V is called **G-invariant** if $\langle gu, gv \rangle = \langle u, v \rangle$ for $g \in G$ and $u, v \in V$. A representation together with a G-invariant inner product is a **unitary representation**. A real representation together with a G-invariant inner product is an **orthogonal representation**.

(2.3.1) Proposition. *A complex representation V of a finite group possesses a G-invariant inner product.*

Proof. Let $b\colon V \times V \to \mathbb{C}$ be any inner product (conjugate-linear in the first variable) and define

$$c(u,v) = \tfrac{1}{|G|} \sum_{g \in G} b(gu, gv).$$

Then c is linear in v, conjugate linear in u, and G-invariant because of the averaging process. Also it is positive definite and $c(u,v) = \overline{c(v,u)}$. □

Let U be a sub-representation of a unitary representation V. Then the orthogonal complement $U^\perp$ is again a sub-representation an $V = U \oplus U^\perp$. This gives another proof that complex representations are semi-simple. Similarly for orthogonal representations. If we choose an orthonormal basis in an n-dimensional unitary representation, then the associated matrix representation is a homomorphism into the unitary group $G \to U(n)$. In terms of matrix representations, 2.3.1 has the interesting consequence that a homomorphism $G \to GL_n(\mathbb{C})$ of a finite group G is conjugate to a homomorphism $G \to U(n)$.

Let V be a complex representation. There is associated the **complex-conjugate representation** $G \times \overline{V} \to \overline{V}$ on the conjugate vector space $\overline{V}$ (the same underlying set and vector addition, but $\lambda \in \mathbb{C}$ now acts as multiplication with $\overline{\lambda}$).

We consider a Hermitian form on V as a bilinear map $\overline{V} \times V \to \mathbb{C}$. Associated is the adjoint $\overline{V} \to V^*$, $v \mapsto (u \mapsto \langle v, u \rangle)$ into the dual vector space. It is an isomorphism of G-representations, in the case of a G-invariant inner product. In terms of characters this means $\chi_V(g^{-1}) = \overline{\chi_V(g)}$. For complex class functions we define a Hermitian form on $Cl(G)$ by

$$(\alpha, \beta) = \tfrac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g). \tag{2.9}$$

The relation $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ shows $(\chi_V, \chi_W) = \langle V, W \rangle$. Therefore the irreducible characters are also an orthonormal basis for this form. Recall the notation $I = \mathrm{Irr}(G; \mathbb{C})$.

Let $C \subset G$ be a representing system for the conjugacy classes. From $|C| = |I|$ we see that $X : C \times I \to K,\ (c, V) \mapsto \chi_V(c)$ is a square matrix. It is called the **character table** of G.

We express the orthogonality relations in terms of the character table. Let $X^* : I \times C \to \mathbb{C}$, $(V, c) \mapsto \overline{\chi_V(c)}$ be the conjugate-transpose and $D : C \times C \to \mathbb{C}$ the diagonal matrix $(c, d) \mapsto \delta_{c,d}|c|$, where $|c|$ denotes the cardinality of the conjugacy class of c. The orthogonality relation $(\chi_V, \chi_W) = \delta_{V,W}$ then reads:

(2.3.2) First orthogonality relation. For irreducible complex representations V and W the relation

$$\sum_{c \in C} |c|\overline{\chi_V(c)}\chi_W(c) = |G|\delta_{V,W}$$

holds. □

In matrix form 2.3.2says $X^*DX = |G|E$ (unit matrix E). This implies

$$XX^*D = XX^*DXX^{-1} = X(|G|E)X^{-1} = |G|E$$

and then $XX^* = |G|D^{-1}$. Let $Z(c) = \{g \in G \mid gcg^{-1} = c\}$ denote the **centralizer** of c in G. Then $|c| = |G/Z(c)|$. We write out the last matrix equation:

(2.3.3) Second orthogonality relation. For $c, d \in C$ the relation

$$\sum_{V \in I} \overline{\chi_V(c)}\chi_V(d) = \delta_{c,d}|Z(c)|$$

holds. □

(2.3.4) Proposition. *Let $(V, \langle -, - \rangle_V)$ and $(W, \langle -, - \rangle_W)$ be unitary representations. Suppose V and W are isomorphic as complex representations. Then they are isomorphic as unitary representations, i.e., there exists a G-morphism $f : V \to W$ such that $\langle f(v_1), f(v_2) \rangle_W = \langle v_1, v_2 \rangle_V$.*

Proof. Let $\varphi : V \to W$ be a G-morphism. We use φ to pull $\langle -, - \rangle_W$ back to V, i.e., we define a second inner product $\langle -, - \rangle'$ on V by $\langle v_1, v_2 \rangle' = \langle \varphi(v_1), \varphi(v_2) \rangle_W$. It suffices to produce a G-morphism $\gamma : V \to V$ such that $\langle \gamma(v_1), \gamma(v_2) \rangle = \langle v_1, v_2 \rangle'$. We choose an orthonormal basis B of V with respect to $\langle -, - \rangle$ and express everything with respect to this basis. Then $\langle -, - \rangle$ becomes the standard inner product. There exists a positive definite Hermitian matrix A such that $\langle u, v \rangle' = \langle u, Av \rangle = u^t Av$. Since $\langle -, - \rangle'$ is G-invariant, $l_g A = A l_g$. Let $C = \sqrt{A}$ be a positive definite Hermitian matrix. The matrix C also commutes with l_g, since it is a limit of polynomials in A. Then C defines a morphism γ with the desired properties. □

(2.3.5) Corollary. *Let $\alpha, \beta \colon G \to U(n)$ be unitary representations which are conjugate in $GL_n(\mathbb{C})$. Then they are conjugate in $U(n)$. The conjugation matrix can be chosen in $SU(n)$, i.e., to have determinant one.* □

We list a few more properties of complex characters.

(2.3.6) Proposition. *Let χ be the character of a complex representation V. Then:*

(1) $\chi(1) = \dim V$.
(2) $|\chi(g)| \le \chi(1)$.
(3) $|\chi(g)| = \chi(1)$ *if and only if l_g is the multiplication by a scalar.*
(4) $\chi(g) = \chi(1)$ *if and only if g is contained in the kernel of V.*

Proof. (1) The trace of the identity is $\dim V$.

(2) Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of l_g. They are roots of unity, and $\chi(g) = \lambda_1 + \cdots + \lambda_n$. Hence $|\chi(g)| = |\sum \lambda_j| \le \sum |\lambda_j| = \chi(1)$.

(3) If equality holds, then $\lambda_1 = \cdots = \lambda_n = \lambda$ and l_g is multiplication by λ.

(4) By (3), l_g is the multiplication by 1, if $\chi(g) = \chi(1)$. □

(2.3.7) Remark. If G has a normal subgroup different from 1 and G, then there exists a nontrivial irreducible character χ and $1 \ne g \in G$ such that $\chi(g) = \chi(1)$. Conversely, from **??** we see, that if χ and g with these properties exist, then G has a nontrivial normal subgroup. We see that one can obtain group theoretic information from the character table. ◇

The values of complex characters are very special complex numbers. The value $\chi_V(g)$ is the sum of the eigenvalues of l_g, and these eigenvalues are $|g|$-roots of unity ($|g|$ order of g). Let $\mathbb{Z}[\zeta]$ be the subring of the field $\mathbb{Q}(\zeta)$ generated by ζ. The **exponent** of a group is the least common multiple of the orders of its elements. Let ζ be a primitive n-root of unity, say $\zeta = \exp(2\pi i/n)$, n the exponent of G. Then χ_V has values in $\mathbb{Z}[\zeta]$. In number theory, the ring $\mathbb{Z}[\zeta]$ is the ring of algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta)$. (An algebraic integer is the root of a monic polynomial with coefficients in $\mathbb{Z}$.)

Problems

1. Express the orthogonality relations **??** for complex representations using the Hermitian form (2.9).

2. Let V, W be orthogonal representations of G which are isomorphic as real representations. Then they are isomorphic as orthogonal representations.

3. Let V, W be real representations. If they are isomorphic, considered as complex representations, then they are isomorphic as real representations. What does this imply for matrix representations?

4. The character table is a square matrix. Determine the absolute value of its determinant.

2.4 Examples

We study in some detail the groups A_4, S_4, A_5. The symmetric group S_n is the permutation group of $[n] = \{1, \dots, n\}$. The alternating group is the normal subgroup of S_n of even permutations. The geometric significance of the groups in question comes from Euclidean geometry: A_4, S_4, A_5 are the symmetry groups of the tetrahedron, octahedron (cube), icosahedron (dodecahedron), respectively (as far as rotations are concerned , i.e., as subgroups of $SO(3)$).

We begin with some general remarks about permutations. Let $\pi \in S_n$. The cyclic group generated by π acts on $[n]$. We decompose $[n]$ into orbits under this action. An orbit has the form

$$(x, \pi(x), \pi^2(x), \dots, \pi^{t-1}(x)), \qquad \pi^t(x) = x$$

where t is the length of the orbit. We call an orbit a **cycle** of the permutation. A permutation can be recovered from its cycles. Therefore we use the cycles to denote the permutation. As an example, the permutation $(318496527) \in S_9$ has the cycles

$$(1, 3, 8, 2), (4), (5, 9, 7).$$

This means, e.g., that $5 \mapsto 9, 9 \mapsto 7, 7 \mapsto 5$. A cyclic permutation of the entries in a cycle does not change its meaning; thus $(5, 9, 7) = (9, 7, 5) = (7, 5, 9)$. In practice it is not necessary to write cycles of length one, since they just describe fixed points of the permutations. The conceptual significance of the cycles is:

(2.4.1) Proposition. *Permutations in S_n are conjugate elements of the group if and only if for each $k \in \mathbb{N}$ they have the same number of cycles of length k.* □

A **partition** of n is a sequence of integers $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1$ and $\sum_{j=1}^{r} \lambda_j = n$. A conjugacy class of S_n is determined by its associated partition; the λ_j are the lengths of the cycles in the permutation. Thus we have found a combinatorial method to determine the number of irreducible complex representations of S_n; it is the number of partitions of n. The partitions $3, 21, 111$ of 3 tell us that S_3 has 3 irreducible representations.

(2.4.2) Proposition. *Suppose $\pi \in S_n$ has $k(j)$ cycles of length j. The the automorphism group of $[n]_\pi$ has order $1^{k(1)} \cdot k(1)! \cdot 2^{k(2)} \cdot k(2)! \cdot \ldots \cdot n^{k(n)} \cdot k(n)!$. This is the order of the centralizer; hence $n!$, divided by this number, is the size of the conjugacy class of π.* □

(2.4.3) Representations of S_4. There exist 5 partitions $1111, 211, 22, 31, 4$. We list representing elements of the conjugacy classes and the cardinality of the conjugacy class in the next table. The second row can be obtained from 2.4.2.

1	(12)	(12)(34)	(123)	(1234)
1	6	3	8	6

We turn to the determination of the character table and to the construction of irreducible representations. Names for the five representations and their characters are V_j, $1 \le j \le 5$. We already know 2 one-dimensional representations, the trivial representation V_1 and the sign-representation V_2. Their characters are easily computed.

Character table of S_4					
	1	(12)	(12)(34)	(123)	(1234)
V_1	1	1	1	1	1
V_2	1	-1	1	1	-1
V_3	2	0	2	-1	0
V_4	3	1	-1	0	-1
V_5	3	-1	-1	0	1

We also know already a three-dimensional representation on the space $V_4 = \{(x_1, x_2, x_3, x_4) \mid \sum x_i = 0\}$ by permutation of coordinates. We check again that it is irreducible by computing its character. The character of the permutation representation on $\mathbb{C}^4$ is easily determined by 2.1.5 to have the values $4, 2, 0, 1, 0$. We have to subtract the character of the trivial representation; the result is given in the table. The computation

$$\langle V_4, V_4 \rangle = \tfrac{1}{24} \sum_g |\chi_{V_4}(g)|^2 = 3^2 + 6 \cdot 1^2 + 3 \cdot (-1)^2 + 8 \cdot 0^2 + 6 \cdot (-1)^2 = 1$$

shows that V_4 is irreducible. The character of $V_5 = V_4 \otimes V_2$ is seen to be as in the table. Thus we found another irreducible representation. We know that the remaining representation must be two-dimensional 1.10.3. It turns out that S_4 has a quotient S_3, the kernel contains (12)(34). We can lift a two-dimensional representation of S_3 to S_4. We lift the analogue of V_4 for S_3. ◇

(2.4.4) Representations of A_5. We begin again with the determination of the conjugacy classes. We use the cycle notation and have to start with even permutations. But now it is only allowed to conjugate with even permutations, and this has the effect that some of the conjugacy classes of S_5 can split in A_5 into two classes.

(2.4.5) Proposition. *Let $c \in A_n$. Then the S_n-conjugacy class of c is contained in A_n. The S_n-conjugacy class of c split into two A_n-conjugacy classes if and only if the centralizers of c in A_n and S_n coincide. This happens if and only if the partition associated to c consists of different odd numbers.* □

Character table of A_5					
	1	(12)(34)	(123)	(12345)	(13524)
V_1	1	1	1	1	1
V_2	3	-1	0	α	β
V_3	3	-1	0	β	α
V_4	4	0	1	-1	-1
V_5	5	1	-1	0	0

Let ζ be a primitive 5-th root of unity; then $-\alpha = \zeta + \zeta^{-1}$ and $-\beta = \zeta^2 + \zeta^{-2}$. The representation V_1 is trivial. V_4 is the permutation representation on $\{(x_1, \ldots, x_5) \mid \sum x_j = 0\}$. The group A_5 has a subgroup $H \cong D_{10}$. The representation V_5 is obtained from the permutation representation $\mathbb{C}(A_5/H)$ by subtracting the trivial representation. The remaining two representations must be three-dimensional, since $60 - 1^2 - 4^2 - 5^2 = 18 = 3^2 + 3^2$. It is possible to determine the characters without construction of the representations; one uses the fact that the representations cannot have a kernel; and that z^j and z^{-j} are conjugate, so that the character values are real and sums of 5-th roots of unity. The group A_5 has an outer automorphism which interchanges $z = (12345)$ and $z^2 = (13524)$; it is obtained by conjugation with (2354); one verifies $(2354) \circ z \circ (4532) = z^2$. The representation V_3 is obtained from V_2 by this automorphism; and V_2 has a realization over $\mathbb{R}$ as orthogonal symmetry group of the icosahedron. ◇

Problems

1. S_4 acts by conjugation on the set of even permutations of order two. Show that this induces a surjection $S_4 \to S_3$.
2. Compute the number of elements in A_5 of a given order.
3. Determine the irreducible representations and the character table for A_4.
4. Decompose the tensor product of irreducible representations for $G = A_4, S_4, A_5$.
5. Show that S_5 has seven conjugacy classes and irreducible complex representations of dimensions $1, 1, 4, 4, 5, 5, 6$.
6. For a partition $(\lambda_1, \ldots, \lambda_r)$ of n let $S(\lambda) = S(\lambda_1) \times \cdots \times S(\lambda_r)$. Set $V(\lambda) = \mathbb{C}(S_n/S(\lambda))$. Decompose these permutation representations in the cases S_4 and S_5 into irreducibles.

2.5 Real and Complex Representations

Let W be a KG-representation. The involution $T\colon W \otimes W \to W \otimes W, x \otimes y \mapsto y \otimes x$ is a morphism of representations. If K has characteristic different from 2, we split $W \otimes W$ into the ± 1-eigenspaces,

$$S^2(W) = (W \otimes W)_+, \qquad \Lambda^2(W) = (W \otimes W)_-.$$

$S^2(W)$ is the second symmetric power of W and $\Lambda^2(W)$ the second exterior power. Suppose now that $K = \mathbb{C}$. If $w_1, \dots, w_n$ is a basis of W, then $g_{ij} = \frac{1}{2}(w_i \otimes w_j + w_j \otimes w_i), i \le j$ is a basis of $S^2(W)$ and $w_{ij} = \frac{1}{2}(w_i \otimes w_j - w_j \otimes w_i), i < j$ is a basis of $\Lambda^2(W)$. From this information we compute the characters

$$\begin{aligned} \chi_{W \otimes W}(g) &= \chi_W(g)^2 \\ \chi_{S^2 W}(g) &= \frac{1}{2}(\chi_W(g)^2 + \chi_W(g^2)) \\ \chi_{\Lambda^2 W}(g) &= \frac{1}{2}(\chi_W(g)^2 - \chi_W(g^2)). \end{aligned}$$

Let W be irreducible. Then $W^* \cong \overline{W}$ is irreducible too. Therefore $\langle W \otimes W, 1 \rangle = \langle W, W^* \rangle$ is 1 if $W \cong W^*$ and 0 otherwise. An isomorphism $W \cong \overline{W}$ exists if and only if χ_W is real-valued. In that case we call W **self-conjugate**. Elements in $\mathrm{Hom}_G(S^2 W, \mathbb{C})$ are symmetric G-invariant bilinear forms, and elements in $\mathrm{Hom}_G(\Lambda^2 W, \mathbb{C})$ are skew-symmetric G-invariant bilinear forms. Hence W carries a non-zero G-invariant bilinear form if and only if W is self-conjugate. Suppose W is self-conjugate. From $\langle W \otimes W, 1 \rangle = \langle S^2 W, 1 \rangle + \langle \Lambda^2 W, 1 \rangle$ we see that there two cases: Either $\langle S^2 W, 1 \rangle = 1$, $\langle \Lambda^2 W, 1 \rangle = 0$ or $\langle S^2 W, 1 \rangle = 0, \langle \Lambda^2 W, 1 \rangle = 1$.

(2.5.1) Proposition. *Let W be irreducible. Then*

$$\sigma(W) = \tfrac{1}{|G|} \sum_{g \in G} \chi_W(g^2) = \begin{cases} 0 & W \ncong W^* \\ 1 & \langle S^2 W, 1 \rangle = 1 \\ -1 & \langle \Lambda^2 W, 1 \rangle = 1 \end{cases}$$

Proof. By the computation above the sum on the left equals $\langle S^2 W, 1 \rangle - \langle \Lambda^2 W, 1 \rangle$. □

Suppose W is the complexification of a real representation U, i.e., $W \cong \mathbb{C} \otimes U = U_{\mathbb{C}}$. Then U is irreducible and carries a symmetric G-invariant $\mathbb{R}$-bilinear form. This form extends to a symmetric G-invariant $\mathbb{C}$-bilinear form on W. Hence in this case $\sigma(W) = 1$. The converse is also true.

(2.5.2) Proposition. *Suppose $W \in \mathrm{Irr}(G, \mathbb{C})$ carries a G-invariant symmetric form. Then W is the complexification of a real representation.*

Chapter 3

The Group Algebra

3.1 The Theorem of Wedderburn

Assume that the characteristic of K does not divide the order of G. Then the group algebra $A = KG$, considered as a left KG-representation, is semi-simple and hence a direct sum of its isotypical parts. Recall that sub-representations of KG are the same thing as left ideals or as submodules.

(3.1.1) Proposition. *Let $A(V)$ denote the V-isotypical part of A belonging to $V \in \mathrm{Irr}(G;K)$. The $A(V)$ is a two-sided ideal, and every two-sided ideal is a direct sum of isotypical parts.*

Proof. Let $V \subset A$ be irreducible and $a \in A$. Then $W = Va$ is a left ideal, and by Schur's lemma $r_a \colon V \to W, x \mapsto xa$ is either zero or an isomorphism. Hence $W \subset A(V)$.

Let V, W be isomorphic irreducible left ideals. Since A is semi-simple, there exists a projection $f \colon A \to V$. Let $s \colon V \to W$ be an isomorphism. Then $fs(x) = fs(x \cdot 1) = x \cdot sf(1) = x \cdot a$, $a = sf(1)$. If $x \in V$, then $f(x) = x$ and hence $sf(x) = s(x) = xa$, i.e., $W = Va$.

Let $B \subset A$ be a two-sided ideal. Let $V \subset B$ be irreducible and also $W \subset A(V)$. Then $W = Va$. Since B is a right ideal, $W \subset B$ and hence $A(V) \subset B$. □

The isotypical parts are therefore the minimal two-sided ideal. A two-sided ideal is itself an associative algebra, with addition and multiplication inherited from A.

(3.1.2) Proposition. *Let $A = A_1 \oplus \cdots \oplus A_r$ be the decomposition into the minimal two-sided ideal. Then $A_i A_j = 0$ for $i \neq j$.*

Proof. Let $I \subset A_i$ and $J \subset A_j$ be irreducible left ideals. Then $A_i \supset IJ \subset A_j$, since A_i, A_j are two-sided. From $A_i \cap A_j = 0$ we see $IJ = 0$. □

Let $1 = e_1 + \cdots + e_r$, $e_j \in A_j$ be the decomposition of the unit element. Then $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$. This follows from

$$\sum e_j = 1 = 1 \cdot 1 = (\sum e_j)(\sum e_k) = \sum e_j e_k = \sum e_j^2.$$

A consequence: e_j is the unit element of the algebra A_j.

Let B be a minimal two-sided ideal of A. A linear subspace V of B is a B-submodule if and only if it is an A-submodule. The map

$$r\colon B \to \mathrm{Hom}_B(B,B), \quad x \mapsto r_x$$

is because of $r_x r_y = r_{yx}$ an anti-isomorphism of algebras. Let $B \cong V^n$, where V is an irreducible submodule of B. Then $\mathrm{Hom}_B(B,B) \cong \mathrm{Hom}_B(V^n, V^n)$. The latter is, by the rules of linear algebra, a matrix algebra. Let $i_j\colon V \to V^n$ be the inclusion of the j-the summand and $p_k\colon V^n \to V$ be the projection onto the k-summand. We associate to $f \in \mathrm{Hom}_B(V^n, V^n)$ the matrix (f_{jk}), $f_{jk} = p_k f i_j \in \mathrm{End}_B(V) = D$. Since V is irreducible, D is a division algebra. Therefore $\mathrm{Hom}_B(V^n, V^n)$ is isomorphic to the matrix algebra $M_n(D)$ of (n,n)-matrices with entries in D. Passage to the transposed matrix is an isomorphism $M_n(D)^\circ \cong M_n(D^\circ)$. (Notation: C° the algebra opposite to C, i.e., order of the multiplication interchanged.) Therefore we have shown in our context:

(3.1.3) Theorem (Theorem of Wedderburn). *The minimal left ideals of the group algebra are isomorphic to matrix-algebras $M_n(D^\circ)$; here $D = \mathrm{End}(V)$ if B is the V-isotypical part, and n is the multiplicity of V in B.* □

If $K = \mathbb{C}$, then the division algebras appearing are just the field $\mathbb{C}$ itself. In the next section we describe the decomposition into matrix algebras in a more explicit manner and relate it to character theory.

3.2 The Structure of the Group Algebra

We assume in this section that K is a splitting field for G of characteristic zero. We write $\dim V = |V|$.

(3.2.1) Proposition. *Let $V \in \mathrm{Irr}(G;K)$. The assignment*

$$t_V\colon \mathrm{Hom}(V,V) \to KG, \quad \alpha \mapsto \tfrac{|V|}{|G|} \textstyle\sum_{g \in G} \mathrm{Tr}(l_g^{-1}\alpha) g$$

is a homomorphism of algebras.

Proof. By comparing coefficients in KG we see that the statement amounts to

$$\frac{|V|}{|G|}\sum_{g\in G}\operatorname{Tr}(l_g^{-1}\alpha)\operatorname{Tr}(l_x^{-1}l_g\beta)=\operatorname{Tr}(l_x^{-1}\alpha\beta)$$

for $\alpha,\beta\in\operatorname{Hom}(V,V)$ and $x\in G$. It suffices to prove this for $x=e$.

We use the relation 2.2.6

$$\frac{|V|}{|G|}\sum_{g\in G}l_g\beta l_{g^{-1}}=\operatorname{Tr}(\beta)\operatorname{id}_V,\qquad \beta\in\operatorname{Hom}(V,V) \tag{3.1}$$

and compute

$$\sigma=\frac{|V|^2}{|G|^2}\sum_{g,u\in G}l_g\beta l_u l_{g^{-1}}l_{u^{-1}}\alpha$$

in two ways. We apply (3.1) to $\sum_u l_u l_{g^{-1}}l_{u^{-1}}$ and use the definition of χ_V; this shows us that σ is equal to

$$\frac{|V|}{|G|}\sum_g\chi_V(g^{-1})l_g\beta\alpha.$$

The endomorphism $\frac{|V|}{|G|}\sum_g\chi_V(g^{-1})l_g$ is the identity on V. Hence $\sigma=\beta\alpha$. We now apply (3.1) to $\sum_g l_g\beta l_u l_{g^{-1}}$ and obtain

$$\sigma=\frac{|V|}{|G|}\sum_u\operatorname{Tr}(\beta l_u)l_{u^{-1}}\alpha.$$

Finally we apply the trace operator to this equation and arrive at

$$\operatorname{Tr}(\beta\alpha)=\frac{|V|}{|G|}\sum_u\operatorname{Tr}(l_u\beta)\operatorname{Tr}(l_{u^{-1}}\alpha),$$

and this was to be shown. □

The homomorphism t_V is moreover a morphism of (G,G)-representations, i.e., one verifies directly from the definitions that $t_V(l_g\alpha l_h)=gt_V(\alpha)h$.

(3.2.2) Proposition. *The image of t_V is the V-isotypical part of KG. The (G,G)-representation $\operatorname{Hom}(V,V)$ is irreducible and the image of t_V is the $\operatorname{Hom}(V,V)$-isotypical part of KG as a (G,G)-representation.*

Proof. The canonical map $V^*\otimes V\to\operatorname{Hom}(V,V)$ is an isomorphism of representations. By 2.1.6, these representations are irreducible. Since t_V is non-zero, t_V is injective. Certainly t_V has an image in the V-isotypical part. We know already that it has dimension $|V|^2=\dim\operatorname{Hom}(V,V)$. Therefore t_V maps isomorphically onto the V-isotypical part. □

The homomorphisms t_V combine to an isomorphism of algebras

$$t\colon \bigoplus_{V\in I}\operatorname{Hom}(V,V)\to KG,\quad (x_V)\mapsto\sum_{V\in I}t_V(x_V).$$

This isomorphism induces an isomorphism of the centers of the algebras. The center of $\operatorname{Hom}(V,V)$ consists of the multiples of the identity. Let $Z(A)$ denote the center of the algebra A. We obtain a homomorphism of algebras

$$\tau_V=\operatorname{pr}_V\circ t^{-1}\colon Z(KG)\to Z(\operatorname{Hom}(V,V))\cong\{\lambda\cdot\operatorname{id}\mid\lambda\in K\}\cong K.$$

(3.2.3) Proposition. $\tau_V(\sum_{g\in G}\alpha(g)g) = |V|^{-1}\sum_{g\in G}\alpha(g)\chi_V(g)$.

Proof. The elements $t_V(\mathrm{id}_V) = e_V$ are, by 3.2.2, a vector space basis of $Z(KG)$. Hence it suffices to verify the assertion for these elements. The verification amounts to $\frac{|W|}{|G|}\frac{1}{|V|}\sum_g \chi_W(g^{-1})\chi_V(g) = \delta_{VW}$, and this we know by 2.2.6. □

(3.2.4) Proposition. *The element $\sum_{g\in G}\alpha(g)g \in KG$ is contained in the center of KG if and only if α is a class function.* □

An element $e \in A$ in an algebra A is called **idempotent**, if it satisfies $e^2 = e$. Idempotents e, f are **orthogonal**, if $ef = fe = 0$. The elements $e_V \in KG, V \in I$ are pairwise orthogonal, central idempotents. A central idempotent is called **primitive** if it is not the sum of two orthogonal (non-zero) idempotents. Since the e_V form a basis of the center of KG, it is easy to verify that the e_V are primitive.

(3.2.5) Proposition. *The multiplication by e_V is in each representation the projection onto the V-isotypical part.* □

(3.2.6) Proposition. *Suppose $V, W \in I(G;\mathbb{C})$. Then the orthogonality relation $\sum_{g\in G}\chi_V(g^{-1})\chi_W(xg) = \frac{|V|}{|G|}\langle V, W\rangle\chi_V(x)$ holds.*

Proof. The relation $e_V e_W = \langle V, W\rangle e_V$ says

$$\frac{|V||W|}{|G|^2}\sum_{g,h}\chi_V(g^{-1})\chi_W(h^{-1})gh = \frac{|V|}{|G|}\langle V, W\rangle\sum_{x\in G}\chi_V(x^{-1})x.$$

Now we compare the coefficients of x^{-1}. □

Chapter 4

Induced Representations

4.1 Basic Definitions and Properties

We compare representations of different groups. The ground field K is fixed.

Let H be a subgroup of G and V an H-representation. Recall the construction $X \times_H Y$ for a right H-set X and a left H-set Y; it is the quotient of the product $X \times Y$ under the equivalence relation $(x, y) \sim (xh^{-1}, hy)$. We denote equivalence classes by their representatives in $X \times Y$. We apply this construction to the right cosets gH, considered as H-sets by right multiplication. We use the bijection $i_g \colon V \to gH \times_H V, v \mapsto (g, v)$ to transport the vector space structure from V to $gH \times_H V$. If we choose another representative $gh \in gH$ of the coset gH, then $i_g l_h = i_{gh}$, and therefore the vector space structure is well-defined. (Although this vector space is just a model of V, we want this model to depend on the coset.) We define a G-action on $\bigoplus_{gH \in G/H} gH \times_H V$; the element $u \in G$ acts as follows

$$gH \times_H V \to ugH \times_H V, \quad (g, v) \mapsto (ug, v).$$

We see that G permutes the summands $gH \times_H V$ transitively. The resulting G-representation is called the **induced representation**, and is denoted by

$$\operatorname{ind}_H^G V = \bigoplus\nolimits_{gH \in G/H} gH \times_H V. \tag{4.1}$$

(4.1.1) Example. Suppose $|G/H| = 2$. Let $h \mapsto A(h)$ be a matrix representation of H. Fix an element $g \in G \smallsetminus H$. Then a matrix representation for ind_H^G is

$$h \mapsto \begin{pmatrix} A(h) & 0 \\ 0 & A(g^{-1}hg) \end{pmatrix}, \qquad gh \mapsto \begin{pmatrix} 0 & A(ghg) \\ A(h) & 0 \end{pmatrix}.$$

Verify this, using the bijections i_e and i_g. $\diamond$

There exist other constructions of the induced representation, from the view point of set-theory or algebra. Therefore we characterize it by a universal property. The bijection $i_e \colon V \to H \times_H V, v \mapsto (e, v)$ preserves the H-action. We thus obtain an H-morphism

$$i_H^G \colon V \to \operatorname{ind}_H^G V.$$

If W is a G-representation, we denote by $\operatorname{res}_H^G W$ the H-representation obtained from W by restricting the group action to H. The universal property is:

(4.1.2) Proposition. *The assignment*

$$\operatorname{Hom}_G(\operatorname{ind}_H^G V, W) \to \operatorname{Hom}_H(V, \operatorname{res}_H^G W), \quad \Phi \mapsto \Phi \circ i_H^G$$

is a natural isomorphism of vector spaces. In terms of dimensions this implies $\langle \operatorname{ind}_H^G V, W \rangle_G = \langle V, \operatorname{res}_H^G W \rangle_H$.

Proof. From the construction of $\operatorname{ind}_H^G V$ we see that a G-morphism from $\operatorname{ind}_H^G V$ is determined by its restriction to the summand $H \times_H V$. Therefore the map in question is injective.

Conversely, given $\varphi \colon V \to \operatorname{res}_H^G W$, we define a G-morphism $\Phi \colon \operatorname{ind}_H^G V \to W$ on the summand $gH \times_H V$ by $(g, v) \mapsto g \cdot \varphi(v)$. Another representative (gh^{-1}, hv) leads to the same value $gh^{-1}\varphi(hv)$, because φ is an H-morphism. Therefore Φ is well-defined, a G-morphism by construction, and $\Phi \circ i_H^G = \varphi$. □

We refer to 4.1.2 as **Frobenius reciprocity**. Suppose $j_H^G \colon V \to \tilde{V}$ is an H-morphism into a G-representation $\tilde{V}$ such that $\Phi \mapsto \Phi \circ j_H^G$ induces a bijection $\operatorname{Hom}_G(\tilde{V}, W) \cong \operatorname{Hom}_H(V, \operatorname{res}_H^G W)$. Then there exists a unique isomorphism $\gamma \colon \operatorname{ind}_H^G V \to \tilde{V}$ of G-representations such that $j_H^G = \gamma \circ i_H^G$. This expresses the fact, that 4.1.2 determines the induced representation. One consequence of this fact is the transitivity of induction:

(4.1.3) Proposition. *Let $A \subset B \subset C$ be groups. Then there exists a canonical C-isomorphism $\operatorname{ind}_B^C \operatorname{ind}_A^B V \cong \operatorname{ind}_A^C V$ for A-representations V, since $i_B^C i_A^B$ has the universal property.* □

Given a G-representation, we often ask whether it can be induced from a subgroup H. From the construction of $\operatorname{ind}_H^G V$ we obtain the following answer.

(4.1.4) Proposition. *Let V be an H-sub-representation of the G-representation W. The subspace $gV \subset W$ depends only on the coset gH. We denote it therefore by gHV. Suppose W is the direct sum of the subspaces gHV. Then the canonical map $\operatorname{ind}_H^G V \to W$ associated by 4.1.3 to the inclusion $V \subset W$ is an isomorphism.* □

An H-morphism $\alpha\colon V_1 \to V_2$ induces a G-morphism

$$\operatorname{ind}_H^G \alpha\colon \operatorname{ind}_H^G V_1 \to \operatorname{ind}_H^G V_2, \quad (g,v) \mapsto (g,\alpha(v)).$$

In this manner ind_H^G becomes a functor from the category of KH-representations to the category of KG-representations. The isomorphism 4.1.3 is compatible with induced morphisms in the variables V and W. In category theory one says that the induction functor ind_H^G is left adjoint to the restriction functor res_H^G.

Induction preserves direct sums; we have a natural isomorphism

$$\operatorname{ind}_H^G(V_1 \oplus V_2) \cong \operatorname{ind}_H^G V_1 \oplus \operatorname{ind}_H^G V_2. \tag{4.2}$$

Let W be a G-representation. Then the bijections

$$gH \times_H (V \otimes \operatorname{res}_H^G W) \to (gH \times_H V) \otimes W, \quad (g, v \otimes w) \mapsto (g,v) \otimes gv$$

combine to a natural isomorphism of G-representations

$$\operatorname{ind}_H^G(V \otimes \operatorname{res}_H^G W) \cong (\operatorname{ind}_H^G V) \otimes W. \tag{4.3}$$

(4.1.5) Example. Let 1_H denote the trivial one-dimensional H-representation. Then $\operatorname{ind}_H^G 1_H$ is the permutation representation $K(G/H)$. The basis element $gH \in K(G/H)$ corresponds to $(g,1) \in gH \times_H K$.

If V happens to be the restriction of a G-representation $V = \operatorname{res}_H^G W$, then

$$\operatorname{ind}_H^G(V) \cong \operatorname{ind}_H^G(1_H \otimes V) \cong (\operatorname{ind}_H^G 1_H) \otimes W \cong K(G/H) \otimes W.$$

In general one can think of $\operatorname{ind}_H^G V$ as a kind of mixture of the permutation representation $K(G/H)$ with V. ◇

We compute the character of an induced representation in the case that K has characteristic zero.

(4.1.6) Proposition. *Let $W = \operatorname{ind}_H^G V$. Then the character of W is given by the formula*

$$\chi_W(u) = \sum_{gH \in F(u,G/H)} \chi_V(g^{-1}ug) = \frac{1}{|H|} \sum_{g \in C(u,H)} \chi_V(g^{-1}ug)$$

where $C(u,H) = \{g \in G \mid g^{-1}ug \in H\}$ and $F(u,G/H) = G/H^u = \{gH \mid ugH = gH\}$. An empty sum is zero.

Proof. Since $u \in G$ sends $gH \times_H V$ to $ugH \times_H V$, we see that the direct summand $gH \times_H V$ contributes to the trace if and only if $ugH = gH$; and in that case l_u is transformed via the canonical isomorphism i_g into l_h, $h = g^{-1}ug$. □

We define a linear map for class functions by the same formula

$$\operatorname{ind}_H^G\colon Cl(H,K) \to Cl(G,K), \quad (\operatorname{ind}_H^G \alpha)(u) = \frac{1}{|H|} \sum_{g \in C(u,H)} \alpha(g^{-1}ug).$$

We leave it to the reader to verify the next proposition. We use the standard bilinear form (2.6) on class functions. The restriction of $\beta \in Cl(G)$ to H is given by composition with $H \subset G$.

(4.1.7) Proposition. *Class functions have the following properties:* $\langle \operatorname{ind}_H^G \alpha, \beta \rangle_G = \langle \alpha, \operatorname{res}_H^G \beta \rangle_H$, $\operatorname{ind}_H^G(\alpha \cdot \operatorname{res}_H^G \beta) = (\operatorname{ind}_H^G \alpha) \cdot \beta$, $\operatorname{ind}_B^C \operatorname{ind}_A^B = \operatorname{ind}_A^C$. □

Problems

1. The character of $\operatorname{ind}_H^G V$ assumes the value 0 at g if g is not conjugate to an element of H.
2. Verify directly that the assignment in 4.1.1 is a homomorphism and that different choices of g lead to conjugate matrix representations.
3. Apply 4.1.1 to the dihedral and quaternion groups D_{2n} and Q_{4n} and compare the result with our earlier constructions.
4. Verify 4.1.7. Verify that $\operatorname{ind}_H^G \alpha$ is a class function.
5. Here is another dual construction of the induced representation, that is also called **coinduction**. The vector space $\operatorname{Map}_H(G,V)$ of H-equivariant maps $G \to V$ carries a G-action $(u \cdot \varphi)(g) = \varphi(gu)$. The decomposition of G into H-orbits, $G = \amalg Hg$, shows $\operatorname{Map}_H(G,V) \cong \bigoplus_{Hg} \operatorname{Map}_H(Hg,V)$. The assignment

$$\alpha\colon \operatorname{Map}_H(G,V) \to \bigoplus gH \times_H V, \quad \varphi \mapsto \textstyle\sum_{gH}(g, \varphi(g^{-1}))$$

is an isomorphism of G-representations; it sends $\operatorname{Map}_H(Hg,V)$ to $g^{-1}H \times_H V$.
6. The induced representation has, of course, a description in terms of modules. The regular representation KG is a left KG-module and a right KH-module. Let V be a left KH-module. Then the tensor product $KG \otimes_{KH} V$ is a left KG-module. Relate this definition to our first definition of the induced representation; in particular explain from this view point the direct sum decomposition (4.1)of the induced representation.
7. Let A and B be groups. An (A,B)-set S is a set S with a left A-action and a right B-action which commute $(as)b = a(sb)$, $(a,b)\ inA \times B, s \in S$. Given a finite (A,B)-set S we associate to an A-representation V the vector space $\operatorname{Map}_A(S,V)$ of A-equivariant maps $\varphi\colon S \to V$. This vector space carries a B-action $(b \cdot \varphi)(s) = \varphi(sb)$. A morphism $\alpha\colon V \to W$ of A-representations yields a morphism $\operatorname{Map}_A(S,V) \to \operatorname{Map}_A(S,W)$, $\varphi \mapsto \alpha \circ \varphi$. Let A-Rep denote the category of finite-dimensional left A-representations (over K). The construction above yields a functor

$$\rho(S)\colon A\text{-}\mathrm{Rep} \to B\text{-}\mathrm{Rep}$$

for each finite (A,B)-set S. Let $\gamma\colon S_1 \to S_2$ be a morphism of (A,B)-sets. Composition with γ yields a morphism

$$\rho(\gamma)\colon \operatorname{Map}_A(S_2,V) \to \operatorname{Map}_A(S_1,V),$$

and the family of these morphisms is a natural transformation $\rho(\gamma)\colon \rho(S_2) \to \rho(S_1)$. Altogether we obtain a contravariant functor

$$\rho\colon A\text{-Set-}B \to [A\text{- Rep}, B\text{- Rep}]$$

of the category A-Set-B of finite (A,B)-sets into the functor category.

The composition of functors $\rho(S)$ is again a functor of the same type.

(4.1.8) Proposition. *Let S be an (A,B)-set and T be a (B,C)-set. Then there exists a canonical isomorphism of functors $\rho(T) \circ \rho(S) \cong \rho(S \times_B T)$.*

One has to provide a natural isomorphism

$$\operatorname{Map}_A(S \times_B T, V) \cong \operatorname{Map}_B(T, \operatorname{Map}_A(S,V))$$

of C-representations. It will be induced by the adjunction isomorphism

$$\operatorname{Map}(S \times T, V) \to \operatorname{Map}(T, \operatorname{Map}(S,V)), \quad \varphi \mapsto \hat{\varphi}, \quad \hat{\varphi}(t)(s) = \varphi(s,t).$$

4.2 Restriction to Normal Subgroups

Let H be a subgroup of G. The g**-conjugate** ${}^gV$ of the H-representation V is a gHg^{-1}-representation with the same underlying vector space and with action

$$gHg^{-1} \times V \to V, \quad (x,v) \mapsto x \cdot_g v = (g^{-1}xg) \cdot v.$$

The representation ${}^gV$ is irreducible if and only if V is irreducible. For $a,b \in G$ the relation ${}^a({}^bV) = {}^{ab}V$ holds, and g-conjugation is compatible with direct sums and tensor products. For $h \in H$ the map $l_h\colon V \to V$ is an isomorphism ${}^{gh}V \to {}^gV$ of gHg^{-1}-representations. The bijections

$$gH \times_H {}^uV \to guH \times_H V, \quad (g,v) \mapsto (gu,v)$$

combine to an isomorphism of G-representations

$$\operatorname{ind}_H^G {}^uV \cong \operatorname{ind}_H^G V.$$

Now suppose that H is a normal subgroup of G, in symbols $H \lhd G$. Then $gHg^{-1} = H$, and ${}^gV$ is again an H-representation which only depends on the coset gH, up to isomorphism. The group G acts on the set $\operatorname{Irr}(H,K)$

of isomorphism classes of KH-representations by $(g,V) \mapsto {}^gV$. This action factors over an action of G/H since ${}^gV$ only depends on the coset gH. Let

$$G(V) = \{g \in G \mid {}^gV \cong V\}$$

be the isotropy group at V of this G-action; it contains H.

Let W be a G-representation and $V \subset \operatorname{res}^G_H W$ be an H-sub-representation. The left translation $l_g\colon {}^gV \to gV$ satisfies

$$l_g(h \cdot_g v) = g(g^{-1}hgv) = hgv = h \cdot l_g v,$$

and this relation shows that gV is an H-sub-representation which is isomorphic to ${}^gV$ by l_g.

Assume, moreover, that W and V are irreducible. The sum of the gV is a G-sub-representation of W, hence equal to W. Since ${}^gV \cong gV$ is irreducible, $\operatorname{res}^G_H W$ is the sum of irreducible H-representations and therefore semi-simple. Thus we have shown:

(4.2.1) Proposition. *The restriction of a semi-simple G-representation to a normal subgroup H is a semi-simple H-representation.* □

Let again W and $V \subset \operatorname{res}^G_H W$ be irreducible. From our analysis of semi-simple representations we know that $\operatorname{res}^G_H W$ is the direct sum of its isotypical parts, and each irreducible sub-representation of $\operatorname{res}^G_H W$ is isomorphic to some ${}^gV$. Let $W(V)$ be the V-isotypical part of $\operatorname{res}^G_H W$. The $gV, g \in G(V)$ are contained in $W(V)$, and $W(V)$ is the sum of these gV. Therefore $W(V)$ is a $G(V)$-sub-representation of W. The inclusion $W(V) \subset \operatorname{res}^G_{G(V)} W$ gives us, by the universal property 4.1.2 of induced representations, a G-morphism

$$\iota\colon \operatorname{ind}^G_{G(V)} W(V) \to W.$$

In our model of the induced representation, ι maps $gG(V) \times_{G(V)} W(V)$ to $gW(V)$. The subspace $gW(V)$ is another isotypical summand of $\operatorname{res}^G_H W$. From 4.1.4 we obtain:

(4.2.2) Proposition. *ι is an isomorphism of G-representations.* □

Suppose $W(V)$ is isomorphic to r copies of V. Then

$$\operatorname{res}^G_H W \cong r \bigoplus\nolimits_{gG(V) \in G/G(V)} {}^gV.$$

The ${}^gV$, $gG(V) \in G/G(V)$ are pairwise non-isomorphic. The integer r is called the **ramification index** of W with respect to the normal subgroup H.

Let $V \in \operatorname{Irr}(H;K)$. The summand $gH \times_H V$ of $\operatorname{ind}^G_H V$ is isomorphic to ${}^gV$; the assignment ${}^gV \to gH \times_H V, v \mapsto (g,v)$ is an isomorphism of H-representations. This shows:

$$\operatorname{res}^G_H \operatorname{ind}^G_H V \cong |G(V)/H| \bigoplus {}^gV. \tag{4.4}$$

The summation is over $gG(V) \in G/G(V)$. We apply Frobenius reciprocity and Schur's lemma to (4.4) and obtain

$$\langle \operatorname{ind}_H^G V, \operatorname{ind}_H^G V \rangle_G = |G(V)/H| \sum \langle V, {}^gV \rangle_H = |G(V)/H| \langle V, V \rangle_H. \quad (4.5)$$

From this relation we deduce:

(4.2.3) Proposition. *Let K be algebraically closed of characteristic not dividing $|G|$. Then $\operatorname{ind}_H^G V$ is irreducible if and only if $G(V) = H$.*

Proof. $\operatorname{ind}_H^G V = W$ is irreducible if and only if $\langle W, W \rangle_G = 1$, and this is, by 4.5, the case if and only if $|G(V)/H| = 1$ and $\langle V, V \rangle_H = 1$. □

(4.2.4) Theorem. *Let K be algebraically closed of characteristic zero. Let $H \lhd G$ and $V \in \operatorname{Irr}(H; K)$. Suppose $\operatorname{ind}_H^{G(V)} V = \bigoplus_{j=1}^k m_j V_j$ with pairwise non-isomorphic $G(V)$-representations V_j. Then:*

(1) $\operatorname{ind}_{G(V)}^G V_j = W_j$ *is irreducible.*
(2) *Let $W \in \operatorname{Irr}(G; K)$ and $\langle V, \operatorname{res}_H^G W \rangle \neq 0$. Then $W \cong W_j$ for some j.*
(3) *The W_j are pairwise non-isomorphic.*
(4) *m_j is the ramification index of W_j with respect to H.*
(5) *Let $I^G(V) = \{W_1, \dots, W_r\} \subset \operatorname{Irr}(G; K)$. Then $I^G({}^gV) = I^G(V)$ and $\operatorname{Irr}(G; K)$ is the disjoint union of the sets $I^G(V)$ where V runs through a representative system of conjugation orbits $\operatorname{Irr}(H; K)/G$.*

Proof. Restriction to H and (4.4) yields

$$\bigoplus_{j=1}^k m_j \operatorname{res}_H^{G(V)} V_j = \operatorname{res}_H^{G(V)} \operatorname{ind}_H^{G(V)} V = |G(V)/H| V. \quad (4.6)$$

Therefore $\operatorname{res}_H^{G(V)} V_j = n_j V$ for some $n_j \in \mathbb{N}$. Frobenius reciprocity yields

$$\begin{aligned} n_j &= \langle n_j V, V \rangle_H = \langle V, \operatorname{res}_H^{G(V)} V_j \rangle_H \\ &= \langle \operatorname{ind}_H^{G(V)} V, V_j \rangle_{G(V)} \\ &= m_j \langle V_j, V_j \rangle_{G(V)} = m_j. \end{aligned}$$

Therefore $\operatorname{res}_H^{G(V)} V_j = m_j V$, and together with (4.6) we obtain

$$|G(V)/H| = \textstyle\sum_{j=1}^k m_j^2. \quad (4.7)$$

Let now $W \in \operatorname{Irr}(G, K)$ be such that

$$\langle \operatorname{ind}_{G(V)}^G V_i, W \rangle_G \neq 0. \quad (4.8)$$

We want to show that $W \cong W_i = \operatorname{ind}_{G(V)}^G V_i$; this shows in particular that W_i is irreducible, since there exist W such that (4.8) holds. By Frobenius reciprocity $0 \neq \langle \operatorname{ind}_{G(V)}^G V_i, W \rangle_G = \langle V_i, \operatorname{res}_{G(V)}^G W \rangle_{G(V)}$. Therefore V_i occurs in $\operatorname{res}_{G(V)}^G W$ and hence $\langle U, V_i \rangle \leq \langle U, \operatorname{res}_{G(V)}^G W \rangle$ for each $G(V)$-representation U. In particular

$$\begin{aligned} \langle V, \operatorname{res}_H^G W \rangle &= \langle \operatorname{ind}_H^{G(V)} V, \operatorname{res}_{G(V)}^G W \rangle \\ &\geq \langle \operatorname{ind}_H^{G(V)} V, V_i \rangle = \langle V, \operatorname{res}_H^{G(V)} V_i \rangle = m_i. \end{aligned}$$

This implies, for each $g \in G$,

$$\langle \operatorname{res}_H^G W, {}^gV \rangle = \langle \operatorname{res}_H^G ({}^{g^{-1}}W), V \rangle = \langle \operatorname{res}_H^G W, V \rangle \geq m_i.$$

Therefore each ${}^gV$ occurs in $\operatorname{res}_H^G V$ at least with multiplicity m_i. From (4.8) we obtain

$$\dim W \leq \dim \operatorname{ind}_{G(V)}^G V_i. \tag{4.9}$$

Since ${}^gV$ occurs in $\operatorname{res}_H^G V$ at least with multiplicity m_i and since there exist $|G/G(V)|$ different ${}^gV$, we see

$$\dim W \geq m_i |G/G(V)| \dim V = |G/G(V)| \dim V_i = \dim \operatorname{ind}_{G(V)}^G V_i. \tag{4.10}$$

From (4.9) and (4.10) we obtain equality of dimensions and therefore $W \cong \operatorname{ind}_{G(V)}^G V_i$, since W occurs in $\operatorname{ind}_{G(V)}^G V_i$. Frobenius reciprocity again yields

$$\langle V, \operatorname{res}_H^G W \rangle = \langle \operatorname{ind}_H^G V, W \rangle = \textstyle\sum_j m_j \langle \operatorname{ind}_{G(V)}^G V_j, W \rangle.$$

Therefore $W \in \operatorname{Irr}(G, K)$ is of the form $\operatorname{ind}_{G(V)}^G V_j$ for some $j \in \{1, \ldots, k\}$ if and only if $\langle V, \operatorname{res}_H^G W \rangle \neq 0$. There are exactly k G-representations of this type if we show $W_i \not\cong W_j$ for $i \neq j$. Suppose that $W_1 \cong W_2$. Then

$$\operatorname{ind}_H^G V = (m_1 + m_2) \operatorname{ind}_{G(V)}^G V_2 + m_3 \operatorname{ind}_{G(V)}^G V_3 + \cdots \tag{4.11}$$

and, together with (4.7) and 4.2.3, we arrive at the contradiction

$$\textstyle\sum_j m_j^2 = |G(V)/H| = \langle \operatorname{ind}_H^G V, \operatorname{ind}_H^G V \rangle \geq (m_1 + m_2)^2 + m_3^2 + \cdots + m_k^2;$$

the inequality $\geq$ is a consequence of (4.11) and $\langle W_i, W_i \rangle = 1$.

The ramification index of W_j with respect to H is $\langle V, \operatorname{res}_H^G W_j \rangle$. By Frobenius reciprocity this is equal to $\langle \operatorname{ind}_H^G V, W_j \rangle = \langle \sum_t m_t W_t, W_j \rangle = m_j$.

Suppose $W \in I^G(V_1) \cap I^G(V_2)$. Then

$$0 \neq \langle \operatorname{ind}_H^G V_1, \operatorname{ind}_H^G V_2 \rangle = \langle V_1, \operatorname{res} \operatorname{ind} V_2 \rangle.$$

Since $\operatorname{res}_H^G \operatorname{ind}_H^G V_2$ contains only conjugates of V_2, we see that V_1 and V_2 are conjugate. Part (2) and $I^G(V) = I^G({}^gV)$ now shows that $\operatorname{Irr}(G; K)$ is the disjoint union of the $I^G(V)$ as stated. □

(4.2.5) Remark. Theorem 4.2.4 gives a kind of recipe for the construction of irreducible G-representations starting from the irreducible representations of a normal subgroup H.

The situation is easy to survey if V happens to be a restriction of a $G(V)$-representation $\tilde{V}$. In that case $\operatorname{ind}_H^{G(V)} V = \operatorname{ind}_H^G 1_H \otimes \tilde{V}$, see **??**. Since $H \lhd G(V)$, the representation $\operatorname{ind}_H^{G(V)} 1_H$ is obtained from the regular representation $K(G(V)/H)$ by composition with the quotient homomorphism $G(V) \to G(V)/H$. The decomposition of the regular representation now determines the V_j in 4.2.4. ◇

The next proposition gives conditions under which the lifting property holds for all irreducible representations of H. For the proof go back to 1.6.1.

(4.2.6) Proposition. *Let K be algebraically closed. Suppose H is an abelian normal subgroup and G the semi-direct product of H and P, i.e. $G = HP$ and $H \cap P = 1$. Then each irreducible H-representation V has an extension to $G(V)$.* □

(4.2.7) Remark. We now combine 4.2.4 - 4.2.6. The hypotheses are as in 4.2.6. The irreducible representations of G are obtained as follows. Start with $\eta \in \operatorname{Irr}(H)$. Let $P_\eta \le P$ be the isotropy group of η under the conjugation action of P on $\operatorname{Irr}(H)$. Extend η to $\tilde{\eta}$ by 4.2.6. Let $U \in \operatorname{Irr}(P_\eta)$ and lift to a representation $\tilde{U}$ of HP_η. Then form $W = \operatorname{ind}_{HP_\eta}^G(\tilde{\eta} \otimes \tilde{U})$. The isomorphism class of W uniquely determines the P-orbit of η and the isomorphism class of the P_η-representation U. ◇

4.3 Monomial Groups

The induced representation of a one-dimensional representation is called a **monomial representation.** A group is called **monomial** if each $V \in \operatorname{Irr}(G;\mathbb{C})$ is monomial.

Let $\rho\colon H \to K^*$ be a one-dimensional representation. A basis of $\operatorname{ind}_H^G \rho$ consists of the $(g_j, 1) = x_j$ where $1 = g_1, \dots, g_r$ is a representative system of G/H. Suppose $gg_j = g_{\sigma(j)}h_j$ with $\sigma \in S_r$ and $h_j \in H$. The computation

$$gx_j = (gg_j, 1) = (g_{\sigma(j)}h_j, 1) = (g_{\sigma(j)}, \rho(h_j)) = \rho(h_j)x_{\sigma(j)}$$

shows: The matrix representation of $\operatorname{ind}_H^G \rho$ with respect to the basis above consists of matrices which have in each row and column exactly one non-zero entry. Matrices of this type are called **monomial**.

A group G is said to be **supersolvable** if there exists a string of normal subgroups $1 = G_0 < G_1 < \ldots < G_r = G$ such hat G_j/G_{j-1} is a group of prime order. Subgroups and factor groups of supersolvable groups are supersolvable. Groups of prime power order are supersolvable. If H is cyclic and G/H supersolvable, then G is supersolvable.

(4.3.1) Theorem. *Supersolvable groups are monomial.*

The proof needs some preparation and will be finished after 4.3.4.

(4.3.2) Proposition. *Suppose G has an abelian normal subgroup A which is not central. Then a faithful irreducible $\mathbb{C}G$-representation is induced from a proper subgroup.*

Proof. Let $W \in \mathrm{Irr}(G;\mathbb{C})$ be faithful and suppose $V \in \mathrm{Irr}(H;\mathbb{C})$ is an H-sub-representation of W. It suffices to show $G(V) \neq G$; see 4.2.2. Suppose $G(V) = G$. Then $\mathrm{res}^G_A W$ is a multiple of the one-dimensional representation V. Therefore each $a \in A$ acts on V as multiplication by a scalar, hence l_a commutes with l_g for each $g \in G$. Since W is faithful this fact implies that a is contained in the center of G. $\Box$

(4.3.3) Lemma. *Let G be a non-abelian supersolvable group. Then G contains a non-central normal abelian subgroup.*

Proof. Let $Z < G$ be the center of G. Since G is supersolvable, so is G/Z. Let $1 \neq H/Z \lhd G/Z$ be a cyclic normal subgroup. Then H is an abelian non-central normal subgroup. $\Box$

We need a formal property of induced representations. Let $\alpha\colon A \to B$ be a homomorphism. We associate to a KB-representation V a KA representation α^*V with the same underlying vector space V and with action

$$A \times V \to V, \quad (a, v) \mapsto \alpha(a) \cdot v.$$

In the case that $\alpha\colon A \subset B$ we have $\alpha^*V = \mathrm{res}^B_A V$. If α is surjective, we say that α^*V is obtained from V by **lifting** the group action along α. We show that induction is compatible with lifting. Consider

$$\begin{tikzcd} \tilde\alpha^{-1}(A) = P \arrow[r, "\subset"] \arrow[d, "\alpha"] & Q \arrow[d, "\tilde\alpha"] \\ A \arrow[r, "\subset"] & B \end{tikzcd}$$

$\tilde\alpha$ surjective and $\alpha = \tilde\alpha|P$. Then

(4.3.4) Lemma. $\tilde\alpha^*(\mathrm{ind}^B_A V) \cong \mathrm{ind}^Q_P(\alpha^*V)$ *for each A-representation V.*

Proof. α induces a bijection $Q/P \to B/A$ by passing to quotients. The isomorphism is induced by $qP \times_P \alpha^*V \to \alpha(q)A \times_A V$, $(q, v) \mapsto (\alpha(q), v)$. $\Box$

Proof. (Of 4.3.1.) Let W be a faithful irreducible $\mathbb{C}G$-representation. If G is abelian, then $\dim W = 1$, and nothing is to prove. Otherwise W is, by 4.3.2 and 4.3.3, induced from a proper subgroup $W = \mathrm{ind}_H^G V$. By induction we can assume that V is monomial, and by transitivity of induction, W is monomial.

If W is not faithful, let L be its kernel, and consider W as G/L-representation $\overline{W}$. By induction again, $\overline{W}$ is monomial. Now use 4.3.4. $\square$

Problems

1. Consider the semi-direct product G of the quaternion group $Q_8 = \langle\, a, b \mid a^2 = b^2, bab^{-1} = a^{-1} \,\rangle$ by $C_3 = \langle\, h \,\rangle$ with respect to the automorphism $hah^{-1} = b, hbh^{-1} = ab$. The faithful representation $G \to SU(2)$

$$a \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad h \mapsto \frac{i-1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

is not monomial: The group G has no subgroup of index 2. The group G has the quotient A_4 and is solvable. (G is called the **binary tetrahedral group**.)

4.4 The Character Ring and the Representation Ring

Let K be a field of characteristic zero. Recall that the characters of the irreducible KG-representations are linearly independent in the ring of class functions $Cl(G;K)$. The additive subgroup $CH(G;K)$ of $Cl(G;K)$ generated by the characters of irreducible representations is therefore a free abelian group of rank $|\,\mathrm{Irr}(G;K)|$. The relation $\chi_{V\oplus W} = \chi_V + \chi_W$ shows that each character is contained in this group. And the relation $\chi_{V\otimes W} = \chi_V\chi_W$ is used to show that $CH(G;K)$ is a subring of $Cl(G;K)$. This ring is called the **character ring** of KG-representations.

The character ring can be constructed formally. It is then called the **representation ring** or **Green ring**. In this context K can be an arbitrary field. Let $R(G;K)^+$ denote the set of isomorphism classes of KG-representations. Direct sum and tensor product induces on $R(G;K)^+$ two composition laws (addition and multiplication), and with these structures $R(G;K)^+$ is almost a commutative ring, except that inverses for the additive structure are missing. In situations like this, there exists a universal ring $R(G;K)$ together with a homomorphism

$$\iota\colon R(G;K)^+ \to R(G;K)$$

of semi-rings which is determined, up to a unique isomorphism, by a universal property: Let $\varphi\colon R(G;K)^+ \to A$ be a homomorphism into an abelian group.

Then there exists a unique homomorphism of abelian groups $\Phi\colon R(G;K) \to A$ such that $\Phi \circ \iota = \varphi$. If, moreover, A is a commutative ring and φ a homomorphism of semi-rings, then the universal homomorphism Φ is a homomorphism of rings. The universal property is used to show that additive constructions with representations extend to the representation ring. Elements in $R(G;K)$ are formal differences $[V]-[W]$ of representations V, W, called **virtual representations**, and $[V]-[W] = [V']-[W']$ if and only if $V \oplus W' \oplus Z \cong V' \oplus W \oplus Z$ for some representation Z.

(4.4.1) Proposition. *If K has characteristic zero, then the homomorphism $R(G;K)^+ \to CH(G;K)$, $V \mapsto \chi_V$ is a model for the universal ring. If the characteristic of K does not divide the order of the group, then $R(G;K)$ has a $\mathbb{Z}$-basis of isomorphism classes of irreducible representations.* □

Typical additive constructions are restriction and induction.

(4.4.2) Proposition. *Suppose $H \le G$. Restriction induces a ring homomorphism $\mathrm{res}^G_H\colon R(G;K) \to R(H;K)$. Induction induces an additive homomorphism $\mathrm{ind}^G_H\colon R(H;K) \to R(G;K)$. The relation* (4.3) *shows $\mathrm{ind}^G_H(x \cdot \mathrm{res}^G_H\, y) = (\mathrm{ind}^G_H\, x) \cdot y$. It implies that the image of ind^G_H is an ideal.* □

(4.4.3) Example. Let $G = C_m = \langle\, a \mid a^m = 1\,\rangle$ and $\rho\colon a \mapsto \exp(2\pi i/m)$ the standard representation. Then $R(G;\mathbb{C})$ is the free abelian group with basis $1, \rho, \rho^2, \dots, \rho^{m-1}$. The multiplicative properties of the ρ^k, namely $\rho^k \otimes \rho^l \cong \rho^{k+l}$, show that the ring $R(G;\mathbb{C})$ is isomorphic to $\mathbb{Z}[\rho]/(\rho^m - 1)$. More formally: For a finite abelian group G with character group G^* the representation ring $R(G;\mathbb{C})$ is isomorphic to the group ring $\mathbb{Z}[G^*]$. ◇

(4.4.4) Example. We determine $R(C_m;\mathbb{Q})$ for $C_m = \langle\, x \mid x^m = 1\,\rangle$. Decompose $x^m - 1 \in \mathbb{Q}[x]$ into irreducible factors $x^m - 1 = \prod_{d|m} \Phi_d(x)$. The cyclotomic polynomial $\Phi_d(x)$ has the primitive d-th roots of unity as its roots. The quotient $V_d = \mathbb{Q}[x]/(\Phi_d(x)$, viewed as a module over the group ring $\mathbb{Q}C_m = \mathbb{Q}[x]/(x^m - 1)$, is an irreducible $\mathbb{Q}C_m$-representation. The $V_d, d|m$ form a $\mathbb{Z}$-basis of $R(C_m, \mathbb{Q})$. There is another $\mathbb{Z}$-basis which consists of the permutation representations $\mathbb{Q}(C_m/C_n) = P_{m/n}$. The representation $P_{m/n}$ contains the irreducible representations which have C_n in its kernel. The kernel of V_d is $C_{m/d}$. Hence $P_k = \sum_{d|k} V_d$. By Möbius-inversion one obtains $V_k = \sum_{d|k} \mu(k/d) P_d$. ◇

(4.4.5) Example. Suppose the characteristic of K does not divide the order of G. We describe $R(C_m;K)$. Let $L = K(\varepsilon)$ be the field extension, ε a primitive m-th root of unity. Then, as in 4.4.3, $R(C_m;L) \cong \mathbb{Z}[\rho]/(\rho^m - 1)$ where $\rho\colon C_m \to L^*$ is given by $\rho(a) = \varepsilon$. Field extension yields an injective homomorphism $\iota\colon R(C_m;K) \to R(C_m;L)$. Let $\Gamma = \mathrm{Gal}(L|K)$ be the Galois group of L over K. An element $\gamma \in \Gamma$ is determined by its value $\gamma(\varepsilon) = \varepsilon^t$; and

t is determined modulo m; hence $\gamma \mapsto t$ yields an injection $\Gamma \subset \mathbb{Z}/m^*$. The group Γ acts on $\{1, \varepsilon, \dots, \varepsilon^{m-1}\}$. If C is an orbit, then $q_C = \prod_{\alpha \in C}(X - \alpha)$ is an irreducible factor of $x^m - 1 \in K[x]$. These irreducible factors correspond to the irreducible KC_m-representations. The group Γ acts on $\mathrm{Irr}(G; L) = \{\rho^t \mid t \in \mathbb{Z}/m\}$ by $(\gamma\rho)(a) = \gamma(\rho(a))$, and hence on $R(G; L)$ by ring automorphisms. The homomorphism ι induces an isomorphism $\iota\colon R(C_m; K) \cong R(C_m; L)^\Gamma$ with the Γ-fixed subring. ◇

Problems

1. Compute $R(D_{2n}; \mathbb{C})$ and study the restriction to $R(C_n; \mathbb{C})$. Compare $R(D_{2n}; \mathbb{R})$ via complexification with $R(D_{2n}; \mathbb{C})$.
2. Let $x \in R(G; \mathbb{C})$ be a unit of finite order. Then the character values of x are roots of unity. This implies that $\langle x, x \rangle = 1$. One concludes that $x = \pm\chi$, where χ is a one-dimensional representation.
3. The group ring $\mathbb{Z}G$ of a finite abelian group G is isomorphic to the representation ring $R(G; \mathbb{C})$. Hence the units of finite order in $\mathbb{Z}G$ are precisely the elements $\pm g$ for $g \in G$. **??**
4. The complexifications of representations induces an injective homomorphism $c\colon R(G; \mathbb{R})^* \to R(G; \mathbb{C})^*$. If $x \in R(G; \mathbb{R})$ is a positive unit of finite order (positive: $x(1) > 0$), then $c(x)$ is a one-dimensional character with real values, hence a homomorphism $G \to \mathbb{Z}^* = \{\pm 1\}$. Hence: $\mathrm{Hom}(G, \mathbb{Z}^*)$ is canonically isomorphic to the group of positive units of finite order in $R(G; \mathbb{R})$. Since these units are rational representations, $R(G; \mathbb{Q})$ has the same units of finite order as $R(G; \mathbb{R})$.

4.5 Cyclic Induction

Let $\mathcal{F}$ be a set of subgroups of G. A general question of induction theory is: For which sets $\mathcal{F}$ is the induction map

$$i_{\mathcal{F}} = \langle \mathrm{ind}_H^G \mid H \in \mathcal{F} \rangle\colon \bigoplus_{H \in \mathcal{F}} R(H; K) \to R(G; K)$$

surjective? We know that the image of $i_{\mathcal{F}}$ is an ideal. Therefore $i_{\mathcal{F}}$ is surjective if and only if the unit element 1_G of $R(G; K)$ is contained in the image of $i_{\mathcal{F}}$. It is also interesting to look for integral multiples of 1_G in the image of $i_{\mathcal{F}}$.

Let K be a field of characteristic zero. In this section we prove Artin's induction theorem which says that $|G| 1_G$ is in the image of $i_{\mathcal{C}}$ for the set $\mathcal{C}$ of cyclic subgroups. The proof is based on a character calculation.

We rewrite the basic orthogonality relation

$$|G| \dim V^G = \textstyle\sum_{g \in G} \chi_V(g). \tag{4.12}$$

Let $C^{\#}$ denote the set of generators of the cyclic group C. Since each element generates a unique cyclic subgroup we can write the right hand side of (4.12) as a sum over the cyclic subgroups C of G

$$|G| \dim V^G = \sum_C (\sum_{g \in C^{\#}} \chi_V(g)). \tag{4.13}$$

We apply this to the cyclic subgroup C itself.

Let $\mu\colon \mathbb{N} \to \mathbb{Z}$ be the Möbius-function, defined inductively by[1] $\mu(1) = 1$ and $\sum_{d|n} \mu(d) = 0$ for $n > 1$. Let f and g be functions from $\mathbb{N}$ into some (additive) abelian group such that $f(n) = \sum_{d|n} g(d)$; then **Möbius inversion** tells us that $g(n) = \sum_{d|n} \mu(n/d) f(d)$.

Note that for each divisor d of $|C| = n$ there exists a unique subgroup $D \leq C$ with $|D| = d$. We obtain by Möbius inversion from (4.13)

$$\sum_{c \in C^{\#}} \chi_V(c) = \sum_{D \leq C} \mu(|C/D|)(\sum_{d \in D} \chi_V(d)). \tag{4.14}$$

The inner sum in (4.14) equals $|D| \dim V^D$. Therefore we obtain altogether:

$$|G| \dim V^G = \sum_C (\sum_{D \leq C} \mu(|C/D|)|D| \dim V^D). \tag{4.15}$$

This equality has the form $|G| \dim V^G = \sum_C a_C \dim V^C$ with suitable integers a_C. We use $\langle\, K(G/H), V \,\rangle_G = \dim V^H$ in (4.15) and see that $\langle\, |G|K(G/G) - \sum_C a_C K(G/C), V \,\rangle_G = 0$ for each representation V. This implies that the left argument of the bracket is zero.

(4.5.1) Proposition. *$|G|[K(G/G)] = \sum_C a_C [K(G/C)]$ in $R(G;K)$. Note that $1_G = [K(G/G)]$ is the unit element in $R(G;K)$.* □

We know that $\mathrm{ind}_C^G \mathrm{res}_C^G \colon R(G;K) \to R(G;K)$ is multiplication by $[K(G/C)]$, see 4.1.5. Hence we obtain from 4.5.1:

(4.5.2) Theorem. *For $x \in R(G;K)$ the identity $\sum_C a_C \mathrm{ind}_C^G \mathrm{res}_C^G x = |G|x$ holds. This implies* **Artin's induction theorem***: $|G|R(G;K)$ is contained in the image of $i_{\mathcal{C}}$.* □

(4.5.3) Theorem. *Let V and W be $\mathbb{Q}G$-representations. Suppose that for each cyclic subgroup $C \subset G$ we have $\dim V^C = \dim W^C$. Then V and W are isomorphic.*

Proof. It suffices to show $\mathrm{res}_C^G V \cong \mathrm{res}_C^G W$ for each cyclic subgroup C of G, since we know that representations are determined by their restriction to cyclic subgroups. From our analysis of irreducible $\mathbb{Q}C$-representations we conclude that they are determined up to isomorphism by fixed point dimensions of subgroups. □

[1] $d|n$ means, d divides n.

(4.5.4) Proposition. *The rank of $R(G;\mathbb{Q})$ equals the number of conjugacy classes of cyclic subgroups of G.*

Proof. Since conjugate subgroups have fixed points of equal dimension we conclude from 4.5.3 that the rank is at most the number of cyclic conjugacy classes.

We show that the permutations representations $U_C = \mathbb{Q}(G/C)$, (C) cyclic conjugacy class, are linearly independent. Suppose $\sum_{(C)} a_C U_C = 0$. Let C be maximal such that $a_C \neq 0$, and let $g \in C$ be a generator. The character of $\sum_{(C)} a_C U_C$ at g is $a_C|G/C^C| \neq 0$; a contradiction. □

(4.5.5) Proposition. *Let $\alpha \in Cl(G;K)$. Suppose for each cyclic subgroup C the restriction $\mathrm{res}^G_C\, \alpha \in R(C;K)$. Then $|G|\alpha \in R(G;K)$.*

Proof. This is a direct consequence of 4.5.2. □

Problems

1. Let $G = A_5$. The virtual permutation representation associated to $[G/C_5] + [G/C_3] + [G/C_2] - [G/C_1]$ realizes $2 \cdot 1_G \in R(G;\mathbb{Q})$.

4.6 Induction Theorems

In order to state further induction theorems we have to specify suitable sets of subgroups. Let p be a prime number. A p-group is a group of p-power order. Let $|G| = p^t q$ with $(p,q) = 1$. Then there exists a subgroup $G(p) \leq G$ of order p^t, and all such groups are conjugate; they are called **Sylow p-groups** of G. A **p-hyperelementary group** H is the semi-direct product of a cyclic group and a p-group P of coprime order; S is a normal subgroup of H and $H/S \cong P$. Let $\mathcal{H}(p,G)$ denote the set of p-hyperelementary subgroups of G. The set $\mathcal{H}(G) = \cup_p \mathcal{H}(p,G)$ is the set of **hyperelementary** subgroups of G. Hyperelementary groups are monomial. A **p-elementary** group H is the direct product $S \times P$ of a cyclic group S and a p-group P of coprime order. We denote by $\mathcal{E}(p,G)$ the set of p-elementary subgroups of G and by $\mathcal{E}(G) = \cup_p calE(p,G)$ the set of **elementary** subgroups of G.

As in the case of Artin's induction theorem, the hyperelementary induction theorem is a consequence of a result about permutation representations.

(4.6.1) Theorem. *Let K be a field of characteristic zero. There exists in $R(G;K)$ a relation of the type $|G/G(p)|1_G = \sum_{E\in\mathcal{H}(p,G)} h_E[K(G/E)]$ with suitable integers h_E.*

We defer the proof to a later section where we deal systematically with such results; see **??**.

(4.6.2) Theorem (Hyperelementary induction). *Let K be a field of characteristic zero. Then $|G/G(p)|R(G;K)$ is contained in the image of $i_{\mathcal{H}(p,G)}$. The induction map $i_{\mathcal{H}(G)}$ is surjective.*

Proof. The first assertion is a consequence of 4.6.1. The integers $|G/G(p)|$, p a divisor of $|G|$, have no common divisor. Hence there exist integers n_p such that $\sum_p n_p|G/G(p)| = 1$. We conclude that 1_G is in the image of $i_{\mathcal{H}(G)}$. □

Hyperelementary groups are supersolvable. Therefore the next theorem is a consequence of 4.6.2.

(4.6.3) Theorem (Monomial induction). *$R(G;\mathbb{C})$ is generated by monomial representations, i.e., each element $x \in R(G;\mathbb{C})$ is of the form $x = [V]-[W]$ with representations V and W which are direct sums of monomial representations.* □

(4.6.4) Proposition. *Let $H = SP$ be a p-hyperelementary group. Then $i_{\mathcal{E}(p,H)}$ is surjective ($K = \mathbb{C}$).*

Proof. By induction on $|H|$ we can assume that irreducible representations of dimension greater than one are in the image of the induction map. Let $\alpha \in X(H)$ be a one-dimensional representation. Consider the elementary subgroup $E = S^P \times P$. We claim: If $\gamma \in X(H)$ occurs in $\mathrm{ind}_E^G \mathrm{res}_E^G \alpha$, then $\alpha = \gamma$ and α occurs with multiplicity one. By Frobenius reciprocity this is a consequence of 1.6.3. Thus modulo representations of dimensions greater that one, each element of $X(H)$ is in the image of the induction map. □

We combine 4.6.2 and 4.6.4 and obtain:

(4.6.5) Theorem (Brauer's induction theorem). *Let $K = \mathbb{C}$. Then $|G/G(p)|R(G;\mathbb{C})$ is contained in the image of $i_{\mathcal{E}(p,G)}$, and $i_{\mathcal{E}(G)}$ is surjective.* □

We now derive an interesting consequence of the monomial induction theorem. The **exponent** $e(G)$ of a group G is the least common multiple of the order of its elements; it divides $|G|$.

(4.6.6) Theorem (Splitting field). *Let ε be a primitive $e(G)$-th root of unity. Then $\mathbb{Q}(\varepsilon)$ is a splitting field for G, i.e., each irreducible $\mathbb{C}G$-representation has a realization with matrices having entries in $\mathbb{Q}(\varepsilon)$.*

Proof. One-dimensional representations of subgroups of G are certainly realizable over $\mathbb{Q}(\varepsilon)$ and therefore also monomial representations, being induced from one-dimensional ones. From 4.6.3 we infer: Let M be a $\mathbb{C}G$-representation.

Then there exist representations U and V, which are realizable over $\mathbb{Q}(\varepsilon)$ and such that $U \cong V \oplus M$. It is now a general fact that this implies: M is realizable over $\mathbb{Q}(\varepsilon)$. See the next proposition. □

Let L be an extension field of K (of characteristic zero). We denote by $V_L = L \otimes_K V$ the extension of a KG-representation to an LG-representation. From character theory we see $\langle V, W \rangle_{KG} = \langle V_L, W_L \rangle_{LG}$.

(4.6.7) Proposition. *Let U and V be KG-representations and M an LG-representation. Suppose $U_L \cong V_L \oplus M$. Then there exists a KG-representation N such that $N_L \cong M$.*

Proof. We fix M. Among all possible isomorphisms $U_L \cong V_L \oplus M$ choose one with V of smallest dimension. We show $V = 0$. Suppose $V \neq 0$. Choose $W \in \mathrm{Irr}(G;K)$ with $\langle W, V \rangle_{KG} > 0$. Then

$$\begin{aligned} \langle U, W \rangle_{KG} &= \langle U_L, W_L \rangle_{LG} = \langle V_L \oplus M, W_L \rangle_{LG} \\ &= \langle V, W \rangle_{KG} + \langle M, W_L \rangle_{LG} > 0. \end{aligned}$$

Therefore W occurs in U, hence $U \cong U' \oplus W$, $V \cong V' \oplus W$ by semi-simplicity. We conclude $U'_L \oplus W_L \cong V'_L \oplus W_L \oplus M$, cancel W_L, and see that V was not of minimal dimension. □

(4.6.8) Proposition. *The restriction $\rho\colon R(G;\mathbb{C}) \to \prod_E R(G;\mathbb{C})$ is an injection as a direct summand (E elementary subgroups).*

Proof. Suppose $1_G = \sum_E \mathrm{ind}_E^G x_E$. Define

$$\lambda\colon R(G) \to \textstyle\prod_E R(E), \quad x \mapsto (res_E^G x \cdot x_E \mid E).$$

Then $\sum_E \mathrm{ind}_E^G (res_E^G \cdot x_E) = \sum_E x \cdot \mathrm{ind}_E^G x_E = x \cdot 1_G = x$. Hence λ is a splitting of the induction.

Dually, define

$$r\colon \textstyle\bigoplus_E R(E) \to R(G), \quad (y_E) \mapsto \sum_E \mathrm{ind}_E^G (y_E \cdot x_E).$$

Then $r\rho(x) = \sum \mathrm{ind}_E^G (\mathrm{res}_E^G x \cdot x_E = x \cdot \sum_E \mathrm{ind}_E^G x_E) = x$. Thus r is a splitting of ρ. □

(4.6.9) Proposition. *Let $\alpha \in Cl(G)$ be such that $\mathrm{res}_E^G \alpha \in R(E)$ for each elementary subgroup E of G. Then $\alpha \in R(G)$.*

Proof. This is a consequence of 4.6.7. □

For arbitrary fields (of characteristic zero) we have results of the type 4.6.7 and 4.6.8, using hyperelementary groups. The proofs are the same.

Problems

1. Verify the elementary induction theorem explicitly for $G = A_5$.
2. The virtual permutation representation associated to $[G/D_5] + [G/D_3] - [G/D_2]$ is the unit element, $G = A_5$.

4.7 Elementary Abelian Groups

We assume that p does not divide the characteristic of K.

(4.7.1) Theorem. *Let V be a faithful KG-representation and $A \lhd G$ an elementary abelian group of rank $n \geq 2$. Then there exists $H \leq A$ such that $|A/H| = p$ and $V^H \neq 0$. The normalizer $N_G H$ is different from G and the canonical map $\mathrm{ind}_{NH}^G V^H \to V$ is an isomorphism.*

The proof needs some preparation. Let $S(A) = \{H \leq A \mid |A/H| = p\}$ be the set of cocyclic subgroups. Consider the following elements in the group algebra KA

$$x_H = |A|^{-1}(p\Sigma_H - \Sigma_A), \qquad y = |A|^{-1}\Sigma_A$$

with $\Sigma_A = \sum_{a\in A}$ and $\Sigma_H = \sum_{h\in H} h$ for $H \in S(A)$.

(4.7.2) Proposition. *The elements x_H and y have the following properties:*
(1) $x_H^2 = x_H$; $y^2 = y$
(2) $x_H x_K = 0$ *for* $H \neq K$; $x_H y = 0$
(3) $1 = y + \sum_{H\in S(A)} x_H$.

Proof. The proof of (1) and (2) is a direct consequence of the relations $\Sigma_A^2 = |A|\Sigma_A$, $\Sigma_H\Sigma_A = |H|\Sigma_A$, $\Sigma_H^2 = \Sigma_H$ and $\Sigma_H\Sigma_K = p^{-2}|A|\Sigma_A$ for $H \neq K$. In order to prove (3), one has to count the number of $H \in S(A)$ which contain $1 \neq a$ and the cardinality of $S(A)$. The latter equals the number $(p^n-1)/(p-1)$ of one-dimensional subspaces of A. The former is the number of subspaces of $A/\langle\, a\,\rangle$ of codimension one. From this information one verifies (3). □

(4.7.3) Proposition. *Let V be a KA-representation. Then $yV = Y^A$ and $x_H V \oplus V^A = V^H$.*

Proof. We already know that multiplication with y is a projection operator onto the fixed point set. The second assertion follows from $|H|^{-1}\Sigma_H = x_h + y$, $x_H y = 0$, and the fact that multiplication by $|H|^{-1}\Sigma_H$ is the projection onto V^H. □

(4.7.4) Proposition. $V = yV \oplus \bigoplus_{H\in S(A)} x_H V$.

Proof. 4.7.2 (3) shows that V is the sum of yV and the $x_H V$, and 4.7.2 (2) shows that the sum is direct. □

(4.7.5) Corollary. *Let* $S(A, V) = \{H \in S(A) \mid V^H \neq 0\}$ *and suppose* $V^G = 0$. *Then* $V = \bigoplus_{H \in S(A,V)} V^H$. *In particular* $S(A, V) \neq \emptyset$. □

Proof. (Of 4.7.1). By 4.7.4, there exists $H \in S(A)$ such that $V^H \neq 0$. Since $gV^H = V^{gHg-1}$, the group G acts on $S(A, V)$ by conjugation. Since $\sum gV^H = V$, the action is transitive, and NH is the isotropy group of $H \in S(A, V)$. The statement is now a special case of 4.2.1. □

We report on group to which 4.7.1 applies.

(4.7.6) Theorem. *Suppose each abelian normal subgroup of the p-group G is cyclic. Then G is a group in the following list.*

(1) *G is cyclic.*
(2) *G is the dihedral group* $D(2^n)$ *of order* 2^n, $n \geq 4$. *It has the presentation* $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{-1} \rangle$.
(3) *G is the semi-dihedral group* $SD(2^n)$ *of order* 2^n, $n \geq 4$. *It has the presentation* $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{2^{n-2}-1} \rangle$.
(4) *G is the quaternion group* $Q(2^n)$ *of order* 2^n, $n \geq 3$. *It has the presentation* $\langle A, B \mid B^2 = A^{2^{n-2}}, BAB^{-1} = A^{-1} \rangle$. □

We have shown that complex representations of p-groups are induced from one-dimensional representations. The virtue of 4.7.1 is that we do not need any hypothesis about the field K, except that we are in the semi-simple case. Thus 4.7.1 applies, e.g., to real or rational representations. It then remains to study the groups in the list 4.7.5. The situation is particularly simple for $p \neq 2$, since then everything is reduced to the cyclic groups.

Chapter 5

The Burnside Ring

5.1 The Burnside ring

Let G be a finite group and $A^+(G)$ the set of isomorphism classes of finite G-sets. We have two composition laws on $A^+(G)$ which give it the structure of a commutative semi-ring: Addition, induced by disjoint union; and multiplication, induced by cartesian product with diagonal action. The **Burnside ring** $A(G)$ of G is the universal ring, the Grothendieck ring, associated to the semi-ring $A^+(G)$. Formally, $A(G)$ is a commutative ring together with a homomorphism $\iota\colon A^+(G) \to A(G)$, such that for each additive homomorphism $\psi\colon A^+(G) \to B$ into an abelian group B there exists a unique homomorphism $\Phi\colon A(G) \to B$ which satisfies $\Psi \circ \iota = \varphi$. If, in addition, B is a commutative ring and ψ respects also multiplication and 1, then Ψ is a ring homomorphism. It is a general algebraic fact that such a universal ring exists. The image of the finite G-set S in $A(G)$ will be denoted $[S]$, i.e., $[S]$ is the image of the isomorphism class of S under ι. Each element of $A(G)$ is the (formal) difference $[S] - [T]$ of two finite G-sets S and T.

Let $H \leq G$. The assignment $S \mapsto |S^H|$ is a homomorphism of semi-rings $A^+(G) \to \mathbb{Z}$. The associated ring homomorphism $\varphi_H\colon A(G) \to \mathbb{Z}$ is called the **Burnside H-mark** of $A(G)$. Conjugate subgroups yield the same mark. We assemble the marks into a single homomorphism. Let $C(G)$ be the ring of all functions $\mathrm{Con}(G) \to \mathbb{Z}$. Considered as a ring, $C(G)$ is the product of $|\mathrm{Con}(G)|$ copies $\mathbb{Z}$. We obtain a ring homomorphism, called **Burnside character** or **mark homomorphism**,

$$\varphi\colon A(G) \to C(G)$$

which assigns to $x \in A(G)$ the map $(H) \mapsto \varphi_H(x)$.

(5.1.1) Proposition. *The additive group of $A(G)$ has as $\mathbb{Z}$-basis the isomorphism classes of transitive G-sets. The homomorphism φ is injective.*

Proof. Each G-set is the disjoint sum of its orbits. Hence the transitive G-sets generate $A(G)$ additively. Suppose $0 \neq x = \sum_{(H)} a_H[G/H]$, $a_H \in \mathbb{Z}$. We choose a maximal (with respect to inclusion) conjugacy class (L) such that its coefficient $a_L \neq 0$. Recall that $G/H^L \neq \emptyset$ if and only if L is subconjugate to H. Therefore $\varphi_L(x) = \sum_{(H)} a_H \varphi_L(G/H) = a_L|G/L^L| \neq 0$. Thus x is not contained in the kernel of φ. A similar argument shows that the $[G/H]$ are linearly independent. □

(5.1.2) Corollary. *Finite G-sets X and Y are isomorphic if and only if they have the same image in $A(G)$. They have the same image in $A(G)$, if for all subgroups H of G their H-fixed point sets have the same cardinality.* □

Recall that $X_{(H)}$ denotes the subset of X of orbits isomorphic to G/H. The cardinality of $|X_{(H)}/G|$ is the number of such orbits. Hence, in $A(G)$,

$$[X] = \textstyle\sum_{(H)} |X_{(H)}/G| \cdot [G/H]. \tag{5.1}$$

In a sum of this type over $(H) \in \mathrm{Con}(G)$ we select from each class (H) a representative H. Recall that $WH = NH/H$ is the automorphism group of the G-set G/H and that this group acts freely on G/H. Therefore each fixed point set G/H^K has cardinality divisible by $|WH|$. This divisibility property shows that the function $x_{(H)} = |WH|^{-1}\varphi(G/H)$ has integral values and defines therefore an element of $C(G)$.

(5.1.3) Proposition. *The $x_{(H)}$, $(H) \in \mathrm{Con}(G)$ are a $\mathbb{Z}$-basis of $C(G)$.*

Proof. The matrix $(K), (L) \mapsto x_{(K)}(G/L) = |WL|^{-1}|G/L^K|$ is triangular with units on the diagonal. (In order to write out a matrix we choose a total order on $\mathrm{Con}(G)$ which refines the partial order on $\mathrm{Con}(G)$ given by subconjugation.) □

(5.1.4) Corollary. *The cokernel of φ is isomorphic to $\prod_{(H)} \mathbb{Z}/|WH|\mathbb{Z}$.* □

In the sequel we often identify $A(G)$ with its image in $C(G)$. Less formally than above we could also define $A(G)$ as the subring of $C(G)$ generated by the functions $(H) \mapsto |S^H|$ for finite G-sets S. (Analogy: Character ring versus representation ring.)

(5.1.5) Corollary. *Let $a \in A(G) \subset C(G)$, $b \in C(G)$ and suppose $a(H) \equiv b(H) \bmod |G|$ for all (H). Then $b \in A(G)$. If we apply this to $a = 0$ we see that $|G|C(G) \subset A(G)$.* □

The interesting structure of $A(G)$ comes from its multiplication. In order to determine the product $[G/K] \times [G/L]$ one has to decompose $G/K \times G/L$ into orbits. The isotropy groups have the form $uKu^{-1} \cap vLv^{-1}$. Actually, one rarely wants to do such a computation.

Let $H \le G$. The induction $X \mapsto G \times_H X$ induces an additive homomorphism

$$\operatorname{ind}_H^G \colon A(H) \to A(G),$$

called **induction** from H to G. If we view a G-set X as an H-set $\operatorname{res}_H^G X$, we obtain a ring homomorphism

$$\operatorname{res}_H^G \colon A(G) \to A(H),$$

called **restriction** from G to H. More generally: Let $\alpha\colon K \to L$ be a homomorphism and view an L-set via α as K-set. This induces a ring homomorphism

$$\alpha^* = A(\alpha)\colon A(L) \to A(K).$$

The isomorphism $G \times_H (X \times Y) \cong (G \times_H X) \times Y$ for H-sets X and G-sets Y yields

$$\operatorname{ind}_H^G(a \cdot \operatorname{res}_H^G b) = \operatorname{ind}_H^G a \cdot b. \tag{5.2}$$

This implies as in the case of the representation ring:

(5.1.6) Proposition. *The image of* ind_H^G *is an ideal of* $A(G)$. □

The Burnside ring codifies combinatorial properties of the lattice of subgroups. It is also a universal object in representation theory. When we assign to a finite G-set S its permutation representation KS over a field K we obtain a ring homomorphism

$$\pi_G\colon A(G) \to R(G;K),$$

see 1.8.5. These homomorphisms are compatible with restriction and induction

$$\pi_G \circ \operatorname{ind}_H^G = \operatorname{ind}_H^G \circ \pi_H, \qquad \pi_G \circ \operatorname{res}_H^G = \operatorname{res}_H^G \circ \pi_H.$$

If K has characteristic zero and if we view $R(G;K)$ as character ring, then the Burnside character is related to the ordinary character:

$$\pi_G(S)(g) = |S^{\langle g \rangle}| = \varphi_{\langle g \rangle}[S] \tag{5.3}$$

(Here $\langle g \rangle$ denotes the cyclic subgroup generated by g; see 2.1.5.)

We determine the Burnside character of $G \times_H S$. Let $\operatorname{ind}_H^G \colon C(H) \to C(G)$ be the additive map which sends $\alpha \in C(H)$ to $\operatorname{ind}_H^G(\alpha) \in C(G)$ defined by

$$(\operatorname{ind}_H^G \alpha)(J) = \sum \{\alpha(g^{-1}Jg) \mid gH \in G/H^J\}.$$

The conjugacy class of $g^{-1}Jg$ does not depend on the representative g of the coset gH. Compare **??**.

(5.1.7) Proposition. $\varphi \circ \operatorname{ind}_H^G = \operatorname{ind}_H^G \circ \varphi\colon A(H) \to C(G)$.

Proof. We use the projection $p\colon G \times_H S \to G/H$. Let $gH \in G/H^J$ and hence $g^{-1}Jg \le H$. The map

$$X^{g^{-1}JG} \to (G \times_H X)^J, \quad x \mapsto (g, x)$$

yields a bijection onto $p^{-1}(gH) \cap (G \times_H X)^J$. □

Problems

1. Let $e(H,G) \in C(G)$ denote the function with value 1 at (H) and value zero otherwise. Use 5.1.7 and show $\mathrm{ind}_H^G\, e(H,H) = |WH| e(H,G)$.
2. Verify (5.2) for $\mathrm{ind}_H^G\colon C(H) \to C(G)$.
3. Verify the so-called **Mackey formula**

$$\mathrm{res}_K^G \circ \mathrm{ind}_H^G = \sum_{KgH} \mathrm{ind}_{^gH \cap K}^K \circ \mathrm{res}_{^gH\cap K}^{^gH} \circ c(g)^*$$

where $c(g)\colon H \to {}^gH$, $h \mapsto ghg^{-1}$. The sum is over the double cosets $KgH \in K\backslash G/H$. Later we deal systematically with such formulae.
4. If X is a G-set and Y an H-set, then $X \times Y$ is naturally a $G \times H$-set. Suppose G and H are finite groups of coprime order. Then $(X,Y) \mapsto X \times Y$ induces a ring isomorphism $A(G) \otimes_{\mathbb{Z}} A(H) \cong A(G \times H)$.
5. The total quotient ring of $A(G)$ (all non zero divisors inverted) is isomorphic to $A(G) \otimes_{\mathbb{Z}} \mathbb{Q}$. The integral closure of $A(G)$ in its total quotient ring is $C(G)$. Hence the inclusion $A(G) \subset C(G)$ is determined by the ring-theoretic properties of $A(G)$ alone. By 5.1.5, we have isomorphisms $A(G)[\frac{1}{|G|}] \cong C(G)[\frac{1}{|G|}]$ and $A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong C(G) \otimes_{\mathbb{Z}} \mathbb{Q}$.
6. The marks are precisely the ring homomorphisms $A(G) \to \mathbb{Z}$.
7. The map $\chi\colon A(G) \to C(G)$ which assigns to each G-set X the function $\chi(X)\colon (H) \mapsto |X^H/WH|$ is an additive isomorphism.
8. The alternating group A_4 has the following conjugacy classes of subgroups and their normalizers.

H	C_1	C_2	C_3	D_2	A_4
NH	A_4	D_2	C_3	A_4	A_4

Verify the multiplication table of the homogeneous sets:

	C_1	C_2	C_3	D_2	A_4
A_4	C_1	C_2	C_3	D_2	A_4
D_2	$3C_1$	$3C_2$	C_1	$3D_2$	
C_3	$4C_1$	$2C_1$	$C_1 + C_3$		
C_2	$6C_1$	$2C_1 + 2C_2$			
C_1	$12C_1$				

We have written H instead of G/H.

5.2 Congruences

We describe $A(G)$ as the kernel of a homomorphism $C(G) \to \prod_{(H)} \mathbb{Z}/|WH|\mathbb{Z}$. We use the counting lemma 1.2.8:

(5.2.1) Lemma. *Let G be a finite group, X a finite G-set, and $\langle g \rangle$ the cyclic group generated by $g \in G$. Then $|G| \cdot |X/G| = \sum_{g \in G} |X^{\langle g \rangle}|$.* □

The counting lemma implies the standard congruence

$$\sum_{g \in G} \varphi_{\langle g \rangle}(X) \equiv 0 \bmod |G| \tag{5.4}$$

for each finite G-set X. Let $nH \in NH/H = WH$ and denote by $\langle n, H \rangle$ the subgroup generated by n and H. Then $(X^H)^{nH} = X^{\langle n,H \rangle}$. We apply 5.2.1 to the WH-sets X^H and obtain the congruence

$$\sum_{nH \in WH} \varphi_{\langle n,H \rangle}(X) \equiv 0 \bmod |WH|. \tag{5.5}$$

Hence $A(G)$ is contained in the kernel of the additive homomorphism

$$\kappa_H : C(G) \to \mathbb{Z}/|WH|\mathbb{Z}, \quad f \mapsto \sum_{nH \in WH} f(\langle n, H \rangle).$$

Let

$$\kappa : C(G) \to H(G) = \prod_{(H)} \mathbb{Z}/|WH|\mathbb{Z}$$

be the product of the κ_H for $(H) \in \mathrm{Con}(G)$.

(5.2.2) Theorem. *The sequence* $0 \to A(G) \xrightarrow{\varphi} C(G) \xrightarrow{\kappa} H(G) \to 0$ *is exact.*

Proof. We know already that φ is injective and that $\kappa \circ \varphi = 0$. Let $x = \sum_{(H)} m_H x_{(H)} \in C(G)$ satisfy the congruences (5.5) . Choose a maximal (H) such that $m_{(H)} \neq 0$. The congruence (5.5) for H shows $m_{(H)} \equiv 0 \bmod |WH|$. Since $|WH| x_{(H)} \in A(G)$, we can remove this summand and obtain, by induction on the number of summands in x, an element in $A(G)$. This proves the exactness at $C(G)$. Since the order of $H(G)$ is the order of the cokernel of φ we conlude that κ is surjective. □

Theorem 5.2.2 describes the subgroup $A(G)$ of $C(G)$ by congruence relations among the values of functions. The standard congruence (5.4) reads

$$\sum_{(C)} |G/NC||C^*|\varphi_C(x) \equiv \bmod |G|; \tag{5.6}$$

here $|C^*|$ is the number of generators of the group C. The sum is over G-conjugacy classes of cyclic subgroups C of G. The congruence (5.5) associated to the subgroup H has the form

$$\textstyle\sum_{(K)} n(H,K)\varphi_K(x) \equiv 0 \bmod |WH| \tag{5.7}$$

with integers $n(H,K)$ and $n(H,H) = 1$; the sum is over G-conjugacy classes (K) of subgroups K such that $H \lhd K$ and K/H is cyclic ($H = K$ is allowed).

The virtue of the congruences is that they allow the construction of elements of $A(G)$ without knowing them as G-sets. Interesting examples are obtained in the next section.

Congruences for the subgroup $A(G) \subset C(G)$ are by no means unique. We make some general remarks concerning congruences. A congruence of type $C(H,m)$ for $A(G)$ is a relation of the form

$$x(H) + \sum_{(K)} \ell(H,K)x(K) \equiv 0 \bmod (m), \tag{5.8}$$

$\ell(H,K) \in \mathbb{Z}$, $(H) < (K)$, which holds for functions $x \in A(G)$.

(5.2.3) Proposition. *Let $(C(H_j,m_j) \mid j \in J)$ be a family of congruences for $A(G)$ with the following property: For each H there exists j with $(H) = (H_j)$, and the smallest common multiple of $\{m_j \mid (H_j) = (H)\}$ is $|WH|$. Then $x \in A(G)$ if and only if* (5.8) *holds for each $(H,m) = (H_j,m_j)$.*

Proof. Let $x = \sum_{(H)} n(H)x_{(H)} \in C(G)$ satisfy the congruences, see 5.1.3. We have to show $n(H) \equiv 0 \bmod |WH|$ for all H. We induct over the number of summands. Suppose (H) is maximal with $n(H) \neq 0$. Let $(H_j) = (H)$. Then $x(H) = n(H)$ and $C(H_j,m_j)$ tells us $n(H) \equiv 0 \bmod m_j$. Thus, by our assumption, $n(H) \equiv 0 \bmod |WH|$. Since $|WH|x_{(H)} \in A(G)$, we can remove this summand. □

Let W_pH denote a Sylow p-group of WH and let $N_pH \leq NH$ be its preimage. If we apply the method above to the pair (N_pH, H) we obtain a congruence $C(H, |W_pH|)$. These congruences satisfy the hypotheses of 5.2.3. Hence they can be used to characterize $A(G) \subset C(G)$. Since the congruences are modulo prime powers, we call them **primary congruences**. They are particularly useful when one studies localizations of the Burnside ring.

Problems

1. Let p be prime number and $G = \mathbb{Z}/p^n$. For each $i \in \{0,\dots,n\}$ there exists a unique subgroup H_i of order p^i, and these comprise all subgroups. The ring $A(\mathbb{Z}/p^n) \subset C(\mathbb{Z}/p^n)$ consists of all functions x which satisfy the congruences

$$x(H_i) \equiv x(H_{i+1}) \bmod p^{n-i}.$$

For each finite G-set X the difference $X \smallsetminus X^{H_1}$ consists of orbits of length p^n, and this yields the congruence in the case $i = 0$.

2. Let G be abelian. Then the product of the homogeneous G-sets in $A(G)$ is $[G/K] \cdot [G/L] = a[G/K \cap L]$ with $a = |G||K|^{-1}|L|^{-1}|K \cap L|$.

3. Let $A \subset \mathbb{Z} \times \mathbb{Z}$ be a subring (with the same unit). Then there exists an integer m such that $A = \{(a, b) \mid a \equiv b \bmod m\}$.

4. Determine the congruences for A_4. Show that $3[A_4/A_4] - [A_4/D_2] - 3[A_4/C_3] + [A_4/C_1]$ realizes the function in $C(A_4)$ which has value 0 for $H \neq A_4$ and value 3 for A_4. Determine the values $|G/K^L|$ in all possible cases.

5. The alternating group A_5 has the conjugacy classes of subgroups and their normalizers displayed in the next table; D_n is the dihedral group of order $2n$.

H	1	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	D_2	D_3	D_5	A_4	A_5
NH	A_5	D_2	D_3	D_5	A_4	D_3	D_5	A_4	A_5

Therefore $A(A_5) \subset C(A_5)$ is the subring of functions z which satisfy the following congruences.

$$
\begin{array}{ll}
(1) & z(H) \text{ arbitrary for } H = D_3, D_5, A_4, A_5 \\
(1) & z(\mathbb{Z}/n) \equiv z(D_n) \bmod 2 \text{ for } n = 2, 3, 5 \\
(2) & z(D_2) \equiv z(A_4) \bmod 3 \\
(3) & z(1) + 15z(\mathbb{Z}/2) + 20z(\mathbb{Z}/3) + 24z(\mathbb{Z}/5) \equiv 0 \bmod 60
\end{array}
$$

The congruence (3) can be replaced by the primary congruences $z(1) \equiv z(\mathbb{Z}/2) \bmod 4$, $z(1) \equiv z(\mathbb{Z}/3) \bmod 3$, $z(1) \equiv z(\mathbb{Z}/5) \bmod 5$.

The ring $A(A_5)$ contains the following units:

1	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/5$	D_2	D_3	D_5	A_4	A_5
a	a	a	a	b	c	d	b	e

Here $a, b, c, d, e \in \{\pm 1\}$, and the second line gives the value of the function z at the element indicated in the first line. The ring $A(A_5)$ contains the idempotents $0, 1, \varepsilon, 1 - \varepsilon$ where ε is represented by the function with value 1 at A_5 and value zero otherwise. The idempotent ε is

$$\varepsilon = [G/G] - [G/A_4] - [G/D_5] - [G/D_3] - [G/1] + [G/\mathbb{Z}/3] + 2[G/\mathbb{Z}/2],$$

in terms of homogeneous spaces.

6. The integers $n(H, K)$ in (5.7) which were obtained from the congruences (5.5) depend on the subgroup structure. We have the equalities

$$
\begin{aligned}
& |NU| \cdot |\{A \mid V \lhd A,\ A/V \text{ cyclic},\ A \sim_G U\}| \\
&= |\{g \in G \mid V \lhd gUg^{-1},\ gUg^{-1}/V \text{ cyclic}\}| \\
&= |\{g \in G \mid g^{-1}Vg \lhd U, U/g^{-1}Vg \text{ cyclic}\}| \\
&= |NV| \cdot |\{B \mid B \lhd U,\ U/B \text{ cyclic},\ B \sim_G V\}|
\end{aligned}
$$

for each pair U, V of subgroups. These yield ($\tilde{\varphi}$ is the Euler function)

$$
\begin{aligned}
n(H, K) &= \tilde{\varphi}(K : H) \cdot |\{A \mid H \lhd A,\ A/H \text{ cyclic},\ A \sim_G K\}| \\
&= \tilde{\varphi}(K : H)|NH||NK|^{-1} \cdot |\{B \mid B \lhd K,\ K/B \text{ cyclic},\ B \sim_G H\}|.
\end{aligned}
$$

As usual, $(K : H) = |K/H|$ denotes the index of H in K. The first equality follows from the construction of the $n(H,K)$. Note that $\tilde{\varphi}(|C|)$ is the numbers of generators of the cyclic group C. In the case that $H \lhd G$ the relations simplifies

$$n(H,K) = \tilde{\varphi}(K : H)|G||NK|^{-1},$$

for cyclic K/H. In the case $H = 1$ we obtain a standard congruence.

5.3 Idempotents

An element e in a ring R is called **idempotent**, if it satisfies $e^2 = e$. Idempotents in $C(G)$ are the functions $\mathrm{Con}(G) \to \mathbb{Z}$ with values in $\{0,1\}$. We use the congruences in order to determine which integral multiples of idempotents are contained in $A(G)$.

Let π be a set of prime numbers and π' the complementary set. We denote by $O^\pi(G)$ the smallest normal subgroup $N \lhd G$ such that G/N is a solvable π-group. In the case that $O^\pi(G) = G$ the group is called **π-perfect**. If $K \sim_G L$ then $O^\pi(K) \sim_G O^\pi(L)$. The group $O^\pi(K)$ is always π-perfect. Let $P_\pi(G) \subset \mathrm{Con}(G)$ denote the set of π-perfect conjugacy classes. For $(J) \in P_\pi(G)$ we set $(J,\pi) = \{(H) \mid (O^\pi(H) = (J)\}$. Then $\mathrm{Con}(G)$ is the disjoint union of the sets (J,π). We decompose $n \in \mathbb{N}$ in the form $n = n(\pi)n(\pi')$, where $n(\pi)$ collects the prime divisors in π. We write $|G|(\pi) = g(\pi)$.

We denote by $e(H) = e(H,G) \in C(G)$ the idempotent with value one at (H) and value zero otherwise. Each idempotent in $C(G)$ is the sum of certain $e(H)$. If $x \in C(G)$, then $y = g(\pi')x$ satisfies the q-primary congruences for $q \in \pi'$. Since $g(\pi')$ is invertible modulo p, $p \in \pi$, the p-primary congruences are satisfied for y if and only if they are satisfied for x.

(5.3.1) Proposition. *Let $e \in C(G)$ be idempotent and $x = g(\pi')e$. Then $x \in A(G)$ if and only if $e(H) = e(K)$ for all pairs (H,K) with $H \lhd K$ and $|K/H| \in \pi$.*

Proof. Let $x \in A(G)$. Then

$$g(\pi')e(H) = \varphi_H(x) \equiv \varphi_K(x) = g(\pi')e(K) \bmod |K/H|.$$

Since $g(\pi')$ is prime to $p \in \pi$ and $e(H) \in \{0,1\}$, the relations $e(H) = e(K)$ and $g(\pi')e(H) \equiv g(\pi')e(K) \bmod |K/H|$ are equivalent. Hence the condition is necessary.

For the converse, we have to show that e satisfies the p-primary congruences for $p \in \pi$. A congruence has the form

$$\sum n(H,L)e(L) \equiv 0 \bmod |W_p(H)|;$$

the sum is over $(L/H), L/H \leq W_p(H)$. Since L/H is a p-group, there exists a series

$$H = H_0 \lhd H_1 \lhd \ldots \lhd H_r = L$$

such that $|H_i/H_{i-1}| = p$. Therefore the $e(L)$ which appear in the congruence all have the same value, hence the congruence is satisfies. □

(5.3.2) Proposition. *The multiple $x = g(\pi')e$ of an idempotent e is contained in $A(G)$ if and only if e is a sum of $e_{(J,\pi)} = \sum_{(H)\in(J,\pi)} e(H)$, for $(J) \in P_\pi(G)$.*

Proof. If $x \in A(G)$ then, by 5.3.1, $e(H) = 1$ if and only if $e(O^\pi(H)) = 1$; hence if $e(H) = 1$, then e has value 1 on $(O^\pi(H))$. This shows that e is a sum of certain $e_{(J,\pi)}$.

We show that $g(\pi')e_{(J,\pi)} \in A(G)$. Let $H \lhd K$, $|K/H| = p \in \pi$. Then $O^\pi(H) = O^\pi(K)$. Therefore H and K are both contained in (J,π) or not. Now we apply 5.3.1. □

By definition we have: $O^p(H)$ is cyclic of order prime to p if and only if H is p-hyperelementary. The **support** $S(e)$ of an idempotent function e is the set of subgroups H such that $e(H) \neq 0$.

(5.3.3) Corollary. *Suppose $e \in C(G)$ is idempotent and $|G/G(p)|e \in A(G)$. Then $S(e)$ contains the set of cyclic subgroups if and only if $S(e)$ contains the set $\mathcal{H}(p,G)$ of p-hyperelementary subgroups.* □

(5.3.4) Proposition. *Suppose $x = \sum_H n(H)[G/H] \in A(G) \subset C(G)$. If $n(H) \neq 0$, then there exist L such that $x(L) \neq 0$ and $(H) \leq (L)$.*

Proof. Let L be maximal with $n(L) \neq 0$ and $(H) \leq (L)$. Then $x(L) = n(L)|WL| \neq 0$. □

(5.3.5) Corollary. *There exist an idempotent $e_p \in C(G)$ with support $S(e_p) = \mathcal{H}(p,G)$ and $x = |G/G(p)|e_p \in A(G)$. The element x is an integral linear combination of $[G/H], H \in \mathcal{H}(p,G)$.* □

(5.3.6) Hyperelementary induction. We now apply the preceding results to hyperelementary induction and prove 4.6.1. If we assign to each finite G-set S the permutation representation $K(S)$, we obtain a ring homomorphism $\pi\colon A(G) \to R(G;K)$. Let K be of characteristic zero. Then we know the character relation $\varphi_{\langle g \rangle}(S) = \chi_{K(S)}(g)$. The element $x_p = |G/G(p)|e_p \in A(G)$ in 5.3.5 therefore satisfies $\pi(x_p) = |G/G(p)| \cdot 1_G \in R(G;K)$. As we explained earlier, the p-hyperelementary induction theorem was a simple consequence of the existence of an element x_p with these properties. ◇

Artin's induction theorem can also be interpreted from this view point, since the idempotent e with value 1 on all cyclic subgroups satisfies $|G|e \in A(G)$.

Let $e \in C(G)$ be an idempotent. Let n be an integer such that $ne = \sum_{(H)} n(H)[G/H] \in A(G)$. Let $H(1), \dots, H(r)$ be the maximal conjugacy classes (H) with $n(H) \neq 0$. We write

$$\mathrm{Im}(e) = \sum_{k=1}^{r} \mathrm{Im}(\mathrm{ind}_{H(k)}^{G}), \qquad \mathrm{Ke}(e) = \bigcap_{H \in S(e)} \mathrm{Ker}(\varphi_H).$$

Under these conditions

(5.3.7) Proposition. $\mathrm{Im}(e) + \mathrm{Ke}(e) \supset nA(G)$.

Proof. The sum on the left is an ideal in $A(G)$. Hence it suffices to show that $n \cdot 1_G$ is contained in this ideal. By construction $x = ne \in \mathrm{Im}(e)$, since $H(k) \in S(e)$. And for $y = ne - x \in A(G)$ and $H \in S(e)$ we have $\varphi_H(y) = n - ne(H) = 0$. □

If $S(e)$ is an open family, then $\mathrm{Ke}(e) = \bigcap_{H \in S(e)} \mathrm{Ker}(\mathrm{res}_H^G)$.

(5.3.8) Theorem. *Let $a(H)$ denote the product of the prime divisors of $H/[H,H]$. Then $ne(H,G) \in A(G)$ if and only if $a(H)|WH|$ divides n.*

Proof. Suppose $me(H,G) \in A(G)$ and $p|a(H)$. Then there exists $K \lhd H$, $|H/K| = p$ and $K \lhd N_pH$. We consider the congruence for the pair (K, N_pH) and apply it to $me(H,G)$. There is a single non-zero summand, and the congruence reads $m \equiv 0 \bmod |N_pH/K|$. If the prime divisor p of $|W_pH|$ does not divide $a(H)$, then the congruence for (H, N_pH) yields $m \equiv 0 \bmod |W_pH|$. Hence the divisibility condition is necessary. In order to show that $a(h)|WH|e(H,G) \in A(G)$, we use $\mathrm{ind}_H^G e(H,H) = |WH|e(H,G)$ and reduce to the case $H = G$. We consider the congruences for $x = a(G)e(G,G)$. The value $x(G)$ appears in a p-primary congruence if and only if $H \lhd G$ and $|G/H| = p$, and the congruence yields $x(G) \equiv 0 \bmod p$. □

Problems

1. Let $G = A_5$. Compute the function in $C(A_5)$ associated to the element $G/D_5 + G/D_3 + G/D_2 - 3G/C_2 + G/C_1$.
2. The finite group H is called **perfect**, if it equals its commutator subgroup. A finite group H has a smallest normal subgroup H_s such that its factor group is solvable. The relation $(H_s)_s = H_s$ holds, and H is perfect if and only if $H = H_s$. An idempotent function $e \in C(G)$ is contained in $A(G)$ if and only if for all $H \leq G$ equality $e(H) = e(H_s)$ holds.

The set of indecomposable idempotents corresponds via $(K) \mapsto e_K$ to the set of perfect conjugacy classes. In particular G is solvable if 0 and 1 are the only

idempotents.
3. Let N be an open family of subgroups of G. Let $p \in \mathbb{Z}$ be a prime and set

$$N^p = \{H \leq G \mid K \lhd H, K \in N, |H/K| \text{ a power of } p\}.$$

The localization at the prime ideal (p) is denoted by an index (p). Let $\mathrm{Ke}(N)$ denote the kernel of the restriction map $A(G)_{(p)} \to \prod_{H \in N} A(H)_{(p)}$ and $\mathrm{Im}(N^p)$ the image of the induction map $\bigoplus_{L \in N^p} A(L)_{(p)} \to A(G)_{(p)}$. Then $\mathrm{Ke}(N) + \mathrm{Im}(N^p) = A(G)_{(p)}$.

5.4 The Mark Homomorphism

We apply linear algebra to the mark homomorphism.

The group $C(G)$ has the standard basis $e_\bullet = (e_{(K)} \mid (K) \in \mathrm{Con}(G))$ consisting of the idempotent functions $e_{(K)} \colon (L) \mapsto \delta_{(K),(L)}$ (Kronecker-delta). Another basis $x_\bullet = (x_{(K)} \mid (K) \in \mathrm{Con}(G))$ is given by the functions obtained in 5.1.3. Let F denote the $\mathrm{Con}(G) \times \mathrm{Con}(G)$-matrix with entries $F(K,L) = |\mathrm{Hom}_G(G/K, G/L)| = |G/L^K|$. Recall the invertible matrix ζ^* with entries $\zeta^*(K,L) = |(K,L)^*| = |WL|^{-1}|G/L^K|$. The function $x_{(L)}$ equals $\zeta^*(-,L)$. The two bases are related by

$$x_{(L)} = \sum_{(K)} \zeta^*(K,L) e_{(K)}, \qquad \zeta_\bullet = e_\bullet \zeta^*. \tag{5.9}$$

Let $\lambda_K \colon C(G) \to \mathbb{Z}/|WK|\mathbb{Z}$, $x_{(L)} \mapsto \delta_{(K),(L)}$ and set $\lambda = (\lambda_K) \colon C(G) \to H(G)$. Then we have as a consequence of 5.1.3:

(5.4.1) Proposition. *The sequence* $0 \to A(G) \xrightarrow{\varphi} C(G) \xrightarrow{\lambda} H(G) \to 0$ *is exact.* □

From (5.9) we obtain by inversion $x_\bullet = \mu^* e_\bullet$, $e_{(L)} = \sum_{(K)} \mu^*(K,L) x_{(K)}$. This yields:

(5.4.2) Idempotent formula. *The idempotent* $e_{(L)} \in C(G)$ *is given as a rational linear combination of the homogeneous* G*-sets by the expansion* $e_{(L)} = \sum_{(K)} |WK|^{-1} \mu^*(K,L)[G/K]$. □

Let $z = \sum_{(K)} n_K x_{(K)} \in C(G)$ be an arbitrary function. By 5.4.1, this function is contained in $\varphi A(G)$ if and only if for all $(K) \leq (G)$ the integer n_K is divisible by $|WK|$. If we write $z = \sum_{(L)} z(L) e_{(L)}$ and insert the idempotent formula, we see that $n_K = \sum_{(L)} \mu^*(K,L) z(L)$. This yields:

(5.4.3) Möbius congruences. *A function* $z \in C(G)$ *is contained in the image of the mark homomorphism if and only if for all* K *the congruence* $\sum_{(L)} \mu^*(K,L) z(L) \equiv 0 \mod |WK|$ *holds.* □

We apply 5.2.3 to the basic Möbius congruence and obtain again a set of p-primary congruences.

Let $\gamma\colon K \to L$ be a surjective group homomorphism. We use γ to view an L-set X as a K-set, $k \cdot x = \gamma(k)x$. This induces a ring homomorphism $\gamma^*\colon A(L) \to A(K)$. On the level of Burnside marks this homomorphism is computed with $\gamma^*\colon C(L) \to C(K)$, $\gamma^*(\alpha)(A) = \alpha(\gamma(A))$.

Recall the **Frattini subgroup** $\Phi(G)$ of G, the intersection of the maximal subgroups of G, and **??**. Let $\gamma\colon G \to G/\Phi(G)$ be the canonical homomorphism. Write $N = \Phi(G)$.

(5.4.4) Proposition. $\gamma^* e(G/N, G/N) = e(G,G)$.

Proof. The relation $\varphi_K \gamma^*(x) = \varphi_{KN/N}(x)$ shows

$$\varphi_K \gamma^* e(G/N, G/N) \neq 0 \quad \Leftrightarrow \quad \varphi_{KN/N} e(G/N, G/N) \neq 0 \quad \Leftrightarrow \quad KN = G.$$

By **??**, the latter only holds for $K = G$. □

Suppose $x \in A(G)$ is expanded in terms of the basis $x = \sum_{(L)} m_L(x)[G/L]$. Then

$$\varphi_K(x) = \textstyle\sum_{(L)} m_L(x)|G/L^K| = \sum_{(L)} m_L(x)|WL|\zeta^*(K,L).$$

By inversion we obtain:

(5.4.5) Orbit formula. $|WK|m_K(x) = \sum_{(L)} \mu^*(K,L)\varphi_L(x)$. □

The multiplicative structure of $A(G)$ depends in a complicated way on the subgroup lattice. Suppose $[G/K][G/L] = \sum_{(A)} n_A^{K,L}[G/A]$. We take U-fixed points in this equation and get $F(U,K)F(U,L) = \sum_{(A)} F(U,A)n_A^{K,L}$. By inversion we obtain:

(5.4.6) Structure constants. $n_A^{K,L} = \sum_{(U)} F^{-1}(A,U)F(U,K)F(U,L)$. □

A **$\mathbb{Z}$-valued additive invariant** for finite G-sets assigns an integer $a(S) \in \mathbb{Z}$ to each finite G-set S such that $a(S \amalg T) = a(S) + a(T)$. By the universal property of $A(G)$, these additive invariants correspond to homomorphisms $A(G) \to \mathbb{Z}$. The group $C(G)$ can be considered from this viewpoint. We identify $C(G)$ with $\mathrm{Hom}(A(G), \mathbb{Z})$. Each finite G-set X yields the additive invariant

$$S \mapsto \sum_{Y \in S/G} |\operatorname{Hom}_G(Y,X)| = \varphi(X,S)$$

and $\varphi(G/H, S) = \varphi_H(S)$. The pairing $(X,S) \mapsto \varphi(S,X)$ is additive in S and X. It induces a bilinear map $A(G) \times A(G) \to \mathbb{Z}$, and its adjoint $A(G) \to C(G), X \mapsto \varphi(X,-)$ is the mark homomorphism.

Problems

1. $\mu^*(K,G) \neq 0$ implies $\Phi(G) \leq K$.
2. $e_(H) = |NH|^{-1} \sum_{K \leq H} |K|\mu(K,H)[G/K]$. The summation is over subgroups and not over conjugacy classes.

5.5 Prime Ideals

Let $(p) \subset \mathbb{Z}$ be a prime ideal. Then the pre-image $q(H,p) = \varphi_H^{-1}(p) \subset A(G)$ is a prime ideal; it only depends on the conjugacy class (H).

(5.5.1) Theorem. *Let $q \subset A(G)$ be a prime ideal. Then*

$$T(q) = \{(K) \mid [G/K] \notin q\}$$

contains a unique minimal element (H). This (H) defines q in the sense that $q = q(H,p)$, and p is the characteristic of $A(G)/q$.

Proof. Since $[G/G] = 1 \notin q$, the set $T(q)$ is not empty. Let (H) be minimal in $T(q)$. Then a relation of type

$$[G/H]x = \varphi_H(x)[G/H] + \sum_{(K)<(H)} a_K[G/K], \qquad a_K \in \mathbb{Z}$$

holds for each $x \in A(G)$. In order to see this, we note that $G/H \times X$ has only isotropy groups which are subconjugate to H. Therefore a relation of this type holds with a constant c, yet to be determined, in place of $\varphi_H(x)$. We apply φ_H to this relation and obtain

$$\varphi_H([G/H]x) = \varphi_H(x)\varphi_H(G/H) = |WH|\varphi_H(x) = |WH|c.$$

Hence $c = \varphi_H(x)$.

By minimality of (H), the $[G/K]$-summands are contained in q. Hence

$$[G/H] \cdot x \equiv \varphi_H(x)[G/H] \bmod q,$$

and since $[G/H] \notin q$, we can divide by $[G/H]$ and obtain $x \equiv \varphi_H(x) \cdot 1 \bmod q$. But this means: $x \in q$ if and only if $\varphi_H(x) \cdot 1 \in \mathbb{Z} \cdot 1 \cap q$, and the latter is the case if and only if $\varphi_H(x) \equiv 0$ modulo the characteristic of $A(G)/q$.

Suppose $[G/K] \in T(q)$ is minimal too. Then

$$0 \not\equiv \varphi_K(G/K) = \varphi_H(G/K) = |G/K^H|,$$

since $q = q(K,p) = q(H,p)$. In particular $G/K^H \neq \emptyset$, and therefore H is subconjugate to K. $\square$

(5.5.2) Remark. The inclusion $A(G) \subset C(G)$ is an integral ring extension, since the cokernel is a finite abelian group [, 5.1]. A general theorem in commutative algebra then says that each prime ideal of $A(G)$ is the intersection of $A(G)$ with a prime ideal of $C(G)$ [, 5.10]. This fact implies immediately that each prime ideal of $A(G)$ has the form $q(H, p)$. $\diamond$

The quotient of $A(G)$ by a prime ideal is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/p$ for a prime number p. Let $\mathrm{Spec}_0(G)$ and $\mathrm{Spec}_p(G)$ be the corresponding set of prime ideals. We associate to each ring homomorphism its kernel and obtain bijections (Hom = set of ring homomorphisms)

$$\mathrm{Spec}_0(G) \cong \mathrm{Hom}(A(G), \mathbb{Z}), \quad \mathrm{Spec}_p(G) \cong \mathrm{Hom}(A(G), \mathbb{Z}/p).$$

(5.5.3) Proposition. *The map* $(H) \mapsto q(H, 0)$ *is a bijection* $\mathrm{Con}(G) \to \mathrm{Spec}_0(G)$.

Proof. Surjectivity follows from 5.5.1. The equality $q(H, 0) = q(K, 0)$ implies $\varphi_H = \varphi_K$. From $|G/K^H| = \varphi_H(G/K) = \varphi_H(G/H) \neq 0$ we conclude that H is subconjugate to K; and conversely. $\square$

We set $\mathrm{Con}_p(G) = \{(H) \mid |WH| \not\equiv 0 \bmod p\}$.

(5.5.4) Theorem. *The map* $(H) \mapsto q(H, p)$ *is a bijection* $\mathrm{Con}_p(G) \to \mathrm{Spec}_p(G)$. *An inverse bijection associates to* q *the minimal element of* $T(q)$.

Proof. Suppose $A(G)/q$ has characteristic p. The minimal element (H) of $T(q)$ satisfies $q = q(H, p)$, and $[G/H] \notin q(H, p)$ implies $(H) \in \mathrm{Con}_p(G)$. This proves surjectivity. Suppose $q(H, p) = q(K, p)$, then, by 5.5.1, $\varphi_H \equiv \varphi_K \bmod p$. From $|G/H^K| \equiv |G/K^K| \not\equiv 0 \bmod p$ we see that K is subconjugate to H; and conversely. $\square$

Different subgroups can define the same prime ideal. It turns out that the only reason for this to happen is given by the next lemma.

(5.5.5) Lemma. *Let* $H \lhd K$ *such that* K/H *is a* p*-group. Then* $q(H, p) = q(K, p)$.

Proof. For each G-set X the K/H-set $X^H \setminus X^K$ consists of orbits of length p^t, $t \geq 1$. Hence $\varphi_H(X) = |X^H| \equiv |X^K| = \varphi_K(X) \bmod p$. $\square$

Let us denote the defining set by $D(q, p) = \{(H) \mid q = q(H, p)\}$. Let $O^p(H)$ be the smallest normal subgroup of H such that the quotient is a p-group. It is a characteristic subgroup. If $H \lhd K$ and K/H is a p-group we therefore have $O^p(H) = O^p(K)$. Let $|WH| \not\equiv 0 \bmod p$; then $H/O^p(H)$ is a p-Sylow group of $WO^p(H)$. A finite group H is called p-perfect if $O^p(H) = H$.

(5.5.6) Proposition. *Let* $q = q(H, p)$ *and* $(H) \in \mathrm{Con}_p(G)$. *Then* $D(q, p)$ *is the set* $E(H) = \{(K) \mid (O^p(H) \leq (K) \leq (H)\}$.

Proof. By 5.5.4, $E(H) \subset D(q,p)$. The sets $E(H)$ form a disjoint decomposition of $\mathrm{Con}(G)$. □

(5.5.7) Proposition. *Let* $\mathrm{Per}_p(G)$ *be the set of* p*-perfect conjugacy classes. Then* $\mathrm{Per}_p(G) \to \mathrm{Spec}_p(G)$, $(H) \mapsto q(H,p)$ *is a bijection.* □

(5.5.8) Proposition. *Let* G *be a* p*-group. Then* $A(G)$ *is a local ring. The unique maximal ideal is* $m = q(1,p)$. *If* $|G| = p^n$, *then* $m^{n+1} \subset pA(G)$. *Therefore the* m*-adic and the* p*-adic topology on* $A(G)$ *coincide.*

Proof. The first assertion follows from our determination of the prime ideals. The abelian group m^{n+1} is generated by products $x_1 \cdots x_{n+1}$, $x_j \in m$. For each such product $p^{-1}\varphi_H(x_1 \cdots x_{n+1}) \in p^n\mathbb{Z}$. By 5.3.4, $(H) \mapsto p^{-1}\varphi_H(x_1 \cdots x_{n+1})$ is an element of $A(G)$. □

5.6 Exterior and Symmetric Powers

There exist a number of constructions with G-sets which provide the Burnside ring with additional structure. We discuss in this section exterior and symmetric powers. They yield the structure of a λ-ring. This structure is then used to derive classical results of elementary group theory in the context of Burnside rings.

Let R be a commutative ring. The structure of a λ-ring on R consists of a sequence $(\lambda_n \colon R \to R \mid n \in \mathbb{N}_0)$ with the properties:

(1) $\lambda^0(x) = 1$

(2) $\lambda^1(x) = x$

(3) $\lambda^n(x+y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$.

Let X be a finite G-set. The n-th exterior power $\Lambda^n(X)$ is the set of subsets of X with n elements and induced G-action $g \cdot A = \{ga \mid a \in A\}$, for $g \in G$ and $A \subset X$. The set $\Lambda^0(X)$ is G-isomorphic to G/G and $\Lambda^1(X)$ is G-isomorphic to X. Moreover, there exists a canonical G-isomorphism

$$\Lambda^n(X \amalg Y) \cong \coprod_{i+j=n} \Lambda^i(X) \times \Lambda^j(Y).$$

It sends $A \in \Lambda^n(X \amalg Y)$ to the product of $A \cap X$ and $A \cap Y$.

The assignment $X \mapsto \Lambda^n(X)$ maps $A^+(G)$ into $A^+(G)$. But since Λ^n is not additive, we cannot use directly the universal property to extend it to the Burnside ring.

We denote by $\Lambda(R) = 1 + tR[[t]]$ the multiplicative group of formal power series $1 + a_1t + a_2t^2 + a_3t^3 + \cdots$ with coefficients $a_i \in R$. We set

$$\lambda_t(x) = \sum_{i \geq 0} \lambda^i(x)t^i \in \Lambda(R),$$

if (λ^n) is a λ-ring structure on R. Then $\lambda_t\colon R \to \Lambda(R)$ is a homomorphism of the additive group of R into the multiplicative group $\Lambda(R)$; this uses the properties (1) and (3).

In an analogous manner we obtain in

$$A^+(G) \to \Lambda(A(G)), \quad [X] \mapsto \sum_{i \geq 0} [\Lambda^i(X)] t^i$$

a homomorphism from the additive semi-group. By the universal property of $\iota\colon A^+(G) \to A(G)$, it extends to a homomorphism

$$\lambda_t\colon A(G) \to \Lambda(A(G)), \quad x \mapsto \sum_{i \geq 0} \lambda^i(x) t^i.$$

In this way we define the λ^i on $A(G)$, and they yield a λ-ring structure on $A(G)$.

(5.6.1) Proposition. *$\varphi_G(\Lambda^k(G/V)) = 1$ for $k = 0, |G/V|$ and $= 0$ for the remaining k.*

Proof. A subset of G/V is a G-fixed point if and only if it is empty or G/V. □

A ring homomorphism $\alpha\colon R \to S$ induces a group homomorphism $\Lambda(\alpha)\colon \Lambda(R) \to \Lambda(S)$: we apply α to the coefficients of a power series. The homomorphism $\varphi_G\colon A(G) \to \mathbb{Z}$ induces $\Lambda(\varphi_G)$, again denoted by φ_G. We can now write 5.6.1 in the form $\varphi_G \lambda_t(G/V) = 1 + t^{|G/V|}$. One can compute $\varphi_G \lambda_t(X)$ from the values on the orbits of X. We can then obtain $\varphi_H \lambda_t(X)$ by first considering X as an H-set.

(5.6.2) Proposition. $\varphi_H \lambda_t(G/1) = (1 + t^{|H|})^{|G/H|}$. □

We use this computation in the proof of 5.6.3.

(5.6.3) Proposition. *Let C be a cyclic group of order n. Then $(\Lambda^d(C/1) \mid d|n)$ is a $\mathbb{Z}$-basis of $A(C)$.*

Proof. Let C_e be the cyclic subgroup of order e of C. By 5.3.2 there exists a relation of the form

$$\Lambda^d(C/1) = \sum_{e|d} a(d,e)[C/C_e], \qquad a(d,e) \in \mathbb{Z}.$$

We have to show that the matrix $a = (a(d,e))$ is invertible. From **??** we infer that $\varphi_{C_e} \Lambda^d(C/1) \neq 0$ implies $e|d$. With respect to the partial order of the divisors of n by size, a is a triangular matrix. Moreover,

$$a(d,d)[C/C_d] = \varphi_{C_d} \Lambda^d(C/1)$$

is the coefficient of $t^{|C_d|}$ in $(1 + t^{|C_d|})^{|C/C_d|}$, and this implies $a(d,d) = 1$. □

The set map $a\colon \mathrm{Con}(G) \to \mathrm{Con}(C_{|G|})$, $(H) \mapsto (C_{|H|})$ induces a ring homomorphism $C(a)\colon C(C_{|G|}) \to C(G)$.

(5.6.4) Theorem. *The homomorphism $C(a)$ induces a homomorphism of subrings $\alpha\colon A(C_{|G|}) \to A(G)$.*

Proof. We use 5.6.2 in order to verify that $C(a)$ sends the function of $\Lambda^q(C/1)$ to the function of $\Lambda^q(G/1)$. Then we use 5.6.3. □

(5.6.5) Remark. Let d be a divisor of $|G|$. Let $C = C_{|G|}$ be a cyclic group of order $|G|$. We set $x_d = \alpha(C/C_{|G|/d})$. Then

$$\varphi_U(x_d) = \varphi_{C_{|U|}}(C/C_{|G|/d}) = \begin{cases} d & |U| \text{ divides } |G|/d \\ 0 & \text{otherwise.} \end{cases}$$

We write $x_d = \sum_{(U)} n(U)[G/U]$. Then the relation $n(U) \neq 0$ implies that $|U|$ divides $|G|/d$: If there would exist U with $n(U) \neq 0$ for other U, then also for each larger group V with $n(V) \neq 0$ the order $|V|$ would not divide $|G|/d$. For a maximal group V of this type we would then have $0 = \varphi_V(x_d) = n(V)|WV|$, a contradiction.

If there exists a group U with $d = |G/U|$, then U is a maximal with $n(U) \neq 0$, and therefore

$$n(U) \cdot |NU/U| = \varphi_U(x_d) = d = |G/U|;$$

hence $n(U) = |G/NU|$ is the number of subgroups which are conjugate to U.◇

(5.6.6) Theorem. *Let d divide $|G|$. Then d is the greatest common divisor of those $|G/U|$ which are multiples of d.*

Proof. By 5.6.5 we have

$$x_d = \sum_{(U),d|(G:U)} n(U)|G/U|.$$

We apply φ_1 and obtain

$$d = \sum_{(U),d|(G:U)} n(U)|G/U|.$$

(The $n(U)$ depend of course on d.) □

(5.6.7) Corollary. *Let $|G| = d \cdot p^t$, p a prime. Then there exist $U \leq G$ with $|U| = p^t$. In particular, there exists a Sylow p-group G_p, i.e. a subgroup G_p of p-power order such that $|G/G_p|$ is prime to p.* □

(5.6.8) Corollary. *If G_p is a Sylow p-group and $H \leq G$ a p-group, then H is subconjugate to G_p. In particular, all Sylow p-groups are conjugate.*

Proof. For a finite G-set X and a p-group $H \leq G$ the congruence $\varphi_H(X) \equiv \varphi_1(X) \bmod p$ holds, since orbits not in X^H have a length which is divisible by p. Since $|G/G_p| \not\equiv 0 \bmod p$, we have $|G/G_p^H| \neq 0$, and this says that H is subconjugate to G_p. □

(5.6.9) Corollary. *Let p^t divide $|G|$. Then the number of subgroups of order p^t is congruent 1 modulo p.*

Proof. Let $d = |G|/p^t$. Then modulo p

$$\begin{aligned} 1 &= \sum_{(U),|U|\,|\,p^t} n(U)d^{-1}|G/U| = \sum_{(U),|U|\,|\,p^t} n(U)|U|^{-1}p^t \\ &\equiv \sum_{(U),|U|=p^t} n(U) \\ &= \sum_{(U),|U|=p^t} |G/NU| = |\{V \leq G \mid |V| = p^t\}|. \end{aligned}$$

The first equality is 5.6.6 and the second a rewriting with $dp^t = |G|$. In the fourth one we use 5.6.5, and in the final one we change summation over conjugacy classes into summation oder subgroups. □

5.6.7 - 5.6.9 are classical results of Sylow and Frobenius. We derive further results of this type.

(5.6.10) Theorem. *Let m divide the order of the group and $E = E_m = \{g \in G \mid g^m = 1\}$. Then $|E| \equiv 0 \bmod m$.*

Proof. Let $|G| = dm$. Then

$$\sum_{g\in G} \varphi_{\langle g \rangle}(x_d) = \sum_{g,d|(G:\langle g \rangle)} d = \sum_{g,|\langle g \rangle|\,|\,m} d = d|E|.$$

By 5.4.2 this number is divisible by $|G| = dm$, hence $|E|$ is divisible by m. □

The Burnside ring $A(G)$ carries a second structure of a λ-ring based on symmetric powers of G-sets.

Let X be a finite G-set and X^r its r-fold cartesian power. The symmetric group S_r acts on X^r by permutation of factors, and this action commutes with the G-action. We obtain therefore an induced G-action on the orbit space $S^r(X) = X^r/S_r$. We call this G-set the r-th symmetric power of X. There exists a canonical isomorphism of G-sets

$$S^r(X \amalg Y) \cong \coprod_{a+b=r} S^a(X) \times S^b(Y).$$

We can therefore proceed as we did with the exterior powers and obtain a homomorphism

$$s_t \colon A(G) \to \Lambda(A(G)), \quad x \mapsto \sum_{i\geq 0} s^i(x)t^i,$$

such that $s^i([X]) = [S^i(X)]$ holds for each finite G-set X. One computes:

(5.6.11) Proposition. $\varphi_G s_t(G/V) = (1 - t^{|G/V|})^{-1}$. □

5.7 Burnside Ring and Euler Characteristic

Later we investigate in detail the relation of the Burnside ring to topological problems. But already at this early stage it is convenient to have a topological definition of $A(G)$ at our disposal.

We work with the category of finite G-CW-complexes X. The fixed point sets X^H are then finite CW-complexes and the Euler characteristic $\chi(X^H) \in \mathbb{Z}$ is defined. We call two such G-complexes X and Y Euler equivalent, if for all $H \leq G$ the equality $\chi(X^H) = \chi(Y^H)$ holds. Let, for the moment, $A'(G)$ denote the set of equivalence classes. This set carries the structure of a commutative ring; addition is induced by disjoint union and multiplication by cartesian product. Basis properties of the Euler characteristic show this to be well defined. It is no longer necessary to apply a Grothendieck construction, since additive inverses are already present: Let K denote a finite complex with trivial G-action and $\chi(K) = -1$; then $[X \times K]$ is the inverse of $[X]$.

We have ring homomorphisms $\varphi_H \colon A'(G) \to \mathbb{Z}$, $[X] \mapsto \chi(X^H)$ and an embedding $\varphi' \colon A'(G) \to C(G)$, in analogy to the case of finite G-sets. Since a finite G-set is a finite G-complex, we have a ring homomorphism $\iota \colon A(G) \to A'(G)$, the identity on representatives. In order to show that ι is an isomorphism we verify the standard congruences for the image of the φ'. For this purpose we use the equivariant Euler characteristic of a G-complex X. Let $R(G)$ denote the complex representation ring of G. The G-action on X makes the homology group $H_i(X;\mathbb{C})$ into a complex G-representation. The alternating sum

$$\chi(G) = \sum_{i \geq 0} (-1)^i [H_i(X;\mathbb{C})] \in R(G)$$

is the equivariant Euler characteristic. Basic properties of homology groups show that

$$\chi_G \colon A'(G) \to R(G)$$

is a ring homomorphism.

The character value $\chi_G(X)(g)$ at $g \in G)$ is the alternating sum of the traces of the maps

$$H_i(l_g) \colon H_i(X;\mathbb{C}) \to H_i(X;\mathbb{C}).$$

This alternating sum is called the Lefschetz index $L(l_g)$ of the left translation l_g. In this section we assume:

(5.7.1) Theorem. $L(l_g) = \chi(X^g)$. □

For each character χ, the relation $\sum_{g \in G} \chi(g) \equiv 0 \bmod |G|$ holds, since $|G|^{-1} \sum_{g \in G} \chi(g)$ is the dimension of the fixed point set (projection operator). If we use this fact for the equivariant Euler characteristic and use 5.7.1 we see that the standard congruences 5.4.2 hold for the image of φ'. As a corollary we obtain that ι is an isomorphism. Henceforth we write $A'(G) = A(G)$. The topological definition gives us more flexibility in the construction of elements of $A(G)$. Also, since no Grothendieck construction is needed, certain constructions with $A(G)$ become easier.

5.8 Units and Representations

We denote by A^* the group of units of a ring A. The units of $C(G)$ are the functions with values in $\{\pm 1\}$.

Suppose $e \in A$ is idempotent. Then $u = 1 - 2e \in A^*$. Conversely, suppose $u \in A^*$ and $1 - u$ is in A divisible by 2, then $e = (1 - u)/2$ is idempotent. If G has odd order, then the cokernel of $A(G) \subset C(G)$ has odd order. If $u \in A(G)^*$, then $1 - u$ assumes the values $0, 2$ and is therefore in $C(G)$ divisible by 2. Thus, if G has odd order, $(1 - u)/2$ is contained in $A(G)$.

(5.8.1) Proposition. *Suppose G is not solvable. Then $A(G)^* \neq \{\pm 1\}$. If G is solvable of odd order, then $A(G)^* = \{\pm 1\}$.*

Proof. If G is not solvable, then there exist idempotents $e \neq 0, 1$ by **??**. The unit $u = 1 - 2e$ is then different from ± 1.

If G has odd order and $u \in A(G)^*$, then we have the idempotent $e = (1 - u)/2$. If G is solvable, then, by (??), $e = \pm 1$, and hence $u = \mp 1$. □

Let $H < G$ have index 2 in G. Then $H \lhd G$, $[G/H]^2 = 2[G/H]$, and we conclude $u(H) = 1 - [G/H] \in A(G)^*$. The element $\frac{1}{2}(1 - u(H)) = \frac{1}{2}[G/H]$ is not contained in $A(G)$.

We construct non trivial units with the help of representation theory. We set $\varepsilon(n) = (-1)^n$, $n \in \mathbb{Z}$.

(5.8.2) Proposition. *Let V be a real representation of G. Then the function $\eta(V) \colon (H) \mapsto \varepsilon(\dim_{\mathbb{R}} V^H)$ is contained in $A(G)^*$.*

Proof. We view the unit sphere $S(V)$ of V as a finite G-complex. Then $\chi(S(V)^H) = 1 - \varepsilon(\dim V^H)$. Hence $1 - [S(V)] \in A(G)$ has the function $\eta(V)$. □

In order to give an algebraic proof, we translate 5.8.2 into an essentially equivalent form. Let $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. Then we can use the complex dimension function $\dim_{\mathbb{C}} V \colon (H) \mapsto \dim_{\mathbb{C}} V_{\mathbb{C}}^H = \dim_{\mathbb{R}} V^H$. A representation of the form $V_{\mathbb{C}}$ has a real character.

(5.8.3) Proposition. *Let V be a complex representation with real character. Then $\eta(V)\colon (H) \mapsto \varepsilon(\dim_{\mathbb{C}} V^H)$ is contained in $A(G)^*$.*

Proof. If the complex representation V has character χ_V, then the complex conjugate representation has the complex conjugate function as character. Therefore a representation has real character if and only if it is isomorphic to its complex conjugate (is self-conjugate). If we decompose such a representation into irreducible ones, then the irreducible summands W which are not isomorphic to its conjugate $\overline{W}$ appear in pairs $W \oplus \overline{W}$. We can discard these summands when we study $\eta(V)$.

If V is a G-representation with real character, then also V^H, considered as WH-representation, has real character. By the congruences of section 4 it therefore suffices to show that for a V with real character the congruence $\sum_{g \in G} \varepsilon(\dim_{\mathbb{C}} V^{\langle g \rangle}) \equiv 0 \bmod |G|$ holds. But this is implied by the next proposition. □

(5.8.4) Proposition. *Suppose V has real character. Then the function $g \mapsto \varepsilon(\dim_{\mathbb{C}} V - \dim_{\mathbb{C}} V^{\langle g \rangle})$ is a character, i.e. a homomorphism $G \to \mathbb{Z}^*$. This character is the determinant representation of V.*

Proof. Let $\zeta_1, \dots, \zeta_n$ denote the eigenvalues of $g \in G$ on V. Then V decomposes as $\langle g \rangle$-representation into $V_1 \oplus \dots \oplus V_n$, and g acts on $V_j = \mathbb{C}$ as multiplication with ζ_j. On the determinant representation $\Lambda^n V \cong \mathbb{C}$ the element g acts as multiplication with $\prod_{j=1}^n \zeta_j$. Since V has real character, the product contains with ζ_j also $\overline{\zeta}_j$. There remains the product $(-1)^k$, where k is the number of eigenvalues -1. The non trivial eigenvalues ($\neq 1$) are those of $V/V^{\langle g \rangle}$. By reasons of parity therefore $k \equiv \dim_{\mathbb{C}} V/V^{\langle g \rangle} \bmod 2$. □

Since $\eta(V \oplus W) = \eta(V)\eta(W)$, we have a homomorphism $\eta\colon RO(G) \to A(G)^*$ from the additive into the multiplicative group. Let $r\colon RU(G) \to RO(G)$ map a complex representation to the underlying real representation. The image is contained in the kernel of η. The induced map $\eta\colon RO(G)/rRU(G) \to A(G)^*$ is in general neither injective nor surjective. The group $RO(G)/rRU(G)$ has exponent 2.

(5.8.5) Proposition. *Let $x \in A(G)^*$ be a positive unit, i.e. $\varphi_{\langle 1 \rangle}(x) > 0$. Then*

$$\eta_x : G \to \mathbb{Z}^*, \quad g \mapsto \varphi_{\langle g \rangle}(x)$$

is a homomorphism.

Proof. $\pi(x) \in R(G;\mathbb{Q})^*$ is a unit of finite order and η_x its character. Now we apply **??**. □

We mention a further result. The proof uses topological methods and will be given later.

(5.8.6) Proposition. *Let V be a complex G-representation and B a regular G-invariant symmetric bilinear form on V. Let $f\colon V \to V$ be a B-orthogonal G-map. Then $(H) \mapsto \det(f^H)$ is contained in $A(G)^*$.* □

5.9 Generalized Burnside Groups

Let G be a topological group. A G-set S is a set S with discrete topology and a continuous left action of G on S. A G-set is called **locally finite** if its orbits are finite sets and if for each subgroup $U \le G$ of finite index in G the cardinality $\varphi_U(S) = |S^U|$ is finite.

Let S be locally finite. Let $m_U(S)$ denote the number of orbits of type G/U in S. Then

$$\varphi_U(S) = \sum_{(V)} m_V(S)\varphi_U(G/V).$$

The sum is taken over conjugacy classes (V) with $(U) \le (V)$. The sum is finite. Hence $m_V(S)$ is finite. Conversely, if S has finite orbits and the numbers $m_V(S)$ are all finite, then S is locally finite.

The disjoint union $S \amalg T$ of (locally) finite G-sets S and T is (locally) finite. The cartesian product $S \times T$ of (locally) finite G-sets S and T is (locally) finite; the orbits of $S \times T$ are isomorphic to orbits of $G/U \times G/V$ for $U \in \mathrm{Iso}(S)$, $V \in \mathrm{Iso}(T)$ and hence finite; moreover $(S \times T)^U = S^U \times T^U$.

Let F be a family of subgroups of finite index in G. Let $A_+(G;\mathcal{F})$ and $A_+^\wedge(G;\mathcal{F})$ denote the set of isomorphism classes of finite and locally finite G-sets with isotropy groups in $\mathcal{F}$, respectively. Disjoint union induces a commutative composition law (addition) in these sets. We let $A(G;\mathcal{F})$ and $A^\wedge(G;\mathcal{F})$ be the associated Grothendieck groups, called **Burnside groups**. If F is multiplicative, then cartesian product of G-sets induces a multiplication, and $A(G;\mathcal{F})$ and $A^\wedge(G;\mathcal{F})$ become commutative $\mathbb{Z}$-algebras, called **Burnside rings**. If $G \in \mathcal{F}$, then G/G represents a unit in this ring. We write $A(G)$, $A^\wedge(G)$, when F is the family of all subgroups of finite index in G. In the case when $\mathcal{F}_1 \circ \mathcal{F}_2 \subset \mathcal{F}_3$, cartesian product induces a bilinear pairing $A(G;\mathcal{F}_1) \times A(G;\mathcal{F}_2) \to A(G;\mathcal{F}_3)$. Thus, if $\mathcal{F}_1 \circ \mathcal{F}_1 \subset \mathcal{F}_1$ and $\mathcal{F}_1 \circ \mathcal{F}_2 \subset \mathcal{F}_2$, then $A(G;\mathcal{F}_2)$ becomes a module over the Burnside algebra $A(G;\mathcal{F}_1)$. Similarly for $A^\wedge$.

We let $[S]$ or simply S denote the image of the G-set S in the Burnside group. The assignment $S \mapsto \varphi_U(S)$ induces an additive map $\varphi_U\colon A^\wedge(G;\mathcal{F}) \to \mathbb{Z}$ which is a ring homomorphism when F is multiplicative. A locally finite G-set is the disjoint union of its G-orbits. The elements in $A^\wedge(A;\mathcal{F})$ are formal linear combinations

$$x = \sum m_V[G/V], \quad m_V \in \mathbb{Z},\ (V) \in (\mathcal{F}).$$

In the case of $A(G;\mathcal{F})$ we use finite sums, and then $\{[G/V] \mid (V) \in (\mathcal{F})\}$ is an additive $\mathbb{Z}$-basis for $A(G;\mathcal{F})$.

We have the function

$$x_{(V)}\colon (\mathcal{F}) \to \mathbb{Z}, \quad (U) \mapsto \frac{1}{|WV|}\varphi_U(G/V).$$

We let $C^\wedge(G;\mathcal{F})$ denote the ring of functions $(\mathcal{F}) \to \mathbb{Z}$, which are formal linear combinations

$$\sum n_V x_{(V)}, \qquad n_V \in \mathbb{Z},\ (V) \in (\mathcal{F}).$$

Note that $x_{(V)}(U) \neq 0$ implies $(U) \leq (V)$, so that for each U only a finite sum matters.

(5.9.1) Lemma. *$C^\wedge(G;\mathcal{F})$ is the group of all functions $(\mathcal{F}) \to \mathbb{Z}$.*

Proof. Let $f\colon (\mathcal{F}) \to \mathbb{Z}$ be given. We try to write f as a sum $f = \sum_{n\geq 1} f_n$, where $f_n \in C(G;\mathcal{F})$ is a linear combination of $x_{(V)}$ with $|G/V| = n$. The f_n are constructed inductively such that $f - \sum_{n=1}^{k} f_n$ assumes the value zero on (U) whenever $|G/U| \leq k$. Suppose k is minimal such that there exists U with $|G/U| = k$ and $f(U) \neq 0$. Then $f_k = \sum f(U)x_{(U)}$, where the sum is taken over $(U) \in (\mathcal{F})$ with $|G/U| = k$. This is the induction step. □

As in the case of a finite group, we have the injective mark homomorphism $\varphi\colon A^\wedge(G;\mathcal{F}) \to C^\wedge(G;\mathcal{F}), x \mapsto ((U) \mapsto \varphi_U x)$. If F is multiplicative, then φ is a ring homomorphism. We have a relation of the type $x_{(H)} \cdot x_{(K)} = \sum n_L x_{(L)}$. The L which occur in this sum lie in $\mathrm{Iso}(G/H \times G/K)$. Hence the sum is finite. Since $\varphi([G/H]) = |WH|x_{(U)}$ and $[G/H][G/K] = \sum m_L[G/L]$, where m_L is the number of orbits of type L in $G/H\times G/K$, we obtain $|WH|\cdot|WK|\cdot|WL|^{-1}m_L = n_L$.

The image of the mark homomorphism φ can be characterized by congruence relations among the values of the functions in $C^\wedge(G;\mathcal{F})$. A congruence relation $C(H,m)$ of type (H,m) for the image of φ is a relation of the form

$$x(H) + \sum_{(K)} n(H,K)x(K) \equiv 0 \bmod (m), \tag{5.10}$$

$n(H,K) \in \mathbb{Z}$, $(H) < (K)$, $(H),(K) \in (\mathcal{F})$, which holds for all functions $x \in \mathrm{Im}\,\varphi$.

(5.9.2) Proposition. *Let $(C(H_j,m_j) \mid j \in J)$, $H_j \in \mathcal{F}$ be a family of congruences for $\mathrm{Im}(\varphi)$ with the following property* S

(S) *For each $H \in \mathcal{F}$ the smallest common multiple of $\{m_j \mid (H_j) = (H)\}$ is $|WH|$.*

Then $x \in \mathrm{Im}(\varphi)$ if and only if (5.10) holds for all $(H,m) = (H_j,m_j)$.

Proof. Let $x = \sum n_V x_{(V)} \in C^\wedge(G;\mathcal{F})$ satisfy the congruences. We have to show that $n_V \equiv 0 \bmod |WH|$. Suppose H is maximal with $n_V \not\equiv 0 \bmod |WH|$. Then the partial sum $y = \sum_{(V)<(K)} n_K x_{(K)} \in \mathrm{Im}(\varphi)$ and $z = x - y$ has the property $z(K) = 0$ for $(H) < (K)$. Now **??**, applied to the (H_j,m_j) with $H_j = H$, implies $z(H) \equiv 0 \bmod |WH|$. But $x(H) = z(H)$ by construction. □

We call the family $((H_j, m_j) \mid j \in J)$ **sufficient**, if it has the property (S) of 5.9.2. We are going to find families of congruences which are sufficient.

Suppose $U \lhd V \le G$ and $|G/U|$ is finite. Let S be a locally finite G-set. We apply **??** to the $H = V/U$-set $M = S^U$. We obtain a relation

$$|H||M/H| = \sum_{vU \in V/U} \varphi_{\langle vU \rangle}(S) \equiv 0 \bmod |V/U|. \tag{5.11}$$

If F is closed, then

$$\sum_{vU \in V/U} x(\langle vU \rangle) \equiv 0 \bmod |V/U| \tag{5.12}$$

is a relation of the type (5.10). We call (**??**) a **standard congruence** $C(U, V)$ for (U, V).

(5.9.3) Proposition. *Let F be a closed family. The set of standard congruences $(C(U_j, V_j) \mid j \in J)$ is sufficient if for each $H \in \mathcal{F}$ the least common multiple of $\{|V_j/U_j| \mid U_j = H\}$ is $|WH|$. Examples for sufficient sets are:*
(1) $(C(H, NH) \mid (H) \in (\mathcal{F}))$.
(2) $(C(H, N_pH) \mid (H) \in (\mathcal{F}))$, *$N_pH/H$ a p-Sylow subgroup of WH, p any prime.* □

If we work with finite G-sets, we need only consider the subring $C(G, \mathcal{F})$ of $C^\wedge(G, \mathcal{F})$ which consists of finite linear combinations $\sum n_V x_{(V)}$. Again, the image of $\varphi\colon A(G, \mathcal{F}) \to C(G, \mathcal{F})$ can be characterized by the same set of congruences.

Problems

1. In order to have a result like 5.9.3 it is not really necessary to assume that F is closed. It is only required that the pairs of subgroups which appear in the congruences are present in $\mathcal{F}$. Thus the standard congruences require:

$$H \in \mathcal{F}, H \lhd K, K/H \text{ cyclic } \Rightarrow K \in \mathcal{F}.$$

And the combinatorial congruences from Möbius inversion require:

$$H \in \mathcal{F}, H \lhd K, K/H \text{ elementary abelian } \Rightarrow K \in \mathcal{F}.$$

In the first case one can even restrict to cyclic p-groups K/H.

2. The Burnside ring $A^\wedge(\mathbb{Z})$ of the additive group $\mathbb{Z}$ consists of all formal sums $\sum_{n=1}^{\infty} a_n[\mathbb{Z}/n]$, $a_n \in \mathbb{Z}$. The multiplication table of the basic elements is $[\mathbb{Z}/m][\mathbb{Z}/n] = (n, m)[\mathbb{Z}/[n, m]]$, where (n, m) denotes the greatest common divisor and $[n, m]$ the least common multiple. The embedding $\varphi\colon A^\wedge(\mathbb{Z}) \to C^\wedge(\mathbb{Z})$ into the ring of all functions $\{n\mathbb{Z} \mid n \in \mathbb{N}\} \to \mathbb{Z}$ is the set of those functions x which satisfy

$$\sum_{j|n} \mu\,(n/j)\, x(j\mathbb{Z}) \equiv 0 \bmod n.$$

Here μ is the ordinary Möbius function of elementary number theory. If one restricts to primary congruences, then this subring is characterized by the set of congruences

$$x(n\mathbb{Z}) \equiv x(n/p\mathbb{Z}) \bmod p^{\nu(p,n)},$$

where $p^{\nu(p,n)}$ is the p-power dividing n. The standard congruences are

$$\sum_{j|n} \varphi\,(n/j)\, x(j\mathbb{Z}) \equiv 0 \bmod n,$$

where φ is the Euler function of elementary number theory.

3. Let $\mathbb{Z}_p$ denote the additive group of p-adic integers (p prime), considered as a compact (profinite) group. Then $A^\wedge(\mathbb{Z}_p)$ consists additively of the formal sums $\sum_{d=1}^{\infty} a_d[\mathbb{Z}/p^d]$ and the congruences are $x(p^d\mathbb{Z}) \equiv x(p^{d-1}\mathbb{Z}) \bmod p^d$.

4. In this section it would have been sufficient to consider profinite groups. If $H_1, \dots, H_r$ are closed subgroups of finite index in G, then $H_1 \cap \dots \cap H_r$ is the isotropy group of $(eH_1, \dots, eH_r)$ in $G/H_1 \times \dots \times G/H_r$ and therefore a closed subgroup of finite index. If K is a closed subgroup of finite index, then the subgroups gKg^{-1} are finite in number and $U = \bigcap_{g \in G} gKg^{-1}$ is a closed normal subgroup of finite index. The subgroup U acts trivially on G/K. Therefore G/K can be considered as a G/U-space. Let $Nor(G)$ denote the set of closed normal subgroups of G of finite index. The **profinite completion** $\hat{G}$ of G is the inverse limit of the G/U. We make this precise. For $U \leq V$ let $p_U^V \colon G/U \to G/V$ denote the canonical quotient map. The group $\hat{G}$ is defined to be the kernel of

$$\kappa\colon \prod_{U \in Nor(G)} G/U \to \prod_{U \leq V; U,V \in Nor(G)} G/V$$

which sends (x_U) to $(x_V - p_U^V x_U)$. By the theorem of Tychonoff, κ is a continuous homomorphism between compact Hausdorff groups and therefore $\hat{G}$ is a compact subgroup of $\prod G/U$. Locally finite G-spaces are essentially the same as locally finite $\hat{G}$-spaces.

Chapter 6

Groups of Prime Power Order

6.1 Permutation Representations

Let G be a finite group and $\mathcal{K}$ a field with characteristic prime to $|G|$. We denote by $R(G;\mathcal{K})$ the Grothendieck ring of finite-dimensional $\mathcal{K}G$-modules with respect to $\oplus$ and $\otimes$ (representation ring). Additively $R(G;\mathcal{K})$ is the free abelian group on isomorphism classes of simple $\mathcal{K}G$-modules (= irreducible representations). We also write $R(G) = R(G;\mathbb{C}), RO(G) = R(G;\mathbb{R})$.

If S is a finite G-set, we have the permutation representation $\mathcal{K}S$ of S. The assignment $[S] \mapsto [\mathcal{K}S]$ yields a ring homomorphism $\pi_G = \pi(G)\colon A(G) \to R(G;\mathcal{K})$. In this chapter we mainly work with $\mathcal{K} = \mathbb{Q}$. The relation to characters is as follows, see 2.1.5:

(6.1.1) Proposition. *Let $\chi_g\colon R(G;\mathbb{Q}) \to \mathbb{Z}$ be the evaluation of characters at $g \in G$. Then $\chi_g \circ \pi(G) = \varphi_{\langle g \rangle}$.* □

We consider the homomorphisms $\pi(G)$ as data of a natural transformation of functors on a category $\mathcal{S}$. The objects of $\mathcal{S}$ are the finite groups. The morphisms from K to L are isomorphism classes of diagrams $(p|i)\colon K \xleftarrow{p} X \xrightarrow{i} L$ with surjective homomorphisms p and injective homomorphisms i. The diagram $(p|i)$ is isomorphic to a diagram $(p'|i')\colon K \leftarrow X' \to L$ if there exists an isomorphism $\sigma\colon X \to X'$ with $p' \circ \sigma = p$ and $i' \circ \sigma = i$. The composition of morphisms is defined by a pullback construction as in **??**.

We make $A(-)$ into a contravariant functor on $\mathcal{S}$. Given $\alpha = (p|i)$ as above (with i an inclusion of a subgroup, for simplicity), we define $A(\alpha)$ as the composition of $p^*\colon A(K) \to A(X)$ with $\mathrm{ind}_X^L\colon A(X) \to A(L)$. One checks that this is well defined on isomorphism classes and functorial. In order to show functoriality, one has to verify $(q')^* \circ \mathrm{ind}_X^L = \mathrm{ind}_C^Y \circ q^*$ There is a similar construction for $R(-;\mathbb{Q})$, and one verifies that the $\pi(G)$ now constitute a natural transformation. If $H \lhd K \leq G$, we have the quotient map $p\colon K \to K/H$ and the inclusion $i\colon K \subset G$. We denote the resulting morphism $A(p|i)$ as **induction from a subquotient**

$$\mathrm{ind}_{K/H}^G\colon A(K/H) \to A(G),$$

and similarly for the representation ring.

6.2 Basic Examples

We need information about permutation representations for some elementary groups. We begin with general remarks about representations. Let V be a $\mathbb{C}G$-module. For each field automorphism γ of $\mathbb{C}$ there exists (up to isomorphism) a unique representation V^γ with character $\chi_{V^\gamma}(g) = \gamma\chi_V(g)$. We call V^γ a **Galois-conjugate** of V.

The characters of $\mathbb{C}G$-modules have values in the cyclotomic field $L = \mathbb{Q}(\zeta_m)$, $\zeta_m = \exp(2\pi i/m)$, $m = |G|$. The representation V^γ depends only on the restriction $\gamma|L$. We identify the Galois-group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)|\mathbb{Q})$ with the group of units $(\mathbb{Z}/m)^*$. The Galois-automorphism $\zeta \mapsto \zeta^k$ corresponds to $k \in (\mathbb{Z}/m)^*$, and its effect on characters is given by $\chi_{W^\gamma}(g) = \chi_W(g^k)$. The corresponding action on the representation ring $R(G)$ is the ring homomorphism Ψ^k (Adams operation).

If V is realized over L, i.e. $V = \mathbb{C} \otimes_L W$ for an LG-module W, we can apply $\gamma \in \mathrm{Gal}(L|\mathbb{Q})$ to the entries of a matrix representation of W over L and obtain W^γ. In this case $V^\gamma = \mathbb{C} \otimes_L W^\gamma$. Each irreducible $\mathbb{C}G$-module can be realized over $L = \mathbb{Q}(\zeta_m)$, $m = |G|$, see 4.6.6.

The relation between complex and rational representations is described by the next theorem. See [**?**, V§14].

(6.2.1) Theorem. *Let $W_1, \dots, W_r$ be the different Galois-conjugates of an irreducible complex representation. Then there exists a natural number n such that $n(W_1 \oplus \cdots \oplus W_r)$ is the complexification of an irreducible rational representation. Each irreducible rational representation arises in this way from a unique Galois-class of irreducible complex representations. The numbers n is called the (rational) Schur index of the irreducible rational representation.* □

We also recall a similar result for real representations. In this case only the complex conjugation automorphism matters. If V is isomorphic to the complex conjugate $\overline{V}$, then V is called self-conjugate. Proofs and more details for the following results can be found in [**?**, II.6].

(6.2.2) Theorem. *If V is a self-conjugate irreducible $\mathbb{C}G$-module, then either V is the complexification of an irreducible $\mathbb{R}G$-module (V of real type) or $2V$ is the complexification of an irreducible $\mathbb{R}G$-module (V of quaternionic type). If $V \not\cong \overline{V}$, then there exists (up to isomorphism) a unique irreducible $\mathbb{R}G$-module U such that $V \oplus \overline{V}$ is the complexification of U. We say, the real Schur index of V is 2 if V is of quaternionic type and 1 otherwise.* $\square$

We don't need these general results for the following examples. But we use the terminology of Galois-conjugates and the Schur index. Also it is clear that if we decompose the complexification of an irreducible rational representation into irreducibles, then this decomposition must be Galois invariant.

The following result from group theory is the reason for studying the groups of 6.2.4 separately. See [**?**, I.14.9, III.7.6].

(6.2.3) Theorem. *Suppose each abelian normal subgroup of the p-group G is cyclic. Then G is a group in the following list.*

(1) *G is cyclic.*
(2) *G is the dihedral group $D(2^n)$ of order 2^n, $n \geq 4$. It has the presentation $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{-1} \rangle$.*
(3) *G is the semi-dihedral group $SD(2^n)$ of order 2^n, $n \geq 4$. It has the presentation $\langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{2^{n-2}-1} \rangle$.*
(4) *G is the quaternion group $Q(2^n)$ of order 2^n, $n \geq 3$. It has the presentation $\langle A, B \mid A^{2^{n-1}} = 1, B^2 = A^{2^{n-2}}, BAB^{-1} = A^{-1} \rangle$.* $\square$

(6.2.4) Theorem. *Each of the groups G in* **??** *has a unique faithful irreducible $\mathbb{Q}G$-module. It arises by induction $\mathrm{ind}_{K/H}^{G}$ from the following data:*

(1) *In case (1) $H = 1$ and K of order p.*
(2) *In cases (2) and (3) $H = \langle B \rangle$ and $K = \langle A^{2^{n-2}}, B \rangle$.*
(3) *In case (4) $H = 1$ and $K = \langle A^{2^{n-2}} \rangle$.*

The Schur index is 2 in case (4) and 1 otherwise. $\square$

(6.2.5) Cyclic groups. We begin with a discussion of cyclic groups. In general, a simple $\mathbb{Q}G$-module is isomorphic to a submodule of $\mathbb{Q}G$ (left ideal). Suppose $G = \mathbb{Z}/m$. Then $\mathbb{Q}G \cong \mathbb{Q}[x]/(x^m - 1)$; here x corresponds to a generator of G. We decompose $x^m - 1$ over $\mathbb{Q}$ into irreducible factors $x^m - 1 = \prod_{d|m} \Phi_d(x)$. The cyclotomic polynomial $\Phi_d(x)$ has the primitive d-th roots of unity as roots. By the chinese remainder theorem

$$\mathbb{Q}(G) \cong \mathbb{Q}[x]/(x^m - 1) \cong \bigoplus\nolimits_{d|m} \mathbb{Q}[x]/(\Phi_d(x)).$$

Since $\Phi_d(x)$ is irreducible, $V_d = \mathbb{Q}[x]/(\Phi_d(x))$ is a simple $\mathbb{Q}G$-module. This gives us all simple $\mathbb{Q}G$-modules. There is a unique faithful one V_m. It decomposes over $\mathbb{C}$ into the Galois-conjugates of a simple faithful one-dimension $\mathbb{C}G$-module. The Schur indices are therefore 1.

We show that $\pi(G)\colon A(G) \to R(G;\mathbb{Q})$ is an isomorphism for cyclic groups $G = \mathbb{Z}/m$. Recall from representation theory that the rank of $R(G;\mathbb{Q})$ is the number of conjugacy classes of cyclic subgroups of G. Therefore $\pi(G)$ is a homomorphism between free abelian groups of the same rank (the number of divisors of m). Thus it suffices to show that $\pi(G)$ is surjective. We can assume by induction that the V_d for $d < m$ are in the image of $\pi(G)$ (lift of non-faithful modules). The regular representation $\mathbb{Q}G$ is the image of $[G/1]$ under $\pi(G)$. Now we use $\mathbb{Q}(G) = \sum_{d|m} V_d$. Hence $\pi(G)$ is surjective.

If $m = p^n$ and if K is the subgroup of order p, then $V_m = \mathrm{ind}_K^G V_p$. In general, by Möbius-inversion, we obtain $V_m = \sum_{d|m} \mu(d)\mathbb{Q}(C_m/C_d)$ if we denote by C_d the subgroup of order d in $G = C_m$. ◇

(6.2.6) Dihedral groups. We consider the group G of order $2m$, $m \geq 3$,

$$D(2m) = \langle\, A, B \mid A^m = 1 = B^2, BAB^{-1} = A^{-1} \,\rangle.$$

Let $\zeta \neq \pm 1$ and $\zeta^m = 1$. The assignments

$$A \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad B \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

yield a two-dimensional irreducible representation $V(\zeta)$ over $\mathbb{C}$; ζ and ζ^{-1} give isomorphic representations. If m is odd, we let $\zeta = \exp(2\pi i k/m)$, $1 \leq k \leq (m-1)/2$. In addition, there are two one-dimensional representations. If m is even, we take $\zeta = \exp(2\pi i k/m)$, $1 \leq k \leq m/2 - 1$. In addition, there are 4 one-dimensional representations. The primitive m-th roots of unity ζ are Galois-conjugate. They yield the faithful irreducibles $V(\zeta)$. We have an action of $D(2m)$ on $\mathbb{Q}[x]/(x^m - 1)$: Let A act by multiplication with x and B by substitution of x^{m-1} for x. There is a similar action on $\mathbb{Q}[x]/(\Phi_m(x))$. This rational module shows that the Schur index of the faithful module is one.

Suppose now that $m = 2^{n-1}$. Let W denote the one-dimensional representation of $K = \langle\, A^{2^{n-2}}, B \,\rangle$ with kernel $\langle\, B \,\rangle$. Let ζ be a primitive 2^{n-1}-th root of unity. By Frobenius reciprocity

$$\langle\, \mathrm{ind}_K^G W, V(\zeta) \,\rangle_G = \langle\, W, \mathrm{res}_K^G V(\zeta) \,\rangle_K,$$

and the right hand side is easily seen to be 1. Thus the rational module $\mathrm{ind}_K^G W$ contains the Galois-conjugates of $V(\zeta)$ exactly once. Therefore, by dimension count, $\mathrm{ind}_K^G W$ is the faithful irreducible $\mathbb{Q}G$-module. ◇

(6.2.7) Quaternion groups. Let $m \geq 2$ and consider the group G of order $4m$

$$Q(4m) = \langle\, A, B \mid A^{2m} = 1, A^m = B^2, BAB^{-1} = A^{-1} \,\rangle.$$

The assignments

$$A \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad B \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

yield for $\zeta^m = -1$, $\zeta \neq -1$ a two-dimensional irreducible complex representation $V(\zeta)$. The numbers ζ have the form $\exp(2\pi ik/2m)$, k odd. Again ζ and ζ^{-1} give isomorphic representations. These representations are faithful and form a Galois-class.

The element $A^m = B^2$ is contained in the center of G. If we factor out $\langle A \rangle$, we obtain $D(2m)$. The irreducible $\mathbb{C}G$-modules are the $V(\zeta)$ or lifted from $D(2m)$.

The Schur index of the faithful module is two. The rational module

$$\mathbb{Q}[x]/(x^m + 1) \oplus \mathbb{Q}[x]/(x^m + 1)$$

with action of A as multiplication by (x, x^{-1}) and $B\colon (p(x), q(x)) \mapsto (-q(x), p(x))$ shows that the Schur index is at most two. The faithful irreducible $\mathbb{C}Q(4m)$-modules are self-conjugate and of quaternionic type. Therefore the real Schur index is two, and the rational Schur index is a multiple of the real Schur index.

Let $m = 2^{n-1}$ and W the non-trivial irreducible $K = \langle A^{2^{n-2}} \rangle$-module. From Frobenius reciprocity

$$\langle \operatorname{ind}_K^G W, V(\zeta) \rangle_G = \langle W, \operatorname{res}_K^G V(\zeta) \rangle_K = 2$$

we conclude as in **??** that $\operatorname{ind}_K^G W$ is the irreducible faithful $\mathbb{Q}G$-module. ◇

(6.2.8) Semi-dihedral groups. We consider the group G of order 2^n, $n \geq 4$,

$$SD(2^n) = \langle A, B \mid A^{2^{n-1}} = 1 = B^2, BAB^{-1} = A^{2^{n-2}-1} \rangle.$$

The assignments

$$A \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & -\zeta^{-1} \end{pmatrix} \qquad B \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $\zeta^{2^{n-2}} = -1$ yield irreducible faithful complex modules. They are Galois-conjugate. The Schur index is one. This can be seen from the rational module $\mathbb{Q}[x]/(x^{2^{n-2}} + 1)$ with action: A multiplication by x and $B\colon p(x) \mapsto -p(x^{2^{n-2}-1})$. ◇

Problems

1. The quaternion group $Q(4m)$ becomes a subgroup of the quaternions of norm 1 if we set $A = \exp(2\pi i/2m) = \zeta_{2m}$ and $B = j$. Left translation of $Q(4m)$ on the quaternions $\mathbb{H}$ yields a quaternionic model for $V(\zeta_{2m})$.

6.3 An Induction Theorem for p-Groups

Here is a basic result about permutation representations and the Burnside ring.

(6.3.1) Theorem. *Let G be a p-group. Then $\pi(G)\colon A(G) \to R(G;\mathbb{Q})$ is surjective.*

The functorial nature of the natural transformation $\pi(-)$ and the simple case $G = \mathbb{Z}/p$ show that 6.3.1 is a consequence of the following more precise induction theorem.

(6.3.2) Theorem. *Let G be a p-group and V a non-trivial simple $\mathbb{Q}G$-module. Then there exist $H \lhd K \le G$ with $|K/H| = p$ such that $V \cong \operatorname{ind}^G_{K/H} W$.*

Proof. Induction over the order of the group. Let V be a non-trivial simple $\mathbb{Q}G$-module.

(1) If V is not faithful, it is obtained by pullback from a $\mathbb{Q}L$-module W along the quotient map $q\colon G \to L = G/K$, K the kernel of V. By induction, $W \cong \operatorname{ind}^L_{A/B} U$. Consider the pre-images under q, $A' = q^{-1}A$ and $B' = q^{-1}B$. We consider the A/B-module U as A'/B'-module U'. Then $V = q^*W = q^* \operatorname{ind}^L_{A/B} U \cong \operatorname{ind}^G_{A'/B'} U'$.

(2) We can now assume by induction that the faithful module is not induced from a proper subgroup. By 4.7.1 it remains to consider p-groups in which each abelian normal subgroups is cyclic. In that case we have verified 6.3.2 in the previous section. □

We say V, comes from K/H, if $V \cong \operatorname{ind}^G_{K/H} W$; and if moreover $|K/H| = p$, we call K/H or $(K,H;W)$ a **source** for V. If (K,H) is a source, then also (gKg^{-1}, gHg^{-1}).

(6.3.3) Proposition. *Let $(K,H;W)$ be a source for V. Then:*

(1) *W is the unique faithful irreducible $\mathbb{Q}K/H$-module,* $\dim W = p-1$, *and* $\dim V = (p-1)|G/K|$.

(2) *The kernel of V is contained in H.* □

The order $|K|$ is independent of the source for V, by **??**. Let U denote V, considered as a faithful representation of G/L. Then the sources of V can be identified with the sources of U (see (1) in the proof of 6.3.2).

(6.3.4) Proposition. *Let V be a faithful $\mathbb{Q}G$-module and $(K,H;W)$ a source of V. Then G has a unique central subgroup of order p, and $K = HZ$. The group H is maximal among the subgroups which do not contain Z.*

Proof. By Schur's lemma, the endomorphism algebra D of V is a division algebra. The action of G on V yields a homomorphism of the center $Z(G)$ of G into the multiplicative group of D. Since V is faithful, this homomorphism is injective.

If $Z(G)$ is not cyclic, then there don't exist faithful irreducible modules. Hence there exists a unique central subgroup Z of order p. If $Z \not\subset K$, then W becomes reducible when induced to KZ, because ZK has a non cyclic center. Hence $Z \subset K$. If $Z \subset H$, then Z would act trivially on $V = \operatorname{ind}_{K/H}^G W$, and this contradicts the injectivity of $Z(G) \to D$. Therefore $K = HZ$.

Suppose $H < L$, $Z \not\subset L$. Then the induction of W along

$$LZ \xleftarrow{\supset} K = HZ \to K/H \cong Z$$

would be reducible, since the induced module would contain the pullback of W along $LZ \to Z$. This shows the maximality of H. $\square$

(6.3.5) Example. Let V be the faithful irreducible $\mathbb{Q}G$-module of one of the exceptional groups 6.2.3. The sources K/H must satisfy:

$$\begin{aligned}
&|K| = p \;\text{ if }\; G \text{ is cyclic} \\
&|K| = 2 \;\text{ if }\; G = Q(2^n), n \geq 3 \\
&|K| = 4 \;\text{ if }\; G = D(2^n), n \geq 3, \quad \text{or if }\; G = SD(2^n), n \geq 4.
\end{aligned}$$

If G is cyclic or generalized quaternion, then Z is the only source for G. In the remaining cases every source is of the form K/H with $K \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. In $SD(2^n)$ there is only one conjugacy class of such subgroups K, and therefore the source is unique up to inner automorphism. In $D(2^n)$ there two conjugacy classes of such subgroups K. Both of them give sources for V, so that we have in this case two conjugacy classes of sources for V. $\diamond$

(6.3.6) Example. Let $M(p)$ be the non-abelian group of order p^3 and exponent $p \neq 2$ []. It has the presentation

$$M(p) = \langle\, x, y, z \mid x^p = y^p = z^p = 1, yz = zy, xz = zx, xyx^{-1} = yz. \rangle$$

The center Z has order p and is generated by z; it equals the commutator subgroup. We have a normal subgroup $A = \langle\, y, z \,\rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$. There are $p+1$ subgroups $A_0, \ldots, A_p$ of index p; they are isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$ and normal subgroups. The group $M(p)$ has a unique Galois-class of faithful irreducible complex representations (they are p-dimensional) and hence a unique faithful irreducible representation V over $\mathbb{Q}$. Moreover, there are p^2 one-dimensional representations over $\mathbb{C}$. The conjugacy classes of sources for V are represented by (A_j, B_j), where $A_j = Z \times B_j$ (the $B_j \leq A_j$ with this property are conjugate in $M(p)$). $\diamond$

6.4 The Permutation Kernel

The homomorphism $\pi_G \colon A(G) \to R(G; \mathbb{Q})$ shows that a certain part of the Burnside ring can be understood by representation theory. Therefore we shall study the kernel $N(G)$ of π_G in more detail. Since π is a natural transformation, we can view $N(-)$ also as a functor on $\mathcal{S}$. In particular, we have induction from subquotients $\operatorname{ind}_{K/H}^G \colon N(K/H) \to N(G)$.

(6.4.1) Theorem. *Let G be a p-group. Then*

$$N(G) = \sum_{K/H} \operatorname{ind}_{K/H}^G N(K/H),$$

where the sum is taken over all subquotients K/H of G which are isomorphic to one of the following groups: $\mathbb{Z}/p \times \mathbb{Z}/p$; $D(2^n)$, $n \geq 3$; $M(p)$.

We prove this theorem by verifying a certain universal property of the natural transformation π. Let $\mathcal{S}_p$ be the full subcategory of p-groups in $\mathcal{S}$.

(6.4.2) Theorem. *Let $\psi \colon A(-) \to Y(-)$ be a natural transformation between $\mathbb{Z}$-functors on $\mathcal{S}_p$. There exists a factorization $\psi = f \circ \pi$ with a natural transformation $f \colon R(-;\mathbb{Q}) \to Y(-)$ if the homomorphism $\psi_G \colon A(G) \to Y(G)$ has a factorization over π_G for the following groups: $\mathbb{Z}/p \times \mathbb{Z}/p$; $D(2^n)$, $n \geq 3$; $M(p)$.*

We apply 6.4.2 to the quotients

$$Y(G) = A(G) / \sum_{K/H} \operatorname{ind}_{K/H}^G N(K/H),$$

where we factor out the right hand side of the asserted equality in 6.4.1. One verifies that these $Y(G)$ constitute a functor which satisfies the assumptions of 6.4.2. The surjection π_G factors over a surjection $\rho_G \colon Y(G) \to R(G;\mathbb{Q})$. The transformation f_G of 6.4.2 is then an inverse to ρ_G. This proves 6.4.1.

Proof. (Of 6.4.2.) Induction over the order of G. We construct a homomorphism $f_G \colon R(G;\mathbb{Q}) \to Y(G)$ such that $f_G \pi_G = \psi_G$. Since π_G is surjective, f_G is uniquely determined by the condition $f_G \pi_G = \psi_G$. We define $f_G(V) \in Y(G)$ for each irreducible $\mathbb{Q}G$-module V and then extend additively. The inductive hypothesis is: f_H is given for all subgroups H of order less than $|G|$ and satisfies $f_H \pi_H = \psi_H$; the f_H form a natural transformation on the subcategory of these groups (referred to as: inductive naturality).

If G is cyclic, π_G is an isomorphism **??**, and we define f_G uniquely by $f_G \pi_G = \psi_G$. By assumption, we also have f_G for the groups $\mathbb{Z}/p \times \mathbb{Z}/p$, $D(2^n)$, and $M(p)$.

Therefore we can assume that G is not one of these groups. Then an irreducible $\mathbb{Q}G$-module V has the form $V = \operatorname{ind}_{K/H}^G W$ with a proper subquotient $|K/H| < |G|$. If f would exist as a natural transformation, we would have $f_G(V) = \operatorname{ind}_{K/H}^G f_{K/H}(W)$, and the latter would not depend on the presentation 6.4.3 of V. Therefore we begin with a verification of this necessary condition.

(6.4.3) Lemma. *The element $y_W = \operatorname{ind}^G_{K/H} f_{K/H}(W) \in Y(G)$ does not depend on the presentation* 6.4.3 *of V.*

Proof. If V is non-trivial, then the induction theorem 6.3.2 and the inductive naturality of f show that there exists a source $(K_0, H_0; W_0)$ of V such that $y_W = y_{W_0}$. This reduces the problem to the case that $(K, H; W)$ is a source of V.

(1) We begin with the case of a faithful module. Let us assume that G contains a normal subgroup $A \cong \mathbb{Z}/p \times \mathbb{Z}/p$. By 6.3.4, there exists a unique central subgroup Z of order p. The normal subgroup A is not contained in the center, since G has a faithful module. We must have $Z \subset A$, for A contains a normal subgroup of G of order p, and this subgroup is then central. The centralizer $C(A)$ of A has index p in G, because the kernel of the conjugation $G \to \operatorname{Aut}(A) \cong GL(2,p)$, $g \mapsto c_g$ has index at most p, and the kernel is different from G since $Z \neq A$.

Let $V = \operatorname{ind}^G_{K/H} W$ with a source $(K, H; W)$. If $K \leq C(A)$, then

$$V = \operatorname{ind}^G_{C(A)} \operatorname{ind}^{C(A)}_{K/H} W = \operatorname{ind}^G_{C(A)} V_0, \quad V_0 = \operatorname{ind}^{C(A)}_{K/H}$$

and hence $y_W = y_{V_0}$.

Suppose $K \not\subset C(A)$. An inclusion $H \subset C(A)$ and 6.3.4 would imply $K = HZ \subset C(A)$. Hence $H \not\subset C(A)$.

Suppose $H \cap A \neq 1$. Then $H \cap A \cong \mathbb{Z}/p$, since $H \cap Z = 1$, and therefore $H \cap A$ is smaller than A. Since $A \lhd G$ also $A \cap H \lhd H$, and a normal subgroup of order p is contained in the center of a p-group. Since the elements of Z and the elements of $H \cap A$ commute with the elements of H, so do the elements of $A = (A \cap H)Z$. But $H \subset C(A)$ was already excluded. Hence $H \cap A = 1$.

Since $H \not\subset C(A)$ we have a surjection $H \to G/C(A) \cong \mathbb{Z}/p$ with kernel $H \cap C(A) = H_0$ of index p in H. We have the subgroup HA of G. The subgroup H_0 is normal therein, because H_0 is normal in H and commutes with A. Since $H \cap A = 1$, we have a split exact sequence

$$1 \to \mathbb{Z}/p \times \mathbb{Z}/p \cong A \to HA/H_0 \to H/H_0 \cong \mathbb{Z}/p \to 1.$$

The group $M = HA/H_0$ is not abelian: Since $H \not\subset C(A)$, the group H/H_0 acts non-trivially on A by conjugation. The classification of the groups of order p^3 shows that M is $D(8)$ or $M(p)$, see [**?**, I(14.10)].

Let W_0 be the unique irreducible $\mathbb{Q}M$-module, see **??** and 6.3.6. We have

$$HA/H_0 \supset HZ/H_0 = K/H_0 \to K/H.$$

We use the fact that (K, H) is a source for W_0, say induced from W. We also have

$$HA/H_0 \leftarrow H_0A/H_0 \to H_0A/H_0Z =: B,$$

and $(B,1)$ is also a source for W_0, say induced by W_1. We can now form $V_1 = \mathrm{ind}_B^{C(A)} W_1$. Therefore the modules W, W_0, W_1, V_1 all give V, and the inductive naturality of f shows

$$y_W = y_{W_0} = y_{W_1} = y_{V_1}.$$

We thus have shown that in each of the subcases there exist $C(A)$-modules which induce up to V. Suppose V_0 and V_1 are any two such $C(A)$-modules. By Clifford theory they are conjugate by some $g \in G$. Let $\pi_{CA}(x_0) = V_0$, $c_g(V_0) = V_1$, $c_g(x_0) = x_1$, $\pi_{CA}(x_1) = V_1$, and

$$y_{V_j} = \mathrm{ind}_{CA}^G f_{CA} V_j = \psi_G \, \mathrm{ind}_{CA}^G x_j.$$

But $\mathrm{ind}_{CA}^G x_1 = \mathrm{ind}_{CA}^G c_g x_0 = c_g \, \mathrm{ind}_{CA}^G x_0 = \mathrm{ind}_{CA}^G x_0$, since inner automorphisms induce the identity on $A(G)$, and therefore $y_{V_0} = y_{V_1}$. This proves **??** in this case.

There remains the case that V does not have a normal subgroup $\mathbb{Z}/p \times \mathbb{Z}/p$. By assumption and the classification of the groups in question, we have to consider the cases: $p = 2$, G semi-dihedral or quaternionic. The sources of V are then unique up to inner automorphism, and one can argue as before.

It remains to consider non-faithful modules. Suppose $V = 1_G$ is trivial. Then $K = G$, $H \lhd G$ and $W = 1_{G/H}$. Since W comes from $1 = 1_{G/G}$, the inductive naturality of f yields $y_W = \mathrm{ind}_{G/G}^G f_{G/G} 1$, and this is uniquely determined.

Let V be non-trivial with kernel $L \neq 1$. We consider V as representation U of G/L. The sources of V can be identified with the sources of U. The induction hypothesis and naturality imply **??** in this case. □

□

We have now defined a homomorphism $f_G \colon R(G;\mathbb{Q}) \to Y(G)$ which sends an irreducible $\mathbb{Q}G$-module V to $\mathrm{ind}_{K/H}^G f_{K/H} W$ whenever $V = \mathrm{ind}_{K/H}^G W$ and K/H is a proper subquotient of G. The proof of 6.4.2 is finished if we have shown:

(6.4.4) Lemma. *For each subquotient K/H of G the diagram*

$$\begin{tikzcd} R(K/H;\mathbb{Q}) \arrow[r, "f_{K/H}"] \arrow[d, "\mathrm{ind}"] & Y(K/H) \arrow[d, "\mathrm{ind}"] \\ R(G;\mathbb{Q}) \arrow[r, "f_G"] & Y(G) \end{tikzcd}$$

Proof. Commutativity holds if we already know that $f_G \pi_G = \psi_G$, hence for the exceptional groups. Conversely, 6.4.4 implies this relation, since each element of $A(G)$ is either induced from a subgroup or lifted from a quotient group.

It suffices to show that both ways map an irreducible $\mathbb{Q}K/H$-module W to the same element. If $K = G$, then this holds by definition of f_G. If $K \neq G$, then we can find a subgroup of index p between K and G. By the inductive naturality it therefore suffices to consider the case $H = 1$ and $|G/K| = p$.

Let W be an irreducible $\mathbb{Q}K$-module. If $\operatorname{ind}_K^G W$ is irreducible, then both compositions agree on W, by construction of f_G.

Let $W = 1_K$. Then $\operatorname{ind}_K^G 1_K = 1_G \oplus W_0$ with an irreducible $\mathbb{Q}G$-module W_0 which is the pullback of an irreducible $\mathbb{Q}G/K$-module U. We compute

$$\begin{aligned} f_G \operatorname{ind}_K^G(W) &= f_G(1_G) + f_G(W_0) \\ &= \operatorname{ind}_{G/K}^G f_{G/K}(1_{G/K}) + \operatorname{ind}_{G/K}^G f_{G/K}(U) \\ &= \operatorname{ind}_{G/K}^G f_{G/K} \operatorname{ind}_{K/K}^{G/K}(1_{K/K}) \\ &= \operatorname{ind}_{K/K}^g f_{K/K}(1_{K/K}) \\ &= \operatorname{ind}_K^G f_K(W). \end{aligned}$$

We have used the inductive naturality of the transformation f.

There remains the case that W is non-trivial and $\operatorname{ind}_K^G W$ reducible. In this case $\operatorname{ind}_K^G W = W_1 \oplus \cdots \oplus W_p$ and $\operatorname{res}_K^G W_j = W$. Suppose $W_1 = \operatorname{ind}_H^G U_1$, $|G/H| = p$. Then $H \neq K$ and, by the Mackey formula,

$$W = \operatorname{res}_K^G \operatorname{ind}_H^G U_1 = \operatorname{ind}_{K\cap H}^K \operatorname{res}_{K\cap H}^H U_1 = \operatorname{ind}_{K\cap H}^K U.$$

Hence $U = \operatorname{res}_{K\cap H}^H U_1$ is irreducible and $\operatorname{ind}_{K\cap H}^H U$ contains U_1. Then we have $\operatorname{ind}_{K\cap H}^H U = U_1 \oplus \cdots \oplus U_p$ and $\operatorname{res}_{K\cap H}^H U_j = U$. Since

$$\begin{aligned} \operatorname{ind}_H^G(U_1 \oplus \cdots \oplus U_p) &= \operatorname{ind}_H^G \operatorname{ind}_{K\cap H}^H U \\ &= \operatorname{ind}_K^G \operatorname{ind}_{K\cap K}^K U \\ &= \operatorname{ind}_K^G W = W_1 \oplus \cdots \oplus W_p, \end{aligned}$$

we can choose the indexing such that $W_j = \operatorname{ind}_H^G U_j$. Now we compute

$$\begin{aligned} f_G \operatorname{ind}_K^G(W) &= \oplus f_G(W_j) = \oplus f_G \operatorname{ind}_H^G(U_j) = \oplus \operatorname{ind}_H^G f_H(U_j) \\ &= \operatorname{ind}_H^G f_H \operatorname{ind}_{K\cap H}^H(U) = \operatorname{ind}_{K\cap H}^G f_{K\cap H}(U) \\ &= \operatorname{ind}_K^G \operatorname{ind}_{K\cap H}^K f_{K\cap H}(U) \\ &= \operatorname{ind}_K^G f_K \operatorname{ind}_{K\cap H}^K(U) = \operatorname{ind}_K^G f_K(W). \end{aligned}$$

There remains the case that W_1 does not come from a proper subgroup by induction. Then 6.2.3 says that the kernel L of W_1 has index p in G. By our assumptions, G has order at least p^3 and therefore $K \cap L = H \neq 1$. Since $\mathrm{res}^G_K W_j = W$, we see that H is the kernel of W_j. Thus W is the pullback of a $\mathbb{Q}K/H$-module U and W_j the pullback of a $\mathbb{Q}G/H$-module U_j and $\mathrm{ind}^{G/H}_{K/H} U = U_1 \oplus \cdots \oplus U_p$. We can now use a computation as before. □

6.5 The Unit-Theorem for 2-Groups

In section I.12 we constructed a homomorphism $\eta\colon RO(G) \to A(G)^*$. The purpose of this section is to prove:

(6.5.1) Theorem. *Let G be a 2-group. Then η is surjective.*

The ring homomorphism $\pi\colon A(G) \to R(G;\mathbb{Q})$ induces a homomorphism between unit groups $\pi^*\colon A(G)^* \to R(G;\mathbb{Q})^*$. A one-dimensional $\mathbb{Q}G$-module, i.e. a homomorphism $\alpha\colon G \to \{\pm 1\}$, represents a unit in $R(G;\mathbb{Q})$.

(6.5.2) Proposition. *Each unit in $R(G;\mathbb{Q})$ has the form $\pm[L]$ for a one-dimensional $\mathbb{Q}G$-module L.*

Proof. Let $u \in R(G;\mathbb{Q})^*$. We can assume that the character satisfies $u(e) = 1$; if not, we consider $-u$. We use the inner product for characters of complex representations $\langle \chi, \psi \rangle = |G|^{-1} \sum_{g \in G} \chi(g)\overline{\psi}(g)$. The value $\langle \chi, \chi \rangle$ is always a natural number. Since u assumes only values ± 1, we must have $\langle u, u \rangle = 1$. We write u as linear combination of complex irreducible characters, $u = \sum_j n_j \chi_j$. Then $\langle u, u \rangle = \sum_j n_j^2$. Hence u is an irreducible character, and $u(e) = \dim u = 1$ shows it to be one-dimensional. □

If L is a one-dimensional real character, we have $\eta(L) \in A(G)^*$, and the image in $R(G;\mathbb{Q})^*$ is the unit L. Moreover $\eta(L \oplus \mathbb{R}) = -\eta(L) \in A(G)^*$ with image $-L \in R(G;\mathbb{Q})^*$. This shows: For each $x \in A(G)^*$ there exists $y \in RO(G)$ such that

$$x \cdot \eta(y) \in N^*(G) = \mathrm{kernel}(A(G)^* \to R(G;\mathbb{Q})^*).$$

Therefore it suffices to show that $N^*(G)$ is in the image of η. In order to do this, we investigate more closely elements in $N(G)$. Write $x \in N(G)$ as a difference of finite G-sets $x = [X_+] - [X_-]$. Then the real permutation modules $V_+ = \mathbb{R}X_+$ and $V_- = \mathbb{R}X_-$ are isomorphic. We determine a G-invariant inner product $B_\pm$ on $V_\pm$ by postulating that the basis $X_\pm$ be orthonormal.

(6.5.3) Lemma. *The exists a G-isomorphism $(V_+, B_+) \to (V_-, B_-)$.*

Proof. Let $\alpha\colon V_+ \to V_-$ be a G-isomorphism. We use α in order to pull back B_- to V_+. Then we have two G-invariant inner products on $V = V_+$. Suppose the second one is given by the positive definite matrix A. Then $v(g)^t Av(g) = A$, if $v(g)$ is the matrix of $l_g : V \to V$. All matrices are formed with respect to the orthonormal basis B_+; in particular $v(g)$ is an orthogonal matrix. Set $C = \sqrt{A}$; here $\sqrt{A}$ is a limit of polynomials in A and therefore commutes with the $v(g)$. Hence C gives an isomorphism of the desired type. □

An element $x \in N(G)$ therefore leads to the following situation:
(1) $x = [X_+] - [X_-]$,
(2) V a real G-representation,
(3) B a G-invariant inner product on V,
(4) X_+ and X_- are G-subsets of V and orthonormal bases.
We now work with such data. Let $\gamma \in \mathrm{Aut}(\mathbb{C})$ be an automorphism of the field $\mathbb{C}$. We complexify $V \otimes \mathbb{C}$ and extend the symmetric bilinear form B on V to a symmetric bilinear form on $V \otimes \mathbb{C}$, still denoted B. Define

$$\varphi_+ : V \otimes \mathbb{C} \to V \otimes \mathbb{C}, \quad \sum_{x \in X_+} a_x x \mapsto \sum_{x \in X_+} \gamma(a_x) x,$$

and define φ_- similarly using the basis X_-. These maps are still G-equivariant and satisfy $\varphi_\pm(av) = \gamma(a)\varphi_\pm(v)$ for $a \in \mathbb{C}$ and $B(\varphi_\pm(v), \varphi_\pm(w)) = \gamma B(v, w)$. The composition $u = \varphi_-^{-1} \circ \varphi_+$ is therefore $\mathbb{C}$-linear and respects B, hence contained in the orthogonal group $O(V, \mathbb{C})$ of the space $(V \otimes \mathbb{C}, B)$. Let $d_H(u)$ be the determinant of u^H on $V^H \otimes \mathbb{C}$. We will compute the function

$$d(u) : \Phi(G) \to \mathbb{C}, \quad (H) \mapsto d_H(u)$$

and show that it is a unit in $N(G)^*$ (in particular the values are in $\{\pm 1\}$). From 9.7.2 we see that $d(u) \in A(G)^*$. Another proof uses the induction theorem **??** and a verification for dihedral groups.

For each subgroup $H \leq G$ let

$$P_H : A(G) \to \mathbb{Q}_+^*$$

be the homomorphism from the additive to the multiplicative group which assigns to each G-set the product of the cardinalities of the H-orbits; i.e. if $\mathrm{res}^G_H\, x = \sum_j n(j)[H/H_j]$, then $P_H(x) = \prod_j |H/H_j|^{n(j)}$. We also use the pairing

$$\mathrm{Aut}(\mathbb{C}) \times \mathbb{Q}_+^* \to \{\pm 1\}, \quad (\gamma, a) \mapsto \langle \gamma, a \rangle = \gamma(\sqrt{a})/\sqrt{a}.$$

(6.5.4) Theorem. *Let $x = X_+ - X_- \in N(G)$, (V, B, X_+, X_-), and $u = \varphi_-^{-1} \circ \varphi_+$ as above. Then $d_H(u) = \langle \gamma, P_H(x) \rangle$.*

Proof. We have the orthonormal basis of V^H which consists of the elements

$$e_S = \frac{1}{\sqrt{|S|}} \sum_{s \in S} s,$$

where S runs through the H-orbits of X_+. By definition, $\varphi_+(e_S) = \langle\, \gamma, |S|\,\rangle e_S$. Let $n = \dim V^H$. The form B induces on the one-dimensional space $\Lambda^n(V^H \otimes \mathbb{C})$ a symmetric bilinear form

$$B_1(v_1 \wedge \ldots \wedge v_n, w_1 \wedge \ldots \wedge w_n) = \det(B(v_i, w_j)).$$

If $S_1, \ldots, S_n$ are the H-orbits of X_+, then $e = e_{S_1} \wedge \ldots \wedge e_{S_n}$ is an orthonormal basis of $\Lambda^n(V^H)$. The map φ_+^H is only γ-linear, hence induces a γ-linear endomorphism $\Lambda^n \varphi_+^H$ of $\Lambda^n(V^H \otimes \mathbb{C})$. We obtain

$$\Lambda^n \varphi_+^H e = \varphi_+^H e_{S_1} \wedge \ldots \wedge \varphi_+^H e_{S_n} = \textstyle\prod_j \langle\, \gamma, |S_j|\,\rangle e = \langle\, \gamma, P_H(X_+)\,\rangle e.$$

Up to sign, e is the unique element with $B_1(e,e) = 1$. If we apply the same procedure to X_- we arrive therefore at $\pm e$, and hence

$$\Lambda \varphi_-^H(e) = \langle\, \gamma, P_H(X_-)\,\rangle e.$$

Altogether we obtain the asserted value for $d_H(u)$. $\square$

We have now constructed for each $\gamma \in \mathrm{Aut}(\mathbb{C})$ a homomorphism

$$U(\gamma)\colon N(G) \to A(G)^*, \quad x \mapsto U(\gamma, x) = ((H) \to \langle\, \gamma, P_H(x)\,\rangle).$$

Since $\pi(C)$ is an isomorphism for cyclic C and elements of $R(G;\mathbb{Q})$ are determined by restriction to cyclic subgroups, we have

$$N(G) = \bigcap_{C \le G} \mathrm{Ker}(\mathrm{res}\colon A(G) \to A(C))$$

and similarly

$$N(G)^* = \bigcap_{C \le G} \mathrm{Ker}(\mathrm{res}\colon A(G)^* \to A(C)^*).$$

Naturality of $U(\gamma)$ with respect to restriction now shows that the image of $U(\gamma)$ is contained in $N(G)^*$. Altogether we have proved:

(6.5.5) Theorem. *There exists a natural transformation $U(\gamma)\colon N(G) \to N(G)^*$ between functors on the category of finite groups and homomorphisms. We have $\varphi_H(U(\gamma,x)) = \langle\, \gamma, P_H(x)\,\rangle$, and $U(\gamma\delta) = U(\gamma)U(\delta)$ for each pair $\gamma, \delta \in \mathrm{Aut}(\mathbb{C})$. For fixed G, the map $U(\gamma)$ only depends on the restriction of γ to the field K which is obtained by adjoining to $\mathbb{Q}$ the $\sqrt{p}$ for prime divisors p of $|G|$.* $\square$

We now consider 2-groups, use γ with $\gamma(\sqrt{2}) = -\sqrt{2}$, and write $U = U(\gamma)$. Note that $U(\gamma, x)(H) = (-1)^{\nu_2 P_H(x)}$ if $P_H(x) = 2^{\nu_2 P_H(x)}$.

(6.5.6) Theorem. *If G is a 2-group, then $U\colon N(G) \to N(G)^*$ is surjective.*

Proof. Set $s\colon N(G)^* \to N(G)$, $u \mapsto u-1$. We will prove that $U \circ s\colon N(G)^* \to N(G)^*$ is bijective. Since $N(G)^*$ is finite, it suffices to show that $U \circ s$ is injective.

Let G be a minimal counter-example to injectivity. Let $u_1, u_2 \in N(G)^*$ have the same image under $U \circ s$. The restrictions of u_1 and u_2 to each proper subgroup H of G are equal. If we interchange the u_i if necessary, we see that $u_1 - u_2 = t_G$, where t_G is the element constructed in I(8.2) with function $t_G(H) = 2$ for $H = G$ and $= 0$ for $H \neq G$. The group G is not cyclic, because $N(G)^*$ is trivial for cyclic groups. The equality $u_1 = t_G + u_2$ implies $s(u_1) = t_G + s(u_2)$, $Us(u_1) = U(t_G)Us(u_2)$, and finally $U(t_G) = 1$. From (2.1) we obtain $d_G U(t_G) = -1$, since $P_G(t_G)$ is an odd power of 2. Hence $U(t_G) = 1 - t_G$, and this contradicts $U(t_G) = 1$. □

Taking the underlying real representation induces an additive homomorphism $r\colon R(G) \to RO(G)$. We denote the cokernel by KO_G^{-1}. (The reason for this notation: In equivariant topological KO-theory we have the suspension isomorphism $KO_G^{-1} \cong \tilde{K}O_G(S^1)$.) From representation theory (2.2) we know that KO_G^{-1} is the $\mathbb{Z}/2$-vector space generated by the irreducible real representations of real type. The homomorphism η factors through a homomorphism $j\colon KO_G^{-1} \to A(G)^*$. The Adams operation Ψ^k on $RO(G)$ induces a homomorphism Ψ^k on KO_G^{-1}.

(6.5.7) Proposition. *Let G be a 2-group. Then the sequence*

$$KO_G^{-1} \xrightarrow{1-\Psi^5} KO_G^{-1} \xrightarrow{j} A(G)^* \to 1$$

is exact.

Proof. We know already that j is surjective 6.5.1. The map Ψ^5 is also represented by a Galois conjugation, and Galois conjugate real representations yield the same unit in $A(G)^*$. Therefore $j \circ (1 - \Psi^5)$ is trivial. Since 5 generates the group $(\mathbb{Z}/2^n)^*/\{\pm 1\}$, we see that the image of $1 - \Psi^5$ consists of the differences of Galois conjugate representations. Hence the cokernel of $1 - \Psi^5$ is the $\mathbb{Z}/2$-vector space with basis the representatives, say $U_1, \dots, U_k$, of the Galois classes of irreducible real representations of real type. Suppose the integral linear combination $U = \sum n_j U_j$ is contained in the kernel of j, i.e. the dimensions $\dim_{\mathbb{R}} U^H$ are even. We have to show that the n_j are even. We have $\langle U \otimes \mathbb{C}, \operatorname{ind}_H^G 1_H \rangle_G = \dim_{\mathbb{R}} U^H$. We know from 6.2.1 that the $\operatorname{ind}_H^G 1_H$ generate $R(G;\mathbb{Q})$. Let W_j be an irreducible $\mathbb{Q}G$-module such that $W_j \otimes \mathbb{C}$ contains $U_j \otimes \mathbb{C}$; then it contains no other $U_l \otimes \mathbb{C}$. The number $\langle U \otimes \mathbb{C}, W_j \otimes \mathbb{C} \rangle = n_j \langle U_j \otimes \mathbb{C}, W_j \otimes \mathbb{C} \rangle$ is even, being a linear combination of fixed point dimensions, and $\langle U_j \otimes \mathbb{C}, W_j \otimes \mathbb{C} \rangle$ is the rational Schur index of W_j. Since U_j is of real type, this Schur index is 1 (see 6.2.1 and 6.2.3). Hence n_j is even. □

6.6 The Elements t_G

Recall the elements t_G: The smallest multiple of the top idempotent $e_G \in C(G)$ which is contained in $A(G)$.

(6.6.1) Theorem. *Let $A \cong (\mathbb{Z}/p)^n$ be an elementary abelian p-group. Then $P_A(t_A) = p^{a(n)}$ and $a(n)$ is the number*

$$\sum_{d=1}^{n}(-1)^d d\binom{n}{d}_p p^{(d-1)(d-2)/2} = (1-p)(1-p)(1-p^2)\cdots(1-p^{n-2}).$$

(We understand $a(1) = 1, a(2) = 1-p$.) Here we use again the quantum binomial coefficient already introduced in the proof of 1.3.5. In particular $a(n) \equiv 1 \bmod p$. If G is a non-cyclic p-group and $\gamma\colon G \to G/\Phi(G)$ the projection onto the Frattini quotient, then $\gamma^* t_{G/\Phi(G)} = t_G$ and $P_G(t_G) = P_{G/\Phi(G)}(t_{G/\Phi(G)})$.

Proof. We begin by showing that the two values for $a(n)$ are equal. We start with the q-identity (1.7) and replace z by zq^{-1}. We differentiate the resulting identity with respect to the variable z and put then $z = 1$. The asserted equality drops out.

We now use the idempotent formula **??**

$$t_A = \frac{p}{|A|}\sum_{K \le G}|K|\mu(K,A)[A/K].$$

In our case, if K has order p^{n-j}, we have

$$\mu(K,A) = (-1)^j p^{j(j-1)/2}.$$

The contribution of the summand $[A/K]$ with $|K| = p^{n-j}$ to P_G is $p^{b(j)}$ with

$$b(j) = p^{1-j}(-1)^j p^{j(j-1)/2}\binom{n}{j}_p j,$$

if one takes into account that there exist $\binom{n}{j}_p$ subgroups of order p^j in A. A simple rewriting now yields the asserted sum-presentation of the number $a(n)$. □

6.7 Products of Orbits

We have a map $P = P(G)\colon A(G) \to \mathrm{Map}(\mathrm{Con}(G), \mathbb{Q}^*_+) = M(G)$; it assigns to X the function $(H) \mapsto P_H(X)$. Recall that $P_H(X) = \prod_{T \in X/H}|T|$. The map P is exponential, $P(x+y) = P(x)P(y)$. Recall that the groups $A(G)$, $N(G)$, and $M(G)$ are the values of functors A, N, and M on the category $\mathcal{A}$. The naturality properties of the homomorphisms P can be expressed as:

(6.7.1) Proposition. *The $P(G)$ constitute a natural transformation $P\colon N \to M$ on the category $\mathcal{B}$. This means that for each finite (L,K)-set S with free K-action the diagram*

$$\begin{tikzcd} N(K) \arrow[r, "P"] \arrow[d, "\rho(S)"] & M(K) \arrow[d, "\sigma(S^\circ)"] \\ N(L) \arrow[r, "P"] & M(L) \end{tikzcd}$$

is commutative.

Proof. We have to verify the commutativity only for generating morphisms of the category. One first checks that the diagram commutes for homomorphisms $\gamma\colon L \to K$ and the induced maps γ^*. Then it remains to verify the commutativity for the ordinary induction. By the double coset formula and the compatibility with restriction and conjugation and by induction over the order of L it suffices to show that the two functions given by the diagram have the same value at L.

Let $x = \sum_j n_j[K/K_j] \in A(K)$. Then $\mathrm{ind}_K^L x = \sum_j n_j[L/K_j]$ and

$$(P(\mathrm{ind}_K^L(x))(L) = \prod_j |L/K_j|^{n_j} = |L/K|^{\sum n_j} \prod_j |K/K_j|^{n_j}.$$

On the other hand, $(\mathrm{mul}_K^L P(x))(L) = P(x)(K) = \prod_j |K/K_j|^{n_j}$. These two values coincide if $\sum n_j = 0$. The sum $\sum_j n_j$ is the number of K-orbits of x. The commutativity of the diagram therefore holds for those x for which $|x/U| = 0$ for all $U \le K$. In general $|x/U| = \dim \mathbb{Q}(x)^U$. We now note that the following are equivalent:

(1) $x \in N(K)$;

(2) For all $U \le K$, $\dim \mathbb{Q}(x)^U = 0$.

(1) $\Rightarrow$ (2). Let $x = X_+ - X_-$ and $\mathbb{Q}(X_+) \cong \mathbb{Q}(X_-)$. Then certainly all fixed point sets have the same dimension.

(2) $\Rightarrow$ (1). The isomorphism type of a $\mathbb{Q}C$-module is determined by the dimensions of fixed point sets of subgroups, if C is cyclic. Hence in general the character is determined by dimensions of fixed points of cyclic subgroups. □

6.8 Exponential Transformations

We need p-adic numbers and p-adic completions. Let $\mathbb{Z}_p$ denote the ring of p-adic integers. It can be defined as the (inverse) limit $\mathbb{Z}_p = \lim_n \mathbb{Z}/p^n$. It is a compact topological ring, and the canonical map $\mathbb{Z} \to \mathbb{Z}_p$ has a dense image. The units of this ring can be obtained as a similar limit $\mathbb{Z}_p^* = \lim(\mathbb{Z}/p^n)^*$. If p

is odd, then $\mathbb{Z}_p^*$ is topologically cyclic: the subgroup generated by $k \in \mathbb{Z}_p^*$ has a dense image (= is a topological generator) if and only if the image in $(\mathbb{Z}/p^2)^*$ is a generator.

The ring $\mathbb{Z}_p$ is a local ring, and $p\mathbb{Z}_p$ is the maximal ideal. The quotient field $\mathbb{Q}_p$ is the field of p-adic numbers.

Let A be a finitely generated abelian group. Then $A \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is called its p-adic completion. This completion can also be defined as $\lim A/p^n A$. The functor $\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is exact on the category of finitely generated abelian groups.

Suppose p is odd. We have an isomorphism of topological groups $\mathbb{Z}_p^* \cong \mathbb{Z}/(p-1) \times \mathbb{Z}_p$. The factor $\mathbb{Z}_p$ corresponds to the units of the form $1+p\mathbb{Z}_p \subset \mathbb{Z}_p^*$. Suppose $k \in \mathbb{Z}_p^*$, $k \equiv 1 \bmod p$, $\lambda \in \mathbb{Z}_p$. Then k^λ can be defined as follows: If λ is the p-adic limit of a sequence (a_n) of integers, then k^λ is the p-adic limit of $((1+p)^{a_n})$; one checks that this is well-defined. The maps $\mathbb{Z}_p \to 1+p\mathbb{Z}_p$, $\lambda \mapsto k^\lambda$ satisfy the usual exponential identities $(kl)^\lambda = k^\lambda l^\lambda$ and $k^{\lambda+\mu} = k^\lambda k^\mu$; for $k = 1 + p$, p odd, the map is an isomorphism, and it induces an isomorphism $p^t\mathbb{Z}_p \to 1 + p^{t+1}\mathbb{Z}_p$ for each $t \in \mathbb{N}$.

The situation is slightly different for $p = 2$. We have a topological isomorphism $\mathbb{Z}_2^* \cong \{\pm 1\} \times \mathbb{Z}_2$. The factor $\mathbb{Z}_2$ corresponds to the multiplicative subgroup $1 + 4\mathbb{Z}_2 \subset \mathbb{Z}_2^*$, and $x \in \mathbb{Z}_2$ is a topological generator if and only if $x \equiv 5 \bmod 8$. Also x is a generator of $\mathbb{Z}_2^*/\{\pm 1\}$ if and only if $x \equiv \pm 3 \bmod 8$. The homomorphism $\lambda \mapsto 5^\lambda$ induces an isomorphism $2^t\mathbb{Z}_2 \to 1 + 2^{t+2}\mathbb{Z}_2$.

Let G be a p-group. We consider the completed Burnside ring $A(G)_p$. It is a local ring with maximal ideal $\{x \mid \varphi_1(x) \equiv 0 \bmod p\}$. The inclusion $A(G) \subset C(G)$ yields an inclusion of topological rings $A(G)_p \subset C(G)_p$ which is still described by the same congruences as the inclusion $A(G) \subset C(G)$. The ring $C(G)_p$ is the ring of functions $\mathrm{Con}(G) \to \mathbb{Z}_p$. We have an induced inclusion of unit groups $A(G)_p^* \subset C(G)_p^*$, and $C(G)_p^*$ is the group of functions $\mathrm{Con}(G) \to \mathbb{Z}_p^*$. We also have the exact sequence

$$0 \to N(G)_p \to A(G)_p \to R(G;\mathbb{Q})_p \to 0.$$

Inside $A(G)_p^*$ we have the group $1 + N(G)_p$.

Let us write $P_H(x) = p^{\nu_p P_H(x)}$. This gives us an additive homomorphism

$$\nu(H)\colon N(G) \to \mathbb{Z}, \quad x \mapsto \nu_p P_H(x) = \nu(H,x).$$

(6.8.1) Lemma. *The image of the homomorphism ν is contained in $(1-p)\mathbb{Z}$.*

Proof. By 6.5.1, this is true for the elements $\mathrm{ind}_H^G\, t_H$ for the non-cyclic subgroups H of G. They generate a subgroup of $N(G)$ of p-power index. □

The previous lemma provides us with the homomorphism

$$\mu(H)\colon N(G) \to \mathbb{Z}, \quad x \mapsto \mu(H,x) = \frac{1}{1-p}\nu_p P_H(x).$$

We use the same symbol for its p-completion. We define an exponential homomorphism

$$\eta(G) = \eta\colon N(G)_p \to C(G)_p^*, \quad \varphi_H \eta(x) = (1+p)^{\mu(H,x)}.$$

From **??** we see that the $\eta(G)$ constitute a natural transformation of functors on the category $\mathcal{B}$.

(6.8.2) Theorem. *Let p be an odd prime. Then the image of η is contained in $1 + N(G)_p$, and the resulting exponential map $\eta\colon N(G)_p \to 1 + N(G)_p$ is a natural isomorphism on the category of p-groups which is compatible with induction.*

(6.8.3) Example. Let $G = \mathbb{Z}/p \times \mathbb{Z}/p$. Then $t_G = p[G/G] - \sum_{j=0}^{p}[G/C_j] + [G/1]$; here $C_0, \dots, C_p$ are the cyclic subgroups of order p. We have $N(G)_p \cong \mathbb{Z}_p$ with basis t_G. We compute $P_G t_G = 1^p \cdot p^{-(p+1)} \cdot p^2 = p^{1-p}$ and therefore $d_G(\eta(t_G)) = 1 + p$; the function $\eta(t_G)$ has values 1 at 1 and C_j. We see that $\eta(t_G) = 1 + t_G \in 1 + N(G)_p$ is a generator and hence η an isomorphism as claimed. $\diamond$

(6.8.4) Example. We consider the group $M = M(p)$; compare **??**. A $\mathbb{Z}$-basis of $N(M)$ consists of the elements $t_M,\ x_1, \dots, x_p,\ y_0$ with

$$\begin{aligned} x_j &= [M/A_j] - [M/A_0] - [M/B_j] + [M/B_0], \\ y_j &= \operatorname{ind}_{A_j}^M t_{A_j} = p[M/A_j] - p[M/B_j] - [M/C] + [M/1]. \end{aligned}$$

The relations $px_j = y_j - y_0$ hold. In terms of functions in $C(G)$ the elements in $N(M)$ have value 0 on the cyclic subgroups $1, C, B_j$ and the other values are given in the following table.

	M	A_0	A_j
t_M	p	0	0
x_j	0	$-p$	p
y_0	0	p^2	0

Write $\overline{N}(G) = N(G)/\sum \operatorname{ind}_{K/H}^G N(K/H)$, $K/H \neq G$. Our computations imply that $\overline{N}(M)$ is the $\mathbb{F}_p$-vector space generated by $x_1, \dots, x_p$. The inclusion $A(M) \subset C(M)$ is given by the following congruences for functions $z \in C(M)$

$$\begin{aligned} z(A_j) &\equiv z(M) \bmod p \\ z(B_j) &\equiv z(A_j) \bmod p \\ z(C) &\equiv \textstyle\sum_{i=0}^{p}(1-p)z(A_j) \bmod p^2 \\ z(1) &\equiv (1-p)z(C) + \textstyle\sum_{j=0}^{p}(1-p)pz(B_j) \bmod p^3. \end{aligned}$$

By a direct calculation one can deduce multiplicative congruences for the inclusion $A(M)_p^* \subset C(M)_p^*$. For this purpose one defines functions $u(H)\colon C(M)^* \to \mathbb{Z}_p^*$ by multiplicative Möbius-inversion

$$\begin{aligned} z(A_j) &= u(A_j)z(M) \\ z(B_j) &= u(B_j)u(A_j)u(M) \\ z(C) &= u(C)\textstyle\prod_j u(A_j)u(M) \\ z(1) &= u(1)\textstyle\prod_j z(B_j)^p \prod_j z(A_j)^{-p} z(C). \end{aligned}$$

The multiplicative congruences are then

$$u(A_j) \equiv 1\,(p),\ u(B_j) \equiv 1\,(p),\ u(C) \equiv 1\,(p^2),\ u(1) \equiv 1\,(p^3).$$

We compute η for the basis elements above and display the exponents of $1+p$ in the following table.

	M	A_0	A_j
t_M	1	0	0
x_j	0	-1	1
y_0	1	p	0

The multiplicative congruences show that η maps our basis of $N(G)_p$ to a basis of $1 + N(M_p)$. $\diamond$

Proof. (Of 6.8.2.) In order to verify that η has an image in $1+N(G)_p$ we use the fact that η is compatible with restriction and induction. We can therefore use the induction theorem **??**. Hence it suffices to consider the cases $G = \mathbb{Z}/p\times\mathbb{Z}/p$ and $G = M(p)$. We have already done this in 6.8.3 and 6.8.4.

In order to show that η_{1+p} is an isomorphism, we use a filtration argument. Let $\mathcal{F}_1$ be the family of all non-cyclic subgroups. Choose a filtration by closed families $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \ldots \supset \mathcal{F}_r$ such that $\mathcal{F}_j \smallsetminus \mathcal{F}_{j+1} = (H_j)$ and $\mathcal{F}_r = (G)$. Let $N(G;\mathcal{F}) \subset N(G)$ be the ideal of functions which have non-zero value only at subgroups in $\mathcal{F}$. Then $\varphi_{H_j}\colon N(G;\mathcal{F}_j)_p/N(G;\mathcal{F}_{j+1})_p \cong a_j\mathbb{Z}_p \subset \mathbb{Z}_p$ with some $a_j \in p\mathbb{N}$. There is a similar isomorphism $1+N(G;\mathcal{F}_j)_p/1+N(G;\mathcal{F}_{j+1})_p \cong 1+a_j\mathbb{Z}_p$. The homomorphism η_k is compatible with this filtration $\eta_k N(G;\mathcal{F}_j)_p \subset 1+N(G;\mathcal{F}_j)_p$. We therefore have to show that η_{1+p} induces an isomorphism on the successive filtration quotients. The integer a_j divides $p|WH_j|$, say $d_j a_j = p|WH_j|$. This is so since $x_j = \mathrm{ind}_{H_j}^G t_{H_j}$ is contained in $N(G;\mathcal{F}_j)$. We know, by 6.6.1, the element $\eta_{1+p}(x_j)$. We conclude that η_{1+p} is on the filtration quotients the homomorphism

$$a_j\mathbb{Z}_p \to 1 + a_j\mathbb{Z}_p, \quad d \mapsto (1+p)^{a(H_j)d/p},$$

where $(1-p)a(H_j) = \nu(H_j, t_{H_j})$, an integer which was computed in 6.6.1and shown to be 1 mod p. $\square$

One has to modify the definitions for 2-groups. In section 6 we have already considered the homomorphism $\eta_{-1}\colon N(G) \to N(G)^*$. Let $N_0(G)$ be its kernel. Again we use $\eta_k\colon N(G)_2 \to 1 + N(G)_2$, $d_H\eta_k(x) = k^{\nu_2 P_H(x)}$.

(6.8.5) Theorem. *Let G be a 2-group. Then $\sqrt{\eta_5}\colon N_0(G)_2 \to 1 + N_0(G)_2$ is defined and an isomorphism.*

(6.8.6) Example. Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$. Then $\eta_5(t_G)(G) = 5^{-1}$, and value 1 otherwise. The element $2t_G$ is contained in $N_0(G)$. In order to obtain a generator of $d_G\colon 1 + N_0(G)_2 \cong 1 + 4\mathbb{Z}_2$, generated by 5^{-1}, we therefore have to use the square root. ◇

(6.8.7) Example. We consider in detail the dihedral group of order 2^n

$$D = D(2^n) = \langle\, A, B \mid A^{2^{n-1}} = 1 = B^2,\ BAB^{-1} = A^{-1}\,\rangle.$$

Let $C(j)$ denote the cyclic subgroup of order 2^j generated by $A^{2^{n-j-1}}$. Moreover we have the subgroups $H(j,0) = \langle\, C(j), B\,\rangle$ and $H(j,1) = \langle\, C(j), AB\,\rangle$. These subgroups are a representative system for the conjugacy classes. In order to simplify the notation, we write elements of $A(D)$ as linear combinations of subgroups instead of homogeneous spaces. We have

$$x_j = \operatorname{ind}^D_{H(j,0)} t_{H(j,0)} = 2H(j,0) - C(j) - 2H(j-1,0) + C(j-1)$$

and similarly for $H(j,1)$. Let

$$y_j = H(j,0) - H(j-1,0) - H(j,1) + H(j-1,1), \quad 1 \le j \le n-2.$$

A $\mathbb{Z}$-basis of $N(D)$ is given by

$$x_j, \quad y_j\ (1 \le j \le n-2), \quad t_D.$$

The relation $2y_j = \operatorname{ind} t_{H(j,0)} - \operatorname{ind}_{H(j,1)}$ holds. These data show that $\overline{N}(D) \cong \mathbb{Z}/2$, generated by y_1. The basis elements have the following functions: t_D value 2 on D and zero otherwise, x_j value 4 on $H(j,0)$ and zero otherwise, y_j value 2 on $H(j,0)$ and value -2 on $H(j,1)$ and zero otherwise. A basis of $N_0(D)$ consists of the following elements

$$2y_j, \quad x_j - t_D\ (1 \le j \le n-2), \quad 2t_D.$$

Note that $|A(D)^*| = 2^{n+2}$, $|N(D)^*| = 2^{n-1}$, and that the displayed elements yield a subgroup of index 2^{n-1} in $N(D)$. In terms of functions, $N_0(D)$ consists of the functions which are zero on cyclic subgroups, and divisible by 4 at other places. This shows that $1 + N_0(D)_2$ consists of all functions on non-cyclic conjugacy classes with values in $1+4\mathbb{Z}_2$. We list η_k of the basis elements and display the exponents of k. The element x_j and $x_{n-1} = t_D$:

$$H \mapsto \begin{cases} -2 & H = H(l,0) \quad\quad j \le l < n-1 \\ -1 & H = H(n-1) \\ \;\;1 & \text{otherwise.} \end{cases}$$

The element y_j:

$$H \mapsto \begin{cases} -1 & H = H(l,0) \quad j \le l < n-1 \\ \;\;1 & H = H(l,1) \quad j \le l < n-1 \\ \;\;1 & \text{otherwise.} \end{cases}$$

From these data one verifies that $\sqrt{\eta_5}$ is an isomorphism. $\diamond$

Proof. (Of 6.8.5.) The condition $x \in \operatorname{kernel} \eta_{-1}$ means that $\nu_2(H,x)$ is divisible by 2, hence $5^{\nu_2(H,x)/2} \in 1+4\mathbb{Z}_2$ is defined. The examples 6.8.6 and 6.8.7 together with the induction theorem **??** show that the image of $\tau = \sqrt{\eta_5}$ is contained in $1 + N_0(G)_2$. By a filtration argument as in the proof of 6.8.2 one shows that τ is an isomorphism. $\square$

Chapter 7

Categorical Aspects

7.1 The Category of Bisets

Let K and L be groups. We have the category $K\text{-}\mathrm{SET}\text{-}L$ of (K,L)-sets. A (K,L)-set S carries a left K- and a right L-action, and these actions commute. We call such objects with an action of two compatible group actions a **biset**. Each (K,L)-set S provides us with a functor

$$\rho(S)\colon L\text{-}\mathrm{SET} \to K\text{-}\mathrm{SET},$$

given on objects by $Y \mapsto S \times_L Y$ and similarly on morphisms. A morphism $\varphi\colon S \to T$ induces a natural transformation $\rho(\varphi)\colon \rho(S) \to \rho(T)$ with values $\rho(\varphi)(Y) = \varphi \times_L Y$. We thus obtain a functor ρ from $K\text{-}\mathrm{SET}\text{-}L$ into the functor category $[L\text{-}\mathrm{SET}, K\text{-}\mathrm{SET}]$.

If S is a (K,L)-set and T an (L,M)-set, then $S \times_L T$ inherits a canonical structure of a (K,M)-set. The associativity

$$(S \times_L T) \times_M Y = S \times_L (T \times_M Y), \tag{7.1}$$

which we treat as identity, shows $\rho(S) \circ \rho(T) = \rho(S \times_L T)$. We formalize this fact and consider the assignment $(T,S) \mapsto S \times_L T$ as a composition in a category.

The category of bisets $\bullet\text{-}\mathrm{SET}\text{-}\bullet$ has as objects the groups. The morphisms from K to L are the objects of $K\text{-}\mathrm{SET}\text{-}L$ (although this is not a set). Composition of morphisms is defined as $T \circ S = S \times_L T$, so that we have $\rho(T \circ S) = \rho(S \circ T)$. In order that this be (strictly) associative, we treat canonical isomorphisms of the type **??** as identity. The K-set K with left and right K-translation is the identity of the object K. This statement also uses a canonical identification.

The category $\bullet\text{-}\mathrm{SET}\text{-}\bullet$ and the categories $K\text{-}\mathrm{SET}\text{-}L$ can be combined into the structure of a 2-category. The morphisms $K \to L$ are the (K,L)-sets. The

2-morphisms between two (K,L)-sets are the (K,L)-equivariant maps, and their ordinary composition is called the vertical composition $*$ of 2-morphisms. If $\alpha\colon S \to S'$ is an (L,K)-map and $\beta\colon T \to T'$ an (M,L)-map, then their horizontal composition is $\alpha \diamond \beta = \alpha \times_L \beta\colon S \times_L T \to S' \times_L T'$. The 2-category axiom

$$(\alpha' * \alpha) \diamond (\beta' * \beta) = (\alpha' \diamond \beta') * (\alpha \diamond \beta)$$

holds by functoriality of the product $\times_L$.

The assignment

$$\begin{array}{rcl} K & \mapsto & K\text{-}\mathrm{SET} \\ S \in K\text{-}\mathrm{SET}\text{-}L & \mapsto & \rho_K^L(S) \\ \varphi\colon S \to S' \in K\text{-}\mathrm{SET}\text{-}L & \mapsto & \rho_K^L(\varphi) \end{array}$$

is a 2-functor from the 2-category $\bullet\text{-}\mathrm{SET}\text{-}\bullet$ to the 2-category CAT of categories. This functor is compatible with coproducts (disjoint union) in the sense that $S \times_L (Y + Y') = S \times_L Y + S \times_L Y'$, and similarly for the variable S. Also composition in $\bullet\text{-}\mathrm{SET}\text{-}\bullet$ is biadditive in the sense that $S \circ T$ is compatible with disjoint union in the variables S and T.

In many cases it is not necessary to work with the 2-category: One can pass to isomorphism classes of bisets.

There is also a contravariant functor. Let $S \in K\text{-}\mathrm{SET}\text{-}L$. Consider the set of K-maps $\sigma(S)(Y) = \mathrm{Hom}_K(S,Y)$ for a K-set Y. It carries the L-action $(l \cdot \varphi)(s) = \varphi(sl)$. The assignment $Y \mapsto \mathrm{Hom}_K(S,Y)$ yields a functor $\sigma(S)\colon K\text{-}\mathrm{SET} \to L\text{-}\mathrm{SET}$. There is a canonical isomorphism $\sigma(T) \circ \sigma(S) = \sigma(S \times_L T) = \sigma(T \circ S)$. It amounts to a natural isomorphism

$$\mathrm{Hom}_K(S \times_L T, Y) \cong \mathrm{Hom}_L(T, \mathrm{Hom}_K(S,Y)).$$

In order to see this, we view the left hand side as the set of maps $\psi\colon S \times T \to Y$ such that $\psi(ks,t) = k\psi(s,t)$ and $\psi(sl,t) = \psi(s,lt)$ for $s \in S, t \in T, k \in K, l \in L$. The isomorphism is then given by the adjunction $\psi \mapsto \psi'$ where $\psi'(t)(s) = \psi(s,t)$.

7.2 Basis Constructions

The functors ρ and σ comprise fundamental constructions with transformation groups.

(7.2.1) Induction. If $K \le L$. Consider L as (L,K)-set by left and right translation. Then $\rho(L)$ is called induction from K to L and is given on objects by $X \mapsto L \times_K X = \mathrm{ind}_K^L X$. $\diamond$

(7.2.2) Restriction. Let $\alpha\colon L \to K$ be ahomomorphism. We consider K via α as left L-set and via right translation as right K-set. Then $\rho(K)$ is given as $X \mapsto \alpha^* X$, where $\alpha^* X$ is obtained from the K-space X by viewing it via α as L-space. In the case that $\alpha\colon L \subset K$ is an inclusion, we call this process restriction $\operatorname{res}^K_L X$. ◇

(7.2.3) Proposition. *The induction functor is left adjoint to the restriction functor. The adjointness means, there is a natural bijection*

$$\mathrm{Hom}_L(\operatorname{ind}^L_K X, Y) \cong \mathrm{Hom}_K(X, \operatorname{res}^L_K Y).$$

It assigns to a K-map $f\colon X \to Y$ the LG-map $L \times_K X \to Y$, $(g,x) \mapsto gf(x)$. □

From 9.7.2 we obtain

$$\operatorname{ind}^G_H((\operatorname{res}^G_H X) \times Y) \cong X \times \operatorname{ind}^G_H Y.$$

The functoriality gives in this context the transitivity of induction and restriction: For $A \le B \le C$ we have natural isomorphism of functors $\operatorname{ind}^C_B \operatorname{ind}^B_A \cong \operatorname{ind}^C_A$, and similarly for res.

(7.2.4) Orbit sets. Let $A \le K$. The right K-set $A\backslash K$ carries a free left $W_K A$-action. The process $A\backslash K \times_K X \cong A\backslash X$ transforms the K-set X into the orbit set $A\backslash X$ with induced action of $W_K A$. ◇

(7.2.5) Fixed point sets. Let $L \le K$ and consider the homogeneous K-set K/L with right $W_L K$-action. Then $\mathrm{Hom}_K(K/L, X) \cong X^L$, $f \mapsto f(eK)$, including the induced $W_L K$-action. The left hand side is the value on G/K of a contravariant Hom-functor on the orbit category, and these bijections make fixed point sets into a contravariant functor $\mathrm{Or}(G) \to \mathrm{SET}$. ◇

(7.2.6) Multiplicative induction. Let $K \le L$ and L the (K,L)-set with translation actions. Then $\sigma(L)$ is called multiplicative induction from K to L. It is given on objects by $X \mapsto \mathrm{Hom}_K(L, X) = \operatorname{mul}^L_K(X)$. ◇

(7.2.7) Proposition. *Multiplicative induction is right adjoint to restriction. There is a natural bijection*

$$\mathrm{Hom}_H(\operatorname{res}^G_H X, Y) \cong \mathrm{Hom}_G(X, rm\ mul^G_H Y)$$

for H-sets Y, G-sets X, $H \le G$. □

It suffices to consider transitive (K,L)-sets S in order to understand more general morphisms. We will show that the basic constructions generate all morphisms. The group K acts as automorphism group on the L-set S. It sends an orbit into an isomorphic one. We can therefore assume that the L-set has the form

$$S = A\backslash L + \cdots + A\backslash L = nA\backslash L.$$

Since S is a transitive set, the group K acts by transitive permutation of the L-orbits. Therefore S has as (K,L)-set the form $S = K \times_B (A\backslash L)$, with $B \leq K$ acting on $A\backslash L$ via a homomorphism $\alpha\colon B \to W_L A$ into the automorphism group of $A\backslash L$. Therefore

$$S \cong (K \times_\alpha W_L A) \times_{W_L A} A\backslash L.$$

This shows how S is composed of basic morphisms.

We remark that $S \times_L X$ is additive in X, whereas $\mathrm{Hom}_K(S,X)$ is multiplicative in X. In special cases this is the distinction between additive and multiplicative induction. Note that fixed points and restrictions are additive and multiplicative as well.

Problems

1. Let $B \leq A$. If S is a finite A-set, then

$$\rho^A_B(S) = \textstyle\coprod_\alpha \{\alpha^{-1}(eB) \mid \alpha \in \mathrm{Hom}_A(S, A/B)\}$$

is a finite B-set. If $\gamma \in C(B)$ is an additive invariant for B-sets, then $S \mapsto \gamma(\rho^A_B(S))$ is an additive invariant for A. The induced homomorphism is $\mathrm{ind}^A_B\colon C(A) \to C(B)$.
2. Show that

$$\mathrm{Hom}_K(L,Y) \to \textstyle\prod_{lK\in L/K}(l_K \times KY), \quad \varphi \mapsto (l, \varphi(l^{-1}))$$

is an isomorphism of L-sets. Hence the multiplicative induction is given, up to isomorphism, by the product $\prod_{x\in L/K} X$, and the L-action permutes in a certain way the factors.
3. There is a dual description of the multiplicative induction. Let X be a K-space and $q\colon G \times_K X \to G/K$ the projection. The sections of p correspond to the G-maps $f\colon G \to X$ with the equivariance property $f(gk) = k^{-1}f(g)$. We have a G-action on the space $\Gamma(q)$ of all sections of p given by $(g\cdot s)(uK) = gs(g^{-1}uK)$. We assign to $\varphi \in \mathrm{Hom}_K(G,X)$ the map $\tilde\varphi\colon g \mapsto \varphi(g^{-1})$. Then $\tilde\varphi(gk) = k^{-1}\tilde\varphi(g)$, and we can view $\tilde\varphi$ as a section. In this way we obtain the G-isomorphism $\mathrm{Hom}_K(G,X) \to \Gamma(q)$, $\varphi \mapsto \tilde\varphi$.

7.3 The Burnside Ring $A(G;S)$

Let S be a finite G-set and G-Set$|S$ the category of finite G-sets over S. Denote by $A^+(G;S)$ the set of isomorphism classes of objects in this category. We define $A(G;S)$ as the Grothendieck group associated to $A^+(G;S)$ with respect to disjoint sum as addition. Additively it is the free abelian group on the set of isomorphism classes of objects of the form $f\colon G/H \to S$. Such an object is determined by $s = f(eH) \in S^H$, and we denote it by (H,s). The set of homogeneous G-sets over S can be identified with $\coprod_{H \le G} S^H$. We have a G-action on the set of objects $g \cdot (K,t) = (gKg^{-1}, gt)$. The orbits of this action correspond to isomorphism classes of homogeneous G-sets over S. Endomorphisms in this category are automorphisms. The automorphism group of (H,s) is WH_s, the isotropy group of $s \in S^H$ under the WH-action. The group $A(G;S)$ carries the structure of a commutative ring; the product is induced by the product in G-Set$|S$, i.e. by fibre product. We introduce the groups $A(G;S)$ in order to express certain formal properties of Burnside rings in a convenient manner.

(7.3.1) Example. The assignment $Y \mapsto (G \times_H Y \to G/H)$ induces an isomorphism between the categories H-Set and G-Set$|(G/H)$. It induces a ring isomorphism $A(H) \cong A(G; G/H)$. $\diamond$

(7.3.2) Example. The groups $A(G;S)$ are additive in the variable S: Restriction to subsets induces a ring isomorphism $A(G; S_1 + S_2) \cong A(G;S_1) \oplus A(G;S_2)$. $\diamond$

We see from 7.3.1 and 7.3.2 that $A(G;S)$ can be reduced to ordinary Burnside rings.

Let $C(G;S)$ denote the ring of $\mathbb{Z}$-valued functions on the set of isomorphism classes $\Phi(G;S)$ of homogeneous G-sets over S. Each $a\colon G/H \to S$ defines a mark homomorphism $\varphi_a\colon A(G;S) \to \mathbb{Z}$ which maps $f\colon X \to S$ to $|\operatorname{Hom}(a,f)|$. The set $\operatorname{Hom}(a,f)$ can be identified with $f^{-1}(s_a) \cap X^H$ where $s_a = a(eH)$. The ring homomorphism φ_a only depends on the isomorphism class of a. As in the case of the ordinary Burnside ring we combine the mark homomorphisms into a single ring homomorphism

$$\varphi\colon A(G;S) \to C(G;S), \quad f \mapsto (a \mapsto \varphi_a(f)), \tag{7.2}$$

and with an analogous proof we obtain:

(7.3.3) Proposition. *The mark homomorphism (7.2) is injective. The group $A(G;S)$ is additively the free abelian group on the isomorphism classes of homogeneous G-sets over S. The cokernel of φ is isomorphic to $\prod |\operatorname{Aut}(a)|$, $a \in \Phi(G;S)$.* $\square$

Problems

1. (Congruences for $A(G;S)$ by Möbius-inversion.) We have the mark isomorphism $A(G;S) \otimes \mathbb{Q} \cong C(G;S) \otimes \mathbb{Q}$. Suppose a function $\xi \in C(G;S) \otimes \mathbb{Q}$ is given. The pre-image in $A(G;S)\otimes\mathbb{Q}$ is a rational linear combination of the isomorphism classes of homogeneous G-sets over S. The coefficient of the isomorphism class (H,s) is:

$$|WH_s|^{-1}\sum_{K,H\le K\le G_s}\mu(H,K)\xi(K,s).$$

For the proof consider the G_s-space $X[s]=f^{-1}(s)\subset X$ for a G-set $f\colon X\to S$ over S. We have $s\in S^H$, hence $H\ge G_s$. In this G_s-space

$$X[s]^H=\coprod_{K,H\le K\le G_s}X[s]_K,$$

and therefore

$$|X[s]_H|=\sum_{K,H\le K\le G_s}\mu(H,K)|X[s]^K|.$$

The number is $|X[s]^K|$ the value of $\varphi_{(K,s)}(X)$.

7.4 The Induction Categories $\mathcal{A}$ and $\mathcal{B}$

Let R be a commutative ring. An R-category is a category where the Hom-sets $\mathrm{Hom}(X,Y)$ carry the structure of a (left) R-module such that composition of morphisms is R-bilinear. An R-functor between R-categories is a functor which is also R-linear on morphism modules. We denote by $\tilde{\mathcal{A}}$ the 2-category with objects the finite groups and morphisms from K to L the finite (K,L)-sets.

(7.4.1) Finite bisets. The category $\mathcal{A}^+$ has as objects the finite groups and $\mathrm{Mor}_{\mathcal{A}^+}(K,L)=\mathcal{A}^+(K,L)$ is the set of isomorphism classes of finite (K,L)-sets. Composition is again defined by $T\circ S=S\times_L T$.

The linearized version $\mathcal{A}$ has as objects the finite groups. The morphism module $\mathcal{A}(K,L)$ is the Grothendieck group associated to $\mathcal{A}^+(K,L)$ with respect to disjoint union. We can view a finite (K,L)-set S as a $K\times L$-set via $((k,l),s)\mapsto ksl^{-1}$. Then the morphism set $\mathcal{A}(K,L)$ can be identified with $A(K\times L)$. ◇

(7.4.2) The induction category. The category $\mathcal{B}$ has as objects the finite groups. The morphism set $\mathcal{B}(K,L)$ is the set of isomorphism classes of finite (K,L)-sets with free L-action. It is easily seen that the composition in $\mathcal{A}$ is compatible with this freeness condition. We obtain the subcategory $\mathcal{B}$ of $\mathcal{A}$. This category plays a fundamental role in representation theory, and we describe its morphism structure in some detail. It suffices to study transitive (K,L)-sets. A (K,L)-set S with free right L-action is, in topological terminology, a right principal L-bundle $S\to S/L$ with left K-action by bundle automorphisms. ◇

Let A be a subgroup of K and $\alpha\colon A\to L$ a homomorphism. Let $K\times_\alpha L$ be the quotient of $K\times L$ under $(ka,l)\sim(k,\alpha(a)l)$ for $a\in A$. With the obvious K- and L-actions this is a transitive (K,L)-set with free L-action.

(7.4.3) Proposition. *A transitive (K,L)-set with free L-action is isomorphic to a set of the form $K \times_\alpha L$. The data $(K, L, \alpha \colon A \to L)$ and $(K, L, \beta \colon B \to L)$ yield isomorphic (K,L)-sets if and only if there exist isomorphisms $c_u \colon A \to B$, $a \mapsto uau^{-1}$ with $u \in K$ and $c_v \colon L \to L$, $l \mapsto vlv^{-1}$ with $v \in L$ such that $c_v\alpha = \beta c_u$.*

Proof. Let S be a transitive set and $s \in S$. Let $A = \{a \in K \mid \exists l \in L, as = sl\}$. Since L acts freely, the relation $as = sl$ associates a unique $l = \alpha(a)$ to a. One verifies that α is a homomorphism. The map $K \times_\alpha L \to S$, $(k,l) \mapsto ksl$ is an isomorphism of (K,L)-sets.

Let $\varphi \colon K \times_\alpha L \to K \times_\beta L$ be an isomorphism. Let $\varphi(e,e) = (u^{-1}, v)$. Then c_u and c_v have the stated properties. Conversely, one verifies that c_u and c_v with properties as stated induce an isomorphism φ with $\varphi(e,e) = (u^{-1}, v)$. $\square$

Let $i \colon A \to K$ be an inclusion and $\alpha \colon A \to L$ a homomorphism. Let $K \times_{(\alpha,i)} L$ be the quotient of $K \times L$ under $(k \cdot i(a), l) \sim (k, \alpha(a) \cdot l)$. This is again a transitive (K,L)-set. The data $j \colon B \to K$ and $\beta \colon B \to L$ define an isomorphic (K,L)-set if and only if there exist $(u,v) \in K \times L$ and an isomorphism $\sigma \colon A \to B$ such that the following diagram commutes

$$\begin{tikzcd} K \arrow[d, "c_u"] & A \arrow[l, "i"'] \arrow[r, "\alpha"] \arrow[d, "\sigma"] & L \arrow[d, "c_v"] \\ K & B \arrow[l, "j"'] \arrow[r, "\beta"] & L. \end{tikzcd}$$

We specify the transitive morphisms in $\mathrm{Mor}(K,L)$ as isomorphism classes of diagrams

$$(\alpha|i) \colon K \xleftarrow{i} A \xrightarrow{\alpha} L,$$

where isomorphism is defined by the data 9.7.2.

Since composition is biadditive, it suffices to determine the composition of the basic morphisms of the type $(\alpha|i)$. We use the following symbols:

$$(\alpha|\,\mathrm{id}) = \alpha_\bullet, \qquad (\mathrm{id}\,|i) = i^\bullet. \tag{7.3}$$

Note that $i^\bullet$ is only defined for injections i. The equality $\alpha_\bullet = \beta_\bullet$ holds if and only if α and β differ by an inner automorphism of L. Similarly for the injections $i^\bullet$. From the definitions one verifies the elementary composition rules

$$(\alpha|i) = \alpha_\bullet i^\bullet, \quad (\alpha\beta)_\bullet = \alpha_\bullet \beta_\bullet, \quad (ij)^\bullet = j^\bullet i^\bullet. \tag{7.4}$$

The assignment $\alpha \mapsto \alpha_\bullet$ yields an injective functor from the category of finite groups and homomorphisms up to inner automorphism. The assignment $i \mapsto i^\bullet$ is a contravariant functor. There are other special cases for which composition is easy. Let

$$\begin{tikzcd} A & & \\ B \arrow[u, "i"] \arrow[r, "p"] & C & \\ Z \arrow[u, "J"] \arrow[r, "P"] & D \arrow[u, "j"] \arrow[r, "q"] & E \end{tikzcd}$$

be a diagram with injections i and j, surjections p and q, and a pullback rectangle. Then

$$(q|j) \circ (p|i) = (qP|iJ), \qquad j^\bullet p_\bullet = P_\bullet J^\bullet. \tag{7.5}$$

Let now

$$A \stackrel{i}{\leftarrow} B \stackrel{\alpha}{\rightarrow} C, \quad C \stackrel{j}{\leftarrow} D \stackrel{\beta}{\rightarrow} E$$

be diagrams with inclusions i and j. Write α as a composition kp with a surjection $p\colon B \to Q$ and an inclusion $k\colon Q \to C$. Suppose $j^\bullet k_\bullet = \sum_s \gamma(s)_\bullet \delta(s)^\bullet$ with injections $\gamma(s), \delta(s)$ (the sum corresponds to disjoint union of transitive sets). Then

$$\begin{aligned} (\beta|j) \circ (\alpha|i) &= \beta_\bullet j^\bullet k_\bullet p_\bullet i^\bullet \\ &= \sum_s \beta_\bullet \gamma(s)_\bullet \delta(s)^\bullet p_\bullet i^\bullet \\ &= \sum_s \beta_\bullet \gamma(s)_\bullet P_\bullet \Delta(s)^\bullet i^\bullet \\ &= \sum_s (\beta\gamma(s)P)_\bullet (i\Delta(s))^\bullet \end{aligned}$$

where the third equality is an application of (7.5) — with a notation suggested by the diagram above. Thus it remains to determine $j^\bullet k_\bullet$ for inclusions j and k by a so-called **double coset formula**.

The composition $j^\bullet k_\bullet$ is represented by the (Q, D)-set C, where the actions are given by left and right translation. The decomposition into transitive sets is precisely the decomposition into double cosets $Q\backslash C/D$. Let $X \subset C$ be a representing system of the double cosets. For $s \in X$ let $\delta(s)\colon Q \cap sDs^{-1} \subset Q$ and $\gamma(s)\colon Q \cap sDs^{-1} \to D, x \mapsto s^{-1}xs$. Then, with these notations,

$$j^\bullet k_\bullet = \sum_{s \in X} \gamma(s)_\bullet \delta(s)^\bullet. \tag{7.6}$$

If we write this in terms of induction and restriction, the double coset formula reads

$$\mathrm{res}^C_Q \, \mathrm{ind}^C_D = \sum_{QsD} \mathrm{ind}^Q_{sDs^{-1}\cap Q} \circ c(s)^* \circ \mathrm{res}^D_{D\cap s^{-1}Qs}. \tag{7.7}$$

Here $c(s)$ is a conjugation $x \mapsto s^{-1}xs$.

Problems

1. Verify **??**, **??**, and **??**.
2. The category of subquotients $\mathcal{U}$. The objects are the finite groups. The morphisms from K to L are the isomorphism classes of diagrams

$$(p|i)\colon K \xleftarrow{i} A \xrightarrow{p} L$$

with injections i and surjections p. The diagram $(q|j)$ is isomorphic to $(p|i)$ if there exists an isomorphism $\sigma\colon A \to B$ such that $j\sigma = i$ and $q\sigma = p$. Composition is defined by pullback as in **??**. We have a functor $\mathcal{U} \to \mathcal{B}$.
3. Let $j\colon D \to C$ be an inclusion and $\alpha\colon B \to Q$ a homomorphism. Then $j^\bullet\alpha_\bullet$ is represented by the (B,D)-set C with right action by translation and left action by translation via α.
4. Let G be a group. Define a $\mathbb{Z}$-category $S(G)$ with objects the finite subgroups of G. The morphism module $\mathrm{Mor}(K,L)$ is the free abelian group on the set of isomorphism classes of diagrams

$$(i|j)\colon K \xleftarrow{j} A \xrightarrow{i} L,$$

where i and j are conjugacy classes of homomorphisms of the form $x \mapsto gxg^{-1}$ with $g \in G$. The group G acts by conjugation on this category.
5. One can define a dual version $\mathcal{S}^\circ$ of $\mathcal{S}$: The morphisms from K to L are the isomorphism classes of (L,K)-sets. There is a contravariant functor $\mathcal{S} \to \mathcal{S}^\circ$; if S is a (K,L)-set, then we have the (L,K)-set S° with action $(k,s,l) \mapsto l^{-1}sk^{-1}$. This extends to 2-categories.
6. Construct a category $\mathcal{C}$ with object the finite groups and with $\mathcal{C}(K,L) = C(K\times L)$ such that the mark homomorphisms constitute a functor $\varphi\colon \mathcal{A} \to \mathcal{C}$.

7.5 The Burnside Ring as a Functor on A

Since $S\times_L X$ is additive in X, the assignment $X \mapsto S\times_L X = \rho(S)(X)$ vinduces an additive homomorphism $A(S)\colon A(L) \to A(K)$ for each finite (K,L)-set S. Since this is also additive in S, the construction extends to a contravariant $\mathbb{Z}$-functor $A\colon \mathcal{A} \to \mathbb{Z}\text{-}\mathrm{Mod}$. We still denote it ρ.

(7.5.1) Proposition. *There exists a unique functor $C\colon \mathcal{A} \to \mathbb{Z}\text{-}\mathrm{Mod}$ such that the mark homomorphisms are a natural transformation $\varphi\colon A \to C$.*

Proof. Since $\varphi\otimes\mathbb{Q}$ is an isomorphism there can exists at most one such functor. It suffices to construct the value of the functor on the basic morphisms: Restriction, Induction, and Orbit space.

Let $H \le G$. The induction homomorphism is $\mathrm{ind}_H^G\colon A(H) \to A(G), X \mapsto G\times_H X$. Define $\mathrm{ind}_H^G\colon C(H) \to C(G)$ by $(\mathrm{ind}_H^G\,\alpha)(J) = \sum\{\alpha(s^{-1}Js) \mid sH \in$

$G/H^J\}$; note: the conjugacy class of $s^{-1}Js$ does not depend on the representative s of the coset sH. Then the diagram

$$\begin{array}{ccc} A(H) & \xrightarrow{\varphi} & C(H) \\ \downarrow{\scriptstyle \mathrm{ind}_H^G} & & \downarrow{\scriptstyle \mathrm{ind}_H^G} \\ A(G) & \xrightarrow{\varphi} & C(G) \end{array}$$

is commutative. For the proof we note: We have the projection $p\colon G \times_H X \to G/H$. Let $sH \in G/H^J$ and hence $s^{-1}Js \le H$. The map

$$X^{s^{-1}Js} \to (G \times_H X)^J, \quad x \mapsto (s,x)$$

yields a bijection onto $p^{-1}(sH) \cap (G \times_H X)^J$.

Let $f\colon K \to L$ be a homomorphism. Viewing an L-set via f as K-set induces a ring homomorphism $f^*\colon A(L) \to A(K)$. The corresponding homomorphism $f^*\colon C(L) \to C(K)$ satisfies $(f^*(\alpha))(J) = \alpha(f(J))$ and $f^*\varphi = \varphi f^*$.

Let $L \lhd G$ be a normal subgroup of G. Then we have an additive homomorphism

$$A(G) \to A(G/L), \quad X \mapsto X/G.$$

It has the following description in terms of marks. Let $p\colon X \to X/L$ and $q\colon G \to G/L$ be the quotient maps. Fix $H \le G/L$, set $B = p^{-1}(X/L^H)$ and $P = q^{-1}(H)$. We consider X and B as P-sets. A P-orbit, which is isomorphic to P/U, is contained in B if and only if $P = LU$. Hence P is a union of P-orbit bundles of X. We have to compute $|B/L|$ in terms of P-marks of $Y = \mathrm{res}^G_P X$. We have the closed family $\mathcal{F} = \{U \le P \mid LU = P\}$. We can now compute $|X/L^H| = |Y(\mathcal{F})/L| = \sum_{(U)\in(\mathcal{F})} |Y(U)/L|$ by Möbius inversion in terms of marks. One uses that $L\backslash G/U$ is a point for $U \in \mathcal{F}$. □

The identity $G \times_H (X \times Y) \cong X \times (G \times_H Y)$ for G-sets X and H-sets Y yields

$$\mathrm{ind}_H^G(\mathrm{res}_H^G(x) \cdot y) = x \cdot \mathrm{ind}_H^G(y). \tag{7.8}$$

Therefore the image of the induction ind_H^G is an ideal in $A(G)$.

For the fixed points of an induction there is a formula which is similar in structure to the double coset formula. Suppose $L \le G \ge K$. Let X be an L-space. We determine $(G \times_L X)^K$ as $W_G K$-space. We have a canonical projection $p\colon (G \times_L X)^K \to G/L^K$. Therefore the $W_G K$-space $(G \times_L X)^K$ is the disjoint union of the $W_G K$-spaces $p^{-1}(A)$ where A runs through the $W_G K$-orbits of G/L^K. We also write $W(G,K) = W_G K$ and $\mathrm{fix}^G_{W(G,K)} X$ for X^K as $W(G,K)$-set. Then the **fixed point formula** reads

$$\mathrm{fix}^G_{W(G,K)} \mathrm{ind}^G_L(X) \cong \coprod \mathrm{ind}^{W(G,K)}_{W(s^{-1}Ls,K)} \circ c(s)^* \circ \mathrm{fix}^L_{W(L,s^{-1}Ks)}(X). \tag{7.9}$$

The summation is over $sL \in W(G,K)\backslash(G/L^k)$. Note that for $sL \in G/L^K$ the inclusion $s^{-1}Ks \leq L$ holds. The decomposition is induced by the inclusions $X^{s^{-1}Ks} \to (G \times_L X)^K$, $x \mapsto (s,x)$ already considered above.

The multiplicative induction induces also a map between Burnside groups. But since multiplicative induction is not additive, we cannot obtain this induced map directly from the universal property. Although an algebraic approach is possible, it is much more convenient to use the topological construction of the Burnside ring. Let S be a finite (K,L)-set and $U \leq L$. We begin with a computation of $\mathrm{Hom}_K(S,X)^U$ for a K-set X.

$$\begin{aligned}\mathrm{Hom}_K(S,X)^U &= \mathrm{Hom}_L(L/U, \mathrm{Hom}_K(S,X)) \\ &= \mathrm{Hom}(S \times_L L/U, X) = \mathrm{Hom}_K(S/U, X) \\ &= \textstyle\prod_j \mathrm{Hom}_K(K/K(j)) = \prod_j X^{K(j)}\end{aligned}$$

if $S/U \cong \coprod_j K/K(j)$ is a decomposition into K-orbits. We write $C(G)^\times$ for the multiplicative monoid of the ring $C(G)$. We define, using the previous definitions,

$$\sigma(S) = C^\times(S)\colon C^\times(K) \to C^\times(L), \quad \alpha \mapsto ((U) \mapsto \textstyle\prod_j \alpha(K(j))). \tag{7.10}$$

Then $\sigma(S)$ describes $X \mapsto \mathrm{Hom}_K(S,X)$ in terms of marks. We note that $\sigma(S)(\alpha \cdot \beta) = \sigma(S)(\alpha)\sigma(S)(\beta)$, i.e. $\sigma(S)$ is a morphism of monoids. The formula **??** makes sense, if we use instead of the $C(G)$ the multiplicative group $\mathrm{Map}(\mathrm{Con}(G), M)$, were M is any multiplicative abelian monoid. A simple verification from the definitions yields:

(7.5.2) Proposition. *We have $\sigma(T) \circ \sigma(S) = \sigma(T \times_L S) = \sigma(S \circ T)$. Hence the σ define a contravariant functor $C^\times$ from $\mathcal{A}^+$ into the category of multiplicative abelian monoids.* □

A topological definition of $A(G)$ can be given as follows. We call finite G-CW-complexes Euler equivalent, if for all $H \leq G$ the fixed point sets X^H and Y^H have the same Euler characteristic. Then $A(G)$ is the set of equivalence classes. Again, disjoint union induces addition and cartesian product multiplication. The computation above, now applied to a K-CW-complex X, shows that the class of $\mathrm{Hom}_K(S,X)$ in $A(L)$ only depends on the class of X in $A(K)$. The topological mark homomorphism φ_H associates to X the Euler characteristic $\chi(X^H)$ of X^H. Later we develop these matters in detail. From this topological definition we obtain a map $\sigma(S)\colon A(K) \to A(L)$. These maps constitute a contravariant functor σ from $\mathcal{A}$ to sets. The computation also shows, that the σ are compatible with the mark homomorphisms, i.e. the mark homomorphisms constitute a natural transformation also for the multiplicative induction functors.

Problems

1. Let $e(J;H) \in C(H)$ be the function which assumes the value 1 at (J) and the value zero otherwise. Then $\mathrm{ind}_H^G\, e(J;H) = |W_G J|/|W_H J| e(J;G)$.
2. Verify **??** in detail.
3. Let $A(G)^*$ denote the group of units in the ring $A(G)$. Since the maps $\sigma(S)\colon A(K) \to A(L)$ for a (K,L)-set S are multiplicative, they induce a homomorphism $\sigma(S)\colon A(K)^* \to A(L)^*$. In this way we obtain a contravariant functor $A^* = \sigma^*$ from $\mathcal{A}$ to abelian groups.

7.6 Representations of Finite Groups: Functorial Froperties

In order to fix the ideas, we consider in this section finite dimensional representations over a fixed ground field $\mathbb{K}$. We denote by $\mathrm{Rep}(G;\mathbb{K}) = \mathrm{Rep}(G)$ the category of such representations of the finite group G. We list several functorial constructions which relate representations of different groups.

(7.6.1) Restriction. Let V be a representation of L and $\varphi\colon K \to L$ a homomorphism. Let $\varphi^* V$ denote the representation space V together with the action $K \times V \to V, (k,v) \mapsto \varphi(k)\cdot v$. We obtain a functor $\varphi^*\colon \mathrm{Rep}(L) \to \mathrm{Rep}(K)$. The special case $\varphi\colon K \subset L$ of an inclusion is called **restriction** from L to K, in symbols res^L_K. ◇

(7.6.2) Induction. Let K be a subgroup of L, in symbols $K \leq L$. We associate to a K-representation V the **induced L-representation** $\mathrm{ind}_K^L V$. If we view a G-representation as a left module over the group ring $\mathbb{K}G$, then $\mathrm{ind}_K^L V = \mathbb{K}L \otimes_{\mathbb{K}K} V$. We obtain a functor $\mathrm{ind}_K^L\colon \mathrm{Rep}(K) \to \mathrm{Rep}(L)$. ◇

(7.6.3) Invariants. Let K be a subgroup of L and V an L-representation. Consider the subspace of K-fixed points V^K. This carries an induced action of the Weyl group $W_L K = N_L K/K$; here $N_L K$ is the normalizer of K in L. In this way we obtain a functor $\mathrm{Rep}(L) \to \mathrm{Rep}(W_L K)$. ◇

(7.6.4) Coinvariants. Let K be a subgroup of L and V an L-representation. Consider the subspace V_0 generated by the elements of the form $v - kv$ for $v \in V$ and $k \in K$. We denote the quotient space V/V_0 by V_K and call it the space of K-coinvariants. Let $n \in N_L K$. Then V_0 is stable under the left translation by n. Thus we obtain an induced action of $N_L K$ on V_K. It induces an action of $W_L K$. ◇

(7.6.5) Induction via bisets. Let K and L be finite groups. Let $\mathbb{K}S$ denote the free $\mathbb{K}$-module on a (K,L)-set S. The actions on S make this into a left $\mathbb{K}K$-module and a right $\mathbb{K}L$-module with commuting actions. We call such objects (K,L)-bimodules. Any such bimodule Σ induces a functor $\tau(\Sigma)\colon \mathrm{Rep}(L) \to \mathrm{Rep}(K)$, given on objects by $V \mapsto \Sigma \otimes_L V$ (the symbol is short-hand for the tensor product over $\mathbb{K}L$). A morphism of (K,L)-bimodules $\varphi\colon \Sigma_1 \to \Sigma_2$ induces a natural transformation $\tau(\varphi)\colon \tau(\Sigma_1) \to \tau(\Sigma_2)$. Thus τ is a functor from the category $K\text{-}\mathrm{Mod}\text{-}L$ of (K,L)-bimodules into the functor category $[\mathrm{Rep}(L), \mathrm{Rep}(K)]$. We write $\tau^L_K(S) = \tau(S)$, if the module Σ is $\mathbb{K}S$ for a (K,L)-set S. We have a natural isomorphism of functors $\tau^L_K(S) \circ \tau^M_L(T) \cong \tau^M_K(S \times_L T)$ due to the isomorphism $\mathbb{K}S \otimes_L \mathbb{K}T \cong \mathbb{K}(S \times_L T)$. ◇

(7.6.6) Coinduction via bisets. Suppose V is a K-representation and S a (K,L)-set. Let $\mathrm{Map}_K(S,V)$ denote the vector space of all K-equivariant maps $S \to V$. We obtain a left L-action on this mapping space by $(l \cdot f)(s) = f(sl)$. The assignment $V \mapsto \mathrm{Map}_K(S,V)$ obviously extends to a functor $\rho(S) = \rho^L_K(S)\colon \mathrm{Rep}(K) \to \mathrm{Rep}(L)$. A morphism of (K,L)-sets $f\colon S \to S'$ induces via composition a natural transformation $\rho(f)\colon \rho(S') \to \rho(S)$. Thus ρ^L_K is a contravariant functor from the category of (K,L)-sets into the functor category $[\mathrm{Rep}(K), \mathrm{Rep}(L)]$. ◇

(7.6.7) Representation groups. Let $R^+(G;\mathbb{K})$ denote the set of isomorphism classes of objects in $\mathrm{Rep}(G;\mathbb{K})$. This is a semi-ring with addition induced by direct sum and multiplication induced by tensor product. Let $R(G;\mathbb{K})$ denote the associated Grothendieck ring (representation ring). Let S be a (K,L)-set. The functors τ^L_K and ρ^L_K are compatible with direct sums and isomorphisms and induce homomorphisms of additive groups

$$t^L_K(S)\colon R(L;\mathbb{K}) \to R(K;\mathbb{K}), \quad r^L_K(S)\colon R(K;\mathbb{K}) \to R(L;\mathbb{K}).$$

The functors t and r are compatible with disjoint union of (K,L)-sets: $t(S \amalg T) = t(S) + t(T)$, $r(S \amalg T) = r(S) + r(T)$. These homomorphisms combine to functors t and r from $\mathcal{A}$ to abelian groups.

In addition to the additive constructions above there also exists a multiplicative induction for representations. Let V be a $\mathbb{K}K$-module. We set

$$m^G_K(V) = \bigotimes_{gK \in G/K} (gK \times_K V);$$

G acts by permutation of the tensor factors. Again it is not obvious that this construction can be extended to a map, called multiplicative induction,

$$m^G_K\colon R(K;\mathbb{K}) \to R(G;\mathbb{K}).$$

Multiplicative induction is compatible with permutation representations, i.e. the diagram

$$\begin{tikzcd} A(K) \arrow[r, "\pi(K)"] \arrow[d, "m_K^G"] & R(K;\mathbb{K}) \arrow[d, "m_K^G"] \\ A(G) \arrow[r, "\pi(G)"] & R(G;\mathbb{K}) \end{tikzcd}$$

is commutative. ◇

Problems

1. Compute the effect of the various induction functors on characters and use this to define corresponding functors on class functions.

7.7 The Induction Categories $\mathcal{A}_G$ and $\mathcal{B}_G$

The category $\mathcal{A}_G$ has as objects the finite G-sets. The morphism module from S to T is $A(G; S \times T)$. The composition is defined by a pullback construction below. We write a representing object $(a,b)\colon X \to S \times T$ also in form of a diagram

$$(b|a)\colon S \xleftarrow{a} X \xrightarrow{b} T.$$

Another diagram $(b'|a')\colon S \leftarrow X' \to T$ is, by definition, isomorphic to $(b|a)$ if there exists an isomorphism $\sigma\colon X \to X'$ such that $a'\sigma = a$ and $b'\sigma = b$. Thus $A(G; S \times T)$ is the free abelian group on the set of isomorphism classes of such diagrams with source a homogeneous G-set. The composition of the morphisms $(\alpha, \beta)\colon X \to S \times T$ and $(\gamma, \delta)\colon Y \to T \times U$ is given via the pullback diagram

$$\begin{tikzcd} Z \arrow[r, "\beta'"] \arrow[d, "\gamma'"] & X \arrow[r, "\alpha"] \arrow[d, "\beta"] & S \\ Y \arrow[r, "\gamma"] \arrow[d, "\delta"] & T & \\ U & & \end{tikzcd}$$

as $(\alpha\beta', \delta\gamma')\colon Z \to S \times U$. Associativity of composition follows from the transitivity of pullbacks.

Let P be a point. Then $A(G; S) = A(G; P \times S)$, and if we view this as a morphism module, then $A(G; -)$ becomes a covariant Hom-functor on $\mathcal{A}_G$.

The identification $A(G;S) = A(G;S \times P)$ makes it into a contravariant Hom-functor. In fact, $\mathcal{A}_G$ is canonically isomorphic to its dual, the isomorphism being given by switching factors $A(G;S \times T) \cong A(G;T \times S)$.

For certain applications it is more natural to consider the full subcategory $\mathcal{B}_G$ with object the homogeneous G-sets. If we identify $H \leftrightarrow G/H$ we obtain instead a category with objects the subgroups of G. There is a canonical functor of this category into $\mathcal{B}$. It is the identity on objects. A morphism $(b|a)\colon G/K \leftarrow G/A \to G/L$ is mapped to $(\beta|\alpha)\colon K \leftarrow A \to L$; here $\alpha(x) = u^{-1}xu$ if $a(eA) = uK$, and similarly for β. The verification of the functor property will be given later in a more general context (and may serve as an exercise at this point)

An additive $\mathbb{Z}$-invariant for finite G-sets over S associates to each $f\colon X \to S$ an integer $a(f) \in \mathbb{Z}$ such that $a(f + g\colon X + Y \to S) = a(f) + a(g)$ and $a(f) = a(g)$ if f and g are isomorphic over S. By the universal property of $A(G;S)$, these invariants correspond bijectively to elements of the group $C(G;S) = \mathrm{Hom}(A(G;S),\mathbb{Z})$. Each $f\colon X \to S$ provides us with an additive invariant. It assigns to $g\colon Y \to S$ the integer

$$\varphi((Y,g),(X,f) = \sum_{P \in Y/G} |\,\mathrm{Hom}((P,g),(X,f))_S|.$$

For each orbit $P \subset Y$ the Hom-set is the set of G-maps $b\colon P \to X$ over S, i.e. the set of G-maps b such that $fb = g$. The expression $\varphi((Y,g),(X,f))$ is additive in (X,f) and (Y,g) and induces therefore a $\mathbb{Z}$-bilinear map

$$\varphi\colon A(G;S) \times A(G;S) \to \mathbb{Z}, \quad (X,f),(Y,g) \mapsto \varphi((Y,g),(x,f)).$$

If we fix the first variable, we obtain a homomorphism, also called φ,

$$\varphi\colon A(G;S) \to C(G;S).$$

Let $\Phi(G;S)$ denote the set of isomorphism classes of homogeneous sets over S. The assignment $(f\colon G/H \to S) \mapsto f(eH)$ is a bijection $\mathrm{Hom}_G(G/H,S) \cong SH$. Elements $x \in S^H$ and $y \in S^K$ define isomorphic objects if and only if there exists $g \in G$ such that $gx = y$ and $K = gHg^{-1}$. We can identify $C(G;S)$ with the group $C(\Phi(G;S),\mathbb{Z})$ of all maps $\Phi(G;S) \to \mathbb{Z}$.

We construct a category $\mathcal{C}_G$ with objects the finite G-sets and with $\mathcal{C}_G(S,T) = C(G;S \times T)$ such that the maps φ constitute a functor $\mathcal{A}_G \to \mathcal{C}_G$. For this purpose we view again $C(G;S \times T)$ as the functions on $\Phi(G;S \times T)$. Let $I \in C(G;S_2 \times S_3)$ and $J \in C(G;S_1 \times S_2)$ be additive invariants. The composition $I \circ J \in \mathrm{Hom}(\Phi(G;S_1 \times S_3),\mathbb{Z})$ is defined to be the additive invariant which assigns to $(a,b)\colon G/H \to S_1 \times S_3$ the value

$$\sum_\sigma I(a,\sigma)J(\sigma,b) = (I \circ J)(a,b).$$

The sum is taken over the $\sigma\colon G/H \to S_2$. This construction is associative. The identity of S is the image of $(\mathrm{id}_S \,|\, \mathrm{id}_S)$ under φ; as a function, it assumes the value 1 at each orbit P of S, i.e. at $(P \subset S | P \subset S)$, and the value zero at the remaining isomorphism classes.

We verify that φ is a functor. Let the following diagram describe a composition of two morphisms.

$$\begin{tikzcd}
Z \arrow[r, "h_3"] \arrow[d, "h_1"] & Y \arrow[r, "g_3"] \arrow[d, "g_2"] & S_3 \\
X \arrow[r, "f_2"] \arrow[d, "f_1"] & S_2 & \\
S_1 & &
\end{tikzcd}$$

We compute the value of the composition at $(a_1, a_3)\colon G/H \to S_1 \times S_3$. It uses the set of $b\colon G/H \to Z$ such that $f_1 h_1 b = a_1$, $g_3 h_3 b = a_3$. Since the square is a pullback such b are in bijection to pairs $b_X : G/H \to X$, $b_Y : G/H \to Y$ such that

$$c = f_2 b_X = g_2 b_Y (= f_2 h_1 b = g_2 h_3 b),$$

with the additional properties

$$f_1 b_X = a_1, \quad g_3 b_Y = a_3.$$

There is no condition on C. If we fix c then we have to form the product of the invariants of X at (a_1, c) and of Y at (c, a_3). Hence

$$\begin{aligned}
&|\operatorname{Hom}((a_1, a_3), (f_1 h_1, g_3 h_3))| = \\
\sum_{c\colon G/H \to S_2} &|\operatorname{Hom}((a_1, c), (f_1, f_2))| \cdot |\operatorname{Hom}((c, a_3), (g_2, g_3))|
\end{aligned}$$

Problems

1. The category $\mathcal{A}_G$ has the additional structure of a tensor category with symmetric braiding. On objects the tensor product is simply the cartesian product $S \times T = S \otimes T$. The tensor product of morphisms is defined by cartesian product of representatives

$$\begin{aligned}
&((f, g)\colon X \to S \times T) \otimes ((f', g')\colon X' \to S' \times T') \\
&= (f \times f', g \times g')\colon X \times X' \to S \times S' \times T \times T').
\end{aligned}$$

This construction is compatible with disjoint union in the variables X and X'; therefore it induces a bilinear map

$$\operatorname{Mor}(S, T) \times \operatorname{Mor}(S', T') \to \operatorname{Mor}(S \otimes S', T \otimes T')$$

which is denoted $(f,g) \mapsto f \otimes g$ on morphisms. Since the product of two pullback squares is again a pullback square one sees that $\otimes$ is compatible with composition

$$(f' \otimes g') \circ (f \otimes g) = f'f \otimes g'g.$$

This tensor product is associative. A point is a neutral object. The symmetric braiding is given by $f \otimes g \mapsto g \otimes f$.

2. We can make C into a tensor category such that φ becomes a tensor functor. We define $C(S,T) \times C(S',T') \to C(S \times S', T \times T')$, $(\alpha,\beta) \mapsto \alpha \otimes \beta$ by

$$(\alpha \otimes \beta)\begin{pmatrix} X \\ \downarrow \\ S \times S' \times T \times T' \end{pmatrix} = \alpha \begin{pmatrix} X \\ \downarrow \\ S \times T \end{pmatrix} \beta \begin{pmatrix} X \\ \downarrow \\ S' \times T' \end{pmatrix}$$

Then one has to verify the naturality $(\alpha \otimes \beta) \circ (\gamma \otimes \delta) = (\alpha \circ \gamma) \otimes (\beta \circ \delta)$.

3. The functor $\varphi\colon \mathcal{A}_G \to \mathcal{C}_G$ is a tensor functor.

Chapter 8

Mackey Functors: Finite Groups

8.1 The Notion of a Mackey Functor

Certain parts of axiomatic representation theory can be based on the notion of a Mackey functor. We give an elementary introduction for finite groups. Later we generalize this to a topological context.

A **bifunctor** $M = (M_*, M^*) \colon C \to D$ between two categories C and D consists of a covariant functor M_* and a contravariant functor M^* which have the same value on objects $M(X) = M_*(X) = M^*(X)$.

A **Mackey functor** $M \colon G\text{-Set} \to R\text{-}\mathrm{Mod}$ is a bifunctor $M = (M_*, M^*)$ with the following properties:
(1) For each pullback in G-Set

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[d, "H"] & S \arrow[d, "h"] \\ T \arrow[r, "f"] & V \end{tikzcd}$$

the relation $F_* H^* = h^* f_*$ holds.
(2) The canonical inclusions $i_j \colon S_j \to S_1 + S_2$ induce an isomorphism $M(S_1) \oplus M(S_2) \to M(S_1 + S_2),\ x_1, x_2) \mapsto M(i_1)_* x_1 + M(i_2)_* x_2$.

We draw some consequences of the axioms. If we apply (2) to the empty sets $S_j = \emptyset$, we conclude $M(\emptyset) = 0$. Let $f\colon X \to Y$ be an isomorphism. Then

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pullback. Therefore f_* and f^* are inverse isomorphisms.The diagrams

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & S \arrow[d, "i"] \\ S \arrow[r, "i"] & S+T \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & T \arrow[d, "j"] \\ S \arrow[r, "i"] & S+T, \end{tikzcd}$$

with the canonical inclusions i, j, are pullbacks. Hence $i^* i_* = \mathrm{id}$ and $j^* i_* = 0$. One uses this to show that $M(S+T) \to M(S) \times M(T),\ x \mapsto (i^*(x), j^*(x))$ is an isomorphism too. Let

$$\begin{tikzcd} \coprod_s G/A_s \arrow[r, "\beta_s"] \arrow[d, "\alpha_s"] & G/L \arrow[d, "\beta"] \\ G/H \arrow[r, "\alpha"] & G/K \end{tikzcd}$$

be a pullback; in the diagram, α_s and β_s denote the restriction to the orbit G/A_s. In this situation axiom (1) says

$$\beta^* \alpha_* = \sum_s \beta_{s*} a_s^*. \tag{8.1}$$

Suppose α and β are projections induced by inclusions $H \le K, L \le K$. Choose a representative $HkL \subset H\backslash K/L$ for the double coset. Let $\alpha_k\colon G/(H \cap kLk^{-1}) \to G/H$ be the projection and let β_k be the composition of the conjugation $G/(H \cap kLk^{-1}) \to G/(k^{-1}Hk \cap L)$ with the projection to G/L. Then (8.1 holds; the sum is now over a representative system for the double cosets k instead over s. Thus axiom (1) is essentially a convenient reformulation of the double coset formula.

Let $\mathrm{Or}(G)$ denote the orbit category of homogeneous sets G/H and G-maps. This category is equivalent to the category $\mathrm{Tran}(G)$ of transitive G-sets. Thus functors from $\mathrm{Or}(G)$ and $\mathrm{Tran}(G)$ to an additive category $\mathcal{D}$ are in bijective correspondence. A functor from $\mathrm{Tran}(G)$ to an additive category $\mathcal{D}$ can be extended to a functor from the category $\mathrm{Set}(G)$ of finite G-sets by taking direct sums over orbits

$$F(X) = \bigoplus_{S \in X/G} F(S).$$

Let $i_S\colon F(S) \to F(X)$ and $p_S\colon F(X) \to F(S)$ be the canonical injection and projection associated to an orbit S of X. Let $\varphi\colon X \to Y$ be a morphism in $\mathrm{Set}(G)$. If $S \subset X$ is an orbit, there exists a unique orbit $T \subset Y$ with $\varphi(S) \subset T$. We define $F(\varphi)\colon F(X) \to F(Y)$ by

$$p_B F(\varphi) i_A = \begin{cases} f(\varphi\colon A \to B) & \text{if } \varphi(A) \subset B \\ 0 & \text{otherwise.} \end{cases}$$

Via this construction, functors $\mathrm{Tran}(G) \to \mathcal{D}$ correspond bijectively to those functors from $\mathrm{Set}(G) \to \mathcal{D}$ which are compatible with coproducts.

(8.1.1) Proposition. *Mackey functors correspond to bifunctors on* $\mathrm{Or}(G)$ *which satisfy the double coset formula* **??**. *We call these functors on* $\mathrm{Or}(G)$ *Mackey functors too.* □

Let G be a group. A *G-category* $\mathcal{C}$ is a category $\mathcal{C}$ together with an action of G by automorphisms, i. e., for each $g \in G$ a functor $c_g\colon \mathcal{C} \to \mathcal{C}$ is given such that $c_g c_h = c_{gh}$ and $c_e = \mathrm{id}(\mathcal{C})$. We write $c_g(X) = {}^gX$ for the values of the functor c_g.

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor from a G-category $\mathcal{C}$ into another category $\mathcal{D}$. A **G-invariance structure** $\zeta_\bullet$ for F is a family $\zeta_\bullet = (\zeta_g\colon F \to F \circ c_g)$ of natural isomorphisms ζ_g such that ζ_e is the identity and

$$\zeta_h(c_g(X)) \circ \zeta_g(X) = \zeta_{hg}(X)$$

for each object X of $\mathcal{C}$. A G-invariance structure $\zeta^\bullet$ for a contravariant functor F is a family of natural transformations $\zeta^g\colon F \circ c_g \to F$ such that $\zeta^g(X)\zeta^h(c_g(X)) = \zeta^{hg}(X)$.

An **equivariant bifunctor** consists of two equivariant functors $(M_*, \zeta_\bullet)$ and $(M^*, \zeta^\bullet)$ such that (M_*, M^*) is a bifunctor and $c^g c_g = \mathrm{id}$.

Let $\mathrm{Sub}(G)$ denote the category of subgroups of G and inclusions. This becomes a G-category via conjugation $c_g\colon H \mapsto gHg^{-1}$.

(8.1.2) Lemma. *The functors* $\varphi\colon \mathrm{Or}(G) \to R\text{-}\mathrm{Mod}$ *correspond bijectively to G-equivariant functors* $\Phi\colon \mathrm{Sub}(G) \to R\text{-}\mathrm{Mod}$ *with the additional property that* $\zeta_h\colon \Phi(H) \to \Phi(H)$ *is the identity if* $h \in H$.

Proof. Given φ, we define $\Phi(H) = \varphi(G/H)$ on objects. If $H \le K$, then $\Phi(H) \to \Phi(K)$ is φ, applied to the projection $G/H \to G/K$, and $\zeta_g\colon \Phi(H) \to \Phi(gHg^{-1})$ is φ, applied to $G/H \to G/gHg^{-1}$, $xH \mapsto xHg^{-1}$. These data yield an equivariant functor with the required properties.

Conversely, suppose (Φ, ζ) is given. We set $\varphi(G/H) = \Phi(H)$. Suppose $f\colon G/H \to G/K$, $xH \mapsto xgK$ with $g^{-1}Hg \subset K$ is a morphism. We define $\varphi(f)$ as the composition

$$\Phi(H) \xrightarrow{\zeta_g^{-1}} \Phi(g^{-1}Hg) \longrightarrow \Phi(K).$$

In order to see that this is well-defined, suppose $f(xH) = xuK$. Then $uK = gK$. We have a commutative diagram

$$\begin{tikzcd}
\Phi(H) \arrow[r, "\zeta_g^{-1}"] \arrow[d, "="] & \Phi(g^{-1}Hg) \arrow[r] \arrow[d, "\zeta_{u^{-1}g}"] & \Phi(K) \arrow[d, "\zeta_{u^{-1}g}"] \\
\Phi(H) \arrow[r, "\zeta_u^{-1}"] & \Phi(u^{-1}Hu) \arrow[r] & \Phi(K),
\end{tikzcd}$$

since the functor is equivariant. But the right most $c_{u^{-1}g} = \mathrm{id}$, by assumption, hence $\varphi(f)$ is well-defined. The functor property for φ is easily verified. □

We unravel the definition of a Mackey functor and give an equivalent definition in more elementary terms. The compatibility with coproducts shows that it suffices (on object level) to define the functor on the finite homogeneous spaces G/H. In this case, we can use the notation $M(G/H) = M(H)$ and think of a functor defined for subgroups H of in G.

For this purpose, we formulate axioms about restriction (R), induction (I), and conjugation (C), and therefore talk about RIC-functors.

An RIC-**functor** for the group G is an equivariant bifunctor $\mathrm{Sub}(G) \to R\text{-}\mathrm{Mod}$ with contravariant G-equivariant functor **restriction** $\mathrm{res}\colon \mathrm{Sub}(G) \to R\text{-}\mathrm{Mod}$ and covariant G-equivariant functor **induction** $\mathrm{ind}\colon \mathrm{Sub}(G) \to R\text{-}\mathrm{Mod}$. Denote the value on H by $M(H)$, and let $\mathrm{res}^L_K\colon M(L) \to M(K)$ and $\mathrm{ind}^L_K\colon M(K) \to M(L)$ denote the induced morphism ($K \le L$). These data are assumed to satisfy the following **double coset formula**: For subgroups L and H of K we have

$$\mathrm{res}^K_L \,\mathrm{ind}^K_H = \sum_{LkH \in L\backslash K/H} \mathrm{ind}^L_{kHk^{-1}\cap L} \circ \zeta(k) \circ \mathrm{res}^H_{H \cap k^{-1}Lk}\,.$$

The sum is taken over a representing system k of double cosets. One verifies that each summand is independent of the choice of k.

By **??**, a RIC-functor determines a Mackey functor, if the hypothesis **??** about the ζ_h holds. This gives the third definition of a Mackey functor.

The definition of a Mackey functor as a bifunctor separates the restriction and the induction process. One can also incorporate the basic pullback axiom (1) into the source category of the functor. Recall the category $\mathcal{A}_G$. Each bifunctor M which satisfies axiom (1) defines a contravariant functor M on $\mathcal{A}_G$. It assigns to a morphism represented by a diagram $(b|a)\colon S \leftarrow X \to T$ the homomorphism $M(b|a) = a_* b^*$. By axiom (1) this is compatible with composition of morphisms; and each bifunctor which satisfies axiom (1) arises in this way from a unique functor on $\mathcal{A}_G$. If we take the additivity axiom (2) into account, we can say (talking about R-functors into a given R-category):

(8.1.3) Proposition. *Mackey functors correspond bijectively to contravariant functors on $\mathcal{B}_G$.* □

8.2 Pairings of Mackey Functors

Let M, N, L be Mackey functors for G, in the guise of bifunctors on G-Set. An **internal pairing** $M \times N \to L$ is a family of bilinear maps

$$M(S) \times N(S) \to L(S), \quad (x, y) \mapsto x \cdot y,$$

one for each finite G-set S, with the following properties: Let $f \colon S \to T$ be a morphism between finite G-sets. Then

$$\begin{aligned} f^*(x \cdot y) &= f^*x \cdot f^*y \\ f_*(x \cdot f^*y) &= f_*x \cdot y \\ f_*(f^*x \cdot y) &= x \cdot f_*y. \end{aligned}$$

A **Green functor** is a Mackey functor U together with a pairing $U \times U \to U$ such that for each S the map $U(S) \times U(S) \to U(S)$ makes $U(S)$ into a commutative ring. Moreover the maps f^* are assumed to be unital ring homomorphisms.

Let U be a Green functor and M a Mackey functor. A pairing $U \times M \to M$ is called the structure of an U-**module** on M if each pairing morphism

$$U(S) \times M(S) \to M(S)$$

is a unital $U(S)$-module structure.

An **external pairing** $M \times N \to L$ consists of a family of bilinear maps

$$M(S) \times N(T) \to L(S \times T), \quad (x, y) \mapsto x \times y,$$

one for each pair S, T of finite G-sets, such that the following holds: Let $f \colon S \to S'$ and $g \colon T \to T'$ be morphisms between finite G-sets. Then

$$\begin{aligned} L^*(f \times g)(x \times y) &= M^*(f)x \times N^*(g)y \\ L_*(f \times g)(x \times y) &= M_*(f)x \times N_*(g)y. \end{aligned}$$

We can rephrase (**??**): The morphisms of an external pairing constitute a natural transformation of bifunctors on G-Set $\times$ G-Set.

We are going to establish a bijection between internal and external pairings. Let $\mu \colon M \times N \to L$ be an internal pairing. Let $p_S \colon S \times T \to S$ be the projection to S and similarly p_T the projection to T. We define bilinear maps

$$\nu(S, T) \colon M(S) \times N(T) \xrightarrow{p_S^* \times p_T^*} M(S \times T) \times N(S \times T) \xrightarrow{\mu} L(S \times T).$$

(8.2.1) Proposition. *The maps $\nu(S, T)$ just defined constitute an external pairing.* $\square$

Conversely, let an external pairing $M \times N \to L$ be given. Let $d_S \colon S \to S \times S$ denote the diagonal map. We define bilinear maps

$$\mu(S) \colon M(S) \times N(S) \to L(S \times S) \xrightarrow{d_S^*} L(S).$$

(8.2.2) Proposition. *The maps $\mu(S)$ just defined constitute an internal pairing.* □

One verifies that the two processes 8.2.1 and 8.2.2 are inverse to each other.

(8.2.3) Proposition. *The bilinear maps $A(G;S) \times A(G;T) \to A(G;S \times T)$ which are the cartesian product on representatives make A_G into a Green functor.* □

The next proposition expresses a universal property of the Green functor A_G.

(8.2.4) Proposition. *Each Mackey functor M is in a canonical way a module over A. The pairing is $A(G;S) \times M(S) \to M(S)$, $([f], x) \mapsto f_* f^* x$.* □

Problems

1. View Mackey functors as functors on the tensor category $\mathcal{A}_G$ and express the axioms of an external pairing.
2. Construct the representation ring as a Green functor. Let $K_G(S;\mathbb{K})$ denote the Grothendieck ring of finite dimensional G-vector bundles over the finite G-set S. If S is a point, then $K_G(S;\mathbb{K}) = R(G;\mathbb{K})$, canonically. We usually skip $\mathbb{K}$ in the notation. If $S = G/H$, then we have the canonical isomorphism

$$R(H) \to K_G(G/H), \quad V \mapsto (G \times_H V \to G/H).$$

We make K_G into a bifunctor. The contravariant part is defined by pullback of bundles. The induced morphisms are ring morphisms. The covariant part is defined as follows. Let $p\colon E \to S$ be a G-vector bundle and $f\colon S \to T$ a G-map. A bundle $q\colon X = f_*(E) \to T$ is defined by specifying the fibre over $t \in T$

$$X_t = q^{-1}(t) = \bigoplus_{s, f(s)=t} E_s, \qquad E_s = p^{-1}(s).$$

The G-action on $X = \coprod_{t \in T}$ is defined by $g\colon X_t \to X_{gt}$, $g(x) = gx$, $x \in E_s$. One verifies that $(E \to S) \mapsto (f_*E \to T$ is a covariant functor on isomorphism classes of G-vector bundles. It is compatible with direct sums and induces an additive homomorphism between the Grothendieck groups. The verification of the pullback property is straightforward. Thus we have a Mackey functor.

This functor incorporates restriction and induction of representations. Let $f\colon G/H \to G/K$ be the map $gH \mapsto guK$ with $u^{-1}Hu \subset K$. Then

$$R(K) \cong K_G(G/K) \xrightarrow{f^*} K_G(G/H) \cong R(H)$$

send the K-representation V to the H-representation $H \times V \to V$, $(h, v) \mapsto (u^{-1}hu)v$, i.e., combines restriction and conjugation. The covariant map induced by $f\colon G/H \to G/G$ is the induction ind_H^G. External tensor product of bundles induces a bilinear map $K_G(S) \times K_G(T) \to K_G(S \times T)$. These maps constitute an external pairing of Mackey functors and make K_G into a Green functor.

The canonical morphism of the Burnside functor into this Green functor $A_G \to K_G$ codifies permutation representations. The ring homomorphism $A(G) = A_G(G/G) \to K_G(G/G) = R(G)$ associates to the finite G-set S the permutation representation $\mathbb{K}(S)$, the free $\mathbb{K}$-module over S with induced G-action.

8.3 Green Categories

We associate to each Green functor U a $\mathbb{Z}$-category Ω_U. It has the property that the U-modules correspond to contravariant $\mathbb{Z}$-functors from Ω_U into abelian groups. Moreover, this category is self-dual.

We fix a Green functor U and call the category to be defined just Ω. The objects of Ω are the finite G-sets. The morphism group is defined to be

$$\mathrm{Mor}_\Omega(S,T) = U(S \times T).$$

The composition of morphisms $(f,g) \mapsto g \circ f$ is defined to be the following bilinear map:

$$U(X \times Y) \times U(Y \times Z) \to U(X \times Y \times Y \times Z) \to U(X \times Y \times Z) \to U(X \times Z).$$

The first map is the external pairing, the second map is contravariantly induced by the diagonal of Y, the third map is covariantly induced by the projection onto $X \times Z$. We have to verify associativity of the composition and the existence of units.

Associativity. By associativity and naturality of the pairing, we that $c \circ (b \circ a)$ for $a \in U(X_1 \times X_2)$, $b \in U(X_2 \times X_3)$, $U(X_3 \times X_4)$ is given by

$$p_{14*}(1 \times d(X_3) \times 1)^* p_{1334*}(1 \times d(X_2) \times 1)^*(a \times b \times c).$$

Here d denotes a diagonal and p a projection and the p-index indicates the indices of the remaining factors. We apply the pullback property to the morphisms in the middle and see that

$$c \circ (b \circ a) = p_* d^*(a \times b \times c)$$

where $p\colon X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_4$ is the projection and $d = 1 \times d(X_2) \times d(X_3) \times 1$ is the diagonal in the X_2- and X_3-factors. This expression for the composition is independent of the bracketing.

Identity. The identity of S is the element $d_*(1_S)$, with diagonal d of S and unit $1_S \in U(S)$. This is seen as follows. The composition of $d_*(1) \in U(Y \times Y)$ with $u \in U(Y \times Z)$ is given by

$$\mathrm{pr}_{13*}(D^* \,\mathrm{pr}_{12}^*\, d_*(1) \cdot D^* \,\mathrm{pr}_{34}^*\, u),$$

where pr_{ab} denotes the projection onto the factors a, b and $D = 1 \times d \times 1$. By the pullback property, we can write

$$\mathrm{pr}_{12}^* d_*(1_Y) = (d \times 1)_* \mathrm{pr}_1^*(1_Y) = (d \times 1)_*(1_{Y \times Z})$$

and by the pairing axiom we obtain

$$u \circ d_*(1) = \mathrm{pr}_{13*}(d \times 1)_*(1_{Y \times Z} \cdot (d \times 1)^* \mathrm{pr}_{23}^* u).$$

But $\mathrm{pr}_{13}(d \times 1) = \mathrm{id}$ and $\mathrm{pr}_{23}(d \times 1) = \mathrm{id}$, and hence $u \circ d_*(1) = u$.

(8.3.1) Example. The Green category associated to A_G is $\mathcal{A}_G$. ◇

(8.3.2) Proposition. *Let U be a Green functor. The maps*

$$\pi(S) \colon A(G; S) \to U(S), \quad [f] \mapsto f_* f^* 1$$

constitute a natural transformation of Green functors.

Proof. The assertion means that the maps are ring homomorphisms and constitute a morphism of Mackey functors.

Let $h \colon S \to T$ be given. Suppose

$$\begin{tikzcd} Z \arrow[r, "H"] \arrow[d, "F"] & X \arrow[d, "f"] \\ S \arrow[r, "h"] & T \end{tikzcd}$$

is a pullback. The following computations show the compatibility with the contravariant morphisms.

$$\pi(S) h^*[f] = \pi(S)[F] = F_* F^* 1$$

$$h^* \pi(T)[f] = h^* f_* f^*(1) = F_* H^* f^*(1) = F_* F^* H^*(1) = F_* F^*(1).$$

The functoriality of U yields directly the compatiblity with the covariant morphisms. Thus we have shown that we have a morphism of functors.

The computations with the data of the pullback above

$$\pi(T)([f][h]) = \pi(T)([fh]) = f_* H_*(1) = (fH)_*(1) = (hF)_*(1)$$

$$\pi(T)[f] \cdot \pi(T)[h] = f_* 1 \cdot h_* 1 = h_*(H^* f_* 1 \cdot 1) = h_* F_* H^*(1) = (hF)_*(1)$$

show the compatibility with the multiplication. □

8.4 Functors from Green Categories

The natural transformation of Green functors $\pi\colon A \to U$ induces a functor between the associated Green categories $\Omega_\pi\colon \Omega_A \to \Omega_U$. We use this functor and associate to a G-map $f\colon S \to T$ between finite G-sets the morphisms $\Omega_\pi(f_\bullet)$ and $\Omega_\pi(f^\bullet)$ which we again denote by $f_\bullet$ and $f^\bullet$. The morphism $f_\bullet\colon S \to T$ in Ω_U is given by $(1,f)_*1_S \in U(S \times T)$ and $f^\bullet$ is given by $(f,1)_*1_S \in U(T \times S)$.

(8.4.1) Proposition. *For each finite G-set S the map $\delta_S\colon U(S) \to \mathrm{Mor}_\Omega(S,S)$, $x \mapsto d_*x$ is a ring homomorphism.*

Proof. The following computation proves the assertion. Note that $U(S)$ is commutative.

$$\begin{aligned} & d_*(x_1x_2) = d_*d^*(x_1 \times x_2) = p_{13*}d_*^2 d^*(x_1 \times x_2) \\ = \quad & p_{13*}(1 \times d \times 1)^*(d \times d)_* = d_*x_2 \circ d_*x_1. \end{aligned}$$

The first equality comes from the relation between external and internal pairings. The second comes from $d = p_{13}d^2$, where p_{13} is the projection onto the factors 1 and 3, and $d^2\colon S \to S^3$ is the diagonal. The third equality comes from the pullback property of a Mackey functor. The forth equality uses the naturality of the external pairing $(d \times d)_* = d_* \times d_*$ and the definition of the composition. □

Proof. Let $f\colon S \to T$ be a morphism between finite G-sets. Then the following equalities hold for elements $a \in U(S)$ and $b \in U(T)$:

$$\begin{aligned} f_\bullet \circ \delta_S a &= (1 \times f)_* d_{S*} a \\ \delta_S a \circ f^\bullet &= (f \times 1)_* d_{S*} a \\ f^\bullet \circ \delta_T b &= (1 \times)^* d_{T*} b \\ \delta_T b \circ f_\bullet &= (f \times 1)^* d_{T*} b \end{aligned}$$

We verify the first identity. The proof of the remaining is along similar lines.

$$\begin{aligned} f \circ \delta_S a &= \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\, d_{S*}a \cdot \mathrm{pr}_{23}^*(1,f)_*1_S) \\ &= \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\, d_{S*}a \cdot (1 \times 1, f)_*\, \mathrm{pr}_2^*\, 1_S) \\ &= \mathrm{pr}_*(1 \times 1, f)_*((1 \times 1, f)^*\, \mathrm{pr}_{12}^*\, d_{S*}a) \\ &= (1 \times f)_* d_{S*} a. \end{aligned}$$

The first equality is the definition, the second uses the pullback property, the third the pairing axioms, and the fourth uses identities between the morphisms involved. □

(8.4.2) Proposition. *For $a \in U(S)$ and $b \in U(T)$ the following identities hold:*

$$\begin{aligned} f_\bullet \circ \delta_S f^*(b) &= \delta_T \circ f_\bullet \\ \delta_S f^*(b) \circ f^\bullet &= f_\bullet \circ \delta_T(b) \\ f_\bullet \circ \delta_S(a) \circ f^\bullet &= \delta_T f_*(a). \end{aligned}$$

Proof. We verify the first one. The remaining are treated similarly. We insert the relevant identity of the previous proposition and use the pullback identity $(1 \times f)_* d_{S*} f^* = (f \times 1)^* d_{T*}$. □

(8.4.3) Theorem. *The $\mathbb{Z}$-functors $\Omega_U \to \mathbb{Z}$-Mod correspond bijectively to U-modules.*

Proof. Suppose a functor $M \colon \Omega \to \mathbb{Z}$-Mod is given. We have to define bilinear maps $U(S) \times M(S) \to M(S)$. Instead, we define the adjoint linear maps

$$U(S) \to \mathrm{Hom}(M(S), M(S))$$

as the composition of δ_S with the functor maps

$$\mathrm{Mor}_\Omega(S,S) \to \mathrm{Hom}(M(S), M(S)), \quad f \mapsto M(f).$$

By definition, these are ring homomorphism and therefore they are a module structure. We verify the pairing axioms. They are a direct consequence of the previous proposition: When we apply the functor M, we obtain the adjoint versions of the axioms for an internal pairing.

Conversely, let a pairing be given. The desired functor consists of linear maps $U(S \times T) \to \mathrm{Hom}(M(T), M(S))$. These are defined as the composition

$$U(S \times T) \times M(T) \xrightarrow{1 \times \mathrm{pr}_T^*} U(S \times T) \times M(S \times T) \to M(S \times T) \xrightarrow{\mathrm{pr}_{S*}} M(S).$$

We verify the functor property. Let $x \in U(S \times T)$ and $y \in U(T \times Y)$ and $z \in M(Y)$. Then, by the definitions,

$$y(x(z)) = \mathrm{pr}_{S*}(x \cdot \mathrm{pr}_T^* \, \mathrm{pr}_{T*}(y \cdot \mathrm{pr}_Y^* \, z)).$$

The index indicates the range factors of the projection; this does not specify the maps uniquely, though. The following diagram displays some of the morphisms we are going to use. The center is a pullback square.

$$\begin{tikzcd} & S \times T \times Y \arrow[r, "\mathrm{pr}_{TY}"] \arrow[d, "\mathrm{pr}_{ST}"] & T \times Y \arrow[r, "\mathrm{pr}_Y"] \arrow[d, "\mathrm{pr}_T"] & Y \\ S & S \times T \arrow[l, "\mathrm{pr}_S"'] \arrow[r, "\mathrm{pr}_T"] & T & \end{tikzcd}$$

We use the pairing axioms and the pullback property to rewrite the element in question.

$$\begin{aligned}
y(x(z)) &= \mathrm{pr}_{S*}(x \cdot \mathrm{pr}_{ST*}\, \mathrm{pr}^*_{TY}(y \cdot \mathrm{pr}^*_Y\, z)) \\
&= \mathrm{pr}_{S*}\, \mathrm{pr}_{ST*}(\mathrm{pr}^*_{ST}\, x \cdot \mathrm{pr}^*_{TY}\, y \cdot \mathrm{pr}^*_{TY}\, \mathrm{pr}^*_Y\, z) \\
&= \mathrm{pr}_{S*}\, \mathrm{pr}_{SY*}(\mathrm{pr}^*_{ST}\, x \cdot \mathrm{pr}^*_{TY}\, y \cdot \mathrm{pr}^*_{SY}\, \mathrm{pr}^*_Y\, z) \\
&= \mathrm{pr}_{S_*}(\mathrm{pr}_{SY*}(\mathrm{pr}_{ST^*}\, x \cdot \mathrm{pr}_{TY*}\, y) \cdot \mathrm{pr}^*_Y\, z) \\
&= \mathrm{pr}_{SY*}((y \circ x) \cdot \mathrm{pr}^*_Y\, z) \\
&= (y \circ x)(z)
\end{aligned}$$

Finally, the computation

$$\mathrm{pr}_{S*}(d_{S*}1 \cdot \mathrm{pr}^*_S\, z) = \mathrm{pr}_{S*}\, d_{S*}1 \cdot z = z$$

shows that the morphism $d_{S*}1$ acts as the identity.

One verifies that the two processes which we have described are inverse to each other. $\square$

We have in particular the contravariant Hom-functor in Ω_U. This corresponds to the U-module U. We mention that $x \in U(S \times T) = \mathrm{Mor}(S, T)$ sends $a \in U(T)$ to $\mathrm{pr}_{S*}(x \cdot \mathrm{pr}^*_T\, a)$. In the case $x = d_{S*}v$ this equals $v \cdot a$.

8.5 Amitsur Complexes

Let $\Omega = \Omega_U$ denote the Green category associated to a Green functor U and let $M \colon \Omega \to R\text{-}\mathrm{Mod}$ be a contravariant functor (= U-module).

For each finite G-set S we obtain a new functor M_S. It is defined on objects by $M_S(T) = M(S \times T)$. Each morphism $f \colon T \to T'$ in Ω has an associated morphism $\mathrm{id}_S \times f \colon S \times T \to S \times T'$. If we apply the same construction to the first variable in the product $S \times T$, we see that each morphism $h \colon S \to S'$ induces a natural transformation (= morphism of U-modules)

$$M_h \colon M_S \to M_{S'}.$$

Altogether we see that $M \colon \Omega \to R\text{-}\mathrm{Mod}$ yields a contravariant functor

$$M_\bullet \colon \Omega \to [\Omega^{op}, R\text{-}\mathrm{Mod}]$$

into the functor category of U-modules.

Recall that a G-map $f \colon S \to T$ yields morphisms $f_\bullet S \to T$ and $f^\bullet \colon T \to S$ in Ω. Let $\mathrm{pr} \colon S \to G/G$ be the projection onto a point. Then we have in particular the morphisms of U-modules

$$\Theta^S = M_{\mathrm{pr}_\bullet} \colon M \to M_S, \qquad \Theta_S = M_{\mathrm{pr}^\bullet} \colon M_S \to M.$$

The U-module M is called **S-injective** (**S-projective**) if $\Theta^S\colon M \to M_S$ ($\Theta_S\colon M_S \to M$) is a split injective (split surjective) morphism of U-modules.
We explain the meaning of these terms.

(8.5.1) Theorem. *The following assertions about M are equivalent:*
(1) *M is S-injective.*
(2) *M is S-projective.*
(3) *M is a direct summand of M_S.*

Proof. Proof. (1), (2) $\Rightarrow$ (3). This is a direct consequence of the definitions.
(3) $\Rightarrow$ (1). By assumption, we have morphisms $\Theta\colon M \to M_S$ and $\Psi\colon M_S \to M$ such that $\Psi\Theta = \mathrm{id}$. We have to find a morphism $\Psi^S\colon M_S \to M$ such that $\Psi^S\Theta^S = \mathrm{id}$. We define $\Psi^S(T)$ by the following commutative diagram

$$\begin{tikzcd}
M(T) \arrow[r, "\mathrm{pr}^*=\Theta^S(T)"] \arrow[d, "\Theta(T)"] & M(S\times T) \arrow[r, "\Psi^S(T)"] \arrow[d, "\Theta(S\times T)"] & M(T) \\
M(S\times T) \arrow[r, "(\mathrm{id}_S\times\mathrm{pr})^*"] & M(S\times S\times T) \arrow[r, "(d_S\times\mathrm{id})^*"] & M(S\times T). \arrow[u, "\Psi(T)"']
\end{tikzcd}$$

The left square commutes, since Θ is a natural transformation. We have $(\mathrm{id}_S \times \mathrm{pr})(d_S \times \mathrm{id}) = \mathrm{id}$, so that the bottom composition is the identity. Since $\Psi(T)\Theta(T) = \mathrm{id}$, we conclude $\Psi^S(T)\Theta^S(T) = \mathrm{id}$.
Claim: the $\Psi^S(T)$ constitute a morphism of U-modules. This is seen as follows. Firstly, Ψ is a morphisms. Secondly; $M_{S\times S} \to M_S$, induced by d_S, is a morphism. Thirdly, if $\Theta\colon M \to N$ is a morphism, then for each S the $\Theta\colon M(S\times ?) \to N(S\times ?)$ constitute a morphism.
(2) $\Rightarrow$ (1). This is proved by a similar reasoning. $\square$

Let S be a finite G-set. We set $S^0 = G/G$ and $S^k = \prod_{i=0}^{k-1}$ for $k \geq 1$. We have the projection $\mathrm{pr}_i\colon S^{k+1} \to S^k$ which omits the i-th factor $(0 \leq i \leq k)$. For each U-module M we have the two chain complexes

$$0 \to M(S^0) \xrightarrow{d^0} M(S^1) \xrightarrow{d^1} M(S^2) \xrightarrow{d^2} \dots$$

$$0 \leftarrow M(S^0) \xleftarrow{d_0} M(S^1) \xleftarrow{d_1} M(S^2) \xleftarrow{d_2} \dots$$

with differentials

$$d^k = \sum_{i=0}^{k} (-1)^i p_i^*, \qquad d_k = \sum_{i=0}^{k} (-1)^i p_{i*}.$$

They are called **Amitsur complexes**.

(8.5.2) Theorem. *Let M be a U-module. Then:*
(1) *For each S the U-module M_S is S-injective and S-projective.*

(2) *If M is S-injective, then the complexes* **??** *and* **??** *are acyclic.*

Proof. (1) We have to construct a splitting of $\Theta^S \colon M_S \to (M_S)_S$ in order to exhibit M_S as S-injective. But Θ^S equals

$$\mathrm{pr}_{23}^* \colon M(S \times T) \to M(S \times S \times T),$$

and a splitting is given by

$$(d_S \times \mathrm{id})^* \colon M(S \times S \times T) \to M(S \times T).$$

Similarly $\Theta_S = \mathrm{pr}_{23*}$ and a splitting is given by $(d_S \times \mathrm{id})_*$.

(2) Let Ψ be a splitting of Θ^s. We construct a chain contraction for **??**. A null homotopy of the identity consists of homomorphisms $s^{k+1} \colon M(S^{k+1}) \to M(S^k)$ which satisfy $s^{k+1}d^k + d^{k-1}s^k = \mathrm{id}$ for $k \geq 0$. We set

$$s^{k+1} = \Psi(S^k) \colon M(S \times S^k) \to M(S^k).$$

Since Ψ is a natural transformation, the diagrams $(0 \leq i \leq k-1)$

$$\begin{tikzcd} M(S \times S^k) \arrow[r, "s^{k+1}"] & M(S^k) \\ M(S \times S^{k-1} \arrow[u, "p_{i+1}^*"] \arrow[r, "s^k"] & M(S^{k-1}) \arrow[u, "p_i^*"'] \end{tikzcd}$$

are commutative. Therefore

$$s^{k+1}d^k + d^{k-1}s^k = s^{k+1} \circ (\sum_{i=0}^{k} (-1)^i p_i^*) + (\sum_{i=0}^{k-1} (-1)^i p_i^*) \circ s^k = s^{k+1} \circ p_0^*,$$

and this is the identity, since Ψ is a splitting of Θ^S.

Since M is S-injective it is also S-projective. Let Γ be a splitting of Θ_S. Then the maps

$$s_k = \Gamma(S^k) \colon M(S^k) \to M(S \times S^k)$$

are a chain contraction of the complex **??**. □

(8.5.3) Example. We explain the meaning of the terms S-injective and S-projective. Suppose M is S-injective. We decompose S into orbits $S = \coprod_j G/H(j)$. Then $M(S) = \prod_j M(G/H(j))$, and $p^* \colon M(G/G) \to M(S)$ consists of the restriction maps $\mathrm{res}^G_{H(j)} \colon M(G/G) \to M(G/H(j))$. The injectivity of p^* thus says that elements of $M(G/G)$ are detected by restriction to the subgroups $H(j)$.

The exactness of the sequence

$$0 \to M(G/G) \xrightarrow{p^*} M(S) \xrightarrow{\mathrm{pr}_1^* - \mathrm{pr}_2^*} M(S \times S)$$

says that we can also characterize the image as a difference kernel. The orbits of $S \times S$ have the form

$$G/K, \qquad K = H(i) \cap gH(j)g^{-1}.$$

An element $(x_j) \in \prod_j M(H(j))$ is contained in the difference kernel if and only if for each pair (i,j) and each $g \in G$ the images of x_i under $\mathrm{res}^{H(i)}_{H(i)\cap gH(j)g^{-1}}$ and of x_j under $M(c_g)\,\mathrm{res}^{H(j)}_{H(j)\cap g^{-1}H(i)g}$ coincide. This coincidence is necessary by the functor property of M. In particular, the Weyl group $WH(j)$ acts on $M(G/H(j))$, and the restriction from $M(G/G)$ to $M(G/H(j))$ is contained in the invariants under this action. For this reason, we sometimes call the whole difference kernel the subgroup of invariant elements.

The difference kernel can be interpreted as an inverse limit. For this purpose we use the category $\mathcal{C}/S$ of homogeneous set G/K over S. Then the projection maps $M(G/G) \to M(G/K)$ yield a map into the inverse limit of the $M(G/K)$ over $\mathcal{C}/S$. The exactness of the sequence above says that this map

$$M(G/G) \to \lim_{\mathcal{C}/S} M(G/K)$$

is an isomorphism.

There are dual results for S-projective M. In this case each element in $M(G/G)$ is a sum of elements induced from $M(G/H(j))$ and the kernel of the induction map $p_*\colon M(S) \to M(G/G)$ is a difference cokernel alias colimit. $\diamond$

The Amitsur complexes can be generalized as follows. Let X and Y be finite G-set. We set

$$a(X,Y) = Y, \quad a_r(X,Y) = X \times a_{r-1}(X,Y) = X^r \times Y.$$

We have morphism

$$d_i^r\colon a_r(X,Y) \to a_{r-1}(X,Y), \qquad 0 \le i < r$$

as follows

$$d_0^r\colon X \times a_{r-1}(X,Y) \to a_{r-1}(X,Y), \qquad \text{projection}$$
$$d_i^r = \mathrm{id}_X \times d_{i-1}^{r-1}, \qquad i > 0.$$

Previously we considered the case that Y is a point. In an additive category we can form the chain complex $a_*(X,Y)$ with differential

$$\partial_r = \sum_{i=0}^{r-1} (-1)^i d_i^r.$$

There are also contravariant versions.

Chapter 9

Induction Categories: An Axiomatic Setting

9.1 Induction categories

We begin with an axiomatic setup. Let $\mathcal{C}$ be a category with a set of isomorphism classes of objects. Let R be a commutative ring. An R-category is a category where the set of morphisms $\mathrm{Hom}(A, B)$ between any two objects A, B carries the structure of a left R-module and where composition

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C), \quad (g, f) \mapsto g \circ f$$

is bilinear. An R-functor $F \colon \mathcal{C} \to \mathcal{D}$ between R-categories is a functor which is R-linear on the morphism modules $F \colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(FA, FB)$. We denote by R-Mod the R-category of left R-modules.

We consider diagrams in $\mathcal{C}$

$$(\beta|\alpha) \colon A \xleftarrow{\alpha} X \xrightarrow{\beta} B.$$

The diagram $(\beta|\alpha)$ is isomorphic to the diagram

$$(\beta'|\alpha') \colon A \xleftarrow{\alpha'} X' \xrightarrow{\beta'} B$$

if there exists an isomorphism $\sigma \colon X \to X'$ in $\mathcal{C}$ such that $\alpha'\sigma = \alpha$, $\beta'\sigma = \beta$. If $\mathcal{C}$ has products, then a diagram $(\beta|\alpha)$ corresponds to a morphism $X \to B \times A$, and isomorphism of diagrams corresponds to isomorphism of objects in the category of objects over $B \times A$.

(9.1.1) Induction categories. An **induction category** $I\mathcal{C}$ for $\mathcal{C}$ is an R-category with the following properties:

(1) $\mathrm{Ob}(\mathcal{C}) = \mathrm{Ob}(I\mathcal{C})$.
(2) For $A, B \in \mathrm{Ob}(\mathcal{C})$ the morphism set $I\mathcal{C}(A,B)$ is the free R-module on the set of isomorphism classes of diagrams in $\mathcal{C}$

$$(\beta|\alpha)\colon A \xleftarrow{\alpha} X \xrightarrow{\beta} B.$$

(3) The following rules hold for the composition in $I\mathcal{C}$:

$$(\alpha|\,\mathrm{id})\circ(\beta|\,\mathrm{id}) = (\alpha\beta|\,\mathrm{id}),\ (\mathrm{id}\,|\gamma)\circ(\mathrm{id}\,|\delta) = (\mathrm{id}\,|\delta\gamma),\ (\beta|\alpha) = (\beta|\,\mathrm{id})\circ(\mathrm{id}\,|\alpha).$$

(4) Suppose $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$ with $n_s \in R$. Then for each s the equality $\alpha\alpha_s = \beta\beta_s$ holds.

If the assignment $(\alpha|\beta) \mapsto (\beta|\alpha)$ extends to an R-functor D from $I\mathcal{C}$ into the dual category $I\mathcal{C}^{op}$ we call $I\mathcal{C}$ an induction category with self-duality. ◇

For the moment the ground ring R will be fixed and is therefore not recorded in the notation of the category. We discuss the axioms.

(9.1.2) The assignment $\alpha \mapsto (\alpha|\,\mathrm{id})$ is a covariant functor $\iota_*\colon \mathcal{C} \to I\mathcal{C}$ which is the identity on objects. ◇

(9.1.3) The assignment $\beta \mapsto (\mathrm{id}\,|\beta)$ is a contravariant functor $\iota^*\colon \mathcal{C} \to I\mathcal{C}$ which is the identity on objects. ◇

(9.1.4) From (9.7.2) and associativity of composition we obtain the rules

$$(\alpha|\,\mathrm{id}) \circ (\beta|\gamma) = (\alpha\beta|\gamma), \quad (\beta|\gamma) \circ (\mathrm{id}\,|\delta) = (\beta|\delta\gamma).$$

The identity of A in $I\mathcal{C}$ is represented by $(\mathrm{id}_A\,|\,\mathrm{id}_A)$; this follows from (9.7.2). Diagrams $(\alpha|\,\mathrm{id})$ and $(\alpha'|\,\mathrm{id})$ are isomorphic if and only if $\alpha = \alpha'$. Therefore ι_* is an embedding of $\mathcal{C}$. We identify $\mathcal{C}$ via ι_* with a subcategory of $I\mathcal{C}$. Similarly, ι^* yields an embedding of the dual category $\mathcal{C}^{op}$ into $I\mathcal{C}$. Since, by (9.7.2), $(\beta|\,\mathrm{id}) \circ (\mathrm{id}\,|\alpha) = (\beta|\alpha)$, we see that the images of ι_* and ι^* span $I\mathcal{C}$. We call β the **covariant** and α the **contravariant component** of $(\beta|\alpha)$. ◇

For an isomorphisms σ in $\mathcal{C}$ we record the following special relations

$$\begin{aligned}(\beta|\alpha) &= (\beta\sigma|\alpha\sigma)\\ (\sigma|\,\mathrm{id}) &= (\mathrm{id}\,|\sigma)^{-1}\\ (\mathrm{id}\,|\beta)\circ(\sigma|\,\mathrm{id}) &= (\mathrm{id}\,|\sigma^{-1}\beta)\\ (\mathrm{id}\,|\sigma)\circ(\alpha|\,\mathrm{id}) &= (\sigma^{-1}\alpha|\,\mathrm{id}).\end{aligned}$$

Proof. The first one stems from the isomorphism definition of diagrams and the second one is a special case of the first one. The third and fourth one are a consequence of the second and axiom (9.7.2ass1.3). □

We display an identity of the type

$$(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s) = \sum_s n_s(\alpha_s|\,\mathrm{id}) \circ (\mathrm{id}\,|\beta_s)$$

in the form of a diagram

$$\begin{tikzcd} \sum_s n_s Z_s \arrow[r, "\alpha_s"] \arrow[d, "\beta_s"] & A \arrow[d, "\alpha"] \\ B \arrow[r, "\beta"] & X \end{tikzcd}$$

and think of it as a replacement for a pullback diagram of (β, α). The identity (**??**) then says that both compositions from B to A yield the same result; here we have to read the vertical morphisms as morphisms in the dual category. In the sequel we refer to this diagram as a **pullback** and call $(n_s, Z_s, \beta_s, \alpha_s)$ the **pull back data** of (β, α). The **transitivity of pullbacks** is implicitly contained in the associativity of composition in a category, if applied to $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) \circ (\beta'|\,\mathrm{id})$. Explicitly, it amounts to the following: Let

$$\begin{tikzcd} \sum_t m_{st} Y_{st} \arrow[r, "\beta_{st}"] \arrow[d, "\beta'_{st}"] & Z_s \arrow[d, "\beta_s"] \\ B' \arrow[r, "\beta'"] & B \end{tikzcd}$$

be the pullback data for (β', β_s); in the summation, the index t runs through some set $I(s)$. Then the diagram

$$\begin{tikzcd} \sum_{s,t} n_s m_{st} Y_{st} \arrow[r, "\alpha_s\beta_{st}"] \arrow[d, "\beta'_{st}"] & A \arrow[d, "\alpha"] \\ B' \arrow[r, "\beta\beta'"] & X \end{tikzcd}$$

displays the pullback data for $(\beta\beta', \alpha)$. In fact, these relations are the basic ones:

(9.1.5) Proposition. *Suppose for each pair $\alpha\colon A \to X$ and $\beta\colon B \to X$ a composition*

$$(1|\alpha) \circ (\beta|1) = \sum_s n_s(\alpha_s|\beta_s)$$

with $n_s \in R$ and $\alpha\alpha_s = \beta\beta_s$ is given such that for each isomorphism σ

$$(1|\sigma) \circ (\beta|1) = (\sigma^{-1}\beta|1), \quad (1|\alpha) \circ (\sigma|1) = (1|\sigma^{-1}\alpha)$$

and such that

$$
\begin{aligned}
((1|\alpha_1) \circ (1|\alpha_2)) \circ (\beta|1) &= (1|\alpha_1) \circ ((1|\alpha_2) \circ (\beta|1)) \\
(1|\alpha) \circ ((\beta_1|1) \circ (\beta|1)) &= ((1\alpha) \circ (\beta_1|1)) \circ (\beta_2|1)
\end{aligned}
$$

whenever these expressions make sense. Then

$$(\alpha_1|\alpha) \circ (\beta|\beta_1) = \sum_s n_s(\alpha_1\alpha_s|\beta_1\beta_s)$$

is a well-defined associative composition and thus yields the structure of an induction category $I\mathcal{C}$. □

We can also specify the preceding proposition in terms of coefficient matrices.

(9.1.6) Proposition. *Write the composition in the form*

$$(1|a)(b|1) = \sum_{(c,d)} \lambda^{a,b}_{c,d}(c|d)$$

where the sum is taken over pairs of morphisms (c, d) *such that* $ac = bd$. *Then these data define an induction category if and only if the following holds:*

$$
\begin{aligned}
\lambda^{m,ab}_{cd,n} &= \sum_w \lambda^{m,a}_{c,w}\lambda^{w,b}_{d,n} \\
\lambda^{ab,m}_{n,cd} &= \sum_w \lambda^{b,w}_{n,d}\lambda^{a,m}_{w,c}
\end{aligned}
$$

Let s, t *be isomorphisms. Then*

$$
\begin{aligned}
\lambda^{a,b}_{cs,ds} &= \lambda^{a,b}_{c,d} \\
\lambda^{s,b}_{s^{-1}b,1} = {} & 1 = \lambda^{a,t}_{1,at^{-1}}
\end{aligned}
$$

and $\lambda^{s,b}_{m,d} = 0 = \lambda^{a,t}_{c,n}$ *if* c *and* d *are not isomorphisms.* □

Suppose the induction category has a self-duality D. Then we obtain: Suppose $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$. Then $(\mathrm{id}\,|\beta) \circ (\alpha|\,\mathrm{id}) = \sum_s n_s(\beta_s|\alpha_s)$.

The commutativity (**??**) implies that the diagram (**??**) is equivalent to the diagram

$$
\begin{tikzcd}
\sum_s n_s Z_s \arrow[r, "\beta_s"] \arrow[d, "\alpha_s"] & B \arrow[d, "\beta"] \\
A\alpha \arrow[r] & X.
\end{tikzcd}
$$

9.2 Pullback Categories

Let C be a category and let C_1, C_2 be subcategories. The three categories have the same objects. We assume that the isomorphisms of C are contained in C_j. The diagrams

$$(b|a)\colon A \xleftarrow{a} X \xrightarrow{b} B$$

with $a \in C_1$ and $b \in C_2$ are the objects of a category $C|(A,B)$. The morphisms $(b|a) \to (b'|a')$ are the morphisms $\sigma \in C$ such that $a'\sigma = a$ and $b'\sigma = b$. We assume that C has (strictly transitive functorial) pullbacks such that in a pullback

$$\begin{tikzcd} Z \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{a}"] & A \arrow[d, "a"] \\ B \arrow[r, "b"] & X \end{tikzcd}$$

with $b \in C_2$ and $a \in C_1$ the morphisms $\tilde{a} \in C_1$ and $\tilde{b} \in C_2$. We define a category $PC = P(C; C_1, C_2)$ with the same objects as C. The class of diagrams $(b|a)$ as above is the class $\mathrm{Mor}_{PC}(A,B)$. Composition is defined as

$$(c|d) \circ (a|b) = (a\tilde{c}|d\tilde{b})$$

where in the diagram

$$\begin{tikzcd} Z \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{c}"] & Y \arrow[r, "d"] \arrow[d, "c"] & C \\ X \arrow[r, "b"] \arrow[d, "a"] & B & \\ A & & \end{tikzcd}$$

the square is a pullback. By assumption (**??**) this is again an allowable diagram. This category structure and the category structure on the diagrams induce on PC the structure of a 2-category. It is called a **pullback category**.

In most cases the vertical structure of the 2-category is not relevant. In that case we define $P_1(C; C_1, C_2)$. The morphisms from A to B are the isomorphism classes of diagrams above. Composition is again defined by the pullback construction, but one can now dispense with the strict transitivity of pullbacks.

9.3 Mackey Functors

Let $\mathcal{C}$ be a category with induction category $I\mathcal{C}$. A **Mackey functor** an $I\mathcal{C}$ is a contravariant R-functor from $I\mathcal{C}$ into R-Mod. A morphism between Mackey functors is a natural transformation. Let $\mathcal{M}(I\mathcal{C})$ denote the R-category of Mackey functors on $I\mathcal{C}$.

A **bifunctor** $M = (M^*, M_*)$ on $\mathcal{C}$ with values in the category $\mathcal{A}$ consists of a covariant functor M_* and a contravariant functor M^* from $\mathcal{C}$ into $\mathcal{A}$ which have the same value on objects. A bifunctor is called **compatible with isomorphisms** if for each isomorphism α the relation $M_*(\alpha)M^*(\alpha) = \mathrm{id}$ holds. A morphism $M \to N$ between bifunctors consists of a family of linear maps $M(S) \to N(S)$, $S \in \mathrm{Ob}(\mathcal{C})$, which constitute a natural transformation of the covariant and the contravariant part. We thus obtain the category of bifunctors.

A Mackey functor M is completely determined by the bifunctor $(M^* = M\iota_*, M_* = M\iota^*)$. This is due to the fact that the images of ι_* and ι^* span $I\mathcal{C}$. By **??**, the bifunctor of a Mackey functor is compatible with isomorphisms. Let $f\colon S \to T$ be a morphism in $\mathcal{C}$. We write $f^* = M\iota_*(f)$ and $f_* = M\iota^*(f)$. The upper index is for contravariant morphisms as in cohomology. Morphisms f_* are sometimes called **restriction maps**, morphisms f^* **transfer maps** or **induction maps**. This terminology comes from representation theory.

An example of a Mackey functor is the contravariant Hom-functor in $I\mathcal{C}$.

(9.3.1) Proposition. *Let M be a Mackey functor and (M_*, M^*) the associated bifunctor. Then this bifunctor is compatible with isomorphisms. If $(\mathrm{id}\,|\alpha)(\beta|\,\mathrm{id}) = \sum_s n_s(\alpha_s|\beta_s)$ in $I\mathcal{C}$, then*

$$\beta^*\alpha_* = \sum_s n_s(\beta_s)_*\alpha_s^*$$

*for any two morphisms $\alpha\colon A \to X$ and $\beta\colon B \to X$ in $\mathcal{C}$. We sometimes call (**??**) the* **double coset formula**. □

(9.3.2) Proposition. *Let (M_*, M^*) be a bifunctor which is compatible with morphisms and satisfies* **??** *for each pair of morphisms. Then there exists a unique Mackey functor M with associated bifunctor (M_*, M^*).*

Proof. We define $M(\beta|\alpha) = \alpha_*\beta^*$. Since the bifunctor is compatible with isomorphisms, this is well-defined on isomorphism classes of diagrams. We extend this definition by R-linearity to the morphism modules of $I\mathcal{C}$. The double coset formula is used to verify that M is compatible with composition. □

Let M, N, and L be Mackey functors for $I\mathcal{C}$. A **bilinear map** or a **pairing** $M \times N \to L$ between Mackey functors is a family of R-bilinear maps

$$M(S) \times N(S) \to L(S), \quad (x, y) \mapsto x \cdot y,$$

one for each object S of $\mathcal{C}$, such that for each morphism $f\colon S \to T$ in $\mathcal{C}$ the following holds:

$$\begin{array}{rcll} L^*f(x \cdot y) & = & (M^*fx) \cdot (N^*fy), & x \in M(T),\, y \in N(T) \\ x \cdot (N_*fy) & = & L_*f((M^*fx) \cdot y), & x \in M(T),\, y \in N(S) \\ (M_*fx) \cdot y & = & L_*f(x \cdot (N^*fy)), & x \in M(S),\, y \in N(T). \end{array}$$

A universal bilinear map $M \times N \to M \Box N$ is called a **tensor product** (or, because of the notation, a **box product**) of M, N. (Universal means, of course, that any other pairing $M \times N \to L$ is obtained from the universal one by composing with a unique morphism $M \Box N \to L$.)

In order to establish the canonical associativity of the box product we define a **trilinear** map $M \times N \times P \to Q$ between Mackey functors as a family of trilinear map

$$M(S) \times N(S) \times P(S) \to Q(S), \quad (x, y, z) \mapsto x \cdot y \cdot z$$

such that

$$f^*(x \cdot y \cdot z) = f^*x \cdot f^*y \cdot f^*z$$

and

$$f_*(f^*x \cdot f^*y \cdot z) = x \cdot y \cdot f_*z$$

and similarly if the two contravariant maps appear at other places. In the same way one defines n-linear maps between Mackey functors.

A **Green functor** A is a Mackey functor $A\colon I\mathcal{C} \to R$-Mod together with a pairing $A \times A \to A$ such that for each object S the pairing map $A(S) \times A(S) \to A(S)$ turns $A(S)$ into an associative R-algebra with unit such that the morphisms $A^*(f)$ preserve the units.

A **left module** over the Green functor A is a Mackey functor M together with a pairing $A \times M \to M$ such that for each object S the pairing map $A(S) \times M(S) \to M(S)$ equips $M(S)$ with the structure of a left unital $A(S)$-module.

9.4 Canonical Pairings

We fix a category $\mathcal{C}$ and an associated induction category $I\mathcal{C}$. For $S \in \mathcal{C}$ let $U(S)$ be the free abelian group on isomorphism classes of objects $\alpha\colon X \to S$ over S. We denote by $[\alpha] \in U(S)$ the element represented by α. We make the assignment $S \mapsto U(S)$ into a Mackey functor. Let $f\colon S \to T$ in $\mathcal{C}$ be given. Then $f_*\colon U(S) \to U(T)$ is defined as composition with f; functoriality $(gf)_* = g_*f_*$ is obvious. Suppose $(\mathrm{id}\,|f) \circ (\alpha|\,\mathrm{id}) = \sum_s n_s(f_s|\alpha_s)$; then we define $f^*[\alpha] = \sum_s n_s[f_s]$. The functoriality $(gf)^* = f^*g^*$ is a direct consequence of the transitivity of pullbacks. Thus we have defined a bifunctor. The double coset formula is again a direct consequence of the transitivity of pullbacks. By **??** we have a Mackey functor U. The following results show its universal character.

(9.4.1) Proposition. *Let M be any Mackey functor. There exists a canonical pairing $U \times M \to M$. If $u = [f\colon X \to S] \in U(S)$ and $x \in M(S)$, then $u \cdot x$ is defined as $f_* f^* x$.*

Proof. We have to verify the axioms of a pairing. Let $h\colon S \to T$ in $\mathcal{C}$ be given. Then

$$h_*(u \cdot h^* x) = h_* f_* f^* h^* x = h_* u \cdot x$$

since $h_* u = [hf]$. Let

$$\begin{tikzcd} \sum_s n_s Z_s \arrow[r, "h_s"] \arrow[d, "f_s"] & S \arrow[d, "h"] \\ X \arrow[r, "f"] & T \end{tikzcd}$$

be the pullback data in $I\mathcal{C}$. Then $h^* u = \sum n_s [h_s]$ and therefore

$$h_*(h^* u \cdot x) = \sum_s n_s h_* h_{s*} h_s^* x = \sum_s n_s f_* f_{s*} h_s^* x = f_* f^* h_* x = u \cdot h_* x.$$

The computation

$$h^*(u \cdot x) = h^* f_* f^* x = \sum_s n_s h_{s*} f_s^* f^* x = \textstyle\sum_s n_s h_{s*} h_s^* h^* x = h^* u \cdot h^* x$$

shows the second axiom of a pairing. □

(9.4.2) Proposition. *The pairing of the previous proposition, applied to $M = U$, makes U into a Green functor and M into a left U-module.*

Proof. The relation $1 \cdot x = x \cdot 1 = x$ is easily seen. We have to verify associativity of the multiplication. Let $u\colon X \to S$ and $v\colon Y \to S$ be given. On the one hand $u \cdot (v \cdot x) = f_* f^* g_* g^* x$. On the other hand

$$(u \cdot v) \cdot x = \sum_t m_t (f f_t)_* (g g_t)^* x = \sum_t f_* f_{t*} g_t^* g^* x = f_* f^* g_* g^* x.$$

Here we have used the pullback data m_t, f_t, g_t of f, g.

The same proof shows that M is a U-module. □

The multiplication in $U(S)$ has the following description. Suppose $(\mathrm{id}\,|\alpha) \circ (\beta|\,\mathrm{id}) = \sum_s n_s (\alpha_s|\beta_s)$. Then $[\alpha][\beta] = \sum_s n_s [\alpha\alpha_s] = \sum_s n_s [\beta\beta_s]$ (compare axiom **??** of an induction category). The identity is represented by id_S. The next proposition is easily verified from the definitions.

(9.4.3) Proposition. *The ring $U(S)$ is canonically isomorphic to a subring of the endomorphism ring $\mathrm{End}_{IC}(S)$, namely as the subring generated by morphisms of the type $(\alpha|\alpha)$ under the map $\alpha \mapsto (\alpha|\alpha)$.* □

(9.4.4) Proposition. *The pairings $\Sigma\colon U \times M \to N$ correspond bijectively to the morphisms $\sigma\colon M \to N$ of Mackey functors.*

Proof. Given a morphism σ, we obtain a pairing $U \times M \to N$ by composing the canonical pairing $\Lambda\colon U \times M \to M$ of (**??**) with σ.

Given a pairing $\Sigma\colon U \times M \to N$ we define

$$\sigma_\Sigma(S)\colon M(S) \to N(S), \quad x \mapsto 1 \cdot x.$$

From the axioms of a pairing it is verified that the $\sigma(S)$ constitute a morphism of Mackey functors.

The two constructions are inverse to each other. □

(9.4.5) Proposition. *Let A be a Green functor. The morphisms*

$$\lambda(S)\colon U(S) \to A(S), \quad [f] \mapsto f_* f^*(1_S)$$

are ring homomorphisms and constitute a morphism of Mackey functors.

Proof. Let n_s, f_s, g_s be the pullback data for f, g. We compute

$$\begin{aligned}
\lambda(S)([f][g]) &= \lambda(S)(\sum_s n_s[ff_s]) \\
&= \sum_s n_s f_* f_{s*} g_s^* g^*(1_S) \\
&= f_* f^* g_* g^*(1_S) \\
&= f_* f^* \lambda(S)(1_s) \\
&= (f_* f^*)(1 \cdot \lambda(S)(g)) \\
&= f_*(f^* 1 \cdot f^* \lambda(S)(g)) \\
&= f_* f^* 1 \cdot \lambda(S)(g) \\
&= \lambda(S)(f) \cdot \lambda(S)(g).
\end{aligned}$$

Moreover $\lambda(S)(1) = \mathrm{id}_* \,\mathrm{id}^*(1) = 1$. The following two computations verify the compatibility with morphisms. Let $h\colon T \to S$ be given, and let (m_t, f_t, h_t) denote the pullback data for (f, h). The contravariant case

$$\begin{aligned}
\lambda(T)(h^*[f]) &= \lambda(T)(\sum_t m_t[h_t]) \\
&= \sum_t m_t h_{t*} h_t^* 1_T = \sum_t m_t h_{t*} h_t^* h^* 1_S \\
&= \sum_t m_t h_{t*} f_s^* f^* 1_S = \sum_t h^* f_* f^* 1_S \\
&= h^* \lambda(S)[f].
\end{aligned}$$

And finally

$$\lambda(T)(k_*[f]) = \lambda(T)([kf]) = k_* f_* f^* k^* 1_T = k_* f_* f^* 1_S = h_* \lambda(S)[f]$$

settles the covariant case. □

9.5 The Projective Induction Theorem

Let $\mathcal{C}$ be a category and $I\mathcal{C}$ an associated induction category. We call any family $\Sigma = (S_j \mid j \in J)$ of objects an **induction system**. We assume that $\mathcal{C}$ ccontains a **point** P. This is a terminal object: Each object S of $\mathcal{C}$ has a unique morphism $p(S)\colon S \to P$.

Let M be a Mackey functor. An induction system Σ leads to a homomorphism $p(\Sigma)$, called **induction morphism**, and $i(\Sigma)$, called **restriction morphism**:

$$p(\Sigma)\colon \bigoplus_{j \in J} M(S_j) \to M(P), \quad (x_j \mid j \in J) \mapsto \sum_{j \in J} p(S_j)_* x_j$$

$$i(\Sigma)\colon M(P) \to \prod_{j \in J} M(S_j), \quad x \mapsto (p(S_j)^* x \mid j \in J).$$

We call the induction system Σ **projective**, if $p(\Sigma)$ is surjective, and **injective**, if $i(\Sigma)$ is injective. Suppose $S, T \in \mathrm{Ob}(\mathcal{C})$. The corresponding pullback will be denoted

$$\begin{CD} \sum_s n_s(S,T) Z_s @>{b_s(S,T)}>> T \\ @V{a_s(S,T)}VV @VV{p(T)}V \\ S @>{p(S)}>> P \end{CD}$$

with $s \in I(S,T)$. Let $\Sigma = (S_j \mid j \in J)$ be an induction system. We have morphisms

$$p(\Sigma, T)\colon \bigoplus_{j,s} M(Z_s) \to M(T), \quad (x(j,s)) \mapsto \sum_{j,s} n_s(S,T) b_s(S,T)_* x(j,s)$$

$$i(\Sigma, T)\colon M(T) \to \bigoplus_{j,s} M(Z_s), \quad x \mapsto (a_s(S,T)^* x \mid j, s).$$

The sums are double sums and $s \in I(S_j, T)$.

(9.5.1) Theorem. *Let A be a Green functor and M a left A-module. Let Σ be a projective induction system for A. Then $p(\Sigma, T)$ is split surjective and $i(\Sigma, T)$ is split injective.*

Proof. Since $p(\Sigma)$ is surjective for A, we can find $x_j \in A(S_j)$ such that

$$\sum_{j \in J} p(S_j)_* x_j = 1 \in A(P).$$

Of course, the sum is essentially finite, so that we can assume without essential restriction, that J is finite. We define a map

$$q(\Sigma, T)\colon M(T) \to \bigoplus_{j,s} M(Z_s), \quad x \mapsto (a_s^* x_j \cdot b_s^* x \mid j, s).$$

Here $a_s^* x_j \in A(Z_s)$, $b_s^* x \in M(Z_s)$ and the dot denotes the pairing $A \times M \to M$. We claim

$$p(\Sigma, T) \circ q(\Sigma, T) = \mathrm{id}_{M(T)}\,.$$

For the proof we use the basic identity

$$\sum_{s \in I(S_j, T)} n_s b_{s*} a_s^* = p(T)^* p(S_j)_*,$$

valid for any Mackey functor, and the properties of a pairing.

$$\begin{aligned}
p(\Sigma, T) q(\Sigma, T) x &= \sum_{j,s} n_s b_{s*}(a_s^* x_j \cdot b_s^* x) \\
&= \sum_{j,s} n_s b_{s*} a_s^* x_j \cdot x \\
&= \sum_{j} p(T)^* p(S_j)_* x_j \cdot x \\
&= p(T)^* (\sum_j p(S_j)_* x_j) \cdot x \\
&= p(T)^* 1 \cdot x \\
&= x.
\end{aligned}$$

Thus $q(\Sigma, T)$ is a splitting for $p(\Sigma, T)$.

A splitting $j(\Sigma, T)$ for $i(\Sigma, T)$ is defined in a dual fashion

$$j(\Sigma, T)\colon \bigoplus_{j,s} M(Z_s) \to M(T), \quad x(j,s) \mapsto \sum_{j,s} n_s b_{s*}(a_s^* x_j \cdot b_s^* x(j,s)).$$

A similar proof as above yields the identity $j(\Sigma, T) i(\Sigma, T) = \mathrm{id}_{M(T)}$. □

An induction theorem for a Mackey functor consists in the determination of a projective induction system. The significance of (**??**) is that a projective induction system for A is also a projective induction system for any A-module. Since A is a module over itself, a projective system for A is also an injective system for A.

We remark that the image of $p(\Sigma)$ in $A(P)$ is always an ideal. This is a general property of Green functors (see ??). Therefore $p(\Sigma)$ is surjective if and only if 1 is in the image of $p(\Sigma)$. If this is the case, then a finite subfamily of Σ suffices. Therefore, if A has a projective system, then also a finite one.

It is not really necessary that the category has a point. Let P be any object of $\mathcal{C}$. We can consider the category $\mathcal{C}/P$ of objects over P. This inherits an induction category $I\mathcal{C}/P$ from $I\mathcal{C}$. More explicitly: An induction system for P consists in a family of morphisms $(p(S_j)\colon S_j \to P \mid j \in J)$. All that matters in the previous proof is another morphism $p(T)\colon T \to P$. Note that $\mathcal{C}/P$ now has a terminal object id_P.

The induction theorem (**??**) has a second part. In it we describe the kernel of $p(\Sigma)$ and the image of $i(\Sigma)$. Let M be a A-module and $\Sigma = (S_j \mid j \in J)$ a projective induction system for A. We write $I(i,j) = I(S_i, S_j)$. In a sum over i,j,s we understand $s \in I(i,j)$; similarly, in a sum over j,s. We have two homomorphisms

$$p_1, p_2\colon \bigoplus_{i,j,s} M(Z_s) \to \bigoplus_{k \in J} M(S_k)$$

defined by

$$p_1(x(i,j,s)) = (\sum_{j,s} n_s a_{s*} x(i,j,s) \mid i \in J)$$

$$p_2(x(i,j,s)) = (\sum_{i,s} n_s b_{s*} x(i,j,s) \mid j \in J).$$

(9.5.2) Theorem. *The sequence*

$$\bigoplus_{i,j,s} M(Z_s) \xrightarrow{p_2 - p_1} \bigoplus_k M(S_k) \xrightarrow{p(\Sigma)} M(P) \to 0$$

is exact.

Proof. We use (**??**) and the notation of its proof. We know already that $p = p(\Sigma)$ is surjective. In this case a splitting is given by $q(\Sigma, P) = q$, defined as $q(z) = (x_j \cdot p(S_j)^* z \mid j \in J)$. We construct a homomorphism q_1 which satisfies

$$(p_2 - p_1)q_1 + qp = \mathrm{id}\,.$$

This identity yields that the kernel of $p(\Sigma)$ is contained in the image of p_2, p_1. Since, by construction, $p(p_2 - p_1) = 0$, we have exactness. We define q_1 as the direct sum $\bigoplus_k q(\Sigma, S_k)$. Note that p_2 is defined as $\bigoplus_k p(\Sigma, S_k)$. Hence, by the proof of (**??**), $p_2 q_1 = \mathrm{id}$. Thus it remains to verify $p_1 q_1 = qp$. This is a computation as in the proof of the previous theorem. □

We also have a dual exact sequence. Its statement uses the following homomorphisms

$$i_1, i_2\colon \bigoplus_k M(S_k) \to \bigoplus_{i,j,s} M(Z_s)$$

defined as

$$i_1(z_k) = (n_s b_s(i,k)^* z_k \mid i, s), \quad i_2(z_i) = (n_s a_s(i,k)^* z_i \mid k, s).$$

(9.5.3) Theorem. *The sequence*

$$0 \to M(P) \xrightarrow{i(\Sigma)} \bigoplus_k M(S_k) \xrightarrow{i_1 - i_2} \bigoplus_{i,j,s} M(Z_s)$$

is exact.

Proof. We already know that i is injective and has a splitting j. Dually to the previous proof we construct a homomorphism j_1 which satisfies the identity $ij + j_1(i_1 - i_2) = \mathrm{id}$. □

9.6 The n-universal Groups

We generalize the universal functor U to a functor in n variables.

Let $S_1, \dots, S_n$ be objects of $\mathcal{C}$. We consider the category $\mathcal{C}/(S_1, \dots, S_n)$ of objects $(a_j : X \to S_j)$ over the family (S_j). A morphism from (X, a_j) to (Y, b_j) is a morphism $f : X \to Y$ such that $b_j f = a_j$ for all j. Let $U(S_1, \dots, S_n)$ denote the free abelian group on the isomorphism classes of objects in $\mathcal{C}/(S_1, \dots, S_n)$. Up to canonical isomorphism, this group is invariant under permutation of the S_j. We make this construction into an n-variable functor in $I\mathcal{C}$, given on objects by $(S_1, \dots, S_n) \mapsto U(S_1, \dots, S_n)$. Fix the first variable. Suppose $f : S \to T$ in $\mathcal{C}$ is given. Then $f_* : U(S, S_j) \to U(T, S_j)$ is given by composition with f in the first component. We clearly have $f_* g_* = (fg)_*$. In order to define $f^* : U(T, S_j) \to U(S, S_j)$ we consider the pullback

$$\begin{tikzcd} S \arrow[d, "f"] & \sum_s n_s Z_s \arrow[l, "f_s"'] \arrow[d, "b_s"] & \\ T & X \arrow[l, "b"'] \arrow[r, "a_j"] & S_j \end{tikzcd}$$

and set

$$f^*(b, a_j) = \sum_s n_s (f_s, a_j b_s).$$

The relation $(fg)^* = g^* f^*$ follows from the transitivity of the pullback. In general we define $(\beta|\alpha)^* = \alpha_* \beta^*$. In order to see that this assignment defines a contravariant R-functor on $I\mathcal{C}$ and that these functors in different variables commute, one uses the transitivity of pullbacks.

The n-universal groups allow a characterization of pairings and n-linear maps. Suppose $P(S) \otimes Q(S) \to R(S)$, $x \otimes y \to x \cdot y$ is a pairing. For objects S_1, S_2, S_3 in $\mathcal{C}$ we define a homomorphism

$$\pi(S_1, S_2, S_3) : U(S_1, S_2, S_3) \to \mathrm{Hom}(P(S_1) \otimes Q(S_2), R(S_3)),$$

which maps the basis element (y_1, y_2, y_3) of $U(S_1, S_2, S_3)$ to

$$x_1 \otimes x_2 \mapsto y_{3*}(y_1^* x_1 \cdot y_2^* x_2).$$

(9.6.1) Proposition. *The homomorphisms $\pi(S_1, S_2, S_3)$ form a natural transformation of functors on $I\mathcal{C}$ in three variables.*

Proof. In order to read the proposition correctly, we have to interprete the variance of the functor in an appropriate manner. When we use the bifunctor language this means: Let $f_3 \colon S_3 \to S_3'$ be a morphism in $\mathcal{C}$. It induces f_{3*} and f_3^* in the third variable of U. Similarly, it induces homomorphisms when R is applied and then the Hom-functor. In the first and second variable we have to compare f_* with the Hom-maps induced by f^*. The compatibility of the π-morphisms with the f_* on the left follows directly from the definitions. The compatibility with the f^* uses the pairing axioms and the double coset formula. □

Conversely, we can characterize pairings by natural transformations. Let a natural transformation $\pi(S_1, S_2, S_3)$ as above be given. Let

$$\pi_S \colon P(S) \otimes Q(S) \to R(S), \quad x \otimes y \mapsto x \cdot y$$

denote the homomorphism which is the image of $(\mathrm{id}, \mathrm{id}, \mathrm{id}) \in U(S, S, S)$.

(9.6.2) Proposition. *The π_S form a pairing $P \times Q \to R$ of Mackey functors.* □

Via (**??**) and (**??**) we obtain a bijection between pairings and natural transformations. We have a similar situation for n-linear maps $P_1 \times \cdots \times P_n \to L$. They correspond to natural transformations

$$U(S_1, \ldots, S_{n+1}) \to \mathrm{Hom}(P_1(S_1) \otimes \cdots \otimes P_n(S_n), L(S_{n+1})).$$

9.7 Tensor Products

In this section we make the category of Mackey functors into a symmetric tensor category[1]. We begin with the construction of the box-product.

Suppose $M_1, \ldots, M_n$ are Mackey functors. We consider $U(S_1, \ldots, S_n, T)$ as a covariant functor in the S_j by using the self duality of $I\mathcal{C}$. We form the tensor product N of this covariant functor over $(I\mathcal{C})^n$ with the contravariant functor $(S_j) \mapsto M(S_1) \otimes \cdots \otimes M(S_n)$. This is, by construction, a Mackey functor. We show that it gives the universal n-linear map.

[1] Also called symmetric monoidal category.

Let $M_1 \times \cdots \times M_n \to L$ be an n-linear map between Mackey functors. Note that $N(T)$ can be defined as a quotient of

$$\tilde{N}(T) = \bigoplus_{(S_j)} U(S_j, T) \otimes M_1(S_1) \otimes \cdots \otimes M_n(S_n).$$

Let $(a_j, b) \in U(S_j, Z)$ denote a basis element. We map $(a_j, b) \otimes x_1 \otimes \cdots \otimes x_n$ to $b_*(a_1^* x_1 \cdot \ldots \cdot a_n^* x_n) \in L(T)$. Here the dots refer to the given n-linear morphism. One verifies

(9.7.1) Proposition. *The linear maps $\tilde{N}(T) \to L(T)$ factor over the quotient $N(T)$ and the resulting maps $N(T) \to L(T)$ constitute a morphism of Mackey functors.* □

We have seen in the previous section that the pairing $M_1 \times \cdots \times M_n \to L$ corresponds to a natural transformation. When we take the adjoint of $\pi(S_1, \ldots, S_n)$ we obtain a homomorphism

$$U(S_j, T) \otimes M_1(S_1) \otimes \cdots \otimes M_n(S_n) \to L(T).$$

The set of these homomorphisms yields the homomorphisms $\tilde{N}(T) \to L(T)$ above; and the fact that the $\pi(S_j)$ form a natural transformation is equivalent to the fact (**??**) that these homomorphisms factor over $N(T)$.

(9.7.2) Proposition. *The canonical maps*

$$M_1(S) \otimes \cdots \otimes M_n(S) \to U(S, S, \ldots, S) \otimes M_1(S) \otimes \cdots \otimes M_n(S) \to N(S)$$

which send $x_1 \otimes \cdots \otimes x_n$ to the class of $(\mathrm{id}, \ldots, \mathrm{id}) \otimes x_1 \otimes \cdots \otimes x_n$ form an n-linear morphism. This is a universal such morphism. □

The last proposition says that N is an n-fold box-product $M_1 \Box \cdots \Box M_n$. One verifies that the two canonical maps of $M(S) \otimes N(S) \otimes P(S)$ into $((M \Box N) \Box P)(S)$ and $(M \Box (N \Box P))(S)$ are both universal trilinear maps. This gives the canonical isomorphism $(M \Box N) \Box P \cong M \Box (N \Box P)$ which satisfies the pentagon axiom for tensor categories.

The functor U is a neutral element for this tensor product. This follows from **??** and **??**.

The symmetric pairing in this tensor category is simply given by the canonical morphism $\tau \colon M \Box N \to N \Box M$ which makes the diagrams

$$\begin{tikzcd}
M(S) \otimes N(S) \arrow[r, "\tau(S)"] \arrow[d] & N(S) \otimes M(S) \arrow[d] \\
M \Box N \arrow[r, "\tau"] & N \Box M
\end{tikzcd}$$

with the twist maps $\tau(S)(x \otimes y) = y \otimes x$ commutative. One verifies the axioms of a braiding.

Problems

1. Here is a slightly more elementary construction of the tensor product (= box-product). The tensor product of Mackey functors M and N is constructed as follows. Let $R(S,T)$ denote the free R-module om the set of morphisms $S \to T$ in $\mathcal{C}$. The group $(M \Box N)(T)$ is a quotient of $\bigoplus_S R(S,T) \otimes M(S) \otimes N(S)$ (tensor products always over R): We factor out the submodule generated by the elements

$$gh \otimes h^* z \otimes y - g \otimes z \otimes h_* y, \quad g \in R(S,T), h \in R(U,S), z \in M(S), y \in N(U)$$

$$g \otimes h_* z \otimes y - gh \otimes z \otimes h^* y, \quad g \in R(U,T), h \in R(S,U), z \in M(S), y \in N(U).$$

This becomes a covariant functor in T from the covariant Hom-functor $R(S,\text{-})$. In order to make it into a contravariant functor, let $\varphi\colon Z \to U$ be given. Suppose $\langle \varphi, h \rangle = \sum_t n_t(\alpha_t, \beta_t)$. Then φ^* maps $h \otimes a \otimes b \in R(S,T) \otimes M(S) \otimes N(S)$ to $\sum_t n_t \alpha_t \otimes \beta_t^* a \otimes \beta_t^* b$. It is shown with the transitivity relation (4) of the double coset decomposition that this is compatible with the equivalence relation. The same rule is used to show functoriality and the double coset formula, so that we have obtained a Mackey functor.

9.8 Internal Hom-Functors

An internal Hom-functor for the category of Mackey functors in $I\mathcal{C}$ assigns to each pair P, Q of Mackey functors another Mackey functor $\mathrm{HOM}(P,Q)$ which is right adjoint to the box product

$$\mathrm{Hom}_{I\mathcal{C}}(N \Box P, Q) \cong \mathrm{Hom}_{I\mathcal{C}}(N, \mathrm{HOM}(P,Q)).$$

Given P, Q we let

$$\Lambda(T) = \mathrm{Nat}_{S_1,S_2}(U(S_1,S_2,T), \mathrm{Hom}_R(P(S_1), Q(S_2))$$

denote the R-module of natural transformations of Mackey functors in the variables S_1, S_2. These modules form a Mackey functor in the variable T. We call this Mackey functor $\mathrm{HOM}(P,Q)$. In order to establish the adjunction (**??**) we take another Mackey functor N and consider a natural transformations $N \to \Lambda$. By adjunction, the natural transformation consists of a family of homomorphisms

$$N(T) \otimes U(S_1,S_2,T) \otimes P(S_1) \to Q(S_2)$$

with properties which ensure a factorization over a natural transformation $N \Box P \to Q$. This assignment is the starting point for the construction of (**??**). A formal consequence of (**??**) is the formal adjunction

$$\mathrm{HOM}(N \Box P, Q) \cong \mathrm{HOM}(N, \mathrm{HOM}(P,Q)).$$