ARITHMETIC OF CALABI–YAU VARIETIES

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Abstract. This is a written-up version of my talk presented at Algebraic Geometry Oberseminar on April 21, 2004 at Mathematics Institute of University of Göttingen. We address the modularity questions of Calabi–Yau varieties of dimension ≤ 3 defined over $\mathbb{Q}$. The up-to-date reference on the modularity of Calabi–Yau varieties is Yui [Yu03].

1. Introduction

Definition 1.1. A smooth projective variety X of dimension d defined over $\mathbb{C}$ is called a *Calabi–Yau* variety if
(i) $H^i(X, \mathcal{O}_X) = 0$ for every i, $0 < i < d$, and
(ii) the canonical bundle $\mathcal{K}_X$ of X is trivial.

Introduce the Hodge numbers

$$h^{i,j}(X) := \dim H^j(X, \Omega^i_X).$$

Then X is a Calabi–Yau variety if

$$h^{i,0}(X) = 0 \quad \text{for every } i, 0 < i < d, \text{ and} \quad h^{0,d}(X) = p_g(X) = 1$$

where $p_g(X)$ is the geometric genus of X.

Example 1.1. • If $d = 1$, a Calabi–Yau variety of dimension 1 is nothing but an elliptic curve (if it is equipped with a rational point).

• If $d = 2$, the conditions $h^{1,0}(X) = 0$ and $p_g(X) = 1$ imply that a Calabi–Yau variety of dimension 2 is a $K3$ surface.

• If $d = 3$, the conditions $h^{1,0}(X) = h^{2,0}(X) = 0$ and $p_g(X) = 1$ imply that a Calabi–Yau variety of dimension 3 is a Calabi–Yau threefold. Since a Calabi–Yau threefold is a Kähler manifold, it is necessary that $h^{1,1}(X) > 0$.

Numerical invariants and Hodge diamonds. Now we define the numerical invariants of Calabi–Yau varieties, and the Hodge diamonds.

• The k-th Betti number of X is

$$B_k(X) = \dim H^k(X, \mathbb{C}) = \dim H^k_{et}(\bar{X}, \mathbb{Q}_\ell).$$

Then $B_k(X) = 0$ for $k > 2d$, and the Poincaré duality implys that

$$B_k(X) = B_{2d-k}(X) \quad \text{for } k, 0 \leq k \leq d.$$

• There are symmetries of Hodge numbers:

$$h^{i,j}(X) = h^{j,i}(X),\ h^{p,q}(X) = h^{d-p,d-q}(X)$$

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where the first identity follows from complex conjugation, and the second from the Serre duality. There is the Hodge decomposition

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X)$$

and the Euler characteristic of X is given by

$$E(X) = \sum_{k=0}^{2d} (-1)^k B_k(X).$$

• The Hodge diamonds of elliptic curves, $K3$ surfaces and Calabi–Yau threefolds are given, respectively, as follows.

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}$$

Then

$$B_0 = B_2 = 1,\ B_1 = 2 \quad \text{and } E(X) = 0.$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Then

$$B_0 = B_4 = 1,\ B_1 = B_3 = 0,\ B_2 = 22 \quad \text{and } E(X) = 24.$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Then

$$B_0 = B_6 = 1,\ B_1 = B_5 = 0,\ B_2 = h^{1,1} = h^{2,2} = B_4,\ B_3 = 2(1 + h^{2,1})$$

and

$$E(X) = 2(h^{1,1} - h^{2,1}).$$

2. The L-series and zeta-function of Calabi–Yau varieties

We are interested in arithmetic and number theoretic properties of Calabi–Yau varieties, and we will consider Calabi–Yau varieties defined over number fields (e.g., $\mathbb{Q}$). Let X be a Calabi–Yau variety of dimension d defined over $\mathbb{Q}$. (By this we mean, the complexification of X is a Calabi–Yau variety.) There exists an integral model for X. Let p be a prime and assume that the reduction $X_p := X \bmod p$ is smooth over $\bar{\mathbb{F}}_p$. Such a prime p is called a *good* prime. For a good prime p, let Frob_p denote the Frobenius morphism on X induced from the p-th power map $x \mapsto x^p$. Let ℓ be a prime $\neq p$. Then Frob_p acts on the ℓ-adic étale cohomology groups $H^i_{et}(\bar{X}_p, \mathbb{Q}_\ell) \simeq H^i(\bar{X}, \mathbb{Q}_\ell)$ for $i = 0, 1, \cdots, 2d$. Let

$$P^i_p(T) := \det\left(1 - \mathrm{Frob}_p^* T \mid H^i_{et}(\bar{X}, \mathbb{Q}_\ell)\right)$$

be the characteristic polynomial of the endomorphism Frob_p^* on the étale ℓ-adic cohomology group, where T is an indeterminate. Then $P^i_p(T) \in 1 + T\mathbb{Z}[T]$ with $\deg P^i_p = B_i$ (the i-th Betti number of X) and its reciprocal roots are algebraic integers with absolute value $p^{i/2}$ (Deligne).

Let $\mathcal{G} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. Then there is an ℓ-adic étale Galois representation

$$\rho^i_{X,\ell} : \mathcal{G} \to GL(H^i_{et}(\bar{X}, \mathbb{Q}_\ell)).$$

In particular, using $\rho^i_{X,\ell}(\mathrm{Frob}_p)$, we define the i-th (cohomological) L-series of X.

Definition 2.1. The i-th (cohomological) L-series of X, $L(X, s) = L(H^i_{et}(\bar{X}, \mathbb{Q}_\ell), s)$, is defined by the Euler product

$$L_i(X, s) := \prod_{p \neq \ell} \det\left(1 - \rho^i_{X,\ell}(\mathrm{Frob}_p)p^{-s}\right)^{-1} \times (\text{similar factor at } p = \ell)$$

where the product runs over good primes. Digressing, we have

$$L_i(X, s) = \prod_{p \neq \ell} P^i_p(p^{-s})^{-1} \times (\text{similar factor at } p = \ell).$$

In particular, if $i = d =$ the dimension of X, we simply write $L(X, s)$ in place of $L_d(X, s)$ and its local factor by $P_p(T)$ instead of $P^d_p(T)$.

3. The modularity of elliptic curves over $\mathbb{Q}$

Now we address the modularity of dimension 1 Calabi–Yau varieties defined over $\mathbb{Q}$. Let E be an elliptic curve over $\mathbb{Q}$, and let Δ denote its discriminant. Then its L-series can be built up by counting the number of $\mathbb{F}_p$–rational points on E.

$$L(E, s) = \prod_p \frac{1}{1 - a(p)p^{-s} + \varepsilon(p)p^{1-2s}} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

where p runs all primes, and

$$a(p) = \begin{cases} p + 1 - \#E(\mathbb{F}_p),\ \varepsilon(p) = 1 & \text{if } p \not| \Delta \\ 0,\ \pm 1,\ \varepsilon(p) = 0 & \text{if } p | \Delta \end{cases}$$

Given a prime p, each local Euler p-factor of $L(E, s)$ can be determined explicitly using the above description. However, there are infinitely many Euler factors. A natural question is:

Is there any global function that determines the L-series $L(E,s)$? The answer to this question comes from a quite different source. Indeed, this will take us to more analytic objects. So we will now define modular groups, modular forms and cusp forms. Let $\mathfrak{H}$ denote the upper-half complex plane and let $SL_2(\mathbb{Z})$ be the group of 2×2 integral matrices with determinant 1 and put $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I_2$ where I_2 denotes the identity matrix of rank 2.

Definition 3.1. Let $N\geq 1$ be an integer, and let

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) \,|\, c\equiv 0 \pmod N\,\} \subset PSL_2(\mathbb{Z})$$

be a congruence subgroup of $PSL_2(\mathbb{Z})$.

Let k be a non-negative integer. A *modular form f of weight* $k\geq 1$ *on* $\Gamma_0(N)$ is a complex-valued holomorphic function on $\mathfrak{H}$ satisfying the following transformation rule:

$$f(\frac{az+b}{cz+d}) = (cz+d)^k f(z) \quad \text{for } z\in\mathfrak{H} \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

A *cusp form f of weight k on* $\Gamma_0(N)$ is a modular form vanishing at all cusps of $\Gamma_0(N)$. In particular, f has a Fourier expansion (at ∞), and we can write

$$f(q) = \sum n = 1^{\infty} a_f(n)\, q^n \quad \text{with } q = e^{2\pi i z}.$$

The L-series of a cusp form f is defined by

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}$$

Now we state the result of Wiles et al. which proves the conjecture of Shimura and Taniyama in the affirmative.

Theorem 3.1. (Wiles et al. [Wi95], [TW95], [BCDT]) *Let E be an elliptic curve over $\mathbb{Q}$. Then E is modular, that is, there is a cusp (new) form f of weight* $2=1+1$ *on* $\Gamma_0(N)$ *such that*

$$L(E,s) = L(f,s) \quad \textit{i.e., } a(n) = a_f(n) \quad \forall n.$$

Here N is the conductor of E.

Remark 3.1. The proof of Wiles et al. is to compare the two 2-dimensional Galois representations arising from E and f. If these two 2-dimensional Galois representations are equivalent mod 3 (or mod 5), then they are equivalent, and establish the Shimura–Taniyama conjecture in the affirmative.

4. The modularity of singular (extremal) $K3$ surfaces

Now we will address the modularity of dimension 2 Calabi–Yau varieties, namely, $K3$ surfaces, defined over $\mathbb{Q}$.

Let X be an algebraic $K3$ surface. Let $NS(X)$ be the Néron–Severi group of X generated by algebraic cycles on X. Then $NS(X)$ is a free finitely generated abelian group. The $\mathbb{Z}$-rank of $NS(X)$ is called the *Picard number* of X and denoted by $\rho(X)$. Since $NS(X)\subseteq H^2(X,\mathbb{Z})\cap H^{1,1}(X,\mathbb{R})$, the Picard number $\rho(X)$ is at most 20. A $K3$ surface X is equipped with the perfect pairing induced by the intersection pairing. Let $T(X) := NS(X)^{\perp}_{H^2(X,\mathbb{Z})}$ be the orthogonal complement of $NS(X)$ in $H^2(X,\mathbb{Z})$ with respect to this perfect pairing. Then $T(X)$ is a lattice of rank $22 - \rho(X)$, and is called the group or the lattice of transcendental cycles on X.

Now we will single out a special class of $K3$ surfaces.

Definition 4.1. A $K3$ surface X is said to be a *singular* (or *extremal*) if $\rho(X) = 20$.

Now we consider a singular $K3$ surface X defined over $\mathbb{Q}$. The L-series of X is defined by

$$L(X,s) = (*) \prod_{p:\text{good}} P_p(p^{-s})^{-1}$$

where the product runs over all good primes (with $(*)$ indicates the factor corresponding to bad primes), and

$$P_p(T) = det(1 - \text{Frob}_p^* T \,|\, H^2_{et}(\bar{X}, \mathbb{Q}_\ell))$$

is an integral polynomial of degree 22 whose reciprocal roots have the absolute value p. The decomposition of the lattices $H^2(X,\mathbb{Z}) = NS(X) \oplus T(X)$ induces the decomposition of the L-series $L(X,s)$:

$$L(X,s) = L(NS(X) \otimes \mathbb{Q}_\ell, s) \cdot L(T(X) \otimes \mathbb{Q}_\ell, s).$$

Remark 4.1. If we pass onto sufficiently large extension K of $\mathbb{Q}$ to ensure that all algebraic cycles are defined over K, then

$$L(NS(X_K) \otimes \mathbb{Q}_\ell, s) = \zeta_K(s-1)^{\rho(X_K)} = \zeta_K(s-1)^{20}$$

where $\zeta_K(s)$ is the Dedekind zeta-function of K. We should remark that the Picard number is an arithmetic invariant which is very sensitive to the field of definition of algebraic cycles. If not all algebraic cycles are defined over a fixed field, e.g., $\mathbb{Q}$, the problem of determining the L-series $L(NS(X) \otimes \mathbb{Q}_\ell), s)$ is still open.

Now we can state the modularity results for singular $K3$ surfaces.

Theorem 4.1. (Shioda and Inose [SI77]) *Every singular $K3$ surface X has a model defined over some number field K, and its Hasse–Weil zeta-function $\zeta(X_K, s)$ is given, up to a finite number of Euler factors, by*

$$\zeta(X_K, s) = \zeta_K(s)\zeta_K(s-1)^{20} L(s-1, \chi^2) L(s-1, \bar{\chi}^2)$$

where $\zeta_(s)$ is the Dedekind zeta-function of K and $L(s,\chi^2)$ is the Hecke L-series with a suitable Grossencharacter χ^2.

A modular (cusp) form is not present in this theorem of Shoida and Inose. The modularity of singular $K3$ surfaces over $\mathbb{Q}$ in our sense has been established by Livné. Livné analyzed the two 2-dimensional Galois representations associated to the rank 2 motive $T(X)$.

Theorem 4.2. (Livné [L95]) *Let X be a singular $K3$ surface defined over $\mathbb{Q}$. Then X is modular. That is, the transcendental part $T(X)$ is modular and*

$$L(T(Y) \otimes \mathbb{Q}_\ell, s) = L(g, s)$$

where g is a cusp form of weight $3 = 2+1$ on some congruence subgroup $\Gamma_1(N)$ or $\Gamma_0(N)$ with a character. Here $\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \,|\, a \equiv d \equiv 1 \pmod{N} \}$. Also the space of cusp forms of weight 3 on $\Gamma_0(N)$ is empty, so we ought to twist forms by a character.

In more representation theoretic formulation, this theorem can be stated as follows.

Theorem 4.3. (Livné [L95]) *Let π be the compatible family of 2-dimensional ℓ-adic Galois representations associated to $T(X)$ and let $L(\pi,s)$ be its L-series. Then there exists a unique cusp form g of weight 3, level the conductor of π and a Dirichlet (odd) character $\varepsilon(p)=(\frac{-d}{p})$ such that $L(\pi,s)=L(g,s)$.*

Remark 4.2. If X is no longer singular (extremal), the modularity question is still open. However, if X has Picard number $\rho(X)=19$, X is equipped with a Shioda–Inose structure and consequently X is either a Kummer surface or a double cover of a Kummer surface ([SI77] and also [Mo90]). So there is the underlying elliptic curve which gives rise to Kummer surface, and the modularity of the transcendental part may be realized by taking the symmetric square of a cusp form of weight 2 associated to the underlying elliptic curve. In fact, this approach has been carried out by Ling Long [Lo03] for a certain family of $K3$ surfaces with Picard number 19.

5. The modularity for rigid Calabi–Yau threefolds over $\mathbb{Q}$

Now we will try to generalize the modularity results to Calabi–Yau threefolds defined over $\mathbb{Q}$. For Calabi–Yau threefolds, the Betti numbers B_2, B_3 are not fixed constants, nor the Euler characteristic. A natural question is:

Is there the absolute constant which bound the absolute value of the Euler characteristic $E(X)$?

The current record for the upper bound for the absolute value of the Euler characteristic is $|E(X)| \leq 960$. This is based on available examples of Calabi–Yau threefolds constructed by physicists using Batyrev's reflexive polytopes. However, Miles Reid claims there should not be such a constant (based on his experience with MMP (Minimal Model Program)).

We first make a crude classification of Calabi–Yau threefolds.

Definition 5.1. A Calabi–Yau threefold X is said to be *rigid* if $h^{2,1}=0$ (so that $B_3=2$). Otherwise, X is said to be *non-rigid*.

Remark 5.1. A rigid Calabi–Yau threefold is indeed the natural generalization of an elliptic curve. The third Betti number $B_3=2$ and accordingly there is a 2-dimensional Galois representation associated to it.

The modularity conjecture for rigid Calabi–Yau threefolds over $\mathbb{Q}$. *Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$. Then X is modular. That is, there exists a cusp form f of weight $4=3+1$ on some $\Gamma_0(N)$ such that*

$$L(X,s)=L(f,s),$$

up to a finite number of Euler factors. Here N is divisible only by primes of bad reduction.

This conjecture was formulated in M-H. Saito and Yui [SaYu01].

Remark 5.2. 1. This conjecture is a special case of the Fontaine–Mazur conjecture that *every irreducible odd 2-dimensional Galois representation "coming from geometry" should be modular, up to a Tate twist.*

2. Also this is a special case of the Serre conjecture about the modularity of the residual mod p 2-dimensional Galois representations attached to certain odd dimensional projective varieties over $\mathbb{Q}$.

3. This may be regarded as a concrete realization of the Langlands Program.

Explicit description of $L(X, s)$. Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$. Then X always has an integral model. Let p be a good prime, and let Frob_p be the Frobenius morphism. Then the characteristic polynomial $P_p(T)$ has the form

$$P_p(T) = 1 - t_3(p)T + p^3T^2 \in 1 + T\mathbb{Z}[T] \quad \text{with } |t_3(p)| \leq 2p^{3/2}$$

Then

$$L(X, s) = (*) \prod_{p:\text{good}} P_p(p^{-s})^{-1} = (*) \prod_{p:\text{good}} \frac{1}{1 - t_3(p)p^{-s} + p^{3-2s}}$$

where $(*)$ stands for the factors corresponding to bad primes.

As for elliptic curves over $\mathbb{Q}$, the local Euler p-factor of $L(X, s)$, that is, essentially $t_3(p)$, can be described in terms of the number of $\mathbb{F}_p$-rational points on X.

For each i, $0 \leq i \leq 6$, let

$$t_i(p) := \mathrm{trace}(\mathrm{Frob}_p^* \,|\, H^i_{et}(\bar{X}, \mathbb{Q}_\ell))$$

be the trace of the Frob_p^* on the étale ℓ-adic i-th cohomology group. Then the Lefschetz fixed point formula asserts that

$$\#X(\mathbb{F}_p) = \sum_{i=0}^{6} (-1)^i t_i(p) = t_0(p) - t_1(p) + t_2(p) - t_3(p) + t_4(p) - t_5(p) + t_6(p).$$

By the Poincaré duality on the étale ℓ-adic cohomology groups, we have

$$\#X(\mathbb{F}_p) = 1 + p^3 + (1+p)t_2(p) - t_3(p) \leq 1 + p^3 + (1+p)ph^{1,1} - t_3(p).$$

The inequality becomes an equality if all cycles in $H^{1,1}(X)$ are defined over $\mathbb{Q}$, in which case, we have

$$t_3(p) = 1 + p^3 + (1+p)p^{1,1} - \#X(\mathbb{F}_p).$$

Theorem 5.1. *Up to date, there are at least* 50 *rigid Calabi–Yau threefolds over* $\mathbb{Q}$ *for which the modularity is established.*

Methods. There are several methods for establishing the modularity for rigid Calabi–Yau (and for non-rigid Calabi–Yau) threefolds defined over $\mathbb{Q}$.

Method 1: The Serre–Faltings criterion.

Method 2: Wiles method.

Method 3: Algebraic correspondence (Tate's conjecture).

Method 4: Conifolds method.

Method 5: Intermediate Jacobians.

(We should remark that Methods 1 and 2 are exclusively for rigid Calabi–Yau threefolds, while the other Methods are applicable for rigid as well as for non-rigid Calabi–Yau threefolds.)

- **The Serre–Faltings criterion**:

Let f be a cusp form of weight 4 on some $\Gamma_0(N)$ and write

$$f(q)=\sum_{n=1}^{\infty} a_f(n)q^n \quad \text{with } q=e^{2\pi i z} \text{ and } a_f(1)=1$$

Suppose that
(Serre) $t_3(p)=a_f(p)$ for all good primes p;
(Faltings) $t_3(p)=a_f(p)$ for finitely many good primes p.

Then invoking the Chebotarev Density Theorem, the semisimplificatons of the two 2-dimensional Galois representations associated to X and f are equivalent, so that $L(X,s)=L(f,s)$, up to a finite number of Euler factors.

• **Wiles method**:
In Wiles' proof of the Shimura–Taniyama conjecture for elliptic curves, prime $\ell=3$ played very crucial role (backed up by prime $\ell=5$). For rigid Calabi–Yau threefolds over $\mathbb{Q}$, a similar criterion has been established.

Theorem 5.2. (Dieulefait and Manoharmayum [DM03], You-Chiang Yi [Yi02]) *Let X be a rigid Calabi–Yau threefold defined over $\mathbb{Q}$. Suppose that X satisfies one of the following two conditions:*
(1) X has good reduction at 3 and 7, or
(2) X has good reduction at 5 and some prime $p\equiv \pm 2 \pmod 5$ with $t_3(p)$ not divisible by 5.
Then X is modular.

• **Algebraic Correspondence (Tate's conjecture)**
Given a rigid Calabi–Yau threefold over $\mathbb{Q}$, its modularity can be established by constructing an algebraic correspondence to a modular rigid Calabi–Yau threefold over $\mathbb{Q}$. This approach is based on the conjecture of Tate that
the isomorphism between two Galois representations is induced by an algebraic correspondence.

First we need to construct rigid Calabi–Yau threefolds over $\mathbb{Q}$, which are associated to modular groups. This has been done by C. Schoen.

Theorem 5.3. (Schoen [Sc81]; Cf. Beauville [Beau82]) *Let $\Gamma\subset PSL_2(\mathbb{Z})$ be a congruence subgroup of finite index, and let $C_\Gamma:=(\mathfrak{H}/\Gamma)^*$ be the modular curve. Let S_Γ be the universal family of elliptic curves over C_Γ. Let $Y:=S_\Gamma\times_{\mathbb{P}^1}S_\Gamma$ be the self-fiber product of S_Γ, and let X be a smooth resolution of Y. Then X is a rigid Calabi–Yau threefold when Γ is one of the following six groups:*

$$\Gamma(3),\ \Gamma_1(4)\cap\Gamma(2),\ \Gamma_1(5),\ \Gamma_1(6),\ \Gamma_0(8)\cap\Gamma_1(4),\ \Gamma_0(9)\cap\Gamma_1(3).$$

Beauville [Beau82] determined explicit defining equations over $\mathbb{Q}$ for the rational elliptic surfaces S_Γ for the above modular groups.

Theorem 5.4. (Verill [Yu03]) *The six rigid Calabi–Yau threefolds constructed in Theorem 5.3 have explicit defining equations over $\mathbb{Q}$. Furthermore, they are all modular. That is, there is a cusp form f of weight 4 on Γ such that*

$$L(X,s)=L(f,s).$$

Now we will give one example of rigid Calabi–Yau threefold over $\mathbb{Q}$ whose modularity is established by constructing an explicit algebraic correspondence to one of the six modular rigid Calabi–Yau threefolds in Theorem 5.4.

Theorem 5.5. (M.H. Saito and N. Yui [SaYu01]) *Let V be the Verrill Calabi–Yau threefold over $\mathbb{Q}$ associated to the root lattice A_3. Then V is realized as a smooth (small) resolution of the hypersurface*

$$(x+y+z+w)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w})-\frac{(t-2)^2}{t}-4=0$$

in $\mathbb{P}^4\times\mathbb{P}^1$, and V is a rigid Calabi–Yau threefold with the Euler characteristic 100.

Let $S_{\Gamma_1(6)}$ be a rational elliptic surface defined by the hypersurface

$$(x+y+z)(xy+yz+zx)=(s+1)xyz$$

associated to $\Gamma_1(6)$. Let Y be the self-fiber product of $S_{\Gamma_1(6)}$ and let $\tilde{Y}$ be a crepant resolution of Y. Then there is an explicit birational transformation defined over $\mathbb{Q}$ from V to $\tilde{Y}$. This birational map is compatible with the Galois action, and induces an equivalence of the associated Galois representations. Consequently,

$$L(V,s)=L(\tilde{Y},s)=\eta(q)^2\eta(q^2)^2\eta(q^3)^2\eta(q^6)^2.$$

Remark 5.3. 1. Verrill [V00] has proved the modularity of V using the Serre–Faltings criterion. Also Dieulefait and Manoharmayum [DM03] and Yi [Yi02] have established the modularity of V using Wiles method. The above proof of Saito and Yui is more geometric along the line of Tate's conjecture.

2. Recently, Schütt [Schu03] has applied the construction of Schoen to twisted self-fiber products $(S_{\Gamma_1(6)}, pr)\times_{\mathbb{P}^1}(S_{\Gamma_1(6)},\pi\circ pr)$ where pr is the natural projection and π is a non-trivial automorphism of $\mathbb{P}^1$ interchanging $0,1,\infty$. Choosing π appropriately, he obtained four more rigid Calabi–Yau threefolds over $\mathbb{Q}$ associated to cusp forms of weight 4 and level $10, 17, 21$ and 73. His method should work for other groups listed in Theorem 5.3.

• **Conifolds Method**:

This method has been developed by Candelas, de la Ossa and Villegas [CDV], and works for one-parameter families of Calabi–Yau complete intersection threefolds. The Picard–Fuchs differential equations of these families turn out to be Gauss hypergeometric series. At conifold points (i.e., assigning special values to the parameter), certain one-parameter families give rise to rigid Calabi–Yau threefolds, and their modularity can be established studying periods of the families.

• **Intermediate Jacobians**:

This method has not yet produced results. I would like to mention this approach hoping that someone might work on it. The (Griffith) *intermediate Jacobian* of a Calabi–Yau threefold X is defined by

$$J^2(X)=\frac{H^{1,2}(X)\oplus H^{0,3}(X)}{H_3(X,\mathbb{Z})}.$$

In particular, if X is rigid, $J^2(X)\simeq\mathbb{C}/\mathbb{Z}^2$ is nothing but a complex torus (an abelian variety of dimension 1). We will now make a conjecture:

A rigid Calabi–Yau threefold over $\mathbb{Q}$ is modular if $J^2(X)$ is defined over $\mathbb{Q}$ (and hence modular).

One class of Calabi–Yau threefolds for which we might be able to get relations between intermediate Jacobians and Calabi–Yau threefolds is the class of rigid Calabi–Yau threefolds of CM type. We say that a rigid Calabi–Yau threefold X is *of CM type* if and only if the intermediate Jacobian $J^2(X)$ is of CM type (as an abelian variety of dimension 1).

Example 5.6. (X. Xarles and N. Yui [XY]) *Let X be a rigid Calabi–Yau threefold defined over a number field F of CM type. Then the intermediate Jacobian $J^2(X)$ is an elliptic curve with CM by an imaginary quadratic field K, and has a model defined over F.*

If χ is a Hecke character associated to $J^2(X)$, and suppose that

$$L(J^2(X),s) = \begin{cases} L(\chi,s) & \text{if } F \text{ does not contain } K \\ L(\chi,s)L(\bar{\chi},s) & \text{if } F \text{ contains } K \end{cases}$$

then

$$L(X,s) = \begin{cases} L(\chi^3,s) & \text{if } F \text{ does not contain } K \\ L(\chi^3,s)L(\bar{\chi}^3,s) & \text{if } F \text{ contains } K \end{cases}$$

Consequently, X is modular.

Remark 5.4. Shoida and Inose [SI77] showed that the isomorphism classes of singular $K3$ surfaces is in one-to-one correspondence with the $SL_2(\mathbb{Z})$-equivalence classes of positive definite even binary quadratic forms. Consequently every singular $K3$ surface has a CM by an imaginary quadratic field (or an imaginary quadratic order). This example may be viewed as a generalization of the result of Shoida and Inose to rigid Calabi–Yau threefolds of CM type. The result of Shoida and Inose and this example provide evidence to the conjecture of Shafarevich that

A variety of CM type is defined over a number field, and its L-series is expressed as a product of the L-series of one-dimensional characters associated to the field of definition of the variety.

6. The modularity of non-rigid Calabi–Yau threefolds over $\mathbb{Q}$

Now we will address the modularity question for non-rigid Calabi–Yau threefolds defined over $\mathbb{Q}$. Let X be a non-rigid Calabi–Yau threefold. Then $h^{2,1} \neq 0$ so that $B_3 \geq 4$. Consequently, the Galois representation associated to X has dimension ≥ 4, and the modularity question becomes increasingly more challenging, though the Langlands Program predicts that there should be some automorphic form(s) determining the L-series and zeta-functions. Up to date, we have constructed only a handful of examples of non-rigid Calabi–Yau threefolds and have established their modularity.

There are several methods to establish the modularity of non-rigid Calabi–Yau threefolds defined over $\mathbb{Q}$.

Method 1: When the B_3-dimensional Galois representation splits into a sum of 2-dimensional Galois representations, apply the method developed for rigid Calabi–Yau threefolds to these rank 2 motives.

Method 2: Construct *modular* non-rigid Calabi–Yau threfolds over $\mathbb{Q}$ (which are the non-rigid analogues of modular rigid Calabi–Yau threefolds discussed in Theorem 5.3 and Theorem 5.4.) To establish the modularity of a given non-rigid Calabi–Yau threefold, construct an algebraic correspondence to a modular non-rigid Calabi–Yau threefold (Tate's conjecture) as Saito and Yui [SaYu01] did for rigid Calabi–Yau threefolds in Theorem 5.5.

We will now give examples illustrating Method 1 and Method 2. For more complete list of examples, see Yui [Yu03].

Example of van Geemen and Nygaard [GN95]**.** Let $Y \subset \mathbb{P}^7$ be the complete intersection defined by

$$\begin{aligned} Y_0^2 &= 2(X_0X_1 + X_2X_3) \\ Y_1^2 &= 2(X_0X_2 + X_1X_3) \\ Y_2^2 &= 2(X_0X_3 + X_1X_2) \\ Y_3^2 &= 2(X_0X_1 - X_2X_3) \end{aligned}$$

The singular locus of Y consists of 16 double points and 4 plane conics intersecting transversally, configured in a square.

Take a resolution X of Y. Then X is a Calabi–Yau threefold with

$$h^{3,0} = 1,\ h^{2,1} = 1,\ h^{1,1} = 41.$$

So $B_3 = 4$, and there is a 4-dimensional Galois representation attached to X. Let $\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL(H^3_{et}(\bar{X}, \mathbb{Q}_\ell))$ be the Galois representation. Let $i = \sqrt{-1}$ and let

$$[i] : (Y_0 : Y_1 : Y_2 : Y_3 : X_0 : X_1 : X_2 : X_3) \mapsto (Y_0 : Y_2 : Y_1 : iY_3 : X_2 : X_3 : X_0 : X_1)$$

be an automorphism of X. Then $[i]$ induces a linear transformation on $H^3_{et}(\bar{X}, \mathbb{Q}_\ell)$. This splits the 4-dimensional Galois representation into a sum of two 2-dimensional Galois representations.

Theorem 6.1. *(a) The L-series $L(X, s)$ is expressed in terms of the Hecke character ψ of $\mathbb{Q}(i)$:*

$$L(X, s) = L(\psi^3, s)L(\psi, s - 1)$$

(b) $L(X, s)$ occurs as a factor in the L-series $L(H^3_{et}(\bar{E}^3, \mathbb{Q}_\ell), s)$ where $E : y^2 = 1 + x^4$ is an elliptic curve with CM by $\mathbb{Z}[i]$.

(c) There exists an algebraic correspondence between X and E^3, which induces the isomorphism of the two Galois representations.

(d) X is realized as a Siegel modular threefold over $\mathbb{Q}$.

Remark 6.1. Recently, K. Hulek and H. Verrill [HV03] have established the modularity of non-rigid Calabi–Yau threefolds over $\mathbb{Q}$ arising from the root lattice A_4. In their example, B_3-dimensional Galois representations also split into sums of 2-dimensional ones.

Now we will construct *modular* non-rigid Calabi–Yau threefolds over $\mathbb{Q}$ as Schoen [Sc81] and Verrill [Yu03] did for the rigid case.

Motivations for the examples of Livné and Yui [LY03]**.** The motivation for these examples come from a rather unexpected source, i.e., a paper of Sun, Tan and Zuo [STZ02] in which they considered Calabi–Yau threefold $f : X \to \mathbb{P}^1$ fibered by semi-stable $K3$ surfaces. Let $S \subset \mathbb{P}^1$ be a finite set of points over which f is non-smooth, and let $\Delta \subset X$ be the pull-back of S. Let $\omega_{X/\mathbb{P}^1}$ be the canonical sheaf. The Kodaira–Spencer maps $\theta^{2,0}$ and $\theta^{1,1}$ are, respectively, defined by

$$f_*\Omega^2_{X/\mathbb{P}^1}(\log\Delta) \xrightarrow{\theta^{2,0}} R^1f_*\Omega^1_{X/\mathbb{P}^1}(\log\Delta)\otimes\Omega^1_{\mathbb{P}^1}(\log S) \xrightarrow{\theta^{1,1}} R^2f_*(\mathcal{O}_{X/\mathbb{P}^1}\otimes\Omega^1_{\mathbb{P}^1}(\log S)^{\otimes 2},$$

and the iterated Kodaira–Spencer map is $\theta^{1,1}\theta^{2,0}$.

Also there is the Arakelov–Yau type inequality for X:

$$\deg f_*\omega_{X/\mathbb{P}^1} \le \deg \Omega^1_{\mathbb{P}^1}(\log S).$$

Sun, Tan and Zuo [STZ02] considered a $K3$-fibered Calabi–Yau threefold for which the Arakelov–Yau inequality reaches the upper bound and becomes an equality. In case of non-rigid Calabi–Yau threefolds, a stronger equality holds.

$$2 = \deg f_*\omega_{X/\mathbb{P}^1} = \begin{cases} \deg \Omega^1_{\mathbb{P}^1}(\log S) & \text{if } X \text{ is rigid} \\ \frac{1}{2}\deg \Omega^1_{\mathbb{P}^1}(\log S) & \text{if } X \text{ is non-rigid} \end{cases}$$

Theorem 6.2. (Sun, Tan and Zuo [STZ02]) *Let $f : X \to \mathbb{P}^1$ be a Calabi–Yau threefold fibered by non-constant semi-stable $K3$ surfaces, reaching the Arakelov–Yau bound.*

(a) If the iterated Kodaira–Spencer map is non-zero, then f has at least four singular fibers. If f has exactly four singular fibers, then X is rigid, and is birational to one of the six rigid Calabi–Yau threefolds in Theorem 5.3 and Theorem 5.4.

(b) If the iterated Kodaira–Spencer map is zero, then f has at least six singular fibers. If f has exactly six singular fibers, then X is non-rigid, the general fibers have Picard number at least 18, and $\mathbb{P}^1 \setminus S \simeq \mathfrak{H}/\Gamma$ for some subgroup of $PSL_2(\mathbb{Z})$ of index 24.

Remark 6.2. The three authors used the terminology “modularity” in the title of their paper to refer to the fact that $\mathbb{P}^1 \setminus S$ is a modular variety.

For the rigid case (a), we have established the modularity in our sense, i.e., for the L-series. We are inspired to consider the modularity of the L-series for the non-rigid case (b).

Examples of Livné and Yui [LY03]**.** Livné and Yui [LY03] have constructed several examples of non-rigid Calabi–Yau threefolds which satisfy the conditions of Theorem 6.2 (b). Our idea of producing such examples is as follows. Let E be an elliptic curve and let Y be a singular $K3$ surface, both defined over $\mathbb{Q}$. We know that both E and Y are modular by the discussions in Section 3 and Section 4, respectively. We consider the product $Y \times E$, though it is not Calabi–Yau. If we choose Y appropriately, then there is an action of ± 1 on the product. Take the quotient $Y \times E/\pm 1$ and let X be its smooth resolution. Then we will show that X is a non-rigid Calabi–Yau threefold satisfying the conditions of Theorem 6.2 (b).

We now choose for Y a special type of $K3$ surfaces, namely, elliptic modular $K3$ surfaces. The concept of elliptic modular surfaces was introduced by Shioda [Sh72], and we will follow his exposition.

Definition 6.1. Let $\bar{\Gamma} \subset PSL_2(\mathbb{Z})$ be a genus zero, torsion-free congruence subgroup of index $\mu < \infty$. Then $\bar{\Gamma}$ can be lifted to a congruence subgroup $\Gamma \subset SL_2(\mathbb{Z})$ of finite index with the property that Γ has no elliptic elements of trance -2. Let $C_\Gamma := (\mathfrak{H}/\Gamma)^*$ be the corresponding modular curve. Consider the automorphism of $\mathfrak{H} \times \mathbb{C}$ defined by

$$(\tau, z) \mapsto (\frac{a\tau + b}{c\tau + d}, \frac{z + m\tau + n}{c\tau + d})$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathfrak{H}$, $z \in \mathbb{C}$ and $(m, n) \in \mathbb{Z}^2$. The quotient of $\mathfrak{H} \times \mathbb{C}$ by this action defines a surface equipped with a morphism to C_Γ. The general fiber for $\tau \in \mathfrak{H}$ is an elliptic curve $E_{\Gamma,\tau}$ corresponding to the lattice $\mathbb{Z} + \tau\mathbb{Z}$. This is called an *elliptic modular surface* associated to Γ, and its geometric genus is given by $p_g = \frac{\mu}{12} - 1$.

We are interested in semi-stable elliptic modular $K3$ surfaces with maximal Picard number 20, that is, singular (extremal) $K3$ surfaces, and semi-stable means that an elliptic fibration has only singular fibers of type I_n, $n > 0$.

Theorem 6.3. *(a)* (Sebbar [Se01]) *There are nine semi-stable elliptic modular $K3$ surfaces with Picard number 20. They correspond to the congruence lifts to $SL_2(\mathbb{Z})$ of genus zero, torsion-free congruence subgroups of $PSL_2(\mathbb{Z})$ of index 24 and 6 cusps with cusp widths adding up to 24.*

index	*Number of cusps*	*Group*	*Cusp widths*	#
12	4	$\Gamma(3)$	$3,3,3,3$	
		$\Gamma_0(4)\cap\Gamma(2)$	$4,4,2,2$	
		$\Gamma_1(5)$	$5,5,1,1$	
		$\Gamma_0(6)$	$6,3,2,1$	
		$\Gamma_0(8)$	$8.2.1.1$	
		$\Gamma_0(9)$	$9,1,1,1$	
24	6	$\Gamma(4)$	$4,4,4,4,4,4$	#1
		$\Gamma_0(3)\cap\Gamma(2)$	$6,6,6,2,2,2$	#2
		$\Gamma_1(7)$	$7,7,7,1,1,1$	#3
		$\Gamma_1(8)$	$8,8,4,2,1,1$	#4
		$\Gamma_0(8)\cap\Gamma(2)$	$8,8,2,2,2,2$	#5
		$\Gamma(8;4,1,2)$	$8,4,4,4,2,2$	#6
		$\Gamma_0(12)$	$12,4,3,3,1,1$	#7
		$\Gamma_0(16)$	$16,4,1,1,1,1$	#8
		$\Gamma(16;16,2,2)$	$16,2,2,2,1,1$	#9

Here

$$\Gamma(8;4,1,2) := \{\pm \begin{pmatrix} 1+4a & 2b \\ 4c & 1+4d \end{pmatrix}, a \equiv c \pmod 2\}$$

and

$$\Gamma(16;16,2,2) := \{\pm \begin{pmatrix} 1+4a & b \\ 8c & 1+4d \end{pmatrix}, a \equiv c \pmod 2\}.$$

(b) (Livné and Yui [LY03], Top and Yui [TY03]) *In each of the nine singular elliptic modular $K3$ surfaces, there is an explicit defining equation over $\mathbb{Q}$ for it.*

(c) (Livné [L95]) *The nine singular elliptic modular $K3$ surfaces over $\mathbb{Q}$ are all modular. (Cf. Theorem 4.2 and Theorem 4.3.)*

Remark 6.3. 1. The cusps correspond to singularities of type I_n, and cusp widths coincide with multiplicities of singular fibers.

2. The groups of index 12 in the Table correspond to rational elliptic modular surfaces. These six groups were first found by Beauville [Beau82] and also by Schoen [Sc81], and then rediscovered by Sebbar [Se01]. These groups are also discussed in Theorem 5.3 where the self-fiber products of these surfaces gave rise to six modular rigid Calabi–Yau threefolds defined over $\mathbb{Q}$ (cf. Theorem 5.4).

Now we will state the main results of Livneé and Yui [LY03].

Theorem 6.4. *Let Y be one of the singular elliptic modular $K3$ surfaces in the Table. Let $T(Y)$ be the transcendental lattice of Y. Let E be an elliptic curve. Consider the product $Y \times E$. We view this product as a family of abelian surfaces over the base modular curve C_Γ. The fiber $A_t = A_{\Gamma,t}$ over each point $t \in C_\Gamma$ is the product of the fiber $E_{\Gamma,t}$ with E.*

(a) The product $Y \times E$ has the Hodge numbers

$$h^{0,3}(Y \times E) = 1,\, h^{1,0}(Y \times E) = 1 = h^{2,0}(Y \times E) \quad \text{and } B_3(Y \times E) = 44,$$

so that $Y \times E$ is not a Calabi–Yau threefold.

(b) The motive $T(Y \times E) = T(Y) \times H^1(E)$ is a submotive of $H^3(Y \times E)$. If both Y and E are defined over $\mathbb{Q}$, this submotive is modular, in the sense that its L-series is associated to cusp forms of weight 3 and 2. That is,

$$L(T(Y \times E), s) = L(g_Y \otimes g_E, s)$$

where g_Y is a weight 3 cusp form associated to $T(Y)$ and g_E is a weight 2 cusp form associated to E.

Now take the quotient $Y \times E/\pm 1$, where we divide each fiber A_t of $Y \times E$ by ± 1 and then blow up the locus of points of order 2 (i.e., a Kummer construction). Let X denote a smooth resolution of $Y \times E/\pm 1$.

Theorem 6.5. *For all groups Γ except for $\Gamma_1(7)$ of index 24 in the Table, the resolution X is a smooth Calabi–Yau threefold, and the following assertions hold for X*

(a) X non-rigid.

(b) The given fibration $f : X \to C_\Gamma$ is semi-stable, with vanishing (iterated) Kodaira–Spencer map.

(c) The Arakelov–Yau equality holds for X, that is, we have

$$deg\, f_*\omega_{X/\mathbb{P}^1} = 2 = \frac{1}{2} deg\, \Omega^1_{\mathbb{P}^1}(log S)$$

where $S \subset \mathbb{P}^1$ is the finite set of six points above which f is non-smooth.

(d) X is modular.

Remark 6.4. 1. We note that in $PSL_2(\mathbb{R})$, the groups Γ of index 24 can be divided into four conjugacy classes: $\{\#1, \#5, \#8\}$, $\{\#2, \#7\}$, $\{\#4, \#6, \#9\}$, and $\{\#3\}$.

2. The groups $\#1, \#2, \#5$ and $\#6$ are all subgroups of $\Gamma(2)$. This guarantees that the 2-torsion points of the fiber A_t are distinct for all $t \in X_\Gamma$, and it follows that the locus of 2-torsion points is smooth and hence so is the blow up X. By the above remark 1, this holds true for all groups except for $\#3 = \Gamma_1(7)$. Also notice that all singular fibers are of type I_n, $n > 1$ with n even for these groups except for $\Gamma_1(7)$.

3. The group $\Gamma_1(7)$ is not a subgroup of $\Gamma(2)$ and the 2-torsion points introduce singularities. K. Hulek studied these singularities. A smooth resolution is no longer a Calabi–Yau theefold. However, its L-series is modular, indeed it is associated to a modular form of weight 3 and two kinds of modular forms of weight 2.

7. Open Problems

Problem 1. For elliptic curves over $\mathbb{Q}$, Shimura constructed a map from the set of Hecke eigenforms of weight 2 with rational Fourier coefficients on $\Gamma_0(N)$ to the isogeny classes of elliptic curves over $\mathbb{Q}$. What Wiles et al. established is that this Shimura map is onto, thereby proving the Shimura–Taniyama conjecture.

For rigid Calabi–Yau threefolds over $\mathbb{Q}$, a natural question is:

Which Hecke eigenforms of weight 4 *on* $\Gamma_0(N)$ *correspond to rigid Calabi–Yau threefolds over* $\mathbb{Q}$*?.*

In fact, this question has been raised by several people, e.g., B. Mazur, D. van Straten, B. van Geemen, K. Hulek.

Problem 2. In Examples of Livné and Yui [LY03], singular $K3$ surfaces Y may be replaced by $K3$ surfaces with smaller Picard numbers, say, 19 or 18. In this case, the modularity for the product $Y \times E$ is still open. If the Picard number is 19, we know that the rank 3 motive $T(Y)$ is self-dual orthogonal via the cup product. By the structure theorem for $K3$ surfaces with Picard number 19 described in Remark 4.2, there should be a cusp form h of weight 2 on $GL(2)$ such that the symmetric square $\mathrm{Symm}^2 h$ should realize $T(Y)$. Then $T(Y \times E)$ may be realized by an autormorphic form on $GL(6, \mathbb{Q})$. In particular, $L(\mathrm{Symm}^2 h \otimes g_E, s)$ has the expected analytic properties.

Problem 3. In case of rigid Calabi–Yau threefolds corresponding to the index 12 groups in the Table, Sun, Tan and Zuo [STZ02] showed that they all reach the Arakelov–Yau bound. We also know that they are all modular by Theorem 5.4. In case of non-rigid Calabi–Yau threeofolds, Livné and Yui [LY03] showd that all eight (except for the case corresponding $\Gamma_1(7)$ as we did not compute it yet) reach the Arakelov–Yau bound and also they are all modular. A question is:

What is the implication of the Arakelov–Yau equalities to the modularity? Is it a necessary condition, or a sufficient condition, or both?

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