THE EIGENCURVE: A BRIEF SURVEY

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In his beautiful paper [Se1], Serre presented the notion of a p-adic analytic family of modular eigenforms. Indeed, in that work, Serre used a single such family: the family of p-adic Eisenstein series. He showed how the mere existence of the Eisenstein family, single-handedly, provides the many congruences (and more) that guarantee the p-adic analyticity of the p-adic zeta function.

We let \( p \) denote a prime number, and \( \mathbb{C}_p \) the completion of an algebraic closure of \( \mathbb{Q}_p \). Consider a p-adic closed disc \( D \) (defined over \( \mathbb{C}_p \)) around a positive integer \( k_0 \). Let \( \mathcal{A} \) denote the ring of p-adic analytic functions over \( D \). For every integer \( k \in D \), there is a specialization map \( \mathcal{A} \rightarrow \mathbb{C}_p \) which is obtained by evaluating a function in \( \mathcal{A} \) at the point \( k \). A p-adic family of modular eigenforms parameterized by \( D \) is a formal \( q \)-expansion

\[
f(q) = \sum_n a_n q^n \in \mathcal{A}[[q]],
\]

such that for all large enough integers \( k \in D \), the specialization of \( f(q) \) at \( k \), i.e.,

\[
f_k(q) = \sum_n a_n(k) q^n \in \mathbb{C}_p[[q]],
\]

is the \( q \)-expansion of an eigenform of weight \( k \). Let us be more precise and specify a level: here we assume that all eigenforms are of level \( \Gamma_1(Np) \), where \( N \) is a positive integer prime to \( p \), and is often referred to as the tame level.

This definition is formulated to capture the situation with the Eisenstein series. The constant term of each p-adic Eisenstein series is a special value of the Riemann zeta function\(^1\), and thus, the analyticity of the function defined by the constant terms alone would provide the p-adic zeta function (that is the p-adic analytic interpolation of those special values). On the other hand, the rest of the \( q \)-expansion coefficients of the Eisenstein series are of quite a simple nature, and the analyticity of the functions they define can be very simply verified. Serre took advantage of this dichotomy: he showed that the latter analyticity will give us the former one for free, by proving that in a p-adic family the constant terms can be calculated from the rest of the coefficients in an analytic manner.

Serre’s work provided the first application of p-adic families, and a dazzling one too. It took, however, more than a decade before further applications of

\(^1\)To be more precise, we shall rather say “the Riemann zeta function devoid of its Euler factor at \( p \).”
$p$-adic families were realized, for the simple reason that it was not so clear how to construct $p$-adic families of modular eigenforms besides the “ur-example” of Eisenstein family. It was Hida who took the first major step in providing families of eigenforms. Hida’s results (e.g., [Hi1, Hi2]), though limited to the case of ordinary modular forms, proved instrumental in many number-theoretical applications (e.g. Greenberg-Steven’s proof of the Mazur-Tate-Teitelbaum conjecture for elliptic curves). Hida also studied $p$-adic analytic families of Galois representations attached to ordinary modular eigenforms. Hida’s approach inspired the work of Mazur and Wiles [MaWi] on families of Galois representations, and consequently, Mazur’s theory of deformation of Galois representations [Ma] which turned out to be a crucial ingredient of Wiles’s proof of Fermat’s Last Theorem.

Another decade had to pass before the “ordinary” restriction which appeared in Hida’s work could be eased in any substantial way. Already, in Serre’s work, it had become clear that one must extend the classical notion of modular forms to include $p$-adic (and later, overconvergent) modular forms. These form $p$-adic Banach (or Frechet) spaces containing the finite-dimensional spaces of classical modular forms, which can be thought of as the $p$-adic interpolation of those classical spaces. Employing a rigid-analytic vision of overconvergent modular forms which was essentially borrowed from earlier work of Katz, Coleman proved the existence of many families: that almost every overconvergent eigenform of finite slope lives in a $p$-adic family. The slope of an eigenform is the $p$-adic valuation of its $U_p$-eigenvalue, and having finite slope is a vast generalization of being ordinary, i.e., to have a $U_p$-eigenvalue which is a $p$-adic unit. Coleman’s important result [Co2] that overconvergent modular forms of small slope are classical, enabled him to prove the existence of an abundance of $p$-adic families of classical modular forms. Coleman’s work was motivated by, and answered, a variety of questions and conjectures that Gouvêa and Mazur had made based on ample numerical evidence (see, for instance, [GoMa]).

The theory reached an aesthetic culmination in Coleman-Mazur’s organization of Coleman’s results (and more) in the form of a geometric object which was labeled the eigencurve. It is a rigid-analytic curve whose points correspond to normalized finite-slope $p$-adic overconvergent modular eigenforms of a fixed tame level $N$. To avoid misleading the reader, we shall point out that here tame level $N$ signifies a wider class of modular forms than those with level $\Gamma_1(Np)$; for instance it consists of all those of level $\Gamma_1(Np^m)$ for all $m \geq 1$. If these modular forms can be put together to form a rigid-analytic variety, it is reasonable to expect their weights to vary $p$-adically analytically as well. We shall not, therefore, restrict ourselves to integral weights. It turns out that for this intuition to work, one should combine data obtained from the weight and the character of the modular form into a single object called the weight-character. The weight-characters will be points on a rigid analytic curve, $W_{N,p}$. 
called the weight space\(^2\) of tame level \(N\). More precisely, the set of \(\mathbb{C}_p\)-valued points of \(W_{N,p}\) is given by
\[
W_{N,p}(\mathbb{C}_p) = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*),
\]
the set of continuous \(\mathbb{C}_p^*\)-valued characters of \(\mathbb{Z}_p^*\). If a modular form of level \(\Gamma_1(Np^m)\) has weight \(k\) and \((\mathbb{Z}/p^m\mathbb{Z})^*\)-character \(\epsilon\), then its weight-character is \(\kappa := \lambda_k \epsilon\), where \(\lambda_k\) is the character \(z \mapsto z^k\), and \(\epsilon\) is now thought of as a character of \(\mathbb{Z}_p^*\) by composing with reduction mod \(p^m\). There are however many more possibilities for a weight-character in general. It is easy to show that the weight space, as a rigid analytic variety, is isomorphic to a disjoint union of finitely many open discs of radius one.

The eigencurve, denoted \(\mathcal{C}_{N,p}\), admits a projection onto the weight space
\[
\mathcal{C}_{N,p} \rightarrow W_{N,p}
\]
which to every overconvergent eigenform associates its weight-character.

How does one construct the eigencurve? There are at least two methods: one method uses deformation rings, and another uses Hecke algebras. The rough idea of the first approach is the following. First, using the multiplicity-one theorem, we replace the normalized eigenform \(f\) by its system of eigenvalues for the Hecke operators. Then, we encode the Hecke system in the following way: the system of eigenvalues of the operators \(T_l\) where \(l\) is a prime not dividing \(Np\) will be encoded by the Galois representation attached to \(f\), and the other eigenvalues will live in a finite-dimensional affine space \(A\). Denote by \(R\) the universal deformation space of all tame-level-\(N\) modular residual Galois (pseudo-)representations (of which there are finitely many). Then the system of Hecke eigenvalues we are looking for will live inside \(R \times A\). The rest is to write down enough equations to specify the locus of \(\mathcal{C}_{N,p}\), and those equation will be formed using enough of the characteristic power series of compact Hecke operators. A technical point is that one may have to further take the nilreduction of the so-obtained space in case it is not reduced. As a result of this construction, one sees that there is a \(p\)-adic family of Galois (pseudo-)representations parameterized by the eigencurve which at a point corresponding to an overconvergent eigenform \(f\), specializes to \(\rho_f\), the Galois representation attached to \(f\).

It is worth noting that in [Se2] Serre, based on Dwork’s work, has developed the functional analysis of compact operators on \(p\)-adic Banach spaces, though it is Coleman’s generalization of this theory in \(p\)-adic families that is applied in the above construction. This is a good opportunity to signal our preference of overconvergent modular forms to \(p\)-adic modular forms (which were constructed in [Se1]): it is to ensure that \(U_p\) (and therefore, every Hecke multiple of it) is a compact operator, and in particular to provide enough defining equations in the above construction.

\(^2\)It would be more appropriate to call it the weight-character space of tame level \(N\).
The Hecke-algebra method is based on the following (roughly-stated) principle: the systems of eigenvalues arising from a commutative algebra of operators $T$ correspond to points of the “space” whose “structure ring” is $T$ (at least under mild conditions). In this naïve form, however, the above principle is not quite useful: in most situations the algebra of operators $T$ is not nearly “good” enough to provide a desirable space. One remedy is replace $T$ with a projective system of better-behaved algebras of operators which in their totality capture the same systems of eigenvalues. To shed some light on this matter, let me sketch how one can use this idea to construct the fibre $C_\kappa$ of the Coleman-Mazur eigencurve $C_{N,p}$ over a weight-character $\kappa \in \mathcal{W}_{N,p}$. The $\mathbb{C}_p$-valued points of $C_\kappa$ correspond to normalized finite-slope Hecke eigenforms occurring in $M^\dagger_\kappa$, the space of overconvergent modular forms of weight-character $\kappa$ and tame level $N$ defined over $\mathbb{C}_p$.

Let $P_\kappa(x)$ denote the characteristic power series of the compact operator $U_p$ acting on $M^\dagger_\kappa$. Let $H_\kappa$ denote the Hecke algebra of $M^\dagger_\kappa$. Let $Z_\kappa$ denote the fibre of the spectral curve over $\kappa$, i.e., the zero locus of $P_\kappa$ in the rigid analytic affine line. Its points correspond to the inverses of non-zero eigenvalues of $U_p$ acting on $M^\dagger_\kappa$. Let $F$ denote a polynomial whose roots form a finite subset (with multiplicities) of these inverse eigenvalues, and set $F^*(x) = x^{\deg F} F(1/x)$. We have a factorization $P(x) = F(x)G(x)$, and by $p$-adic Riesz theory, as developed by Serre in [Se2], we can find a corresponding decomposition $M^\dagger_\kappa = M_F \oplus M_G$ such that $F^*(U_p)$ is zero on $M_F$ and invertible on $M_G$. Let $T_F$ denote the algebra of operators of $M_F$ induced by $H_\kappa$. Then, it is easy to see that $T_F$ is the ring of functions on an affinoid rigid analytic space $C_{\kappa,F}$. This space is going to be the part of $C_\kappa$ accounting for eigenforms for which the inverse of the $U_p$-eigenvalue is a root of $F(x)$. It turns out that varying $F$, we can glue the various $C_{\kappa,F}$’s along $Z_\kappa$ and construct $C_\kappa$. In essence, the Hecke-algebra construction of $C_{N,p}$ is an implementation of the above procedure in $p$-adic families over the weight space, though it is much more technically involved. Among other things, it requires Coleman’s generalization of Serre’s Fredholm theory and Riesz theory used in the above simplified construction.

The geometric picture provided by the eigencurve is so fascinating that it might make us forget, for a moment perhaps, how little we know about the eigencurve as a geometric object. There are many basic questions that are still unresolved; for instance, we still don’t know whether the eigencurve has a finite or infinite number of connected components. Or whether it is smooth or proper over the weight space. Progress is being made, however. Recent work of Buzzard and Kilford [BuKi] shows us that the 2-adic eigencurve indeed looks quite simple at the boundary of the weight space: it is the disjoint union of infinitely many $p$-adic annuli. Using this, Buzzard and Calegary [BuCa] have shown that the 2-adic eigencurve is proper over the weight space.

\[^3\text{Far from the set of classical weigh-characters that is!}\]
Another direction in which research in this area is taking place is the investigation of a similar theory for various reductive groups, other than GL$_2, \mathbb{Q}$. Buzzard has axiomatized and generalized Coleman-Mazur’s Hecke-algebra construction of the eigencurve in [Bu2] (see also Chenevier’s work [Ch1]). With this machinery at hand, at least two important steps have to be taken in the case of a general reductive group $G$. First, one needs to formulate a definition of overconvergent automorphic forms and Hecke algebras for the reductive group in question, and also a definition of $p$-adic analytic families of these objects. Equipped with an appropriate definition, one can apply Buzzard’s general machinery to produce an eigenvariety, a rigid analytic variety whose points would correspond to a certain class of overconvergent automorphic forms for $G$. The second step, is to prove a “classicality criterion”; one which decides when an overconvergent automorphic form is classical, and hence enables us to identify (at least to some extent) the classical locus on the eigenvariety. This will allow us to use $p$-adic methods to obtain results about classical automorphic forms.

In the case of GL$_2, \mathbb{Q}$, for instance, these two steps were taken, respectively, in [Co1, Co2]. Let me mention, very briefly, some of the progress that has been made for groups other than GL$_2, \mathbb{Q}$. Based on an idea of Stevens [St], Buzzard [Bu1] developed the theory of overconvergent automorphic forms for the group of units of a quaternion algebra over $\mathbb{Q}$ which is ramified at infinity, completing the two steps alluded to earlier in this case. Later, following [Bu1] closely, Buzzard [Bu2] and Yamagami [Ya] (with some technical differences) studied the case of a quaternion algebra over a totally real field which is ramified at all infinite primes (i.e., is definite). Chenevier [Ch1] also used this idea to study the case of some unitary groups which are twisted forms of GL$_n, \mathbb{Q}$. These constructions are combinatorial in nature (the relevant “Shimura variety” is zero-dimensional), and some of the results such as the classicality criterion follow with considerably less effort than in Coleman’s approach which is geometric in nature. The picture in these cases, however, is more intrinsic since unlike in Coleman’s work, it doesn’t wholly depend on the existence of a specific $p$-adic family (the Eisenstein family). In fact this dependence has been a main obstacle in creating an equally complete picture in other geometric situations. In [Kas1, Kas2] we investigated the above-mentioned first step in two cases: the group of units of a quaternion algebra over $\mathbb{Q}$ which is split at infinity, and the unitary group arising from a twist of a quaternion algebra over a totally real field $F \neq \mathbb{Q}$ which is split at one infinite place. In both cases the relevant Shimura variety is PEL and one-dimensional, and we use Coleman’s geometric approach. In [Kas4] we also prove a result which specializes to an analogue of Coleman’s classicality criterion for both cases. The proof, however, does not follow Coleman’s, and is based on our more conceptual method presented in [Kas3]. In fact the

\[4\]In other words, a moduli space of abelian varieties with polarization, endomorphism by a certain order, and level structure.
results of [Kas4] allow one to implement our two steps for Shimura curves which are not necessarily PEL; for instance one can study the case of the group of units of a quaternion algebra over a totally real field which is split at one place at infinity. This is possible in light of our recent general approach to the theory of canonical subgroups [GoKa] which avoids the possible modulithetoric properties of Shimura curves, and relies solely on their formal and rigid geometric features. In the case of $\text{GL}_{2,F}$, where $F$ is a totally real field, Kisin and Lai [KiLa] have undertaken the study of the first step of the above strategy. The classicality criterion is however still missing in that context (though they provide a consolation classicality result for some applications). We are hopeful that the method in [Kas3] will be able to address this question at least after acquiring a better understanding of the canonical subgroups of Hilbert-Blumenthal abelian schemes.

As hinted at before, a common shortcoming of the (geometric) methods of [Kas1, Kas2, KiLa] is their incomplete treatment of the weights. This is because in these cases, one does not have an analogue of the Eisenstein family which is “full” with respect to the weight. A partial remedy comes with the lifting of a Hasse invariant over the special fibre of the Shimura variety to characteristic zero to create a family in a one-dimensional direction of the weight space. This underlines the need for a more intrinsic definition of overconvergent automorphic forms, even in the classical case of $\text{GL}_{2,\mathbb{Q}}$. In fact even more is desirable: to have a $p$-adic overconvergent analogue of the classical formalism of automorphic forms and automorphic representations of general reductive groups. A promising approach is Emerton’s more conceptual construction of the eigencurve [Em] using the theory of locally $p$-adic analytic representations of $p$-adic reductive groups developed by Schneider and Teitelbaum [ScTe1, ScTe2, ScTe3, ScTe4]. The approach is quite general, but it is less concrete\textsuperscript{5} than Coleman-Mazur’s and technical difficulties can arise for many groups. We Would like to mention the very recent approach of Iovita and Stevens [IoSt] where overconvergent modular forms of a general weight (and families thereof) are interpreted as sections of certain sheaves over rigid analytic regions in modular curves. There is also a general approach to the construction of the eigenvarieties by E. Urban (yet unpublished).

With an appropriate general theory at our disposal, it is tempting to conjecture that the various Langlands functoriality principles would $p$-adically interpolate to rigid analytic maps between eigenvarieties of the corresponding groups. For instance, Chenevier [Ch2] has shown the existence of a $p$-adic Jacquet-Langlands correspondence between overconvergent automorphic forms

\textsuperscript{5}Progress in [BuCa, BuKi] depends on the rather concrete nature of Coleman-Mazur’s construction.
for $GL_2 \mathbb{Q}$, and those for a definite quaternion algebra over $\mathbb{Q}$. The proof proceeds by taking a $p$-adic “closure” of the classical Jacquet-Langlands correspondence\(^6\).

Investigation of the Langlands philosophy in $p$-adic families is emerging as one of the mainstream directions of research in number theory for the years to come.

**References**


\(^6\)It is tantalizing to imagine a “$p$-adic” proof which combined with classicality criteria on both sides would give us a new proof of the classical Jacquet-Langlands correspondence.


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