SURFACES OVER NON–ALGEBRAICALLY CLOSED FIELDS

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Abstract. This talk is organized as follows. In the first part I remind some facts on classification of surfaces over arbitrary fields. And in the second part I illustrate this ideas on conic bundles and Del Pezzo surfaces of degree 4.

1. Classification of Surfaces over Arbitrary Fields

A smooth surface $S$ over the field $k$ is called minimal if every birational $k$–map $S \rightarrow S'$ to the smooth surface is an isomorphism. The following classification theorem of minimal surfaces over an arbitrary field is due to Iskovskikh[1] and Manin[2].

Theorem 1.1. Let $S$ be a smooth proper surface defined over an arbitrary field $k$. There is a sequence of contractions $S \rightarrow S_1 \rightarrow \cdots \rightarrow S_n = S'$ such that $S'$ and its Picard number $\rho(S')$ satisfy exactly one of the following conditions:

1. $K_{S'}$ is numerically effective (nef);
2. $\rho(S') = 2$ and $S'$ is a conic bundle over a curve $C$;
3. $\rho(S') = 1$ and $S'$ is a Del Pezzo surface, i.e., $-K_{S'}$ is ample.

Recall that for a smooth surface $S$ over the field $k$ the Neron–Severi group $\text{NS}(S)$ is equipped with the natural action of the Galois group $\text{Gal}(\bar{k}/k)$, preserving the intersection pairing and the canonical class $K_S$. In other words this action defines a representation $\rho$ of $G = \text{Gal}(\bar{k}/k)$ in the group of automorphisms of $\text{NS}(S)$ preserving intersection pairing and $K_S$. Then $G = \text{Im} \rho$ is called the splitting group of $S$.

Denote by $\bar{S}$ the surface $S \times_{\text{Spec} \ k} \text{Spec} \bar{k}$ over the algebraic closure of $k$.

Lemma 1.2. If $K_S$ is nef, then $\bar{S}$ is minimal.

We see that such surfaces have the same geometry of minimal models as over the closed field. However, for the classification over non-closed fields we can use arithmetic methods.

Definition 1.3. A variety $Y$ is called a form of $X$ over the field $k$ if $\bar{Y} \cong \bar{X}$.

We have the following general result.

Theorem 1.4. The set of all forms of $X$ is in 1–1 correspondence with the set $H^1(\text{Gal}(\bar{k}/k), \text{Aut}_k(X))$.

2. Conic Bundles and Del Pezzo Surfaces

We say that the conic bundle $\pi_S : S \rightarrow C$ is minimal if every birational map $S \rightarrow S'$ over $C$ to the smooth surface is an isomorphism, this means that we cannot blow-down a line in a fiber of $\pi_S$. Conic bundle is minimal if and only if $\rho(S) = 2$, i.e., if $\text{NS}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$.
Definition 2.1. Let $X$ be a variety over the finite field $\mathbb{F}_q$ and $N_d$ be the number of points of degree 1 on $X \otimes \mathbb{F}_{q^d}$. The zeta–function of $X$ is defined as

$$Z_X(t) = \exp(\sum_{d=1}^{\infty} \frac{N_dt^d}{d}).$$

Theorem 2.2. Let $\pi_S : S \to C$ be a minimal conic bundle and $a_r$ be the number of degenerate fibers over points of degree $r$. Then

$$(2.1) \quad Z_S(t) = Z_C(t)Z_C(qt)\prod_{r} (1 + qt^r)^{-a_r}.$$  

Every birational isomorphism of minimal conic bundles over $C$ is a composition of elementary transformations. The zeta–function of the minimal conic bundle $S$ is determined by the birational isomorphism class of $S$. And the latter is determined by the generic fiber of $\pi_S$. It is a form of $\mathbb{P}^1$ over the function field $K = k(C)$ of a curve $C$.

Let us recall some facts about Del Pezzo surfaces of degree less than 4. We use the notation of Theorem 2.5.

Lemma 2.3. For a form $P$ of $\mathbb{P}_K^1$ there exists a smooth conic bundle $\pi : S \to C$ such that the fiber over the generic point of $C$ is $P$.

If $v$ is a closed point of $C$ we have $\text{Br}(K_v) \cong \mathbb{Q}/\mathbb{Z}$ and an exact sequence:

$$(2.2) \quad 0 \longrightarrow \text{Br}(K) \xrightarrow{\gamma} \bigoplus_v \text{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.$$  

We say that the degenerate fiber of $\pi_S$ splits if we can blow-down the line in this fiber. To describe $\gamma(P)$ for the form $P$ of $\mathbb{P}_K^1$ we need the following lemma.

Lemma 2.4. Let $\pi_S : S \to C$ be a conic bundle. The invariant of a corresponding form of $\mathbb{P}_K^1$ at a closed point $v$ of $C$ is 1/2 if and only if the fiber of $\pi_S$ over $v$ is degenerate and does not split.

From this follows

Theorem 2.5. We use the notation of Theorem 2.2. A conic bundle with zeta–function (2.1) exists if and only if there exists a set of points $x_1, \ldots, x_s$ on the curve $C$, where $s$ is even, such that exactly $a_r$ points have degree $r$.

Such a set of points always exists for a large $q$.

Definition 2.6. The degree of Del Pezzo surface $S$ is $\deg S = (K_S, K_S)$.

We can use previous results for the classification of Del Pezzo surfaces of degree 4. Let us recall some facts on Del Pezzo surfaces of degree less than 4.

- $\deg S = 9$: then $S = \mathbb{P}^2$;
- $\deg S = 8$: then $S = \mathbb{P}^1 \times \mathbb{P}^1$ or is a blow up of $\mathbb{P}^2$ in a point;
- $\deg S = 7$: then $S$ is not minimal;
- $\deg S = 6$ or $\deg S = 5$ and $k = \mathbb{F}_q$: then $S$ is not minimal;

It is known that if $S$ is rational then

$$(2.3) \quad Z_S(t) = (1-t)^{-1}P(t)(1-q^2t)^{-1},$$

where $P(t) = \det(1 - qtF_r^*)$ characteristic polynomial of the weighted Frobenius action on $\text{NS}(S)$. In [M1, IV.9] Manin proved, using calculations of Swinnerton–Dyer, that there are 6 possibilities for a zeta–function of a minimal Del Pezzo surface of degree 4.

We have more complete result for surfaces of degree 4.
**Theorem 2.7.** Let $S$ be a minimal Del Pezzo surface of degree 4. Then the zeta–function of $S$ is one of the functions from Manin’s list.

Surfaces with such zeta–functions exist whenever $q > 3$.

**Sketch of the proof.** For a rational surface we have $\text{deg } X = (K_X, K_X) = 10 - \rho(X)$ (see [M1, IV.2.4]). Suppose $S$ be a minimal conic bundle (i.e., $\text{Pic}(S) \cong \mathbb{Z} \oplus \mathbb{Z}$) with degenerate fibers over the points $x_i$. Since $\text{deg } S = 4$ we have $f = \rho(\tilde{S}) - 2 = \sum_{i=1}^{s} \text{deg } x_i = 4$. By Theorem 2.5 $s$ is even. Obviously, we have only three possibilities for the numbers $\text{deg } x_i$:

1. $s = 4$, $\text{deg } x_i = 1$;
2. $s = 2$, $\text{deg } x_1 = 1$, $\text{deg } x_2 = 3$;
3. $s = 2$, $\text{deg } x_1 = \text{deg } x_2 = 2$.

From the formula (2.1) we get three possible zeta–functions.

At the same time, such surfaces do exist if $q > 3$. Indeed, let $X$ be a conic bundle with the zeta–function. It can be shown that if $X$ is not a Del Pezzo, then the elementary transformation at a $k$-point of $X$, not lying on an exceptional line (such a point exists if $q > 3$), gives us a Del Pezzo surface $S$. The zeta–function of $S$ is $Z_X(t)$.

Let us consider the second case. Suppose $S$ is not a conic bundle, i.e., $\text{Pic}(S) \cong \mathbb{Z}$. Let $\tilde{S}$ be a blow up of $S$ at a point of degree 1. Then $\tilde{S}$ is a Del Pezzo of degree 3 and a minimal conic bundle of degree 3. Hence $f = 5$, and we again have three possibilities for $\text{deg } x_i$:

1. $s = 4$, $\text{deg } x_i = 1$ for $i = 1, 2, 3$ and $\text{deg } x_4 = 2$;
2. $s = 2$, $\text{deg } x_1 = 2$, $\text{deg } x_2 = 3$;
3. $s = 2$, $\text{deg } x_1 = 1$, $\text{deg } x_2 = 4$.

This gives us three other zeta–functions.

Let us construct surfaces with such zeta–functions. We start with a minimal conic bundle $X$ of degree 3. In fact, it is a non–minimal Del Pezzo surface of degree 3. After the contraction of an exceptional line we get a Del Pezzo surface $S$ of degree 4. Since $\rho(X) = 2$, we see that $\rho(S) = 1$ and $S$ is minimal. This completes the proof.

**References**


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