Mathematisches Institut, Seminars
(Y. Tschinkel, ed.), p. –
Universität Göttingen, 2004-05

∞–GROUPOIDS, STACKS, AND SEGAL CATEGORIES

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Résumé. Motivated by descent problems for K–theory and derived categories, and inspired by insights in the theory of infinite loop spaces, I introduce a robust theory of (∞, n)–categories and (∞, n)–stacks, and I formulate several important results and conjectures within this framework.

These are heavily revised and reorganized notes for a five–part series of talks I gave at the Mathematisches Institut Göttingen, in early December 2004. I thank Y. Tschinkel and M. Spitzweck for making my visit possible. I thank M. Spitzweck and G. Racinet for many discussions during which I was able to revise my presentation of the facts introduced in these notes. Finally, I thank J. P. May for inviting me to the University of Chicago for a week, where many of the revisions to these notes were made.

December 2004.

1. Descent Problems for ∞–Categories.

Let us be glad we don't work in algebraic geometry.—J. F. Adams.

I should begin by apologizing for what will be at times very elementary and occasionally rigorless notes. When one is attempting to introduce a new formalism or piece of machinery, it is nothing more than good citizenship to begin by motivating an audience who otherwise would not have taken an interest in the formalism. This will be my sole focus in this talk. I intend to discuss two problems that exhibit a need for a more "complete" theory of descent. I will rather frequently recite theorems and definitions that are well–known to everyone, and I will almost certainly offend members of the audience by omitting important but (for my purposes, at least) irrelevant details. Nevertheless, I would like to open with two problems that should be of interest to Göttingeners.

***1.1* (A naïve view of K–theory).** As I was originally to present this talk in a seminar ostensibly intended for talks on L–functions and the like, I feel I should begin by giving a brief history of an interesting problem, whose origins lie in the theory of ζ–functions. In fact, this problem is by now very probably solved, but it may come as little surprise that there are aspects of the proof that seem less than ideal.

***1.1.1*.** Suppose :
F a number field with
r_1 real embeddings and
r_2 pairs of complex embeddings;
$\mathcal{O}_F$ the ring of integers of F;
c_F the class numberof F.
Recall that the Dedekind ζ–function

$$\zeta_F(s) = \sum_{0\neq\mathfrak{a}\lhd\mathcal{O}_F} \frac{1}{\#(\mathcal{O}_F/\mathfrak{a})}$$

is a natural generalization of Riemann's ζ–function $\zeta = \zeta_{\mathbf{Q}}$, and is convergent for $\Re s > 1$. As with Riemann's ζ–function, ζ_F :
— can be extended to a meromorphic function on all of $\mathbf{C}$,
— has a simple pole at $s = 1$, and
— satisfies a beautiful functional equation relating $\zeta_F(s)$ and $\zeta_F(1 - s)$.

***1.1.2*.** Examination of this functional equation reveals the following interesting facts :
— ζ_F has a zero of order $r_1 + r_2 - 1$ at the origin.

— The first nonzero coefficient in a Taylor expansion about an integer $1-n$ for $n>0$ is the *special value* of ζ_F at $1-n$, denoted $\zeta^\star(1-n)$. Even for $n=1$, this special value contains some remarkable arithmetic information about the field F, as one can see from Dirichlet's Analytic Class Number Formula. Dirichlet defined a *regulator map*

$$\rho_F^D : \mathscr{O}_F^\times/\mu_F \to \mathbf{R}^{r_1+r_2-1},$$

which is a logarithmic embedding of the lattice $\mathscr{O}_F^\times/\mu_F$ into the vector space $\mathbf{R}^{r_1+r_2-1}$; the covolume of the image lattice is the *Dirichlet regulator* DR_F. Dirichlet's Analytic Class Number Formula then states that

$$\zeta^\star(0) = -\frac{c_F}{\#\mu_F} DR_F.$$

— The order of vanishing of ζ_F at $1-n$, for $n>1$, is

$$d_n = \begin{cases} r_1+r_2 & \text{if } n \text{ is odd}\,; \\ r_2 & \text{if } n \text{ is even.} \end{cases}$$

Moreover the values of ζ_F at the positive integers are determined by its special values at the negative integers.

1.1.3. Since they determine the values of the ζ function at all integers, one may well expect that the special values $\zeta^\star(1-n)$ contain even more arithmetic data about F for $n>1$. To try to generalize Dirichlet's formula appropriately, it seems necessary to introduce some other arithmetic invariants of rings, and to play with them. This leads us to K–theory, to which we now turn.

1.1.4. Suppose R is a commutative, unital ring. Grothendieck defined the K–theory $K_0(R)$ to be the free abelian group generated by the isomorphism classes of finitely generated projective R–modules, *modulo* the relation that for any short exact sequence

$$0 \to M \to P \to N \to 0,$$

$[M]+[N]=[P]$. Since any short exact sequence of finitely generated projective R–modules is split, this relation amounts to the equation $[M]+[N]=[M\oplus N]$. One easily verifies that two projective R–modules are equal as elements in $K_0(R)$ iff they are stably isomorphic, in the sense that there exists an isomorphism between $M\oplus R^{\oplus n}$ and $N\oplus R^{\oplus n}$ for some integer $n \geqslant 0$.

Example 1.1.5. Easy : If R is a principal ideal domain or a local ring, the rank function gives rise to an isomorphism $K_0(R)\to\mathbf{Z}$.

Hard : If R is a Dedekind domain, then $K_0(R)$ is isomorphic to $\mathbf{Z}\oplus\mathrm{Cl}(R)$, where of course $\mathrm{Cl}(R)$ denotes the Dedekind class group of R.

1.1.6. For the purposes of reinterpreting and generalizing Dirichlet's formula, I actually require much more than K_0. The failure of certain moduli spaces to be representable in algebraic geometry shows that it is rarely enough to work with isomorphism classes of objects. One must be, in some sense, conscientious of their automorphisms. Since K_0 measures the failure of the uniqueness of representatives of stable isomorphism types of projective modules, there is a similar loss of information here ; one would like also a measurement of the failure of the uniqueness of these isomorphisms. In other words, one should like to work not with a group, but perhaps with a groupoid whose objects are "stable" projective modules, and whose morphisms are somehow "stable" isomorphism between them. An invariant, K_1, would then be the "fundamental group" of this groupoid, classifying equivalence classes of automorphisms.

1.1.7. Historically, a definition for K_1 actually appeared before that of K_0, as K_1 arises quite naturally in cobordism theory, and its definition is quite simple. The most efficient definition is as the abelianization of the infinite general linear group :

$$K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)].$$

The commutator $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is, by a lemma of Whitehead, equal to the subgroup $E(R)$ of $\mathrm{GL}(R)$ generated by the elementary matrices.

Example 1.1.8. Easy : The determinant yields an isomorphism

$$K_1(R) \twoheadrightarrow R^\times \oplus \mathrm{SL}(R)/E(R).$$

Hard : If R is the ring of integers in a number field, then in fact $SL(R)/E(R)$ is trivial. (This follows from the Bass–Milnor–Serre solution of the congruence subgroup problem for SL_n.) Thus for a number field F, $K_1(\mathscr{O}_F) = \mathscr{O}_F^\times$.

1.1.9. It is now possible to reinterpret Dirichlet's formula, viz. :

$$\zeta^\star(0) = -\frac{|K_0(\mathscr{O}_F)^{tors}|}{|K_1(\mathscr{O}_F)^{tors}|} DR_F.$$

1.1.10. It was Quillen who realized that K_0 and K_1—or even the groupoid of which they are π_0 and π_1—would not suffice. Quillen realized that an automorphism might be equivalent to another in a multitude of ways, and these differences are important, for example, to give suitable long exact sequences. To make all of this work, Quillen said, it is necessary to work with some much more robust : a simplicial set. The 0–cells of such a simplicial set should be the "stable" projective modules, the 1–cells should be the "stable" isomorphisms between them, the 2–cells should be equivalences between these isomorphisms, and so on. In other words, Quillen was looking for an ∞–groupoid, which he imagined as a simplicial set. From this, he believed, it should be possible to extract the K–groups as the homotopy groups of this simplicial set. A variant of the following definitions, which appeared in print for the first time in a paper by Waldhausen, was in fact known to Quillen long before his Q construction.

Definition 1.1.11. A *category with cofibrations* $\mathbf{A} = (\mathbf{A}, \mathrm{cof}\,\mathbf{A})$ consists of a pointed category $\mathbf{A}$ and a subcategory $\mathrm{cof}\,\mathbf{A}$ that contains $\iota\mathbf{A}$, whose morphisms are called *cofibrations* such that the following axioms hold.

— For any object X of $\mathbf{A}$, the unique morphism $\star \to X$ is a cofibration.

— For any cofibration $X \to Y$ and any morphism $X \to Z$, the pushout

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \to & Y \sqcup^X Z \end{array}$$

exists, and the canonical morphism $Z \to Y \sqcup^X Z$ is a cofibration.

Definition 1.1.12. A morphism $Y \to Z$ of $\mathbf{A}$ for which there exists a cofibration $X \to Y$ such that the square

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ \star & \to & Z \end{array}$$

is a pushout square is called a *fibration*, and in this case, such an object Z (which is unique up to a canonical isomorphism) will be suggestively denoted by Y/X, and the sequence $X \to Y \to Y/X$ is called a *cofibration sequence.*

Example 1.1.13. Any exact category is a category with cofibrations in which the cofibrations are the exactly the admissible monomorphisms. Thus for a ring R, the category of finitely generated projective R–modules is a category with cofibrations.

Example 1.1.14. A pointed category with all finite colimits is a category with cofibrations in which every morphism is a cofibration.

Definition 1.1.15. For any nonnegative integer p, let p denote also the category $[0 \to 1 \to \ldots \to p]$. I will use the functor category p^1, which is sometimes referred to as the "arrow category" of p. Its objects are pairs (i, j) with $0 \leqslant i \leqslant j \leqslant p$. Suppose now $\mathbf{A}$ a category with cofibrations; then a functor $X : p^1 \to \mathbf{A}$ is said to be a *p–filtered object of* $\mathbf{A}$ iff the following axioms are satisfied.

— For every $0 \leqslant j \leqslant p$, $X(j, j) = \star$.

— For every $0 \leqslant i \leqslant j \leqslant k \leqslant p$, the morphism $X(i,j) \rightarrowtail X(i,k)$ is a cofibration.

— For every $0 \leqslant i \leqslant j \leqslant k \leqslant p$, the square

$$\begin{array}{ccc} X(i,j) & \rightarrowtail & X(i,k) \\ \downarrow & & \downarrow \\ X(j,j) & \rightarrowtail & X(j,k) \end{array}$$

is a pushout.

Thus the sequence $X(i,j) \rightarrowtail X(i,k) \rightarrowtail X(j,k)$ is a cofibration sequence, and to give a p–filtered object of $\mathbf{A}$ is to give a sequence of cofibrations

$$X(0,1) \rightarrowtail X(0,2) \rightarrowtail \cdots \rightarrowtail X(0,p)$$

together with a choice of subquotients $X(i,j) = X(0,i)/X(0,j)$.

Definition 1.1.16. For any nonnegative integer p and any category with cofibrations $\mathbf{A}$, let $S_p\mathbf{A}$ denote the full subcategory of the functor category $\mathbf{A}^{p^1}$ spanned by the p–filtered objects. This gives a simplicial category $S_\bullet\mathbf{A}$, i.e., a simplicial object in the category of categories. Thus $\operatorname{Obj} S_\bullet\mathbf{A}$ is a simplicial set, with a unique 0–cell. (It is canonically pointed.) One then defines $K(\mathbf{A})$, the K–theory of the category with cofibrations A as the simplicial set $\operatorname{Ex}^\infty\Omega(\operatorname{Obj} S_\bullet\mathbf{A})$.

***Exercise 1.1.17* (Easy).** Use Kan's loop description of the fundamental group of a space with only one zero simplex to show that if R is a ring, and $\mathbf{A}$ is the category with cofibrations of finitely generated projective R–modules, then $K_0(R)$ is in fact isomorphic (naturally, in fact) to $\pi_0 K(\mathbf{A}) = \pi_1 \operatorname{Obj} S_\bullet\mathbf{A}$.

Example 1.1.18. Hard : In fact $K_1(R)$ is isomorphic to $\pi_1 K(\mathbf{A})$. Probably the quickest way to write a proof is to observe the following.
Key : Quillen's K–theory of an exact category (using the Q–construction) and the one I have given for the corresponding category with cofibrations are naturally homotopy equivalent.

Definition 1.1.19. One thus defines the higher algebraic K–theory of a ring R by $K_i(R) = \pi_i K(\mathbf{A})$, where $\mathbf{A}$ is the category with cofibrations of finitely generated projective R–modules.

1.1.20. This description, though perhaps a little abstract, has several benefits, which together make it my favorite approach to algebraic K–theory. I list some.

— This approach clearly exhibits the categorical origins of higher K–theory. We see that K–theory has nothing to do with additivity.

— This description can be used to give more than a mere simplicial set : $K(\mathbf{A})$ is in fact an infinite loop space. This can be easily seen from this construction. Indeed, by naturality, the $S_\bullet$ construction extends to (multi)simplicial categories with cofibrations. One can define the cofibrations of $S_\bullet\mathbf{A}$ in a very natural way, and therefore it is possible to iterate the construction. Now simply let $S_\bullet^{(m)}\mathbf{A}$ be the application of $S_\bullet$ to $S_\bullet^{(m-1)}\mathbf{A}$. It is easy to see that the inclusion of the 1–cell Obj $\mathbf{A}$ gives rise to a natural map

$$S^1 \wedge \mathrm{Obj}\,\mathbf{A} \to S^1 \wedge \mathrm{Obj}\,S_\bullet\mathbf{A}.$$

It follows that the spaces $(\mathbf{K}'\mathbf{A})_m = \mathrm{diag}(S_\bullet^{(m)}\mathbf{A}$ define a spectrum, and, by permuting the application of the $S_\bullet$, one effortlessly makes it a symmetric spectrum. The K–theory spectrum $\mathbf{KA}$ is then the loop spectrum of this spectrum. $\mathbf{KA}$ is quickly seen to be an Ω–spectrum, and so it follows that $K\mathbf{A}$ is the infinite loop space of this spectrum.
— Still more is true : if $\mathbf{A}$ is a symmetric monoidal category, a multiplicative structure on $\mathbf{KA}$ is induced, making $\mathbf{KA}$ into a symmetric ring spectrum.
— This construction can also be used in a subtler way (yielding the $wS_\bullet$ construction), which can be applied to Waldhausen categories, which come equipped with certain "weak equivalences." Rather than give the technical definition, let it suffice to say that the full subcategory of cofibrant–fibrant objects of a pointed, proper closed model category forms a Waldhausen category, and all known examples of Waldhausen categories can be constructed in this way. This machine can therefore be applied to the category of *bounded chain complexes* in an abelian category A with an injective model structure, in which the weak equivalences are quasi–isomorphisms, and the cofibrations are degreewise admissible monomorphisms. Again this $S_\bullet$–construction of Waldhausen can be made, and the resulting K–theory is, by a theorem of Waldhausen, Gillet, and Thomason, homotopy equivalent to the K–theory of A, viewed as a category with cofibrations as above.

1.1.21. Borel demonstrated that for a number field F, $K_m(\mathscr{O}_F)$ for m even and positive is finite.

1.1.22. Generalizing Dirichlet's regulator map, Borel constructed higher regulator maps

$$\rho_F^B : K_{2n-1}(\mathscr{O}_F) \to \mathbf{R}^{d_n}$$

for $n > 0$ (where, recall, d_n is the order of vanishing of ζ_F at $1-n$), and showed that the kernel is finite, and the image is a lattice in $\mathbf{R}^{d_n}$, whose covolume is the *Borel regulator* BR_F. When $n = 0$, the Borel regulator and the Dirichlet regulator correspond. Borel used this regulator to show that :

— K_{2n-1} has rank d_n, and

— $\zeta_F^\star(1-n) = q_n BR_F$ for some rational number q_n.

This last result is clearly a weakened generalization of Dirichlet's theorem.

Conjecture 1.1.23 **(Lichtenbaum).** *Lichtenbaum proposed in 1971 a strict generalization of Dirichlet's theorem, namely,*

$$\zeta_F^\star(1-n) = \pm \frac{\# K_{2n-1}(\mathscr{O}_F)^{tors}}{\# K_{2n-2}(\mathscr{O}_F)^{tors}} BR_F,$$

for any $n > 0$, up to a power of 2.

1.1.24. As a consequence of Wiles' proof of the main conjecture of Iwasawa theory, we have the following result. Suppose that F is a totally real number field, n a positive even integer. Then

$$\zeta_F^\star(1-n) = \pm \frac{\#(\prod_p H^2_{\text{ét}}(\mathscr{O}_F, \mathbf{Z}_p(n)))}{\# H^0(F, \mathbf{Q}/\mathbf{Z}(n))}$$

up to powers of 2. Thus a good approach might be to make use of some relationship between K–theory and étale cohomology.

1.1.25. Such a relationship already exists : Grothendieck's general theory of Chern classes gives the étale Chern characters for each prime number p

$$\chi^p_{i,n} : K_{2n-i}(\mathscr{O}_F[\tfrac{1}{p}]) \twoheadrightarrow H^i_{\text{ét}}(\mathscr{O}_F, \mathbf{Z}_p(n)),$$

which have been shown to be surjective for $i = 1, 2$, $2n > i$, and either p odd or $\sqrt{-1} \in F$ by Soulé and Dwyer–Friedlander.

Conjecture 1.1.26 **(Quillen–Lichtenbaum).** *The étale Chern character $\chi^p_{1,n}$ is an isomorphism if p is an odd prime.*

1.1.27. An immediate corollary of the Quillen–Lichtenbaum conjecture is that the Lichtenbaum conjecture is true for a totally real number field. It would therefore be very nice to have a proof of this conjecture. In fact, the Quillen–Lichtenbaum conjecture is a consequence of the Kato conjecture, which is the assertion that a particular Galois symbol is always an isomorphism. Nevertheless, there is something unsettling about this conjecture. The Chern character is produced by a general result in homological algebra, which is somewhat mysterious. Fortunately, we can rephrase this conjecture in a manner that is more concrete, and more general.

Conjecture 1.1.28 **(Quillen–Lichtenbaum, General Concrete Version)** *Suppose F a field, G_F its absolute Galois group of cohomological dimension d. Suppose ℓ is a prime different from the characteristic of F. Then there is a natural morphism of ring spectra*

$$\mathbf{K}F \xrightarrow{\sim} (\mathbf{K}\overline{F})^{G_F} \to (\mathbf{K}\overline{F})^{hG_F},$$

where $(\mathbf{K}\overline{F})^{hG_F}$ is the homotopy fixed point set $\underline{\mathrm{Mor}}_G(EG, \mathbf{K}\overline{F})$. The statement of the conjecture is : the induced morphism

$$\widehat{\mathbf{K}F}_\ell \to \widehat{(\mathbf{K}\overline{F})^{hG_F}}_\ell$$

induces a "co-d" weak equivalence, i.e., it induces an isomorphism on all homotopy groups π_i for $i > d$. Since there is a spectral sequence converging to $\pi_{p+q}((\mathbf{K}\overline{F})^{hG_F})$ whose E_2 term is

$$E_2^{p,q} = H^{-p}(G_F, \pi_q(\widehat{(\mathbf{K}\overline{F})}_\ell)),$$

this spectral sequence must converge to $\pi_{p+q}(\widehat{\mathbf{K}F}_\ell)$ for $p + q > d$.

1.1.29. A simple question is the following : Is this spectral sequence in any sense a descent spectral sequence for the Galois descent of K–theory ? It seems to be, but how can one make this precise ?

1.2 **(Chain complexes when the base varies).** I have been informed that in the model category seminar, the projective model structure on the category of unbounded R–modules for a ring R has been constructed. I will denote this category $\mathbf{Cplx}(R)$. I am interested in what happens when the base varies.

1.2.1. Suppose $f : R \to S$ a homomorphism of rings. Then the functor

$$f^\star = - \otimes_R S : \ \mathbf{Cplx}(R) \to \mathbf{Cplx}(S)$$

has a right adjoint $f_\star$, which is the forgetful functor. Observe that $f_\star$ preserves objectwise epimorphisms and quasi–isomorphisms, so $(f^\star, f_\star)$ defines a Quillen adjunction.

Exercise 1.2.2. Show that a ring homomorphism f induces a Quillen equivalence $(f^\star, f_\star)$ iff it is an isomorphism.

1.2.3. One can think of the assignment $R \mapsto \mathbf{Cplx}(R)$ as a kind of "presheaf" in model categories on the category of affine schemes. Such a presheaf is sometimes called a *left Quillen presheaf.* The assignment $R \mapsto \mathbf{D}(R) = \mathrm{Ho}\,\mathbf{Cplx}(R)$ is a kind of "presheaf" in categories. One can ask whether such a thing is a *stack*. This would indicate, in particular, that it is possible to "glue" complexes up to quasi–isomorphism.

Definition 1.2.4. Suppose C a category. Then a *pseudofunctor* L on C (taking values in the category of categories) consists of the following data :

— an assignment of a category LX to any object $X \in C$,

— an assignment of a functor $Lf : LX \to LY$ to any morphism $f : X \to Y$ in C, and

— an assignment of a natural isomorphism $\gamma_{g,f} : L(gf) \to Lg \circ Lf$ to any pair of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in C,

subject to the following axioms :

— for any morphism $f : X \to Y$ in C, $\gamma_{f,\mathbf{1}_X} = \mathbf{1}_{Lf} = \gamma_{\mathbf{1}_Y,f}$, and

— for any triple of morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

the diagram

$$\begin{array}{ccc} L(h \circ g \circ f) & \xrightarrow{\gamma_{h\circ g,f}} & L(h \circ g) \circ Lf \\ {\scriptstyle \gamma_{h,g\circ f}} \downarrow & & \downarrow {\scriptstyle \gamma_{h,g} \cdot Lf} \\ Lh \circ L(g \circ f) & \xrightarrow[Lh \cdot \gamma_{g,f}]{} & Lh \circ Lg \circ Lf \end{array}$$

commutes.

Definition 1.2.5. A *stack* (in categries) L on a site (T, τ) is a contravariant pseudofunctor on T satisfying the following properties.

— L is a *separated prestack* : for any object X of $\mathscr{S}$, and for any pair of objects $F, G \in LX$, the presheaf of sets on the site $\mathscr{S}/X$ that assigns to any $U \xrightarrow{i} X$ the set $\mathrm{Mor}_{LU}(F|_U, G|_U)$ is a sheaf (i.e., one can glue morphisms).

— For any covering family $\{U_i \to U\}_{i\in I}$, any family of objects $\{F_i \in LU_i\}_{i\in I}$, and any family of isomorphisms $\theta_{ij} : F_i|_{U_i\times_U U_j} \to F_j|_{U_i\times_U U_j}$ satisfying the *cocycle condition*, i.e., the commutativity of

$$\begin{array}{ccccc}
F_k|_{U_j\times_U U_k}|_{U_i\times_U U_j\times_U U_k} & \xrightarrow{\theta_{jk}} & F_j|_{U_j\times_U U_k}|_{U_i\times_U U_j\times_U U_k} & \xrightarrow[\sim]{} & F_j|_{U_i\times_U U_j\times_U U_k} \\
\downarrow{\scriptstyle\sim} & & & & \uparrow{\scriptstyle\sim} \\
F_k|_{U_i\times_U U_j\times_U U_k} & & & & F_j|_{U_i\times_U U_j}|_{U_i\times_U U_j\times_U U_k} \\
\uparrow{\scriptstyle\sim} & & & & \downarrow{\scriptstyle\theta_{ij}} \\
F_k|_{U_i\times_U U_k}|_{U_i\times_U U_j\times_U U_k} & & & & F_i|_{U_i\times_U U_j}|_{U_i\times_U U_j\times_U U_k} \\
{\scriptstyle\theta_{jk}}\downarrow & & & & \downarrow{\scriptstyle\sim} \\
F_i|_{U_i\times_U U_k}|_{U_i\times_U U_j\times_U U_k} & & \xrightarrow{\sim} & & F_j|_{U_i\times_U U_j\times_U U_k}
\end{array}$$

there exist $F \in LU$ and isomorphisms $\theta_i : F|_{U_i} \to F_i$ such that the diagram

$$\begin{array}{ccccc}
F|_{U_i}|_{U_i\times_U U_j} & \xrightarrow[\sim]{} & F|_{U_i\times_U U_j} & \xleftarrow[\sim]{} & F|_{U_j}|_{U_i\times_U U_j} \\
{\scriptstyle\theta_i}\downarrow & & & & \downarrow{\scriptstyle\theta_j} \\
F_i|_{U_i\times_U U_j} & & \xrightarrow[\theta_{ij}]{} & & F_j|_{U_i\times_U U_j}
\end{array}$$

commutes (i.e., one can glue objects).

1.2.6. Is, then, the assignment $R \mapsto \mathbf{D}(R)$ a stack on the Zariski site of affine schemes ? Consider what this would mean for an affine scheme X and an affine open cover $\{U_0, U_1, U_2, U_3\}$ thereof. Unpacking the above definition carefully, we see that in order for $\mathbf{D}$ to be a stack, then, given the following data :

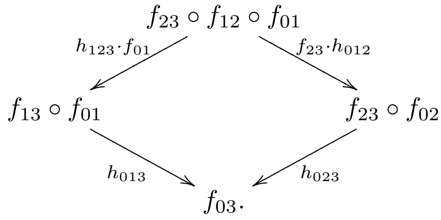
— for each $0 \leqslant i \leqslant 3$, complexes of $H^0(\mathscr{O}_{U_i})$–modules $C_i^\bullet$,

— for each $0 \leqslant i \leqslant j \leqslant 3$, a quasi–isomorphism $f_{ij} : C_i^\bullet|_{U_j} \xrightarrow[\sim]{} C_j^\bullet|_{U_i}$, and

— for each $0 \leqslant i \leqslant j \leqslant k \leqslant 3$, a chain homotopy $h_{ijk} : f_{jk} \circ f_{ij} \simeq f_{ik}$,

there must exist a complex of $H^0(\mathscr{O}_X)$–modules $C^\bullet$ such that $C^\bullet|_{U_i}$is quasi–isomorphic to C_i. But observe that if this were the case, then the homotopies

h_{ijk} could be chosen so that the following square of homotopies commutes :



Imagine the resulting combinatorics if we had had an open cover of cardinality 5. Then there would be 6 diagrams not given as part of the data ! This certainly seems to be too much to hope for, and it is.

Exercise 1.2.7. Show that $\mathbf{D}$ is not a stack. (Hint : an easy example can be given using an open cover of cardinality 4 of $\mathbf{A}^4_k$.) Grothendieck said of the objects of the derived category that they were “de nature essentiellement non recollables.”

1.2.8. The problem with $\mathbf{D}$ is, in a very precise sense, the same problem as the problem with K_0. We have lost “higher homotopical” data in passing to K_0 and the to the derived category. But how do write down a descent condition for left Quillen presheaves ?

***1.3* (Toward ∞–groupoids).** Having given two motivating examples, I now turn to the design of a suitable theory of ∞–categories that will yield the proper setting in which to formulate (and, eventually, answer) the questions of descent. The notion of an ∞–category is one that, for the moment, I intend to use heuristically ; morally, an ∞–category consists of a set[(1)] of objects, a set of morphisms between any two objects, sets of 2–morphisms between any two morphisms, 3–morphisms between any two 2–morphisms, and so on. The n–morphisms should be composable, up to a coherent $(n+1)$–morphism, and the composition law should be associative up to a natural coherent $(n+1)$–morphism.

I shall consider only (∞, n)–categories—i.e., ∞–categories in which the i–morphisms are invertible up to an $(i+1)$–morphism for all $i > n$—, as these are the kinds of ∞–categories that typically arise in algebraic geometry. Making this notions precise for $n = 0$ will be the focus of this section. In order

(1) Here I neglect any set–theoretic difficulties, which are aptly handled by the use of closed model categories and Grothendieck universes.

to motivate the definition, it is convenient to discuss and interpret the nerve construction of Grothendieck.

1.3.1. Let **Cat** denote the category of (small) categories; observe that Δ is isomorphic to the full subcategory of **Cat** spanned by the categories $p = [\ 0 \to 1 \to \ldots \to p\ ]$. Composing the ordinary enrichment functor

$$\mathrm{Mor} : \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Set}$$

with the natural functor

$$\Delta^{\mathrm{op}} \times \mathbf{Cat} \to \mathbf{Cat}^{\mathrm{op}} \times \mathbf{Cat},$$

yields, by adjunction, a functor $\mathbf{Cat} \to s\mathbf{Set}$, called the *nerve functor*, $\nu_\bullet$.

Proposition 1.3.2. *The functor $\nu_\bullet$ is fully faithful and has a left adjoint.*

Proof. One recovers a category C from its nerve $\nu_\bullet C$ in the following manner. Its set of objects is $\nu_0 C$; for any two objects X and Y of C, the set $\mathrm{Mor}_C(X,Y)$ is the fibre of the morphism $(d_1, d_0) : \nu_1 C \to \nu_0 C \times \nu_0 C$ over the pair (X,Y). The compostion law is given by the composite map

$$\nu_1 C \times_{\nu_0 C} \nu_1 C \xrightarrow{\sim} \nu_2 C \xrightarrow{d_1} \nu_1 C.$$

It is now easy to check that the natural map

$$\mathrm{Mor}_{\mathbf{Cat}}(C,D) \to \mathrm{Mor}_{s\mathbf{Set}}(\nu_\bullet C, \nu_\bullet D)$$

is a bijection. ○

Lemma 1.3.3. *If C is a small category, then $\nu_\bullet C$ is a 2–coskeleton. That is, for any simplicial set $X_\bullet$, the natural map*

$$\mathrm{Mor}_{s\mathbf{Set}}(X_\bullet, \nu_\bullet C) \to \mathrm{Mor}_{s\mathbf{Set}}(\mathrm{sk}_2\, X_\bullet, \nu_\bullet C)$$

is a bijection.

Proof. Using the previous proposition, one sees that this lemma holds when $X_\bullet$ is a standard simplicial set Δ^n. Since any simplicial set is a colimit of standard simplicial sets, the result follows. ○

Proposition 1.3.4. *The nerve of a small category C is a Kan simplicial set iff C is a groupoid. Moreover, a functor $C \to D$ between groupoids is an equivalence iff the induced morphism $\nu_\bullet C \to \nu_\bullet D$ is a weak equivalence.*

Proof. By the lemma, it suffices to verify the Kan condition for morphisms $\Lambda^n[k] \twoheadrightarrow \nu \bullet C$ when $n \leqslant 3$. Proceeding case by case, one sees that this is equivalent to the invertibility of any morphism. The final sentence of the proposition follows from the observation that for fibrant simplicial sets, homotopies and simplicial homotopies are the same. ○

***Theorem 1.3.5* (Thomason). Cat** *is a closed model category, wherein a functor $F : C \twoheadrightarrow D$ is a weak equivalence or a fibration iff the induced morphism*

$$\mathrm{Ex}^2 \nu_\bullet C \twoheadrightarrow \mathrm{Ex}^2 \nu_\bullet D$$

of simplicial sets is so.

About the Proof. Thomason demonstrated this directly, by presenting generating sets of cofibrations and of trivial cofibrations, and using the small object argument. Unfortunately, Thomason's proposed class of cofibrations is not stable under retracts, an error noticed and repaired in a recent note of Cisinski. ○

1.3.6. The interpretation I have in mind for these facts is the following. Fibrant simplicial sets are models for *weak ∞–groupoids*; a 0–simplex of a simplicial set should be viewed as an object of the ∞–groupoid; a 1–simplex y is a 1–isomorphism from $d_1 y$ to $d_0 y$; a 2–simplex is a 2–morphism; etc. Of course, this particular avatar of the notion of n–morphism may be unfamiliar, since under this interpretation, n–morphisms do not have only a source and target, but have $(n+1)$ faces instead.

The role of the Kan condition is to guarantee composability. Indeed, given two 1–morphisms of an ∞–groupoid, $A \twoheadrightarrow B \twoheadrightarrow B$, there should exist a "composite up to homotopy," $A \twoheadrightarrow C$; more precisely, for any pair of 1–simplices x_1 and x_2 with $d_0 x_1 = d_1 x_2$, there should exist a 2–simplex x with $d_1 x = x_1$ and $d_2 x = x_2$—the composite up to homotopy is then $d_0 x$. This is precisely the Kan condition in dimension 2. In higher dimensions, the Kan condition guarantees *up–to–homotopy composability* for certain configurations of $(n+1)$ n–morphisms. This leads to the following definition.

Definition 1.3.7. A *(weak) ∞–pregroupoid* is a simplicial set. A *(weak) ∞–groupoid* is a fibrant simplicial set.

Example 1.3.8. If R is a ring (commutative, unital), and $\mathbf{A}$ is the category with cofibrations of finitely generated projective R–modules, then $K(\mathbf{A})$ is an ∞–groupoid

***1.4* (S–categories and simplicial localization).** **S**–categories are categories enriched over simplicial sets, and therefore fibrant **S**–categories can be viewed as models for certain $(\infty, 1)$–categories. The simplicial localization construction of Dwyer and Kan gives a canonical way to convert categories with weak equivalences into **S**–categories.

Definition 1.4.1. An **S**–*category* is a category enriched over $s\mathbf{Set}$. Functors that preserve the simplicial structure are called **S**–*functors*. The category of small **S**–categories is denoted $\mathbf{S} - \mathbf{Cat}$.

1.4.2. Any ordinary category can be viewed as an **S**–category in the obvious way. This defines a functor

$$\mathbf{Cat} \to \mathbf{S} - \mathbf{Cat}$$

whose left adjoint is the functor that to any **S**–category $\mathbf{C}$ assigns the category whose objects are exactly those of $\mathbf{C}$ and whose Mor–set from an object X to an object Y is the set $\pi_0 \operatorname{Mor}_{\mathbf{C}}(X, Y)$. For brevity, denote this left adjoint simply by π_0.

Definition 1.4.3. An **S**–equivalence is an **S**–functor $F : \mathbf{C} \to \mathbf{D}$ that satisfies the following conditions.

— (Full faithfulness) For any objects X and Y of $\mathbf{C}$, the induced morphism or simplicial sets $\operatorname{Mor}_{\mathbf{C}}(X, Y) \to \operatorname{Mor}_{\mathbf{D}}(FX, FY)$ is a weak equivalence.

— (Essential surjectivity) The induced functor $\pi_0\mathbf{C} \to \pi_0\mathbf{D}$ is essentially surjective.

***Proposition 1.4.4* (Bergner).** *There exists a cofibrantly generated closed model structure on* $\mathbf{S} - \mathbf{Cat}$ *with the following properties.*

— *The weak equivalences are exactly the* **S**–*equivalences.*

— *The fibrations are those* **S**–*functors* $F : A \to B$ *such that*

– *the induced morphisms* $\operatorname{Mor}_A(x, y) \to \operatorname{Mor}_B(Fx, Fy)$ *are fibrations for every pair of objects* (x, y)*, and*

– *for any object* a' *of* A*, any object* b *of* B*, and any equivalence* $e : Fa' \to b$*, there exists an object* a *of* A *and an equivalence* $d : a' \to a$ *such that* $Fd = e$.

About the Proof. Such a model structure has been believed to exist for some time now. Dwyer and Kan sketched a faulty proof, by providing generating sets of cofibrations and of trivial cofibrations. Unfortunately, their suggested generating trivial cofibrations are not weak equivalences. Bergner repaired this fault in the overture to her thesis, providing correct generating sets of cofibrations and of generating cofibrations and proving the existence of this model structure directly, by means of the small object argument. ◯

Corollary 1.4.5. *The fibrant objects of* $\mathbf{S} - \mathbf{Cat}$ *are exactly the categories enriched in* ∞*–groupoids.*

1.4.6. A category enriched in ∞–groupoids can be viewed as an $(\infty, 1)$–category, wherein the n–morphisms are the $(n-1)$–cells of the Mor–sets. Observe that the morphisms of such $(\infty, 1)$–categories are strictly composable, and their composition law is strictly associative, so one might correctly call these *strict* $(\infty, 1)$*–categories* (or, more precisely, but altogether less linguistically practical, "weak–∞–strict–1–categories"). The basic source of examples is the simplicial localization, to which I now turn.

Definition 1.4.7. A *quasihomotopical category* $\mathbf{C} = (\mathbf{C}, w\mathbf{C})$ consists of a category $\mathbf{C}$ and a full subcategory $w\mathbf{C}$—whose morphisms are called *weak equivalences*—satisfying the two-out-of-three axiom. A functor $F : \mathbf{C} \to \mathbf{D}$ is *homotopical* iff for any weak equivalence f of $\mathbf{C}$, Ff is a weak equivalence of $\mathbf{D}$.

Definition 1.4.8. Suppose $\mathbf{C} = (\mathbf{C}, w\mathbf{C})$ is a quasihomotopical category. For any pair of objects X and Y in $\mathbf{C}$ and any odd number $n > 0$, I define a category $w\,\mathrm{Mor}^n_{\mathbf{C}}(X, Y)$. The objects of $w\,\mathrm{Mor}^n_{\mathbf{C}}(X, Y)$ are strings of morphisms

$$X = X_0 \leftarrow X_1 \rightarrow X_2 \leftarrow \cdots \rightarrow X_{n-1} \leftarrow X_n = Y$$

such that each morphism $X_{2i} \leftarrow X_{2i+1}$ is contained in $w\mathbf{C}$. Morphisms between two such sequences are simply commutative diagrams of the form

$$\begin{array}{ccccccccc} & & X_1 & \rightarrow & X_2 & \leftarrow \cdots \rightarrow & X_{n-1} & & \\ & \swarrow & \downarrow & & \downarrow & & \downarrow & \nwarrow & \\ X & & & & & & & & Y \\ & \nwarrow & \downarrow & & \downarrow & & \downarrow & \swarrow & \\ & & X'_1 & \rightarrow & X'_2 & \leftarrow \cdots \rightarrow & X'_{n-1} & & \end{array},$$

wherein the vertical maps are in $w\mathbf{C}$. The hammock localization of $\mathbf{C}$ is the $\mathbf{S}$–category $L\mathbf{C}$ whose objects are exactly those of $\mathbf{C}$, with

$$\mathrm{Mor}_{L\mathbf{C}}(X, Y) = \mathrm{Ex}^{\infty} \operatorname{colim}_n \nu_{\bullet}(w\,\mathrm{Mor}^n_{\mathbf{C}}(X, Y))$$

for any objects X and Y.

1.4.9. This construction is clearly pseudofunctorial with respect to homotopical functors.

It is obvious that $\pi_0 L\mathbf{C}$ is the Gabriel–Zisman localization $\mathbf{C}[w\mathbf{C}^{-1}]$. Moreover, there is a canonical $\mathbf{S}$–functor $\mathbf{C} \to L\mathbf{C}$ that is universal in a sense that is more or less immediate.

1.4.10. A priori, this construction seems nightmarish, because the colimit in question seems unmanageable. Fortunately, for model categories, it is not necessary to compute $w\,\mathrm{Mor}^n$ for $n > 3$.

Lemma 1.4.11 **(Dwyer–Kan).** *If $\mathbf{M}$ is a closed model category (*CMC*), then for any two objects X and Y of $\mathbf{M}$, the canonical morphism*

$$\nu_\bullet(w\,\mathrm{Mor}^3_{\mathbf{M}}(X,Y)) \to \mathrm{Mor}_{L\mathbf{M}}(X,Y)$$

is a weak equivalence.

About the Proof. This is a more general fact that holds for any quasi-homotopical category possessing a 3–*arrow calculus*, in the sense of Dwyer and Kan. It follows easily from the existence of functorial factorizations. ◯

1.4.12. Recall that if $\mathbf{M}$ is a CMC, then $s\mathbf{M}$ and $c\mathbf{M}$ each have a *Reedy closed model structure*. The *cosimplicial resolution* functor $q^\bullet$ is the composite

$$\mathbf{M} \to c\mathbf{M} \to (c\mathbf{M})_c$$

and, dually, the *simplicial resolution functor* $r_\bullet$ is the composite

$$\mathbf{M} \to s\mathbf{M} \to (s\mathbf{M})_f.$$

Lemma 1.4.13. *Suppose X and Y objects of a* CMC *$\mathbf{M}$. Then the functors*

$$\mathrm{Mor}_{\mathbf{M}}(q^\bullet X, -) : \mathbf{M}_f \to s\mathbf{Set}$$

and

$$\mathrm{Mor}_{\mathbf{M}}(-, r_\bullet Y) : \mathbf{M}_c \to s\mathbf{Set}$$

are homotopical.

Scholium 1.4.14 **(Dwyer–Kan).** *Suppose $\mathbf{M}$ a* CMC*. There are $\mathbf{S}$–equivalences*

$$\begin{array}{ccc} L(\mathbf{M}_{cf}) & \to & L(\mathbf{M}_f) \\ \downarrow & & \downarrow \\ L(\mathbf{M}_c) & \longrightarrow & L\mathbf{M} \end{array}$$

and there are natural weak equivalences of the simplicial sets

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathbf{M}}(q^{\bullet}X, rY) & & \mathrm{Mor}_{\mathbf{M}}(qX, r_{\bullet}Y) \\
\searrow & & \swarrow \\
& \operatorname{diag}\mathrm{Mor}_{\mathbf{M}}(q^{\bullet}X, r_{\bullet}Y) & \\
& \uparrow & \\
& \operatorname{hocolim}_{(m,n)}\mathrm{Mor}_{\mathbf{M}}(q^{m}X, r_{n}Y) & \\
& \downarrow & \\
& \nu_{\bullet}w\,\mathrm{Mor}^{3}_{\mathbf{M}}(X, Y) & \\
& \downarrow & \\
& \mathrm{Mor}_{L\mathbf{M}}(X, Y), &
\end{array}$$

where q and r are the cofibrant and fibrant replacement functors in $\mathbf{M}$*.*

About the Proof. This chain of weak equivalences follows quickly from the Bousfield–Kan theory of homotopy colimits. ○

Corollary 1.4.15. *If* $\mathbf{M}$ *is a simplicial closed model category, then the* $\mathbf{S}$*–category* $\mathbf{M}_{cf}$ *is* $\mathbf{S}$*–equivalent to* $L\mathbf{M}$*.*

Definition 1.4.16. Suppose A an $\mathbf{S}$–category, κ a regular cardinal. Then A is *κ–presentable* if A is cocomplete, and if there exists a κ–small sub–$\mathbf{S}$–category A_c spanned by κ–compact objects that generates A by κ–filtered colimits. A is said to be *presentable* if it is κ–presentable for some regular cardinal κ.

Theorem 1.4.17 **(Simpson)*.*** *Suppose A an* $\mathbf{S}$*–category. Then the following are equivalent.*

— *There exists a cofibrantly generated* CMC $\mathbf{M}$ *such that A is* $\mathbf{S}$*–equivalent to* $L\mathbf{M}$*.*

— *A is presentable.*

About the Proof. This is a very technical result, whose proof requires the use of the theory of Segal 1–categories, which I will introduce momentarily. ○

Example 1.4.18. For any ring R, the category $\mathbf{Cplx}(R)$ of unbounded chain complexes of R–modules is a combinatorial closed model category. The simplicial localization $L\mathbf{Cplx}(R)$ is therefore a presentable $\mathbf{S}$–category.

1.5 **(Bousfield localization)*.*** There are two canonical ways to add weak equivalences to a closed model category in a minimal way. The existence proofs are often distractingly technical, so I shall not have much to say about them.

Definition 1.5.1. Suppose $\mathbf{M}$ a CMC, and suppose C a class of morphisms in $\mathbf{M}$.

— The *left Bousfield localization* of $\mathbf{M}$ with respect to C is a CMC $L_C\mathbf{M}$ equipped with a left Quillen functor $\mathbf{M} \to L_C\mathbf{M}$ that is initial among all left Quillen functors $F : \mathbf{M} \to \mathbf{N}$ with the property that for $f \in C$, Ff is a weak equivalence.

— The *left Bousfield localization* of $\mathbf{M}$ with respect to C is a CMC $R_C\mathbf{M}$ equipped with a right Quillen functor $\mathbf{M} \to L_C\mathbf{M}$ that is initial among all left Quillen functors $F : \mathbf{M} \to \mathbf{N}$ with the property that for $f \in C$, Ff is a weak equivalence.

***Proposition 1.5.2* (Cole).** *Suppose M a category with two model structures $\mathbf{M}$ and $\mathbf{M}'$ such that $w\mathbf{M} \subset w\mathbf{M}'$ and $\mathrm{fib}\mathbf{M} \subset \mathrm{fib}\mathbf{M}'$. Then the right Bousfield localization $R_{w\mathbf{M}'}\mathbf{M}$ exists; the underlying category of $R_{w\mathbf{M}'}\mathbf{M}$ is again M, and $w(R_{w\mathbf{M}'}\mathbf{M}) = w\mathbf{M}'$ and $\mathrm{fib}(R_{w\mathbf{M}'}\mathbf{M}) = \mathrm{fib}\mathbf{M}$.*

Likewise, if N is a category with two model structures $\mathbf{N}$ and $\mathbf{N}'$ such that $w\mathbf{N} \subset w\mathbf{N}'$ and $\mathrm{cof}\mathbf{N} \subset \mathrm{cof}\mathbf{N}'$, then the left Bousfield localization $L_{w\mathbf{N}'}\mathbf{N}$ exists; the underlying category of $L_{w\mathbf{N}'}\mathbf{N}$ is again N, and $w(L_{w\mathbf{N}'}\mathbf{N}) = w\mathbf{N}'$ and $\mathrm{cof}(L_{w\mathbf{N}'}\mathbf{N}) = \mathrm{cof}\mathbf{N}$.

Proof. This proof is left as an easy exercise in the axioms of closed model categories. ○

Definition 1.5.3. Suppose $\mathbf{C}$ a quasihomotopical category; suppose S a class of morphisms thereof; and suppose K a class of objects thereof.

— An object X of $\mathbf{C}$ is $\mathbf{S}$*–local* (resp., $\mathbf{S}$*–colocal* if for any element $A \to B$ of S, the induced morphism $\mathrm{Mor}_{L\mathbf{C}}(B, X) \to \mathrm{Mor}_{L\mathbf{C}}(A, X)$ (resp., the induced morphism $\mathrm{Mor}_{L\mathbf{C}}(X, A) \to \mathrm{Mor}_{L\mathbf{C}}(X, B)$) is a weak equivalence. The class of $\mathbf{S}$–local objects is denoted $S - \mathrm{loc}$, and the class of $\mathbf{S}$–colocal objects is denoted $S - \mathrm{col}$.

— A morphism $C \to D$ of $\mathbf{C}$ is said to be K*–local* if for any element Y of K, the induced morphism $\mathrm{Mor}_{L\mathbf{C}}(D, Y) \to \mathrm{Mor}_{L\mathbf{C}}(C, Y)$ (resp., the induced morphism $\mathrm{Mor}_{L\mathbf{C}}(Y, C) \to \mathrm{Mor}_{L\mathbf{C}}(Y, D)$) is a weak equivalence. The class of $\mathbf{S}$–local morphisms is denoted $S - \mathrm{loc}$, and the class of $\mathbf{S}$–colocal morphisms is denoted $S - \mathrm{col}$.

— For P a class either of objects or of morphisms of $\mathbf{M}$, I define the *left hull* $\mathrm{lh}(P)$ of P as the class $(P - \mathrm{loc}) - \mathrm{loc}$ and the *right hull* $\mathrm{rh}(P)$ as the class $(P - \mathrm{col}) - \mathrm{col}$.

Lemma 1.5.4. *If P is a class either of objects or of morphisms of a* CMC $\mathbf{M}$*, then $P \subset \mathrm{lh}(P)$, and if S is a class of morphisms, the left Bousfield localization $L_S\mathbf{M}$ is naturally isomorphic to the left Bousfield localization $L_{\mathrm{lh}(S)}\mathbf{M}$.*

Dually, $P \subset \mathrm{rh}(P)$, and if S is a class of morphisms, the right Bousfield localization $R_S\mathbf{M}$ is naturally isomorphic to the right Bousfield localization $R_{\mathrm{rh}(S)}\mathbf{M}$.

Definition 1.5.5. Suppose $\mathbf{M}$ a CMC, κ a regular cardinal. Then a class of morphisms S in C is *κ–sequential* if S is closed under colimits of κ–sequences, i.e., if for any κ–sequence of cofibrations X_α in $\mathbf{M}$, and any sequence of morphisms $X_\alpha \twoheadrightarrow Y$ of S, the morphism from the transfinite composition

$$\mathrm{colim}_\alpha X_\alpha \twoheadrightarrow Y$$

is a morphism of S as well. If the set of fibrations is κ–sequential, then one says simply that $\mathbf{M}$ *has κ–sequential fibrations.*

***Theorem 1.5.6* (Christensen–Isaksen).** *If $\mathbf{M}$ is a right proper* CMC *and K is a set of objects thereof such that there exists a regular cardinal κ with the following properties :*

— $\mathbf{M}$ *has κ–sequential fibrations, and*

— *each element of K is κ–small relative to the cofibrations,*

then the right Bousfield localization $R_{K-\mathrm{col}}\mathbf{M}$ exists ; the underlying category is the same as the underlying category of $\mathbf{M}$, and $w(R_{K-\mathrm{col}}\mathbf{M}) = K - \mathrm{col}$ and $\mathrm{fib}(R_{K-\mathrm{col}}\mathbf{M}) = \mathrm{fib}\mathbf{M}$. The cofibrant objects of $R_{K-\mathrm{col}}\mathbf{M}$ are exactly the $K - \mathrm{col}$–colocal objects.

About the Proof. This was proved under rather more restrictive hypotheses (right properness and cellularity) in Hirschhorn's book. The proofs there are easily translated, mutatis mutandis, to this more general setting. ◯

Corollary 1.5.7. *If $\mathbf{M}$ is a cofibrantly generated* CMC*, and K is a set of objects of $\mathbf{M}$, then the right Bousfield localization $R_{K-\mathrm{col}}\mathbf{M}$ exists.*

Definition 1.5.8. A CMC is *combinatorial* if it is cofibrantly generated and presentable.

***Theorem 1.5.9* (Smith).** *Suppose $\mathbf{M}$ a left proper combinatorial* CMC*, S a set of morphisms of $\mathbf{M}$. Then the left Bousfield localization $L_S\mathbf{M}$ exists ; the underlying category of $L_S\mathbf{M}$ is that of $\mathbf{M}$, and $w(L_S\mathbf{M}) = \mathrm{lh}(S)$ and $\mathrm{cof}(L_S\mathbf{M}) = \mathrm{cof}(\mathbf{M})$.*

About the Proof. This was proved under different hypotheses (left properness and cellularity) in Hirschhorn's book. The critical Bousfield–Smith cardinality argument carries over with only slight modification. ◯

1.5.10. Combinatoriality is a flexible condition that is often satisfied for categories constructed from the category of simplicial sets or from that of CW complexes. For categories constructed from categories of more general topological spaces, Hirschhorn's condition of cellularity is better–behaved. Unfortunately, there are examples of left proper combinatorial CMCs that are not cellular, and there are examples of left proper cellular CMCs that are not combinatorial, so neither setting contains the other.

***Theorem 1.5.11* (Lurie).** *Suppose A is a presentable $\mathbf{S}$–category, and suppose S a set of morphisms of A. Then the full sub–$\mathbf{S}$–category $L_S A$ of $\mathbf{S}$–*local objects*—i.e., objects X such that for any $Y \to Z$ in S, the morphism* $\mathrm{Mor}_A(Z,X) \to \mathrm{Mor}_A(Y,X)$ *is a weak equivalence—is a reflexive subcategory of A.*

About the Proof. This is the first major result of the overture to Lurie's thesis, rewritten for $\mathbf{S}$–categories. ◯

Lemma 1.5.12. *Suppose* $\mathbf{M}$ *a* CMC*, S a set of morphisms thereof. Then* $L(L_S\mathbf{M})$ *is equivalent to* $L_S(L\mathbf{M})$*, and this equivalence is compatible with the "localization" morphisms.*

Proof. The proof is left as a nearly trivial exercise. ◯

***1.6* (The question of descent).** I now formulate the central questions of descent for simplicial sets and left Quillen presheaves, and I comment upon the answers.

1.6.1. It will be convenient to recall the following key fact. Suppose $\mathbf{M}$ a cofibrantly generated closed model category, and suppose C a small category. Then the functor category $\mathbf{M}^C$ is a cofibrantly generated closed model category in which the fibrations and weak equivalences are defined objectwise. This closed model structure on $\mathbf{M}^C$ will be called the *projective* closed model structure. Combinatoriality, left properness, and right properness are all inherited by $\mathbf{M}^C$ from $\mathbf{M}$.

1.6.2. The natural precursor for the theory of descent I introduce here is ordinary sheaf theory. Let (T,τ) be a site, which, for the sake of exposition, I will assume has enough points. Let $\mathbf{PSh}(T)$ be the category $\mathbf{Set}^{T^{\mathrm{op}}}$ of presheaves (of sets) on T, and let $\mathbf{Sh}(T,\tau)$ be the full subcategory thereof spanned by the sheaves. The category **Set** has its trivial closed model structure, and therefore $\mathbf{PSh}(T)$ has a projective closed model structure.

Theorem 1.6.3. *The category* $\mathbf{PSh}(T)$ *has a left proper, combinatorial closed model structure in which the following conditions hold.*

— *Every morphism is a cofibration.*

— *The fibrant objects are precisely the sheaves.*

— *The weak equivalences are precisely those morphisms* $F \to G$ *such that for any point* x *of the site* (T, τ)*, the induced morphism* $F_x \to G_x$ *of sets is a bijection.*

— *Between any two sheaves* F *and* G*, two morphisms* $F \to G$ *are homotopic iff they are equal.*

Proof. Let $\mathbf{Cov}(T, t)$ be the class of morphisms $\coprod_\alpha U_\alpha \to X$ (viewed as morphisms of presheaves) for all τ–covering families $\{\ U_\alpha \to X\ \}$ of T. Using the smallness of T, one can find easily a subset H_τ thereof such that the left Bousfield localization $L_{H_\tau}\mathbf{PSh}(T)$ of the projective closed model structure with respect to H_τ is the left Bousfield localization $L_{\mathbf{Cov}(T,\tau)}\mathbf{PSh}(T)$ with respect to $\mathbf{Cov}(T, \tau)$. It is immediate from the characterization of left Bousfield localizations that the listed conditions hold. ◯

Corollary 1.6.4. *The composite functor*

$$\mathbf{Sh}(T, \tau) \to \mathbf{PSh}(T) \to \operatorname{Ho} L_{\mathbf{Cov}(T,\tau)}\mathbf{PSh}(T)$$

is an equivalence of categories.

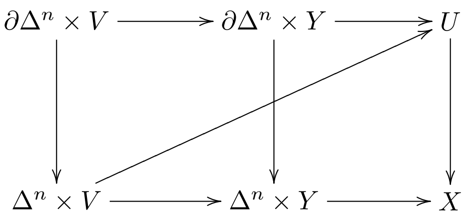
1.6.5. The task is then to generalize this description of sheaves to stacks in ∞–groupoids. Since the equivalences between ∞–groupoids are not isomorphisms, the descent data should give not isomorphisms on overlaps, but equivalences. Additionally, the category of ∞–groupoids, i.e. of fibrant simplicial sets, does not itself have a closed model structure ; instead, it is the class of fibrant objects within the closed model category of ∞–pregroupoids. For presheaves of sets, this distinction was invisible to us because every set is fibrant.

Observe that the category of presheaves of ∞–pregroupoids on T, $s\mathbf{SPr}(T) = s\mathbf{Set}^{T^{\mathrm{op}}}$ has its projective closed model structure, which is left proper and combinatorial. The fibrant objects of this closed model structure are those presheaves of ∞–pregroupoids X such that for any object U of T, $X(U)$ is an ∞–groupoid. The aim now is to left–Bousfield–localize the projective model structure in a manner that is similar to the localization performed for ordinary presheaves. The weak equivalences of the resulting closed model structure should be the τ–local weak equivalences, and the fibrant objects should be exactly those objects satisfying a descent condition. In order to guarantee that the weak equivalences are as described, it is necessary to force the fibrant objects to satsify a descent condition with respect to *hypercoverings*, not merely coverings. The distinction is a subtle one, as many familiar topologies have the *Brown–Gersten property*, which ensures that the localization with respect to coverings and with respect to hypercoverings are in fact the same.

Definition 1.6.6. A *hypercovering* of an object X of T is a simplicial presheaf U along with a morphism $U \to X$ such that U_p is a coproduct of representables for any $p > 0$, and the morphism $U \to X$ is a *local trivial fibration*, i.e., for any object Y of T, any $n \geqslant 0$, and any commutative square

$$\begin{array}{ccc} \partial\Delta^n \times Y & \to & U \\ \downarrow & & \downarrow \\ \Delta^n \times Y & \to & X \end{array}$$

there exists a covering sieve R of Y such that for every $V \to Y$ in R, there is a lift :



Theorem 1.6.7. *There is a left proper, combinatorial closed model structure on $s\mathbf{SPr}(T)$ such that the following conditions are satisfied.*

— *The cofibrations are precisely the projective cofibrations.*

— *The fibrant objects are exactly those presheaves of ∞–groupoids F such that for any hypercovering $U \to X$ of any object X of T, the induced morphism*

$$FX \to \operatorname{holim}\left[\prod_{\alpha_0} FU_0^{\alpha_0} \rightrightarrows \prod_{\alpha_1} FU_1^{\alpha_1} \overset{\to}{\underset{\to}{\to}} \dots\right]$$

is an equivalence of simplicial sets, where the $U_n^{\alpha_n}$ are the representing objects of the functors of which U_n is the coproduct.

— *The weak equivalences are exactly those morphisms $F \to G$ such that for any point x of T, the induced morphism $F_x \to G_x$ is a weak equivalence of ∞–pregroupoids.*

This closed model structure is called the local projective *closed model structure.*

Proof. Let $\mathbf{Hypercov}(T,\tau)$ be the class of hypercoverings of objects of T. Using the smallness of the site, one verifies that there exists a subset H of $\mathbf{Hypercov}(T,\tau)$ such that the left Bousfield localization $L_{H_\tau}(s\mathbf{SPr}(T)$ of the projective closed model structure with respect to H is the left Bousfield localization $L_{\mathbf{Hypercov}(T,\tau)}(s\mathbf{SPr}(T)$ with respect to $\mathbf{Hypercov}(T,\tau)$. The listed properties follow immediately from the characterization of left Bousfield localizations. ◯

Definition 1.6.8. An *∞–stack* or *stack in ∞–groupoids* is a fibrant object of the local projective closed model category of presheaves of ∞–pregroupoids.

1.6.9. Now it is possible to ask the question : is the algebraic K–theory functor K—a presheaf of ∞–groupoids—an ∞–stack for the étale topology ? for any other interesting topologies ?

The answer turns out to be a very qualified yes. First of all, Thomason showed that algebraic K–theory does satisfy Zariski descent. For smooth schemes, he also demonstrated that K–theory satisfies Nisnevich descent. Over a field with resolution of singularities, Haesemeyer showed that *homotopy* K–theory satisfies cdh descent. Finally, K–theory does not itself satisfy étale descent, but its failure to do so is in a sense bounded : Thomason showed that *Bott–inverted* K–theory satisfies descent.

This is still not quite enough to prove the Beilinson–Lichtenbaum conjecture. The Beilinson–Lichtenbaum spectral sequence does not converge below the line $p+q=d$, where d is the cohomological dimension of the absolute Galois group of the ground field. It therefore cannot be a descent spectral sequence for K–theory as a presheaf of ∞–groupoids alone. The central idea, due to Gunnar Carlsson, is to view K instead as a presheaf of ring objects in a certain category of *Mackey functors*. This approach, while promising, seems not to have generated much attention, despite the fact that it may very well lead toward a conceptual proof of the Beilinson–Lichtenbaum conjecture. The details of the set–up are perhaps better left for another day.

1.6.10. To study our second question, namely, whether the left Quillen presheaf **Cplx** satisfies descent, one may be tempted to try to use our work on stacks in ∞–groupoids to design a notion of $(\infty,1)$–stacks. In particular, one might study presheaves of **S**–categories on T. The category $(\mathbf{S}-\mathbf{Cat})^{T^{\mathrm{op}}}$ of all such presheaves on T has its projective closed model structure, and one can try to take the left Bousfield localization of this projective closed model structure with respect to all hypercoverings. The fibrant objects of the resulting closed model category would be the presheaves of **S**–categories that satsify descent, and the weak equivalences would be the local **S**–equivalences.

Unfortunately, this seems to require an ability to view an ∞–groupoid G as an **S**–category X_G in such a way that the objects of X_G are precisely those of G and the n–morphisms of X_G are precisely those of G. This is impossible, because the 1–morphisms of ∞–groupoids are only composable up to a 2–morphism, whereas the 1–morphisms of **S**–categories are strictly composable. It is therefore unclear how to view hypercoverings as **S**–functors. In order to continue, therefore, it is necessary to design a theory of $(\infty, 1)$–categories wherein the 1–morphisms are only composable up to a 2–morphism.

2. Delooping machines and (∞, n)–categories

Maybe at times I like to give the impression, to myself and hence to others, that I am the easy learner of things in life, wholly relaxed, "cool" and all that—just keen for learning, for eating the meal and welcome smilingly whatever comes with it's message, frustration and sorrow and destructiveness and the softer dishes alike. This of course is just humbug, an image d'Epinal which at whiles I'll kid myself into believing I am like. Truth is that I am a hard learner, maybe as hard and as reluctant as anyone.—A. Grothendieck.

2.1 (Infinite loop spaces and delooping machines)

I will give what I hope is convincing evidence that the study of infinite loop spaces in algebraic topology has the support of a powerful and deep theory, and that this theory has pleasant formal properties.

I will begin this subsection by discussing loop spaces, giving their nice properties, and giving a few examples. Then I will introduce the Stasheff associahedra, A_∞–spaces, infinite loop spaces, and finally Segal's theory of delooping machines. It will be seen that an infinite hierarchy of higher homotopies must be dealt with, and it will also be seen that there are two ways of managing this hierarchy : the Stasheff, or operadic, method, which deals more or less explicitly with the combinatorics of the homotopies, and the Segal, or simplicial, method, which hides the combinatorics behind the theory of simplicial sets. One can today look at these simple constructions through more sophisticated eyes, and this is exactly what I will do here.

2.1.1. To fix ideas, I will work in the category $\mathscr{T}$ of *pointed, compactly generated* spaces (with basepoint–preserving continuous maps). Spaces in $\mathscr{T}$ are those spaces X that have the following pleasant properties :

— X is *weakly Hausdorff* ; that is, the image of any continuous map from a compact Hausdorff space to X is closed in X.

— X is *Kelley* ; that is, any subset U of X that is *compactly open*—in the sense that U has open inverse image under any continuous map from a compact Hausdorff space to X—is in fact open.

The category $\mathscr{T}$ itself has extremely nice properties :

— Any space Y can be made into a Kelley space kY (with the same underlying point set) by *declaring* that a subset U is open in kY iff U is compactly open in Y. It can then be made into a compactly generated space wY by taking the maximal weak Hausdorff quotient. This defines a functor w from the category of all pointed spaces to $\mathscr{T}$.

— $\mathscr{T}$ has all small limits and colimits, which are given by taking the limits and colimits in the category of all topological spaces and then applying w. As a rule, I will remove w from the notation, as quite often it is not even necessary to apply it.

— Though the product in $\mathscr{T}$ is the product $- \times -$, where the basepoint is given by the point whose coordinates are the basepoints of the factors, there is also a kind of "tensor product" given by the smash product $- \wedge -$. $\mathscr{T}$ is *closed monoidal category* with this product, in the sense that applying w to a set of continuous, basepoint–preserving maps with the compact–open topology gives an internal $\underline{\mathrm{Mor}}$ functor (where the basepoint is the constant map) so that $\mathrm{Mor}(X \wedge Y, Z) = \mathrm{Mor}(X, \underline{\mathrm{Mor}}(Y, Z)$.

— The geometric realization of any pointed simplicial set is an object of $\mathscr{T}$; this defines a functor $| - |$ from the category $\mathscr{S}$ of pointed simplicial sets to $\mathscr{T}$.

— The geometric realization functor is left adjoint to the functor that sends a compactly generated space to its total singular complex. This pair forms a Quillen equivalence between $\mathscr{S}$ and $\mathscr{T}$ with their usual model structures (where in particular the weak equivalences are given by maps that induce isomorphisms on all homotopy groups), and so these functors descend to quasi–inverses between the homotopy categories $\mathrm{Ho}\,\mathscr{S}$ and $\mathrm{Ho}\,\mathscr{T}$.

When I write the word 'space,' I mean an object of $\mathscr{T}$, and when I write 'map,' I mean a continuous, basepoint–preserving map, unless otherwise noted.

$\mathscr{T}$ is really the largest category of pointed spaces that is so well–behaved formally.

2.1.2. With the conventions above, the *loop space* ΩX of a space $X \in \mathscr{T}$ is the space $\underline{\mathrm{Mor}}(S^1, X)$, where of course S^1 is a *pointed* circle. The loop space can be thought of as a flabby version of the Poincaré fundamental group of X ; it comes with *higher homotopies*. Indeed, the connected components of ΩX form a group, and that group is naturally isomorphic to $\pi_1 X$. More generally, one can check easily that $\pi_i(\Omega X)$ is naturally isomorphic to $\pi_{i+1}X$ for any $i \geqslant 0$.

Everyone who has seen the fundamental group defined knows that ΩX is not quite a topological group, even though its connected components are. One can compose any two loops by running through one and then the other, each at double speed. More precisely, the *pinching map* $S^1 \twoheadrightarrow S^1 \vee S^1$ given by the identification of a non–basepoint with the basepoint induces a natural transformation from $\underline{\mathrm{Mor}}(S^1 \vee S^1, -)$ to $\underline{\mathrm{Mor}}(S^1, -)$, so that there is a map $\Omega X \times \Omega X \twoheadrightarrow \Omega X$, which we can call μ. But the constant loop, which is the basepoint $\star$ of ΩX, is not a strict unit for this mutliplication law : neither $\mu(\gamma, \star)$ nor $\mu(\star, \gamma)$ is *equal* to γ, since, for example, $\mu(\gamma, \star)$ is the loop that dawdles at the basepoint for half the time around the circle, and then rushes through γ at double speed. Nor is this multiplication law is strictly associative : given three loops α, β, and γ, the loop $\mu(\alpha, \mu(\beta, \gamma))$ runs through γ and β both at quadruple speed and then runs through α at half speed, whereas $\mu(\mu(\alpha, \beta), \gamma)$ runs through γ at half speed and then runs through β and α at quadruple speed. (One can actually tidy up this part of the situation by looking at what are called "Moore loops" to make ΩX equivalent to a topological monoid, but I needn't go into this here.) Finally, note that the inverse loop is not a strict inverse.

The structure of ΩX is described by saying that μ gives ΩX the structure of a group in the homotopy category of spaces. That is, one has the following :

— *The basepoint is a homotopy unit.* If '$\star$' denotes the inclusion of the basepoint into ΩX, then $\mu \circ (\star \times \mathbf{1})$ and $\mu \circ (\mathbf{1} \times \star)$ are both homotopic to the identity on ΩX.

— *The multiplication is homotopy–associative.* $\mu \circ (\mu \times \mathbf{1})$ and $\mu \circ (\mathbf{1} \times \mu)$ are homotopic maps from $\Omega X \times \Omega X \times \Omega X$ to ΩX.

— *There are homotopy inverses.* If 'ν' denotes the self–map on ΩX that sends a loop to its inverse, and Δ is the diagonal map $\Omega X \twoheadrightarrow \Omega X \times \Omega X$, then $\mu \circ (\mathbf{1} \times \nu) \circ \Delta$ and $\mu \circ (\nu \times \mathbf{1}) \circ \Delta$ are each homotopic to the map that factors through $\star$.

Serre called a space satisfying the first and second of these of these a *homotopy–associative H–space*, in honor of Hopf. We will, by an abuse of terminology, use the somewhat handier expression *H–space* to refer to such objects, which, morally, are spaces Z with maps $Z \times Z \twoheadrightarrow Z$ that give monoids in the homotopy category. The third condition might seem like a key fact about loop spaces, but in fact it is something of a red herring : it can be done away with altogether by looking more closely at the first two conditions, which is precisely what I will do.

Serre used the loop space quite well : he remarked that if one took the usual unit interval I with basepoint at (say) 0, one could design a continuous map from the space $\Pi X = \underline{\mathrm{Mor}}(I, X)$ of paths in X to X just by evaluation at 1. This map is a very good map ; to be precise, it is a *fibration* in the sense of Serre. The fibre over the basepoint of X is the loop space ΩX. Moreover ΠX is contractible : one can write the contraction explicitly as shrinking along the paths. These facts indicate the possibility of an analogy with the universal principal G–bundle over the classifying space $\mathrm{B}G$ of a topological group G : morally, X can be thought of as the classifying space of the H–space ΩX, and ΠX can be thought of as the universal torsor for ΩX. From this point of view, ΩX is the more primitive object, and X is built from it.

Of course the endofunctor $\Omega = \underline{\mathrm{Mor}}(S^1, -)$ has, by closedness, a left adjoint $\Sigma = - \wedge S^1$, called *suspension.* This means that for any X there are maps $X \to \Omega\Sigma X$ (the unit) and $\Sigma\Omega X \to X$ (the counit) such that the induced maps $\Sigma X \to \Sigma\Omega\Sigma X$ and $\Omega\Sigma\Omega X \to \Omega X$ are left– and right–invertible, respectively, with the obvious inverses.

Example 2.1.3. Obviously, it does not take a lot of effort to find examples of loop spaces. Here are some spaces that have the homotopy type of loop spaces :

— $\mathbf{Z}$, viewed as a discrete topological group with basepoint 0, is homotopy equivalent to the loop space of S^1 : each of the connected components of ΩS^1 is contractible !

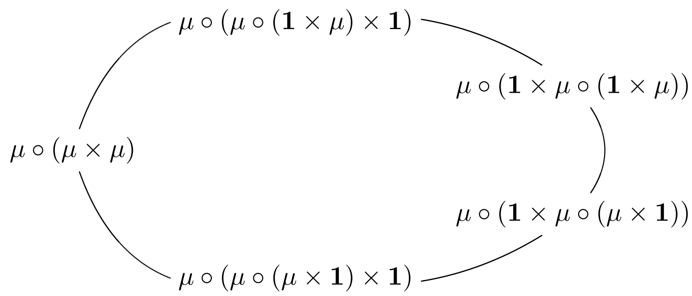
— S^1 has the homotopy type of the loop space of $\mathbf{P}^{\infty}_{\mathbf{C}}$.

— Generalizing the above two examples, *any Eilenberg–Mac Lane complex $K(\pi, n)$ has the homotopy type of $\Omega K(\pi, n+1)$.* To check this, just note that the Eilenberg–Mac Lane complexes are CW complexes, and that there is a weak equivalence $K(\pi, n) \to \Omega K(\pi, n+1)$, which therefore must be a homotopy equivalence.

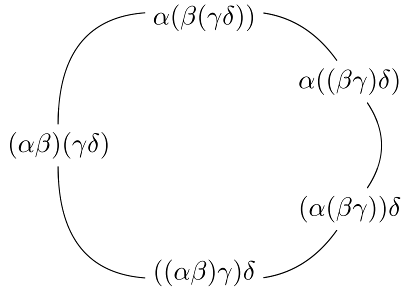
— *Any topological group that has the homotopy type of a CW complex has the homotopy type of a loop space.* Suppose G is such a group, and construct its classifying space $\mathrm{B}G$ as a CW complex. The *universal G–torsor* is then a contractible space EG along with a principle G–bundle $EG \to \mathrm{B}G$ that is universal in the sense that for any space X, the pullback map from $\underline{\mathrm{Mor}}(X, \mathrm{B}G)$ to the space of principle G–bundles over X is a bijection. In particular, the long exact sequence of a fibration tells us that G has the homotopy type of $\Omega \mathrm{B}G$. So in the case where X is a space that already has the homotopy type of a classifying space, X really is the classifying space of ΩX, and ΠX really is the universal torsor for ΩX.

Exercise 2.1.4. If one adopts the attitude that loop spaces are more important than regular spaces, one can go into the business of sniffing out loop spaces. The discussion above indicates that a good place to begin is with H–spaces. But this is not the end of the story : give an example of an H–space (in the sense described above) that is not a loop space. (Hint : By **??**, you know already that such an H–space will *not* be a topological group. After you have found a H–space that you think is not a loop space, you can show that your space fails to satisfy the adjunction property of **??**.) You can even construct such an H–space by imposing a new multiplication law on a CW complex with the homotopy type of a topological group ! So here is a place where you cannot rely on the ordinary homotopy category to do the work for you.

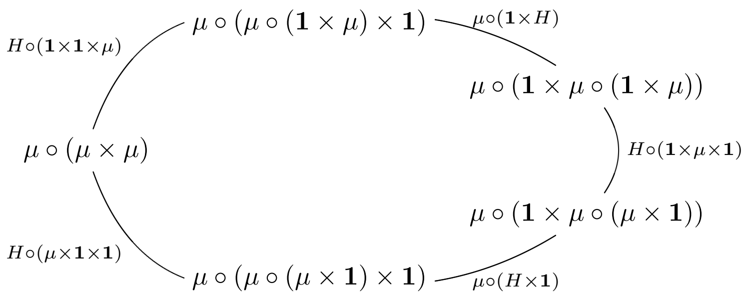
2.1.5. If one wishes to find an intrinsic characterization of loop spaces, one can start by looking at H–spaces. The exercise above indicates that loop spaces are not run–of–the–mill H–spaces. Looking more deeply at the problem, we can see that there is a little ambiguity in the axioms for an H–space. An H–space must have a multiplication law that gives a monoid in the homotopy category, but without any restrictions on the homotopies *themselves*. In particular, let us have a look at the homotopy associativity axiom : fix an H–space X and a homotopy H from $\mu \circ (\mu \times \mathbf{1})$ to $\mu \circ (\mathbf{1} \times \mu)$. We can easily think of five maps $(\Omega X)^{\times 4} = \Omega X \times \Omega X \times \Omega X \times \Omega X \to \Omega X$, which we can arrange as vertices of a pentagon :



This picture corresponds to the five ways of bracketing four loops $\alpha, \beta, \gamma, \delta \in \Omega X$ in order :

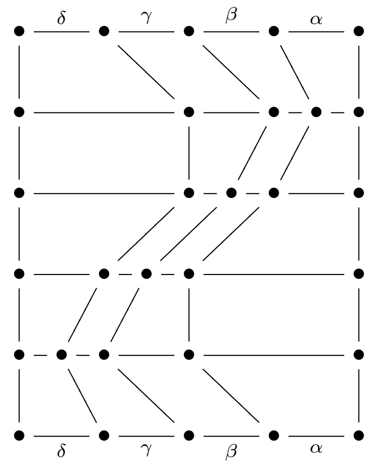


The edges of this pentagon in fact refer to homotopies :



The shape of the diagram is no accident. By pasting these homotopies together, we build a map $S^1 \times (\Omega X)^{\times 4} \to \Omega X$. But now a moment's reflection is all that is required to see that these homotopies commute, in the sense that the corresponding map $S^1 \to \underline{\mathrm{Mor}}((\Omega X)^{\times 4}, \Omega X)$ (which need *not* respect basepoints, and so is not really a map in our category) is actually null–homotopic ;

here is the picture :



with the top and bottom edges identified, of course. Graphically, one just straightens the vertical paths out in a continuous way. In other words, the map $S^1 \times (\Omega X)^{\times 4} \to \Omega X$ can be extended to a map $e^2 \times (\Omega X)^{\times 4} \to \Omega X$, where e^2 is a 2–cell (or disk).

This need not have happened ; nothing in the axioms of an H–spaces forces this null–homotopy to exist. But for loop spaces, we have this condition, which one can think of as a *secondary homotopy condition*. One can carry on like this, obtaining a tertiary homotopy condition, etc.

2.1.6. Stasheff, inspired by Sugawara's earlier recognition principle for loop spaces, defined spaces K_m for each $m > 1$, called the *associahedra* or the *Stasheff polytopes*, as $(m-2)$–cells e^{m-2} with a prescribed subdivision of the boundary S^{m-3} into a polyhedron with b_m vertices, where b_m is the number of ways to form pairwise brackets about m letters in a fixed order. So K_2 is a point and K_3 is a unit interval ; not much work can be done at the boundary. Then he defined *A_m–spaces* for every $m > 1$ inductively as follows : an A_2–space is simply an H–space Z ; from the map

$$K_2 \times Z^{\times 2} = Z^{\times 2} \xrightarrow{M_2} Z$$

one constructs a map

$$(\partial K_3) \times Z^{\times 3} = S^0 \times Z^{\times 3} \xrightarrow{\partial M_3} Z$$

by sending $(0, \lambda, \mu, \nu)$ to $M_2(M_2(\lambda, \mu), \nu)$ and sending $(1, \lambda, \mu, \nu)$ to $M_2(\lambda, M_2(\mu, \nu))$.

Now an A_m–space is an A_{m-1}–space Z such that the map

$$(\partial K_m) \times Z^{\times m} \xrightarrow{\partial M_m} Z$$

constructed (in a completely combinatorial way, the general description of which is unenlightening) from the maps $Z^{\times n} \xrightarrow{M_n} Z$ for $n < m$, extends to a map

$$K_m \times Z^{\times m} \xrightarrow{M_m} Z$$

Of course an A_∞*–space* is a space Z with maps

$$K_m \times Z^{\times m} \xrightarrow{M_m} Z$$

for all $m > 1$ which give X the structure of an A_n–space for any $n > 1$.

Exercise 2.1.7. Find an explicit formula for b_m in terms of m. Can you say how many faces (i.e., $(m-3)$–cells ∂K_m must have ?

2.1.8. K_2 is, as we have already said, a point, and K_3 is an interval. K_4 is the solid *Stasheff pentagon* drawn above, and K_5 is a solid polyhedron with 14 vertices, 21 edges, and 9 faces (3 quadrilaterals and 6 pentagons). A nice rotatable graphic of K_5 is available from

`http ://www.labmath.uqam.ca/ chapoton/stasheff.html.`

Theorem 2.1.9 **(Stasheff).** *An H–space Z is a loop space iff Z is an A_∞–space and $\pi_0 Z$ is a group. Moreover Ω induces an equivalence of categories between the category of connected spaces and the category of A_∞–spaces whose connected components form a group.*

*About the Proof***.** The essence of the idea is to construct and use a classifying space and universal torsor for an A_∞–space, following the intuition suggested above. ○

2.1.10. Stasheff's associahedra combine to form what is called an *operad*, a notion due essentially to Boardman, Vogt, and May. I'll briefly describe this notion in a heuristic way. Operads can live in any symmetric monoidal category, but there is no reason to use that level of generality ; I'll stick to my category $\mathscr{T}$. An operad $\mathscr{X}$ here consists of objects X_j for $j \geqslant 0$, an action of the symmetric group Σ_j on each X_j, and maps from $X_{j_1} \times \cdots \times X_{j_n}$ to $X_{j_1+\cdots+j_n}$ which are associative, unital, and equivariant in a precise sense, which I'll not go into here.

It's much better to see an example of an operad in action. For some $n > 0$, one has the *little n–cubes operad* $\mathscr{P}(n)$, which acts on *n–fold loop spaces,* i.e., spaces of the form $\Omega^n X$ for some space X. To begin, build a subspace $P_j(n) \subseteq \underline{\mathrm{Mor}}(S^n, \bigvee_{m=1}^{m=j} S^n)$ by only considering those maps from $S^n = I^n/\partial I^n$ to the bouquet of m copies of S^n that lift to maps that are constant outside m cubes C_i in I^n, each of which is mapped linearly onto the i-th copy of I^n. Σ_j acts on $P_j(n)$ in a natural way, and one can check that the $P_j(n)$ form an operad $\mathscr{P}(n)$.

Now we can see that $\mathscr{P}(n)$ acts on ΩX : each point

$$S^n \to \bigvee_{m=1}^{m=j} S^n$$

of $P_j(n)$ induces a map

$$(\Omega^n X)^{\times j} = \underline{\mathrm{Mor}}(\bigvee_{m=1}^{m=j} S^n, X) \to \underline{\mathrm{Mor}}(S^n, X) = \Omega^n X$$

and we have our action.

An *E_n–space* is a space along with an action of $\mathscr{P}(n)$. So n–fold loop spaces are E_n–spaces. J. P. May proved the analogue to **??**, which essentially says that :

— Ω^n, viewed as a functor to E_n–spaces whose connected components form a group, has a left adjoint B^n,

— that the components $X \to \Omega^n \mathrm{B}^n X$ of the unit are equivalences of E_n–spaces whenever $\pi_0 X$ is a group, and

— that the components $\mathrm{B}^n \Omega^n Z \to Z$ of the counit are homotopy equivalences whenever Z is $(n-1)$–connected.

Example 2.1.11. Further still, one can try to give a recognition principle for *infinite loop spaces,* i.e., spaces of the form $\Omega^\infty X = \operatorname{colim} \Omega^m X$. This is indeed possible, but before we discuss this, we should remark on some interesting infinite loop spaces :

— Eilenberg–Mac Lane spaces are always infinite loop spaces, since $K(\pi, n)$ always has the homotopy type of $\Omega K(\pi, n+1)$.

— Bott Periodicity states that $\mathbf{Z} \times \mathrm{B}U$ has the homotopy type of its double loop space $\Omega^2(\mathbf{Z} \times \mathrm{B}U)$, and that $\mathbf{Z} \times \mathrm{B}O$ has the homotopy type of its 8–fold loop space $\Omega^8(\mathbf{Z} \times \mathrm{B}O)$. Hence both $\mathbf{Z} \times \mathrm{B}U$ and $\mathbf{Z} \times \mathrm{B}O$ are infinite loop spaces.

— The *classifying space* of a category $\mathscr{C}$ is a space assigned to $\mathscr{C}$: the classifying space of $\mathscr{C}$ is then the geometric realization $\mathrm{B}\mathscr{C} = |\nu_\bullet \mathscr{C}|$. The classifying space of a *symmetric monoidal* category is always an infinite loop space.

2.1.12. Note that the little $(n-1)$–cubes operad embeds into the little n–cubes operad. Then the *little cubes operad* $\mathscr{P}(\infty)$ is the colimit of the $\mathscr{P}(n)$ for all n. An *E_∞–space* is a space equipped with an action of $\mathscr{P}(\infty)$. It clearly acts on an infinite loop space $\Omega^\infty X$, and now one can prove an analogue of **??** for infinite loop spaces, which essentially says that a space is an infinite loop space iff it is an E_∞–space.

2.1.13. I can try an alternative strategy for understanding loop spaces and infinite loop spaces that is a little more accessible : I can try to talk simplicial sets into keeping track of classes of formulas like $\mu \circ (\mu \times \mathbf{1})$ and $\mu \circ (\mathbf{1} \times \mu)$ for me. This is Segal's approach to loop spaces and infinite loop spaces, introduced through a privately–distributed manuscript called "Homotopy–Everything H–Spaces," sketched by Anderson, and continued in Segal's famous paper, "Categories and Cohomology Theories." If Z is an H–space, then one can take a variant of the above construction by viewing Z as a monoid in the homotopy category, which is in particular a category $\mathscr{Z}$, and we can then take the nerve $\nu_\bullet \mathscr{Z}$. This can be viewed as a simplicial object in $\mathrm{Ho}\,\mathscr{T}$ with the property that $[n]$ is sent to $Z^{\times n}$, the first degeneracy map is the multiplication map μ, and the rest of the face and degeneracy maps are given by obvious products of μ, $\mathbf{1}$, and inclusions. If one finds a lift of this to a simplicial object in $\mathscr{T}$, one can form a classifying space, and proceed using the yoga of Stasheff and Sugawara.

Suppose $\Lambda_\bullet$ a simplicial object of $\mathscr{T}$. Then one has n face maps $\Lambda_n \to \Lambda_1$, which can be combined to form a single map $\Lambda_n \to (\Lambda_1)^{\times n}$. Segal calls $\Lambda_\bullet$ a *special* simplicial space if these maps are homotopy equivalences for every $n \geqslant 1$.

This is clearly a very elegant way to organize arbitrarily high homotopies, and it is a point of view that informs a large part of the applications to algebraic geometry that we will encounter here.

Theorem 2.1.14 **(Segal).** *A space Z is an A_∞–space iff there exists a special simplicial object $\Lambda_\bullet$ in $\mathscr{T}$ such that $\Lambda_1 = Z$.*

2.1.15. In this situation, Λ_n is homotopy equivalent to $Z^{\times n}$, *and* Λ_n is playing the role of the product $P_j \times Z^{\times n}$. Segal's approach is in some sense doubly nice : not only is it refreshingly easy to describe the parameter spaces (especially since the Λ_n are pulling double duty !), but the ugly combinatorics are hidden behind the pleasant opaque veneer of simplicial sets.

This same trick will work for infinite loop spaces, but one needs a bigger category. Define Γ to be the category whose objects are finite sets, and whose morphisms are maps between the power sets of these finite sets that preserve disjoint unions. A Γ–*space* is then a contravariant functor from Γ to $\mathscr{T}$. Now Δ embeds into Γ in an obvious way : any nondecreasing map f from $\{0, 1, \ldots, m\}$ to $\{0, 1, \ldots, n\}$ gives a morphism from the power set of $\{0, 1, \ldots, m\}$ to that of $\{0, 1, \ldots, n\}$ by sending any one–point set $\{i\}$ to the set $\{0 \leqslant j \leqslant n \mid f(i-1) <$

$j \leqslant f(i)\}$. Hence a Γ–space gives rise to a simplicial space. Now a *special* Γ–space is a Γ–space that gives rise to a special simplicial space in this way.

***Theorem 2.1.16* (Segal's Delooping Machine).** *A space Z is an E_∞–space iff there exists a special* Γ*–space P such that $P(\star) = Z$.*

Exercise 2.1.17. The Segal approach is so elegant that I can without hesitation give the following exercise : read Stasheff's proof of **??**, and use this strategy to devise a proof of the theorems of Segal.

2.1.18. I've given two ways of dealing with the hierarchies of higher homotopies in the associativity of the product of loops in loop spaces and infinite loop spaces. Segal's delooping machine is certainly much easier to define and prove theorems with, but it has the unfortunate disadvantage of not being very explicit. Γ–spaces are very flabby objects, and explicit computations could be very difficult. Nevertheless, for our applications, Segal's theory has the nicest formal properties, and his delooping machine will be used heavily (sometimes in disguise) in the sequel.

***2.2* (Weak (∞, n)–categories).** Beginning with the theory of ∞–pregroupoids and ∞–groupoids, I now use Segal's delooping machine to produce a theory of (∞, n)–categories. The idea is that an $(\infty, 1)$–category is a multi-object version of a Segal space, i.e., $(\infty, 1)$–categories are to Segal spaces what categories are to monoids.

Presumably, any delooping machine can be used, in an altogether analogous way, to produce a theory of (∞, n)–categories that, by the May–Thomason Uniqueness Theorem, is equivalent (in an appropriate sense) to the one I introduce here. This idea has not yet been made precise, but Toën's work on an axiomatization of the theory of $(\infty, 1)$–categories inspires hope.

Definition 2.2.1. I define Segal n–precategories recursively : suppose that one has defined the following :

— a category $\mathbf{Se}-(n-1)-\mathbf{PC}$ of Segal $(n-1)$–precategories, and

— a fully faithful functor c_{n-1} from **Set** to $\mathbf{Se}-(n-1)-\mathbf{PC}$, the objects of the essential image of which are called constant.

Then I define the following :

— the category $\mathbf{Se}-n-\mathbf{PC}$ of *Segal n–precategories*, the full subcategory of the category $\underline{\mathrm{Mor}}(\Delta^{\mathrm{op}}, \mathbf{Se}-(n-1)-\mathbf{PC})$ of simplicial objects in the category of Segal $(n-1)$–precategories spanned by those simplicial objects $[p] \mapsto A_p$ such that A_0 is a constant Segal $(n-1)$–precategory, and

— the fully faithful functor c_n from **Set** to $\mathbf{Se} - (n-1) - \mathbf{PC}$, defined by sending a set S to the constant simplicial object taking its value at $c_{n-1}(S)$, so that a *constant* Segal n–category is a constant simplicial object of $\mathbf{Se} - (n-1) - \mathbf{PC}$, taking its value on a constant Segal $(n-1)$–precategory.

2.2.2. One can "dévisser" this definition : a Segal n–precategory is nothing more than a multisimplicial set

$$(\Delta^{\mathrm{op}})^{\times(n+1)} \longrightarrow \mathbf{Set}$$

$$(m_1, \ldots, m_{n+1}) \mapsto A_{(m_1,\ldots,m_{n+1})}$$

that satisfies Tamsamani's *constancy condition*, i.e., for any $(m_1, \ldots, m_i) \in (\Delta^{\mathrm{op}})^{\times i}$, if $m_i = 0$, then the functor $(m_{i+1}, \ldots, m_{n+1}) \mapsto A_{(m_1,\ldots,m_{n+1})}$ is a constant functor $(\Delta^{\mathrm{op}})^{\times(n-i+1)} \to \mathbf{Set}$.

Hence, forming the quotient Θ^{n+1} of $(\Delta^{\mathrm{op}})^{\times(n+1)}$ by identifying

$$(m_1, \ldots, m_{n+1}) = (m_1, \ldots, m_{i-1}, 0, \ldots, 0),$$

whenever $m_i = 0$, then one can define a Segal n–precategory as a functor from Θ^{n+1} to **Set**.

2.2.3. Now to define Segal n–categories. Suppose, for the sake of discussion, that A is a Segal 1–precategory. One should expect A to be a Segal 1–category provided that A is a good model for an ∞–category, all of whose n–morphisms are invertible if $n > 1$. If x and y are elements of the set A_0, then one should probably expect that one can define the simplicial set of morphisms from x to y. The two face maps from [0] to [1] in Δ induce a morphism from A_1 to $A_0 \times A_0$ by taking the product. One should think of this morphism as sending a morphism to the pair consisting of its source and target. So a composition law should resemble a morphism from $A_1 \times_{A_0} A_1$ to A_1, where the morphisms from A_1 to A_0 consist of the morphism from A_1 to $A_0 \times A_0$, composed with the first and second projection, respectively. This not quite what I have. I have the obvious variant of the Segal map from **??** : $A_p \to A_1 \times_{A_0} \cdots \times_{A_0} A_1$ (where the maps alternate : source, target, source, target, ...), and I have the obvious map $A_p \to A_1$ (induced by sending [1] to $[p]$ via the inclusion of $\{0, p\}$ into $[p]$). Instead of defining a *strict* map from $A_1 \times_{A_0} \cdots \times_{A_0} A_1$ to A_1 making the diagram commute, one requires instead that the Segal map is a weak equivalence of simplicial sets. Hence composition is only defined up to homotopy.

Definition 2.2.4. I define Segal n–categories recursively : let the category $\mathbf{Se}-0-\mathbf{Cat}$ of *Segal* 0*–categories* be the category $\mathbf{Se}-0-\mathbf{PC}$ of simplicial sets. For any Segal 0–category A, let $\tau_{\leqslant 0}A$ be the discrete category whose objects are $\pi_0(A)$ (i.e., the connected components of A). Now suppose that one has defined the following :

— a full subcategory $\mathbf{Se}-(n-1)-\mathbf{Cat}$ of $\mathbf{Se}-(n-1)-\mathbf{PC}$, whose objects are called Segal $(n-1)$–categories,

– for any object $A \in \mathbf{Se}-(n-1)-\mathbf{Cat}$, a category $\tau_{\leqslant 0}A$, called the homotopy category of A,

— a subcategory of $\mathbf{Se}-(n-1)-\mathbf{Cat}$, whose objects are the same and whose morphisms are called equivalences.

Then I define the following :

— the category $\mathbf{Se}-n-\mathbf{Cat}$ of *Segal n–categories*, the full subcategory of $\mathbf{Se}-n-\mathbf{PC}$ spanned by those Segal n–precategories A such that for any $[p] \in \Delta^{\mathrm{op}}$, A_p is an $(n-1)$–category, and the Segal morphism $A_p \twoheadrightarrow A_1 \times_{A_0} \cdots \times_{A_0} A_1$ is an equivalence of $(n-1)$–categories,

— for any object $A \in \mathbf{Se}-n-\mathbf{Cat}$, the *homotopy category* $\tau_{\leqslant 0}A$ is the category whose set of objects is A_0 and whose set of morphisms is the set of isomorphism classes of objects of $\tau_{\leqslant 0}A_1$ (with the obvious sources and targets)

— a morphism $A \twoheadrightarrow B$ of Segal n–categories is a *Dwyer–Kan equivalence* iff it induces a Dwyer–Kan equivalence of categories $\mathrm{Ho}\,A \twoheadrightarrow \mathrm{Ho}\,B$ and, for any $[p] \in \Delta^{\mathrm{op}}$, an equivalence of Segal $(n-1)$–categories $A_p \twoheadrightarrow B_p$.

Exercise 2.2.5. Show that $\mathrm{Ho}\,A$ is actually a category ; that is, show that compositions exist, and composition is associative.

2.2.6. One has now a kind of strictification endofunctor $\mathbf{SeCat}$ of $\mathbf{Se}-n-\mathbf{PC}$, along with a natural transformation σ from the identity functor to $\mathbf{SeCat}$ such that :

— for any Segal n–precategory A, $\mathbf{SeCat}(A)$ is a Segal n–category ;

— for any Segal n–precategory A, σ_A induces a bijection between A_0 and $\mathbf{SeCat}(A)_0$;

— for any Segal n–category A, σ_A is a Dwyer–Kan equivalence of n–categories ; and

— for any Segal n–precategory A, $\mathbf{SeCat}(\sigma_A)$ is a Dwyer–Kan equivalence of n–categories.

2.2.7. Though it would take me somewhat far afield to define them, the theory of Segal n–categories is sufficiently rich to deal with the usual universal constructions in category theory, such as the notions of fullness, faithfulness, limit, colimit, adjunction, Kan extensions (though I have not yet seen this in the literature), Yoneda embeddings, etc.

Example 2.2.8. There are tons of examples. I'll just list a few.

— Any set is a Segal n–category in the obvious way. This is not an interesting example.

— Carlos Simpson demonstrated that the category $\mathbf{Se} - (n-1) - \mathbf{Cat}$ can be viewed naturally as a Segal n–category.

— Any $\mathbf{S}$–category can in particular be viewed as a Segal 1–category.

2.2.9. Observe that Segal 0–categories are the same as ∞–pregroupoids. Preference for the latter terminology is justified and generalized by means of the existence of the free closed model structures introduced below.

2.3 (Closed model structures for (∞, n)–precategories)

I briefly describe two closed model structures on the category of Segal n–precategories.

Scholium 2.3.1. *For every $n \geqslant 0$, there is a cofibrantly generated, internal closed model structure on* $\mathbf{Se} - n - \mathbf{PC}$*. These model structures possess the following properties.*

— *The model structure on* $\mathbf{Se} - 0 - \mathbf{PC}$ *is the model structure on ∞–pregroupoids.*

— *For any n, the cofibrations of this model structure are precisely the monomorphisms.*

— *Any fibrant object is a Segal n–category, but not conversely.*

— *A morphism $A \twoheadrightarrow B$ of Segal n–categories is a weak equivalence iff it is a Dwyer–Kan equivalence.*

This closed model structure will be called the Hirschowitz–Simpson closed model structure.

*About the Proof***.** This is proved using a complicated induction and the small object argument. A similar theorem for the Tamsamani–Simpson theory of n–categories appeared earlier. ◯

2.3.2. In fact, not all *objectwise fibrant* Segal n–categories—Segal n–categories A such that A_p is a fibrant Segal $(n-1)$–category for every $p \in \Delta^{\mathrm{op}}$—are fibrant. There are in some sense very few fibrant objects in the Hirschowitz–Simpson closed model structure, and they are very difficult to characterize. This motivates the following result of mine.

Scholium 2.3.3. *For every $n \geqslant 0$, there is a proper, combinatorial closed model structure on* $\mathbf{Se} - n - \mathbf{PC}$*. These model structures possess the following properties.*

— *The model structure on* $\mathbf{Se} - 0 - \mathbf{PC}$ *is the Quillen closed model structure on ∞–pregroupoids.*

— *For any $n \geqslant 0$, the identity functor is a left Quillen equivalence from this model structure to the Hirschowitz–Simpson closed model structure.*

— *The cofibrations are certain "free" cofibrations.*

— *For any $n \geqslant 0$, the fibrant objects in this model structure are precisely those Segal n–categories A such that for every $p \in \Delta^{\mathrm{op}}$, A_p is fibrant in the free closed model structure on* $\mathbf{Se} - (n-1) - \mathbf{PC}$*.*

— *For any $n \geqslant 0$, a morphism $A \twoheadrightarrow B$ of Simpson* $\mathbf{Se} - n - \mathbf{PC}$*–categories is a weak equivalence in this model structure iff it is a Dwyer–Kan equivalence.*

About the Proof. A version of this theorem was proved for $n = 1$ by J. Bergner in her thesis. The general case appears in my thesis. The proof proceeds by induction on n. For $n = 0$, there is nothing to say. To construct the desired closed model structure on $\mathbf{Se} - n - \mathbf{PC}$, one defines a kind of projective closed model structure, takes a certain left Bousfield localization to ensure that the fibrant objects are exactly the objectwise fibrant Segal n–categories, and then takes a right Bousfield localiztion to ensure that the weak equivalences between fibrant objects are exactly the Dwyer–Kan equivalences. ◯

Definition 2.3.4. A Segal n–precategory is called a *Simpson (∞, n)–category* if it is a fibrant object of the free closed model structure. In particular, Simpson $(\infty, 0)$–categories are exactly ∞–groupoids.

2.3.5. It is my view that Simpson (∞, n)–categories are a better model for (∞, n)–categories. By induction, they are Segal n–categories A such that the simplicial set $A(m_1, \ldots, m_n)$ is fibrant for any $(m_1, \ldots, m_n) \in \Theta^n$.

It should be noted that the ease with which one can characterize fibrant objects in the free closed model structure has come at a price : it is not an internal closed model structure ; there is no homotopically correct internal Mor functor in the free closed model structure.

2.3.6. The diagonal functor $\mathbf{Se} - n - \mathbf{PC} \twoheadrightarrow s\mathbf{Set}$ can be viewed as the assignment that formally inverts all i–morphisms for $i \leqslant n$.

Lemma 2.3.7. *The diagonal functor* $\mathbf{Se} - n - \mathbf{PC} \twoheadrightarrow s\mathbf{Set}$ *is a left Quillen functor for both the Hirschowitz–Simpson and the free closed model structures. Its right adjoint is denoted $\Re$.*

***2.4* (Descent for (∞, n)–stacks).** In this section, I explain descent for presheaves of (∞, n)–categories, using the free closed model structure. This follows closely the discussion of stacks in ∞–groupoids. Again let (T, τ) be a site with enough points. Let $(\infty, n) - \mathbf{PS}(T)$ be the category of presheaves of Segal n–precategories. Recall that since the free closed model structure is combinatorial, $(\infty, n) - \mathbf{PS}(T)$ has a projective closed model structure.

Theorem 2.4.1. *There is a left proper, combinatorial closed model structure on $(\infty, n) - \mathbf{PS}(T)$ such that the following conditions are satisfied.*

— *The cofibrations are precisely the projective cofibrations.*

— *The fibrant objects are exactly those presheaves of Simpson (∞, n)–categories F such that for any hypercovering $U \to X$ of any object X of T, the induced morphism*

$$FX \to \operatorname{holim} \left[\prod_{\alpha_0} FU_0^{\alpha_0} \rightrightarrows \prod_{\alpha_1} FU_1^{\alpha_1} \substack{\to\\\to\\\to} \dots\right]$$

is an equivalence of simplicial sets, where the $U_n^{\alpha_n}$ are the representing objects of the functors of which U_n is the coproduct.

— *The weak equivalences are exactly those morphisms $F \to G$ such that for any point x of T, the induced morphism $F_x \to G_x$ is a Dwyer–Kan equivalence of Simpson (∞, n)–categories.*

This closed model structure is called the local projective *closed model structure.*

Proof. By using the functor $\mathbf{R}$, any hypercovering can be seen as a morphism of Segal n–precategories. Now, as before, one verifies that the left Bousfield localization of the projective closed model structure with respect to all hypercoverings exists and satisfies the listed properties. ○

Definition 2.4.2. An (∞, n)*–stack* is a fibrant object of the local projective closed model structure on $(\infty, n) - \mathbf{PS}(T)$.

2.4.3. Finally, one can put forward the following question : is the assignment $\operatorname{Spec} R \mapsto L\mathbf{Cplx}(R)$ an $(\infty, 1)$–stack for any familiar topologies on the category of affine schemes ? The answer is yes, but it turns out not to be so easy to verify this. Indeed, in the first version of *Descente pour les n–Champs* in 1998, Hirschowitz and Simpson gave a faulty proof of this. This result had to be partially retracted, and the authors were able to verify descent (for any subcanonical topology) only for the presheaf of positive complexes $\operatorname{Spec} R \mapsto L\mathbf{Cplx}^+(R)$.

In my forthcoming note *Redintegration of a Theorem of Hirschowitz and Simpson*, I demonstrate that, as a consequence of a generalized tilting theorem of Schwede and Shipley, for any ring R, the closed model category $HR - \mathbf{Mod}$

of modules over the Eilenberg–Mac Lane E_∞ ring spectrum HR is Quillen equivalent to the closed model category $\mathbf{Cplx}(R)$ of unbounded complexes, and, moreover, this Quillen equivalence gives rise to a *functorial* equivalence of Dwyer–Kan simplicial localizations $L(HR-\mathrm{Mod}) \to L\mathrm{Cplx}(R)$. In lieu of $\operatorname{Spec} R \mapsto L\mathbf{Cplx}(R)$, one can therefore study the presheaf of Simpson $(\infty,1)$–categories $A \mapsto L(A-\mathbf{Mod})$ on the opposite model category $\mathscr{A}_{H\mathbf{Z}}$ of the model category of $H\mathbf{Z}$–algebras, and pull back this presheaf along the Eilenberg–Mac Lane functor H. Using Toën and Vezzosi's theory of *homotopical algebraic geometry*, I observe that for any subcanonical topology τ on the category of affine schemes, there is a topology on $\operatorname{Ho}\mathscr{A}_{H\mathbf{Z}}$ such that the Eilenberg–Mac Lane functor H is a continuous morphism of *model sites*, and the 1–prestack $A \mapsto L(A-\mathbf{Mod})$ is a 1–stack. A corollary is that the Hirschowitz–Simpson descent result for positive complexes can be extended to unbounded complexes, as it was originally envisioned.

Theorem 2.4.4. *The* 1*–prestack* $\operatorname{Spec} R \mapsto L\mathbf{Cplx}(R)$ *is a* 1*–stack for the category of affine schemes equipped with any subcanonical topology.*