# The Arithmetic of Prym varieties in Genus 3

Nils Bruin (Simon Fraser University)

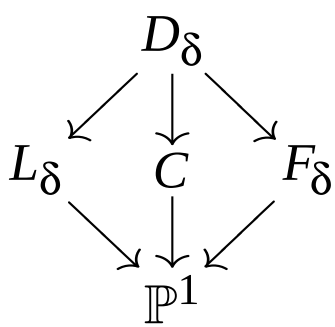
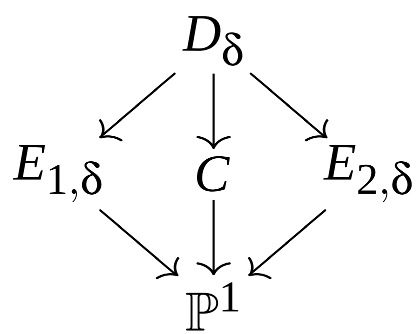
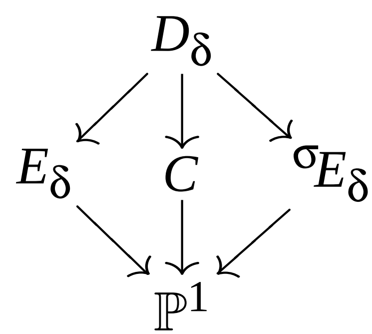
## Introduction: Hyperelliptic case

**Hyperelliptic curve:** $C \stackrel{x}{\to} L$ where $\deg(x) = 2$ and $L$ is a genus $0$ curve.

If $Cstackrelx{\to}\mathbb{P}^1$ and the genus $g_C = 3$:

$$C : y^2 = f(x) = f_8x^8 + f_7x^7 + \cdots + f_0$$

Some unramified double covers of $C$:

| $f(x) = f_2(x)f_6(x)$ | $f(x) = f_4(x)g_4(x)$ | $f(x) = f_4(x)^{\sigma}f_4(x)$ |
|---|---|---|
|  |  |  |
| $L_\delta : \delta y_1^2 = f_2(x)$ | $E_{1,\delta} : \delta y_1^2 = f_4(x)$ | $E_\delta : \delta y_1^2 = f_4(x)$ |
| $F_\delta : \delta y_2^2 = f_6(x)$ | $E_{2,\delta} : \delta y_2^2 = g_4(x)$ | $\langle\sigma\rangle = \mathrm{Gal}(\mathbb{Q}(\sqrt{d})/\mathbb{Q})$ |

**Main Descent Theorem:** We need only finitely many $D_\delta$ to cover $C(\mathbb{Q})$.

## Finding rational points on $C$ over $K=\mathbb{Q}$ (Chabauty)

*Coleman, Flynn, Poonen, Schaefer, Stoll, Wetherell, B.*

- Embed $C$ in an Abelian variety $J$
- Using that $J(\mathbb{Q}) \subset \overline{J(\mathbb{Q})} \subset J(\mathbb{Q}_p)$:
$$\#C(\mathbb{Q}) \leq \#\left(C(\mathbb{Q}_p) \cap \overline{J(\mathbb{Q})}\right)$$
- Use $p$-adic analytic methods to determine the right hand side.
- In Brauer-Manin language, use $J(\mathbb{Q}) \subset \overline{J(\mathbb{Q})} \subset J(\mathbb{A}_\mathbb{Q})$:
$$\#C(\mathbb{Q}) \leq \#\left(C(\mathbb{A}_\mathbb{Q}) \cap \overline{J(\mathbb{Q})}\right)$$

**If this fails:** Try same approach for $D_\delta(\mathbb{Q})$ instead

**Computationally easier:** Determine $L_\delta(\mathbb{Q})$, $F_\delta(\mathbb{Q})$ or $E_{i,\delta}(\mathbb{Q})$.

If $E_\delta$ is over $\mathbb{Q}(\sqrt{d})$: Determine $\{P \in E_\delta(\mathbb{Q}) : x(P) \in \mathbb{P}^1(\mathbb{Q})\}$.

## What Abelian Varieties do arise?

**Albanese variety:** Universal abelian variety $C \hookrightarrow \mathrm{Alb}(C)$ over $\overline{\mathbb{Q}}$.

**Jacobian variety:** For curves, $\mathrm{Jac}(C) \simeq \mathrm{Alb}(C)$ and $\mathrm{genus}(C) = \dim \mathrm{Jac}(C)$.

**Covers of curves:** $\pi : D \to C$ induces

$$\pi_* : \mathrm{Jac}(D) \to \mathrm{Jac}(C)$$

If $\deg \pi = 2$ and $\pi$ is unramified then $g_D = 2g_C - 1$

$$0 \to \mathrm{Ker}(\pi_*) \to \mathrm{Jac}(D) \xrightarrow{\pi_*} \mathrm{Jac}(C) \to 0$$

We have that $\mathrm{Ker}(\pi_*)$ has 2 components.

$$\mathrm{Prym}(D/C) := \mathrm{Ker}(\pi_*)^0; \quad \dim \mathrm{Prym}(D/C) = g_C - 1$$

For $C$ hyperelliptic genus $3$ we get:

$$\begin{aligned} \mathrm{Prym}(D/C) &= \mathrm{Jac}(F) && \text{or} \\ \mathrm{Prym}(D/C) &= \mathrm{Jac}(E_1) \times \mathrm{Jac}(E_2) && \text{or} \\ \mathrm{Prym}(D/C) &= \Re_{\mathbb{Q}(\sqrt{d})/\mathbb{Q}} \mathrm{Jac}(E) \end{aligned}$$

**Simplification:** We can map $D$ to $\mathrm{Prym}(D/C)$ by projection.

## The non-hyperelliptic case

**Non-hyperelliptic curve of genus $3$:** Smooth plane quartic.

**With an unramified double cover:**

$$C: \quad Q_1(u,v,w)Q_3(u,v,w) = Q_2(u,v,w)^2$$

$$D_\delta: \begin{cases} Q_1(u,v,w) &= \delta r^2 \\ Q_2(u,v,w) &= \delta rs \\ Q_3(u,v,w) &= \delta s^2 \end{cases}$$

$$\pi: \begin{array}{ccc} D & \to & C \\ (u:v:w:r:s) & \mapsto & (u:v:w) \end{array}$$

$$\iota: \begin{array}{ccc} D & \to & D \\ (u:v:w:r:s) & \mapsto & (u:v:w:-r:-s) \end{array}$$

**Finding $C(\mathbb{Q})$:** Determining $\mathrm{Jac}(C)(\mathbb{Q})$ is really hard.

**Covering collection:** If $C$ over $\mathbb{Q}$ then WLOG $Q_i \in \mathbb{Z}[u,v,w]$.

Need only consider $\delta = \pm p_1 \cdots p_r$ where $p_i$ are such that:

$$Q_1 \equiv Q_2 \equiv Q_3 \equiv 0 \pmod{p_i} \text{ has a solution.}$$

# A closer study of $D$

Consider the quadratic forms / symmetric matrices:

$$M_1 = \left(\begin{array}{c|cc} Q_1 & & \\ \hline & -\delta & 0 \\ & 0 & 0 \end{array}\right) \; M_2 = \left(\begin{array}{c|cc} Q_2 & & \\ \hline & 0 & -\frac{1}{2}\delta \\ & -\frac{1}{2}\delta & 0 \end{array}\right) \; M_3 = \left(\begin{array}{c|cc} Q_3 & & \\ \hline & 0 & 0 \\ & 0 & -\delta \end{array}\right)$$

Their linear span:

$$\Lambda = \{\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3\} \simeq \mathbb{P}^2; \; D_\delta = \mathrm{Var}(\Lambda)$$

The locus of singular quadrics:

$$\Gamma \;=\; \{M \in \Lambda : \mathrm{rk}(M) \leq 4\} \;:\; \det(\lambda_1 M_1 + \lambda_2 M_2 + \lambda_3 M_3) = 0$$

$$\Gamma \;=\; \underset{\det(\lambda_1 Q_1 + \lambda_2 Q_2 + \lambda_3 Q_3) = 0}{\Gamma^+} \;\cup\; \underset{4\lambda_1\lambda_3 - (\lambda_2)^2 = 0}{\Gamma^-}$$

- $\dim \Gamma = 1$
- $\{M \in \Lambda : \mathrm{rk}(M) \leq 3\} = \mathrm{Sing}(\Gamma)$
- $\{M \in \Gamma^- : \mathrm{rk}(M) \leq 3\} = \Gamma^+ \cap \Gamma^-$

## Geometry of some special divisors on $D$

[Arbarello-Cornalba-Griffiths-Harris VI, Excercises F]

**Variety of special divisor classes:**

$$W_4^1 = \{\mathfrak{d} \in \mathrm{Pic}^4(D) : l(\mathfrak{d}) \geq 2\}$$

**Residuality:**

$$\mathfrak{d} \mapsto [\kappa_D] - \mathfrak{d} : W_4^1 \to W_4^1$$

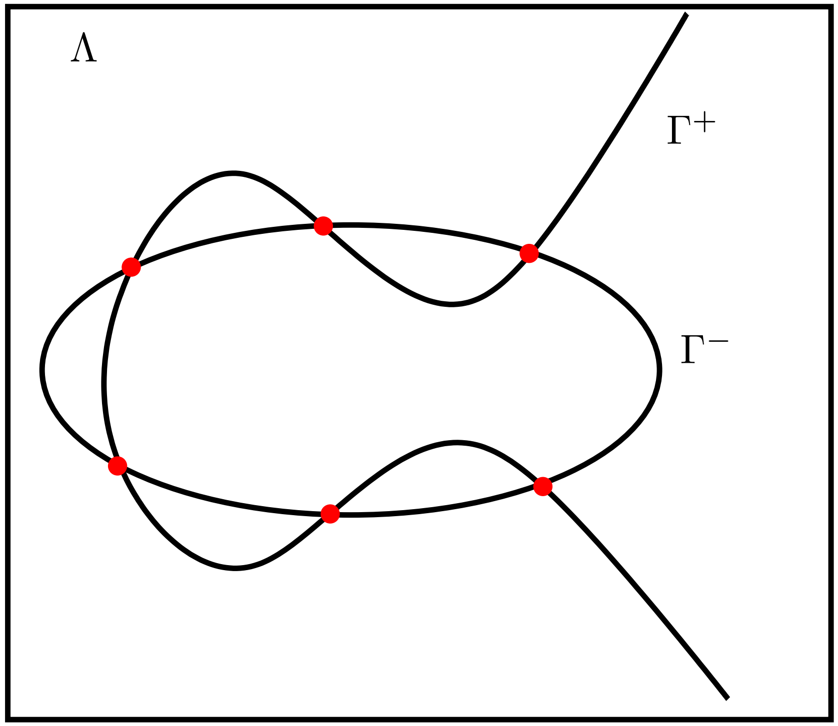
**Characterisation:** *Effective* divisors $[\sum_{i=1}^4 P_i] \in W_4^1$ are of the form

$V \cap D,$ where $V \subset \mathbb{P}^4$ is a plane.

- If $\deg(V \cap D) = 4$ then $\Lambda|_V$ is a pencil, i.e. $\exists M \in \Lambda : V \subset M$.
- A quadric in $\mathbb{P}^4$ containing a plane is singular: $W_4^1 \to \Gamma$.
- A quadric $M \subset \mathbb{P}^4$ of rank 4 has two systems of planes (It's a cone over a quadric in $\mathbb{P}^3$).
- If $V_1, V_2 \subset M$ belong to opposite systems, then $[(V_1 \cap D) + (V_2 \cap D)] = [\kappa_D]$.
- $W_4^1 \to \Gamma$ is a double cover, with $(\mathfrak{d} \mapsto \kappa_D - \mathfrak{d}) \in \mathrm{Aut}(W_4^1/\Gamma)$.

## Decomposition of $W_4^1$

We have $W_4^1$ as a double cover of $\Gamma$:




Let $F_\delta$ be the component of $W_4^1$ over

$$\Gamma^- : 4\lambda_1\lambda_3 = (\lambda_2)^2; \text{ parametrically: } (\lambda_1 : \lambda_2 : \lambda_3) = (1 : 2x : x^2)$$

For some $\tilde{\delta}$:

$$F_\delta : y^2 = -\tilde{\delta}\det(Q_1 + 2xQ_2 + x^2Q_3)$$

## **Description of** $\mathrm{Prym}(D/C)$

**Note:** If $V \subset M \in \Gamma^-$ then $\pi(V)$ is a line. Hence, $\pi_*(V \cap D) \in \mathrm{Pic}_C$ is $[\kappa_C]$.

$$\begin{array}{rccc} \pi_* : & F_\delta & \longrightarrow & \mathrm{Pic}^4(C) \\ & \mathfrak{d} & \longmapsto & [\kappa_C] \end{array}$$

**Embedding**:

$$\begin{array}{ccc} \mathrm{Jac}_F & \hookrightarrow & \mathrm{Jac}_D \\ [p_1 + p_2 - \kappa_F] & \mapsto & p_1 + p_2 - [\kappa_D] \end{array}$$

It follows that $\pi_*(\mathrm{Jac}_F) = 0$, so

$$\mathrm{Jac}_F \hookrightarrow \mathrm{Prym}(D/C).$$

Since $\mathrm{Jac}_F$ is of the right dimension, equality must hold.

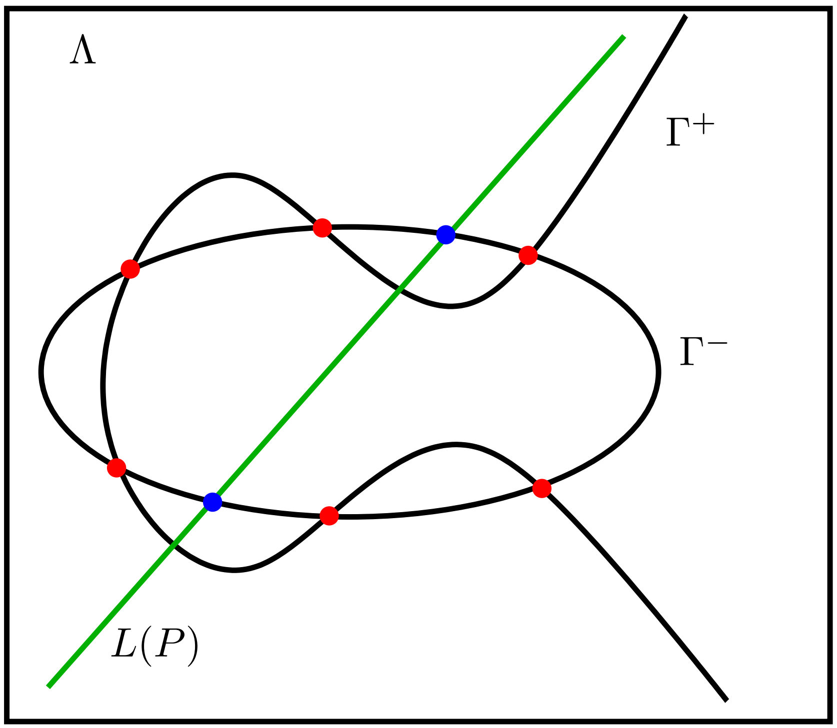
## **Mapping** $D$ **into** $\mathrm{Prym}(D/C)$

A simple mapping:

$$
\begin{array}{rcl}
D & \longrightarrow & W_4^1 + W_4^1 \subset \mathrm{Pic}^8(D) \\
P & \mapsto & \sum\{[\mathfrak{d}] \in F : \mathfrak{d} \geq 2P\}
\end{array}
$$

**In terms of** $\Lambda$**:** Given $P$:

$$L(P) := \{M \in \Lambda : T_P(D) \subset M\}$$



## Determining $C$ as a curve in $\mathrm{Kum}_F$

We map

$$\begin{array}{rcccl} D & \longrightarrow & \mathrm{Jac}(F) & = & \mathrm{Prym}(D/C) \\ P & \longmapsto & \displaystyle\sum_{\substack{[\mathfrak{d}] \in F \\ \mathfrak{d} \geq 2P}} [\mathfrak{d} - \kappa_D] & = & 2[P - \iota P] \end{array}$$

This map is injective outside the singular locus of the image.

$$\begin{array}{ccc} D_\delta & \longrightarrow & \mathrm{Jac}_{F_\delta} \\ \pi \downarrow & & \downarrow k \\ C & \xrightarrow{\ \phi\ } & \mathrm{Kum}_{F_\delta} \end{array}$$

**Computation** of $\phi : C \to \mathrm{Kum}_{F_\delta}$: Use interpolation.

**Application to rational points:** Determine

$$\phi(C)(\mathbb{Q}_p) \cap \overline{k(\mathrm{Jac}(F)(\mathbb{Q}))} \text{ in } \mathrm{Kum}(\mathbb{Q}_p) \text{ to bound } \pi(D_\delta(\mathbb{Q})) \subset C(\mathbb{Q}).$$

## Application: Chabauty

Consider:

$$C : (-4u^2 - 4vw + 4w^2)(2u^2 + 4uv + 4v^2) = (-2u^2 + 2uw - 4vw + 2w^2)^2$$

We have

$$\bigcup_{\delta \in \{\pm 1, \pm 2, \pm 5, \pm 10\}} \pi(D_\delta(\mathbb{Q})) = C(\mathbb{Q}).$$

Local considerations show $D_\delta(\mathbb{Q}) = \emptyset$ for $\delta \neq 1$. Component of $W_4^1$:

$$F : y^2 = x^5 + 8x^4 - 7x^3 - 7/2x^2 + 5x - 1$$

$$\mathrm{Jac}_F(\mathbb{Q}) = \langle \mathfrak{g} \rangle = \langle [(2\sqrt{2} - 2, 17\sqrt{2} - 25) + (-2\sqrt{2} - 2, -17\sqrt{2} - 25) - 2\infty] \rangle$$

$$\begin{aligned} \mathrm{Kum}_F : \quad & 11k_1^4 - 28k_1^3k_2 + 70k_1^3k_3 + 4k_1^3k_4 + 32k_1^2k_2^2 - 164k_1^2k_2k_3 - 10k_1^2k_2k_4 + 171k_1^2k_3^2 + \\ & 14k_1^2k_3k_4 + 4k_1k_2^3 - 20k_1k_2^2k_3 + 14k_1k_2k_3^2 + 14k_1k_2k_3k_4 + 14k_1k_3^3 - 32k_1k_3^2k_4 - \\ & 4k_1k_3k_4^2 + k_2^2k_4^2 - 2k_2k_3^2k_4 + k_3^4 = 0 \end{aligned}$$

Equation for $C \subset \mathrm{Kum}_F$:

$$\begin{aligned} \phi : \quad & 429136k_1^4 + 1330784k_1^3k_3 + 567232k_1^3k_4 - 159200k_1^2k_2^2 - 2866016k_1^2k_2k_3 + 33440k_1^2k_2k_4 + 4248768k_1^2k_3^2 + \\ & 27552k_1^2k_3k_4 + 881664k_1^2k_4^2 + 288072k_1k_2^3 - 777432k_1k_2^2k_3 - 256928k_1k_2^2k_4 + 244832k_1k_2k_3^2 + \\ & 907424k_1k_2k_3k_4 - 745472k_1k_2k_4^2 + 593152k_1k_3^3 - 991488k_1k_3^2k_4 + 357440k_1k_3k_4^2 + 573440k_1k_4^3 + 34895k_2^4 - \\ & 69720k_2^3k_3 + 1120k_2^3k_4 + 151704k_2^2k_3^2 - 364448k_2^2k_3k_4 + 226032k_2^2k_4^2 - 251552k_2k_3^3 + 569376k_2k_3^2k_4 + \\ & 10752k_2k_3k_4^2 - 315392k_2k_4^3 + 156704k_3^4 - 167552k_3^3k_4 - 283136k_3^2k_4^2 + 200704k_3k_4^3 + 114688k_4^4 = 0 \end{aligned}$$

## Application: Chabauty (continued)

If $P \in D_1(\mathbb{Q}) \subset \mathrm{Jac}_F(\mathbb{Q})$, then

$$P = n\mathfrak{g} \text{ for some } n \in \mathbb{Z}$$

**Base change** to $\mathbb{F}_{13}$: If $k(n\mathfrak{g}) \in C \pmod{13}$ then $n = \pm 1 \pmod{10}$.

**13-adically:** $\phi(N) = \phi(k((1+10N)\mathfrak{g})) = \phi(k(\mathfrak{g} + \mathrm{Exp}(N\mathrm{Log}(10\mathfrak{g}))))$.

$$\phi(N) = \phi_0 + \phi_1 N + \phi_2 N^2 + \cdots \in \mathbb{Z}_{13}[[N]] \text{ with } \nu_{13}(\phi_i) \geq i$$

**Observation:** $\phi(k(\mathfrak{g})), \phi(k(11\mathfrak{g})) \pmod{13^2}$ determine $\phi_0, \phi_1 \pmod{13^2}$.

**Fact:** $\phi(k(\mathfrak{g})) = 0$ and $\phi(k(11\mathfrak{g})) \neq 0 \pmod{13^2}$; therefore $\nu_{13}(\phi_1) = 1$.

**Theorem** (Straßmann): If $f(z) = \sum_n f_n z^n \in \mathbb{Z}_p[[n]]$ is convergent on $\mathbb{Z}_p$ and $\nu_p(f_N) < \nu_p(f_n)$ for all $n > N$ then

$$\#\{z \in \mathbb{Z}_p : f(z) = 0\} \leq N.$$

**Corollary:** $D_1(\mathbb{Q}) = \{\mathfrak{g}, -\mathfrak{g}\}$ and $C(\mathbb{Q}) = \{(0:1:0)\}$.

**Other application** (Brauer-Manin type obstruction)

Consider the everywhere locally soluble curve

$$C : (v^2 + vw - w^2)(uv + w^2) = (u^2 - v^2 - w^2)^2.$$

We have

$$\bigcup_{\delta \in \{\pm 1, \pm 2\}} \pi(D_\delta(\mathbb{Q})) = C(\mathbb{Q})$$

and only $D_1$ is everywhere locally soluble.

We compute

$$F : y^2 = x^6 + 2x^5 + 15x^4 + 40x^3 - 10x$$

and

$$\mathrm{Jac}_F(\mathbb{Q}) = \langle \mathfrak{g}_1, \mathfrak{g}_2 \rangle = \langle [\infty^+ - \infty^-], [((4\sqrt{41} - 41)/205, \ldots) + \cdots] \rangle$$

# Using congruence information

We find

| $p$ | $\mathrm{Jac}_F(\mathbb{Q}) \pmod{p}$ | relations |
|---|---|---|
| 7 | $\mathbb{Z}/55\mathbb{Z}$ | $55\mathfrak{g}_1 \equiv 0,\ \mathfrak{g}_2 \equiv 15\mathfrak{g}_1$ |
| 11 | $\mathbb{Z}/93\mathbb{Z}$ | $93\mathfrak{g}_1 \equiv 0,\ \mathfrak{g}_2 \equiv 47\mathfrak{g}_1$ |

**Intersecting** $C(\mathbb{F}_p)$ **with** $k(\mathrm{Jac}_F(\mathbb{Q}) \pmod{p})$**:**

$$\begin{array}{lclcl} D(\mathbb{Q}) & \subset & \{\pm 9\mathfrak{g}_1, \pm 22\mathfrak{g}_1, \pm 23\mathfrak{g}_1\} & + & \langle 55\mathfrak{g}_1, \mathfrak{g}_2 - 15\mathfrak{g}_1\rangle \\ D(\mathbb{Q}) & \subset & \{\pm 33\mathfrak{g}_1\} & + & \langle 93\mathfrak{g}_1, \mathfrak{g}_2 + 46\mathfrak{g}_1\rangle \end{array}$$

**Deeper information at** $11$**:** gives $11 \cdot 2$ congruence classes modulo:

$$\langle 11 \cdot 93\mathfrak{g}_1, 11 \cdot (\mathfrak{g}_2 + 46\mathfrak{g}_1)\rangle$$

**Intersection:**

$$\langle 55\mathfrak{g}_1, \mathfrak{g}_2 - 15\mathfrak{g}_1, 11 \cdot 93\mathfrak{g}_1, 11 \cdot (\mathfrak{g}_2 + 46\mathfrak{g}_1)\rangle = \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle$$

Combining the information:

$$\begin{array}{lclclcl} \text{from } 7 & : & D(\mathbb{Q}) & \subset & \{0, \pm\mathfrak{g}_1, \pm 2\mathfrak{g}_1\} & + & \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle \\ \text{from } 11^2 & : & D(\mathbb{Q}) & \subset & \{\pm 4\mathfrak{g}_1\} & + & \langle 11\mathfrak{g}_1, \mathfrak{g}_2 - 4\mathfrak{g}_1\rangle \end{array}$$

**Corollary:** $D(\mathbb{Q}) = \emptyset$ and hence $C(\mathbb{Q}) = \emptyset$.