THE ANDRÉ-OORT CONJECTURE AND MANIN-MUMFORD

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1. Introduction

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<th>Abelian varieties</th>
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<td>$A = \mathbb{C}^n/\Gamma +$ polarization</td>
<td>$S = \Gamma \backslash X +$ complex structure</td>
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<td>$X$ hermitian symmetric space</td>
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<td>$X \simeq G(\mathbb{R})/Z_G(\mathbb{R}) K_\infty$,</td>
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<td>$G_\mathbb{Q}$ a $\mathbb{Q}$-reductive group</td>
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<td>$X = B + P$, $P \in A_{\text{tors}}$, $B$ abelian subvariety</td>
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<td>$S_H = \Gamma \cap H(\mathbb{R}) \backslash X_H$, $H_\mathbb{Q} = T H^\text{der}_\mathbb{Q}$</td>
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The conjecture on each side says: A component of the Zariski closure of a set of special points is a special subvariety. Equivalently: Let $Y \subseteq S$ be a subvariety; then there exist special subvarieties $Z_1, \ldots, Z_r$ such that every special subvariety $Z \subseteq Y$ is contained in $\bigcup_{i=1}^r Z_i$.

Raynaud proved the Manin-Mumford conjecture. The GRH implies the André-Oort conjecture (Klinger-Yafaev and Ullmo-Yafaev, using some results of Clozel-Ullmo and the strategy of Edixhoven-Yafaev).

Equidistribution plus Galois implies André-Oort and Manin-Mumford.

Define strongly special subvarieties $X$ as follows:

$X = B$ abelian subvariety | special subvariety associated with a semisimple Mumford-Tate group

Equidistribution of strongly special subvarieties:

Fourier series | theorems of Ratner and Margulis-Dani

Classification (and equidistribution) of special subvarieties such that $\#\{Z^\sigma : \sigma \in \text{Gal}(\mathcal{K}/K)\}$ is bounded, where $K$ is a number field over which $S$ is defined (Ullmo-Yafaev).

$Z = P + B$ with ord$(P)$ bounded | “$T$-special subvarieties” with finitely many $T$’s, |
| where $T$ is the center of the Mumford-Tate group |

2. Hindry’s proof of Manin-Mumford revisited

2.1. Equidistribution. Let $A = \mathbb{C}^n/\Gamma$. Let $\mu_A$ be the associated normalized Haar measure. Let $Z = B + P$ and let $\mu_Z$ be the associated probability measure.
Proposition 2.1. Let $Z_n = B_n + P_n$ be a sequence of special subvarieties with $\text{ord}(P_n)$ bounded. Then there exists a special subvariety $Z = P + B$ and a subsequence $Z_{n_k}$ such that $\mu_{Z_{n_k}}$ converges weakly to $\mu_Z$ and $Z_{n_k} \subseteq Z$ for all sufficiently large $k$.

Sketch of proof. Let $G_Q = \mathbb{Q}^n$. Let $\Gamma = \mathbb{Z}^n$. Let $X = \mathbb{Z}^n \setminus \mathbb{R}^n$. Let $\pi : \mathbb{R}^n \to X$ be the projection. A (weakly) special subvariety of $X$ is by definition of the form $Z = \pi(H \otimes \mathbb{R})$ for a $\mathbb{Q}$-vector subspace of $H_Q \subseteq G_Q$. Such a $Z$ is endowed with a canonical probability measure. □

Lemma 2.2. Let $Z_n$ be a sequence of (weakly) special subvarieties. Then there exists a (weakly) special subvariety $Z \subset X$ such that $\mu_{Z_{n_k}}$ converges weakly to $\mu_Z$ and $Z_{n_k} \subseteq Z$ for all $k \gg 1$.

Proof. Weyl’s criterion say that we need to check only that for all complex characters $\chi$ of $X$,
$$\int_X \chi_{\mu_{Z_{n_k}}} \to \int_X \chi_{\mu_Z}.$$ □

We have $A = \Gamma \setminus \mathbb{C}^m \sim \mathbb{Z}^{2m} \setminus \mathbb{R}^{2m}$. The abelian subvarieties of $A$ are a subset of the set of weakly special subvarieties of $\mathbb{Z}^{2m} \setminus \mathbb{R}^{2m}$. We need only check that if $B_n$ is a sequence of special subvarieties of $A$ such that $\mu_{B_{n_k}} \to \mu_Z$, then $Z$ is in fact an abelian subvariety.

2.2. Galois orbits. Let $K$ be a number field. Suppose $A$ is defined over $K$. There exists a finite extension $K'$ of $K$ such that any abelian subvariety $B$ of $A$ is defined over $K'$. Without loss of generality, assume that $K' = K$.

Proposition 2.3 (Serre). For all $\epsilon > 0$, there exists a constant $C = C(A, K, \epsilon) > 0$ such that for all $x \in A_{\text{tors}}$ with $\text{ord}(x) = n$,
$$\# \{x^\sigma : \sigma \in \text{Gal}(\overline{K}/K)\} \geq C n^{1-\epsilon}.$$ Let $B$ be an abelian subvariety. Then there exists an abelian subvariety $B'$ such that $A = B + B'$ with $\#(B \cap B') \ll 1$ (Bertrand); i.e. bounded in terms of $A$ only. We say that the description of $Z$ as $B + P$ is normalized if $P \in B'$ (given by Bertrand). We assume this from now on.

Remark 2.4. Let $Z_n = B_n + P_n$ (in normalized form). Then $\# \{Z_n^\sigma : \sigma \in \text{Gal}(\overline{K}/K)\} \to \infty$ if and only if $\text{ord}(P_n) \to \infty$.

Sketch of proof of Manin-Mumford using Equidistribution + Galois. Let $Y \subset A$ be a subvariety and let $Z_n = B_n + P_n \subset X$ be a Zariski dense (in $Y$) sequence of special subvarieties. Up to passing to a subsequence, at least one of the following occurs:

1. $\text{ord}(P_n)$ is bounded ($\mu_{Z_n}$ is equidistributed to $\mu_Z$, $Z_n \subset Z \subset Y$, so $Z = Y$, so $Y$ is special)
2. $\# \{Z_n^\sigma : \sigma \in \text{Gal}(\overline{K}/K)\} \to \infty$; then using Galois plus a characterization of special subvarieties, we have for all $n \gg 1$ that there exists a special subvariety $Z'_n$ with $Z_n \subseteq Z'_n \subset Y$. □
Characterization of special subvarieties: Let $X \subset A$ be a subvariety, and let $d \geq 2$ be an integer; if $[d]X \subset X$, then $X$ is a torsion subvariety (main technical moral tool in Klinger-Yafaev).

**Theorem 2.5 (Serre).** There exists $c = c(A, K)$ such that for all $x \in A_{\text{tors}}$ with $p \nmid \text{ord}(x)$, there exists $\sigma \in \text{Gal}(\overline{K}/K)$ such that

$$[p^c]x = x^\sigma.$$  

(Main use of GRH in the Shimura case.)

Suppose $Z_n \subset Y$ where $Z_n$ is special and $\#\{Z_n^\sigma : \sigma \in \text{Gal}(\overline{K}/K)\} \to \infty$. Choose $p \nmid \text{ord}(p_n)$ with $Z_n = P_n + B_n$, $p \ll \log^A(n)$, $Y \supset Y \cap [p^c]Y \supset Z_n$. Let $Y' = Y \cap [p^c]Y$. So $\deg Y' \ll \log^B(n)$. If $Y \cap [p^c]Y$ is not proper, then $Y'$ is special, and $Z_n \subset Y' = Y$. If $Y \cap [p^c]Y$ is proper, repeat with $Y'$ in place of $Y$, and iterate. Note that $Z_n = Y^{(i)} \cap [p^c]Y^{(i)}$ is not possible since $\deg(Z_n) \gg$.

We want to explain how, under GRH, Equidistribution + Galois gives the André-Oort conjecture.

**Definition 2.6.** A Shimura datum $(G_Q, X)$ where $G_Q$ is a $\mathbb{Q}$-reductive group and $X$ is a $G(\mathbb{R})$-conjugacy class of morphisms

$$h_0: S \to G_r$$

where $S = \mathbb{C}^\times$. Let $X^+$ be a connected component of $X$. Then $X^+$ is a hermitian symmetric space. If $K \subset G(A_f)$ is a compact open subgroup, then

$$\text{Sh}_K(G, X) = G(Q)\backslash X \times G(A_f)/K = \coprod_{i=1}^r \Gamma_i \backslash X^+$$

where $G(A_f) = \coprod_{i=1}^r G(Q)\alpha_i K$ and $\Gamma_i = G(Q) \cap \alpha_i K \alpha_i^{-1}$ is an arithmetic lattice. A component $S = \Gamma \backslash X^+ \subset \text{Sh}_K(G, X)$ is called a Shimura variety. Then $S$ has the structure of a quasi-projective scheme. Baily-Borel: If $\Gamma$ is torsion-free, then $S$ is smooth.

**Definition 2.7.** Call $(H_Q, X_H) \subset (G_Q, X)$ a sub-Shimura datum if $H_Q \subset G_Q$ induces $X_H \subset X$; i.e.,

$$h_0: S \to G_r$$

factors through $H_r$, and $X_H$ is the $H(\mathbb{R})$-conjugacy class of this $h_0 \in X$. A sub-Shimura variety $Z$ of $S$ is a component of the image of $\text{Sh}_{K \cap H(A_f)}(H, X_H) \hookrightarrow \text{Sh}_K(G, X)$.

**Remarks 2.8.** (1) Such a $Z$ is equipped with a canonical probability measure $\mu_Z$.

(2) $Z$ does not determine $(H, X_H)$: one can change $H$ to $\gamma H \gamma^{-1}$ for any $\gamma \in \Gamma$. We can choose for $H_Q$ the “Mumford-Tate” group on $X_H$; i.e., the smallest $\mathbb{Q}$-algebraic subgroup $H_Q'$ of $G_Q$ such that for all $h \in X_H$, $h: S \to H_r$ factors through $H_Q'$. Then $(H_Q', X_H) \subset (H_Q, X_H)$ also is a sub-Shimura datum. We fix a Shimura variety $S = \Gamma \backslash X^+$ associated to $(G_Q, X)$. We suppose $G_Q = G_Q^{\text{ad}}$.

**Definition 2.9.** Special points are special subvarieties of dimension 0, i.e., defined by a sub-Shimura datum $(H, X_H)$ with $H_Q$ a $\mathbb{Q}$-torus.

**Definition 2.10.** A strongly special subvariety is defined by $(H_Q, X_H)$ with $H_Q$ semisimple.
Definition 2.11. Fix a \( \mathbb{Q} \)-torus \( T \subset G_\mathbb{Q} \). A \( T \)-special subvariety is associated to \((H_\mathbb{Q}, X_H)\) with \( H_\mathbb{Q} = T \cdot H_\mathbb{Q}^{der} \).

In the case where \( T = \{1\} \), \( T \)-special is the same as strongly special.

Theorem 2.12 (Clozel, Ullmo). Let \( Z_n \) be a sequence of strongly special (or \( T \)-special) subvarieties. Then there exists a strongly special (or \( T \)-special) subvariety \( Z \) such that there exists a subsequence \( Z_{n_k} \) with

1. \( \mu_{Z_{n_k}} \) converges weakly to \( \mu_Z \): i.e., for all \( f \in C_b^0(S) \), \( \int_S f \mu_{Z_{n_k}} \to \int_S f \mu_Z \).
2. \( Z_{n_k} \subset Z \) for \( k \gg 1 \).

Galois orbits: Let \( F \) be a number field over which \( S \) is defined.

Theorem 2.13 (Ullmo, Yafaev). There exists \( B > 0 \) such that for all \( N > 0 \) and all \((H, X_H) \subset (G, X)\) with \( H_\mathbb{Q} = T_\mathbb{Q} H_\mathbb{Q}^{der} \) with \( \dim T > 0 \), and every associated sub-Shimura variety \( Z \),

\[
\#\{Z^\sigma : \sigma \in \text{Gal}(\bar{F}/F)\} \gg B^{N}|T_\mathbb{Q}^m/K_T|(|\log |\text{disc}(L_T)|)^N.
\]

\( K = \prod K_p, K_T = T(A_f) \cap K = \prod K_{T,p}, K_{T,p}^m \) is the maximal compact open subgroup of \( T(A_f) \), \( i(T) = \#\{p \text{ such that } K_{T,p}^m \neq K_{T,p}\} \), \( L_T \) is the splitting field of \( T \).

Theorem 2.14. Let \( Z_n \) be a sequence of special subvarieties of \( S \) such that \( \#\{Z_n^\sigma : \sigma \in \text{Gal}(\bar{F}/F)\} \) is uniformly bounded. Then there exists a torus \( T \) and a subsequence \( Z_{n_k} \) such that \( Z_{n_k} \) is a \( T \)-special subvariety.

Corollary 2.15 (Alternative: Galois/ergodic). Let \( Z_n \subset Y \subset S \) be a Zariski dense (in \( Y \)) sequence of special subvarieties. Then after passing to a subsequence, at least one of the following occurs:

1. \( \mu_{Z_n} \to \mu_Z, Z_n \subset Z \subset Y \) for all \( n \gg 1 \). Then \( Y = Z \) is special.
2. \( \#\{Z_n^\sigma : \sigma \in \text{Gal}(\bar{F}/F)\} \to \infty \). Then Klinger-Yafaev prove (under GRH) that for \( n \gg 1 \), there exists a subvariety \( Z'_n \) with \( Z_n \subsetneq Z'_n \subset Y \).

Proof of Theorem 2.14: Ratner’s theory. Fix \( G_\mathbb{Q} \) a semisimple group over \( \mathbb{Q} \). Let \( \Omega = \Gamma \backslash G(\mathbb{R})^+ \) for an arithmetic lattice \( \Gamma \subset G(\mathbb{Q}) \). We define \( \mathcal{X} \) to be the set of connected closed Lie subgroups \( H \subset G(\mathbb{R})^+ \) such that

1. \( H \cap \Gamma \) is a lattice in \( H \), so in particular \( \Gamma \backslash \Gamma H \simeq H \cap \Gamma \backslash H \) is endowed with a canonical \( H \)-invariant probability measure \( \mu_H \).
2. \( L(H) \) be the subgroup of \( H \) generated by unipotent 1-parameter subgroups of \( H \). Then \( L(H) \) acts ergodically on \( H \cap \Gamma \backslash H \) with respect to \( \mu_H \): i.e., any \( L \)-invariant \( \mu_H \)-measurable set in \( H \cap \Gamma \backslash H \) has measure 0 or 1.

Properties:

1. Let \( H \in \mathcal{X} \). Then there exists a \( \mathbb{Q} \)-algebraic group \( H_\mathbb{Q} \subset G_\mathbb{Q} \) such that \( H = H_\mathbb{Q}(\mathbb{R})^+ \):
   - \( H_\mathbb{Q} \) is the Mumford-Tate group of \( H \) (the smallest \( \mathbb{Q} \)-subgroup of \( G_\mathbb{Q} \) such that \( H \subset H_\mathbb{Q}(\mathbb{R})^+ \)).
2. If \( H = H_\mathbb{Q}(\mathbb{R})^+ \in \mathcal{X} \) \( (H_\mathbb{Q} \simeq R.H_1, \cdots, H_n) \), then \( R \) is unipotent (for all \( i \in \{1, \ldots, n\} \) \( H_i(\mathbb{R}) \) is not compact.
3. Let \( L \) be a subgroup generated by unipotent 1-parameter subgroups. Then there exists a unique \( H \in \mathcal{X} \) such that \( L \subset L(H) \subset H \) and \( L \) acts ergodically on \( H \cap \Gamma \backslash H \).
Theorem 2.16 (Ratner). Let $L$ be a group generated by unipotents. Let $H = \text{MT}(L)(\mathbb{R})^+$. Then $\Gamma\backslash G = \Gamma\backslash H = H \cap \Gamma\backslash H$.

Let $P(\Omega)$ be the set of probability measure on $\Omega = \Gamma\backslash G$ and let $Q(\Omega) = \{\mu_H : H \in \mathcal{X}\} \subset P(\Omega)$.

Theorem 2.17 (Mozer-Shah). $Q(\Omega)$ is compact: If $\mu_n \in Q(\Omega)$, there exists $H \in \mathcal{X}$ and a subsequence $\mu_{n_k} \to \mu_H$ and $H_{n_k} \subset H$ for all $k \gg 1$.

Principle of proof. Let $S = \Gamma\backslash G$ and $\Omega = \Gamma\backslash G \leftrightarrow \mathcal{X}$. Let $Z_n$ be a sequence of strongly special subvarieties corresponding to $(H_n, Q_n, X_n)$. (Deligne’s definition) $\implies H_n, Q_n \subset X$. $\mu_{H_n} \to \mu_H$ for some $H \in \mathcal{X}$, $H_n \subset H$ for $n \gg 1$.

Problem: Prove that $H_Q$ is related to Shimura varieties.

Lemma 2.18. One can reconstruct $(H, X_H)$ from $H_n^{\text{der}} \subset H_Q$.

Pass from $\Gamma\backslash G \to S = \Gamma\backslash G$. $\mu_{Z_n} = \pi_{X_n} \ast \mu_{H_n}, x_n \in X_{H_n}$ for all $x \in X$.

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