Negative values of truncations to $L(1, \chi)$

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Abstract. For fixed large $x$ we give upper and lower bounds for the minimum of $\sum_{n \leq x} \chi(n)/n$ as we minimize over all real-valued Dirichlet characters $\chi$. This follows as a consequence of bounds for $\sum_{n \leq x} f(n)/n$ but now minimizing over all completely multiplicative, real-valued functions $f$ for which $-1 \leq f(n) \leq 1$ for all integers $n \geq 1$. Expanding our set to all multiplicative, real-valued multiplicative functions of absolute value $\leq 1$, the minimum equals $-0.4553 \cdots + o(1)$, and in this case we can classify the set of optimal functions.

1. Introduction

Dirichlet’s celebrated class number formula established that $L(1, \chi)$ is positive for primitive, quadratic Dirichlet characters $\chi$. One might attempt to prove this positivity by trying to establish that the partial sums $\sum_{n \leq x} \chi(n)/n$ are all non-negative. However, such truncated sums can get negative, a feature which we will explore in this note.

By quadratic reciprocity we may find an arithmetic progression (mod $4 \prod_{p \leq x} p$) such that any prime $q$ lying in this progression satisfies $\left( \frac{q}{p} \right) = -1$ for each $p \leq x$. Such primes $q$ exist by Dirichlet’s theorem on primes in arithmetic progressions, and for such $q$ we have $\sum_{n \leq x} \left( \frac{q}{n} \right)/n = \sum_{n \leq x} \lambda(n)/n$ where $\lambda(n) = (-1)^{\Omega(n)}$ is the Liouville function. Turán [6] suggested that $\sum_{n \leq x} \lambda(n)/n$ may be always positive, noting that this would imply the truth of the Riemann Hypothesis (and previously Pólya had conjectured that the related $\sum_{n \leq x} \lambda(n)$ is non-positive for all $x \geq 2$, which also implies the Riemann Hypothesis). In [Has58] Haselgrove showed that both the Turán and Pólya conjectures are false (in fact $x = 72,185,376,951,205$ is the smallest integer $x$ for which $\sum_{n \leq x} \lambda(n)/n < 0$, as was recently determined in [BFM]). We therefore know that truncations to $L(1, \chi)$ may get negative.

Let $F$ denote the set of all completely multiplicative functions $f(\cdot)$ with $-1 \leq f(n) \leq 1$ for all positive integers $n$, let $F_1$ be those for which each $f(n) = \pm 1$, and $F_0$ be those for which each $f(n) = 0$ or $\pm 1$. Given any $x$ and any $f \in F_0$ we may find a primitive quadratic character $\chi$ with $\chi(n) = f(n)$ for all $n \leq x$ (again, by using...
quadratic reciprocity and Dirichlet’s theorem on primes in arithmetic progressions) so that, for any \( x \geq 1, \)

\[
\min_{\chi \text{ a quadratic character}} \sum_{n \leq x} \frac{\chi(n)}{n} = \delta_0(x) := \min_{f \in \mathcal{F}_0} \sum_{n \leq x} \frac{f(n)}{n}.
\]

Moreover, since \( \mathcal{F}_1 \subset \mathcal{F}_0 \subset \mathcal{F} \) we have that

\[
\delta(x) := \min_{f \in \mathcal{F}} \sum_{n \leq x} \frac{f(n)}{n} \leq \delta_0(x) \leq \delta_1(x) := \min_{f \in \mathcal{F}_1} \sum_{n \leq x} \frac{f(n)}{n}.
\]

We expect that \( \delta(x) \sim \delta_1(x) \) and even, perhaps, that \( \delta(x) = \delta_1(x) \) for sufficiently large \( x \).

Trivially \( \delta(x) \geq -\sum_{n \leq x} \frac{1}{n} = -\log x + \gamma + O(1/x) \). Less trivially \( \delta(x) \geq -1 \), as may be shown by considering the non-negative multiplicative function \( g(n) = \sum_{d \mid n} f(d) \) and noting that

\[
0 \leq \sum_{n \leq x} g(n) = \sum_{d \leq x} f(d) \left[ \frac{x}{d} \right] \leq \sum_{d \leq x} \left( x \frac{f(d)}{d} + 1 \right).
\]

We will show that \( \delta(x) \leq \delta_1(x) < 0 \) for all large values of \( x \), and that \( \delta(x) \to 0 \) as \( x \to \infty \).

**Theorem 1.** For all large \( x \) and all \( f \in \mathcal{F} \) we have

\[
\sum_{n \leq x} \frac{f(n)}{n} \geq -\frac{1}{(\log \log x)^2}.
\]

Further, there exists a constant \( c > 0 \) such that for all large \( x \) there exists a function \( f = f_x \in \mathcal{F}_1 \) such that

\[
\sum_{n \leq x} \frac{f(n)}{n} \leq -\frac{c}{\log x}.
\]

In other words, for all large \( x \),

\[
-\frac{1}{(\log \log x)^2} \leq \delta(x) \leq \delta_0(x) \leq \delta_1(x) \leq -\frac{c}{\log x}.
\]

Note that Theorem 1 implies that there exists some absolute constant \( c_0 > 0 \) such that \( \sum_{n \leq x} f(n)/n \geq -c_0 \) for all \( x \) and all \( f \in \mathcal{F} \), and that equality occurs only for bounded \( x \). It would be interesting to determine \( c_0 \) and all \( x \) and \( f \) attaining this value, which is a feasible goal developing the methods of this article.

It would be interesting to determine more precisely the asymptotic nature of \( \delta(x), \delta_0(x) \) and \( \delta_1(x) \), and to understand the nature of the optimal functions.

Instead of completely multiplicative functions we may consider the larger class \( \mathcal{F}^\ast \) of multiplicative functions, and analogously define

\[
\delta^\ast(x) := \min_{f \in \mathcal{F}^\ast} \sum_{n \leq x} \frac{f(n)}{n}.
\]

**Theorem 2.** We have

\[
\delta^\ast(x) = \left( 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{x}} \frac{\log t}{t+1} dt \right) \log 2 + o(1) = -0.4553 \ldots + o(1).
\]
If \( f^* \in \mathcal{F}^* \) and \( x \) is large then
\[
\sum_{n \leq x} \frac{f^*(n)}{n} \geq \frac{1}{(\log \log x)^{\frac{1}{5}}},
\]
unless
\[
\sum_{k=1}^{\infty} \frac{1 + f^*(2^k)}{2^k} \ll (\log x)^{-\frac{1}{40}}.
\]
Finally
\[
\sum_{n \leq x} \frac{f^*(n)}{n} = \delta^*(x) + o(1)
\]
if and only if
\[
\left( \sum_{k=1}^{\infty} \frac{1 + f^*(2^k)}{2^k} \right) \log x + \sum_{3 \leq p \leq x/(1+\sqrt{e})} \sum_{k=1}^{\infty} \frac{1 - f^*(p^k)}{p^k} + \sum_{x/(1+\sqrt{e}) \leq p < x} \frac{1 + f^*(p)}{p} = o(1).
\]

2. Constructing negative values

Recall Haselgrove’s result [Has58]: there exists an integer \( N \) such that
\[
\sum_{n \leq N} \frac{\lambda(n)}{n} = -\delta
\]
with \( \delta > 0 \), where \( \lambda \in \mathcal{F}_1 \) with \( \lambda(p) = -1 \) for all primes \( p \). Let \( x > N^2 \) be large and consider the function \( f = f_x \in \mathcal{F}_1 \) defined by \( f(p) = 1 \) if \( x/(N+1) < p \leq x/N \) and \( f(p) = -1 \) for all other \( p \). If \( n \leq x \) then we see that \( f(n) = \lambda(n) \) unless \( n = p\ell \) for a (unique) prime \( p \in (x/(N+1), x/N] \) in which case \( f(n) = \lambda(\ell) = \lambda(n) + 2\lambda(\ell) \). Thus
\[
\sum_{n \leq x} \frac{f(n)}{n} = \sum_{n \leq x} \frac{\lambda(n)}{n} + 2 \sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sum_{\ell \leq x/p} \frac{\lambda(\ell)}{\ell} = \sum_{n \leq x} \frac{\lambda(n)}{n} - 2\delta \sum_{x/(N+1) < p \leq x/N} \frac{1}{p}.
\]

A standard argument, as in the proof of the prime number theorem, shows that
\[
\sum_{n \leq x} \frac{\lambda(n)}{n} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{\zeta(2s+2) x^s}{s} ds \ll \exp(-c \sqrt{\log x}),
\]
for some \( c > 0 \). Further, the prime number theorem readily gives that
\[
\sum_{x/(N+1) < p \leq x/N} \frac{1}{p} \sim \log \left( \frac{\log(x/N)}{\log(x/(N+1))} \right) \ll \frac{1}{N \log x}.
\]

Inserting these estimates in (2.1) we obtain that \( \delta(x) \leq -c/\log x \) for large \( x \) (here \( c \approx \delta/N \), as claimed in Theorem 1.

Remark 2.1. In [BFM] it is shown that one can take \( \delta = 2.0757641 \cdots 10^{-9} \)
for \( N = 72204113780255 \) and therefore we may take \( c \approx 2.87 \cdot 10^{-23} \).
3. The lower bound for $\delta(x)$

**Proposition 3.1.** Let $f$ be a completely multiplicative function with $-1 \leq f(n) \leq 1$ for all $n$, and set $g(n) = \sum_{d \mid n} f(d)$ so that $g$ is a non-negative multiplicative function. Then

$$\sum_{n \leq x} f(n) = \frac{1}{x} \sum_{n \leq x} g(n) + (1 - \gamma) \frac{1}{x} \sum_{n \leq x} f(n) + O\left( \frac{1}{(\log x)^{1/2}} \right).$$

**Proof.** Define $F(t) = \frac{1}{t} \sum_{n \leq t} f(n)$. We will make use of the fact that $F(t)$ varies slowly with $t$. From [GS03, Corollary 3], we find that if $1 \leq w \leq x/10$ then

$$|F(x)| - |F(x/w)| \ll \left( \frac{\log 2w}{\log x} \right)^{1-\frac{\gamma}{2}} \log \left( \frac{\log x}{\log 2w} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}.$$

We may easily deduce that

$$|F(x) - F(x/w)| \ll \left( \frac{\log 2w}{\log x} \right)^{1-\frac{\gamma}{2}} \log \left( \frac{\log x}{\log 2w} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}} \ll \left( \frac{\log 2w}{\log x} \right)^{1/2}.$$

Indeed, if $F(x)$ and $F(x/w)$ are of the same sign then (3.2) follows at once from (3.1). If $F(x)$ and $F(x/w)$ are of opposite signs then we may find $1 \leq v \leq w$ with $|\sum_{n \leq x/v} f(n)| \leq 1$ and then using (3.1) first with $F(x)$ and $F(x/v)$, and second with $F(x/v)$ and $F(x/w)$ we obtain (3.2).

We now turn to the proof of the Proposition. We start with

$$\sum_{n \leq x} g(n) = \sum_{n \leq x} f(d) \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d \leq x} \frac{f(d)}{d} \sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\}.$$

Now

$$\sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\} = \sum_{j \leq x/x(j+1)<d\leq x/j} f(d) \left( \frac{x}{d} - j \right) = \sum_{j \leq \log x \atop x/j \leq x/(j+1)} x \int_{x/(j+1)}^{x/j} \frac{f(d)dt}{t^2} + O\left( \frac{x}{\log x} \right).$$

From (3.2) we see that if $j \leq \log x$, and $x/(j+1) < t \leq x/j$ then

$$\sum_{x/(j+1)<d \leq t} f(d) = \left( t - \frac{x}{(j+1)} \right) \frac{1}{x} \sum_{n \leq x} f(n) + O\left( \frac{x \log (j+1)}{j (\log x)^{1/2}} \right).$$

Using this above we conclude that

$$\sum_{d \leq x} f(d) \left\{ \frac{x}{d} \right\} = \left( \sum_{n \leq x} f(n) \right) \sum_{j \leq \log x} \left( \log \left( \frac{j+1}{j} \right) - \frac{1}{j+1} \right) + O\left( \frac{x (\log \log x)^2}{(\log x)^{1/2}} \right).$$

Since $\sum_{j \leq J} (\log (1+1/j) - 1/(j+1)) = \log (J+1) - \sum_{j \leq J+1} 1/j+1 = 1 - \gamma + O(1/J)$, when we insert (3.4) into (3.3) we obtain the Proposition. □
Set $u = \sum_{p \leq x} (1 - f(p))/p$. By Theorem 2 of A. Hildebrand [Hil87] (with $f$ there being our function $g$, $K = 2$, $K_2 = 1.1$, and $z = 2$) we obtain that

$$\frac{1}{x} \sum_{n \leq x} g(n) \gg \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \ldots \right) \sigma_-(\exp \left( \sum_{p \leq x} \max(0, 1 - g(p)) \right)) + O(\exp(-(\log x)^\beta)),$$

where $\beta$ is some positive constant and $\sigma_-(\xi) = \xi\rho(\xi)$ with $\rho$ being the Dickman function. Since $\max(0, 1 - g(p)) \leq (1 - f(p))/2$ we deduce that

$$\frac{1}{x} \sum_{n \leq x} g(n) \gg (e^{-u} \log x)(e^{u/2}\rho(u/2)) + O(\exp(-(\log x)^\beta))$$

$$\gg e^{-ue^{u/2}}(\log x) + O(\exp(-(\log x)^\beta)),$$

since $\rho(\xi) = \xi^{-\xi+o(\xi)}$.

On the other hand, a special case of the main result in [HT91] implies that

$$\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll e^{-\kappa u},$$

where $\kappa = 0.32867\ldots$. Combining Proposition 3.1 with (3.5) and (3.6) we immediately get that $\delta(x) \geq -c/(\log \log x)^L$ for any $\xi < 2\kappa$. This completes the proof of Theorem 1.

**Remark 3.2.** The bound (3.5) is attained only in certain very special cases, that is, when there are very few primes $p > x^{-\xi}$ for which $f(p) = 1 + o(1)$. In this case one can get a far stronger bound than (3.6). Since the first part of Theorem 1 depends on an interaction between these two bounds, this suggests that one might be able to improve Theorem 1 significantly by determining how (3.5) and (3.6) depend upon one another.

4. Proof of Theorem 2

Given $f^* \in \mathcal{F}^*$ we associate a completely multiplicative function $f \in \mathcal{F}$ by setting $f(p) = f^*(p)$. We write $f^*(n) = \sum_{d|n} h(d)f(n/d)$ where $h$ is the multiplicative function given by $h(p^k) = f^*(p^k) - f^*(p^{k-1})$ for $k \geq 1$. Now,

$$\sum_{n \leq x} \frac{f^*(n)}{n} = \sum_{d \leq x} \frac{h(d)}{d} \sum_{m \leq x/d} \frac{f(m)}{m} \leq \sum_{d \leq (\log x)^6} \frac{h(d)}{d} \sum_{m \leq x/d} \frac{f(m)}{m} + O\left( \log x \sum_{d > (\log x)^6} \frac{|h(d)|}{d} \right).$$

Since $h(p) = 0$ and $|h(p^k)| \leq 2$ for $k \geq 2$ we see that

$$\sum_{d > (\log x)^6} \frac{|h(d)|}{d} \leq (\log x)^{-2} \sum_{d \geq 1} \frac{|h(d)|}{d} \ll (\log x)^{-2}.$$
Further, for $d \leq (\log x)^{5}$, we have (writing $F(t) = \frac{1}{\ell} \sum_{n \leq t} f(n)$ as in section 3)
\[
\sum_{x/d \leq n \leq x} \frac{f(n)}{n} = F(x) - F(x/d) + \int_{x/d}^{x} \frac{F(t)}{t} \, dt = \frac{\log d}{x} \sum_{n \leq x} f(n) + O\left(\frac{1}{(\log x)^{\frac{5}{2}}}\right),
\]
using (3.2). Using the above in (4.1) we deduce that
\[
\sum_{n \leq x} f(n) \sum_{d \leq (\log x)^{6}} \frac{h(d)}{d} - \frac{1}{x} \sum_{n \leq x} f(n) \sum_{d \leq (\log x)^{6}} \frac{h(d) \log d}{d} + O\left(\frac{1}{(\log x)^{\frac{5}{2}}}\right).
\]
Arguing as in (4.2) we may extend the sums over $d$ above to all $d$, incurring a negligible error. Thus we conclude that
\[
\sum_{n \leq x} f(n) = H_0 \sum_{n \leq x} f(n) + H_1 \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{1}{(\log x)^{\frac{5}{2}}}\right),
\]
with
\[
H_0 = \sum_{d=1}^{\infty} \frac{h(d)}{d}, \quad \text{and} \quad H_1 = - \sum_{d=1}^{\infty} \frac{h(d) \log d}{d}.
\]
Note that $H_0 = \prod_{p} (1 + h(p)/p + h(p^2)/p^2 + \ldots) \geq 0$, and that $H_0, |H_1| \ll 1$.

We now use Proposition 3.1, keeping the notation there. We deduce that
\[
\sum_{n \leq x} f(n) = H_0 \frac{1}{x} \sum_{n \leq x} g(n) + \left((1 - \gamma)H_0 + H_1\right) \frac{1}{x} \sum_{n \leq x} f(n) + O\left(\frac{1}{(\log x)^{\frac{5}{2}}}\right).
\]
If $H_0 \geq (\log x)^{-\frac{1}{6}}$ then we may argue as in section 3, using (3.5) and (3.6). In that case, we see that $\sum_{n \leq x} f(n)/n \geq -1/(\log \log x)^{\frac{5}{2}}$. Henceforth we suppose that $H_0 \leq (\log x)^{-\frac{1}{5}}$. Since
\[
H_0 \asymp 1 + \frac{h(2)}{2} + \frac{h(2^2)}{2^2} + \ldots \ll 1 + \frac{f^*(2)}{2} + \frac{f^*(2^2)}{2^2} + \ldots,
\]
we deduce that (note $h(2) = 0$)
\[
\sum_{k=2}^{\infty} \frac{2 + h(2^k)}{2^k} \ll (\log x)^{-\frac{1}{5}}.
\]
This proves the middle assertion of Theorem 2.

Writing $d = 2^k \ell$ with $\ell$ odd,
\[
H_1 = - \sum_{\ell \text{ odd}} \frac{h(\ell)}{\ell} \sum_{k=0}^{\infty} \frac{h(2^k)}{2^k} (k \log 2 + \log \ell)
\]
\[
= - \log 2 \left(\sum_{k=1}^{\infty} \frac{kh(2^k)}{2^k}\right) \sum_{\ell \text{ odd}} \frac{h(\ell)}{\ell} + O\left((\log x)^{-\frac{1}{5}}\right)
\]
\[
= 3 \log 2 \prod_{p \geq 3} \left(1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \ldots\right) + O\left(\frac{\log \log x}{(\log x)^{\frac{11}{10}}}\right).
\]
where we have used (4.4) and that
\[ \sum_{k=1}^\infty k h(2^k)/2^k = -3 + O(\log \log x / (\log x)^{\frac{3}{2}}). \]
Using these observations in (4.3) we obtain that
\[
\sum_{n \leq x} \frac{f^*(n)}{n} = H_0 \frac{1}{x} \sum_{n \leq x} g(n) + 3 \log 2 \prod_{p \geq 3} \left( 1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots \right) \frac{1}{x} \sum_{n \leq x} f(n) + o(1)
\geq 3 \log 2 \prod_{p \geq 3} \left( 1 + \frac{h(p)}{p} + \frac{h(p^2)}{p^2} + \cdots \right) \frac{1}{x} \sum_{n \leq x} f(n) + o(1).
\]

Let \( r(\cdot) \) be the completely multiplicative function with \( r(p) = 1 \) for \( p \leq \log x \), and \( r(p) = f(p) \) otherwise. Then Proposition 4.4 of [GS01] shows that
\[
\frac{1}{x} \sum_{n \leq x} f(n) = \prod_{p \leq \log x} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{f(p)}{p} \right)^{-1} \frac{1}{x} \sum_{n \leq x} r(n) + O\left( \frac{1}{(\log x)^{\frac{1}{2}}} \right).
\]

Since \( f(2) = -1 + O(H_0) \) we deduce from (4.5) and the above that
\[
\sum_{n \leq x} \frac{f^*(n)}{n} \geq \log 2 \prod_{p \geq 3} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{f(p)}{p} \right) + \frac{1}{x} \sum_{n \leq x} r(n) + o(1).
\]

One of the main results of [GS01] (see Corollary 1 there) shows that
\[
\frac{1}{x} \sum_{n \leq x} r(n) \geq 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{t+1} dt + o(1) = -0.656999 \ldots + o(1),
\]
and that equality here holds if and only if
\[
\sum_{p \leq x^{1/(1+\sqrt{\varepsilon})}} \frac{1 - r(p)}{p} + \sum_{x^{1/(1+\sqrt{\varepsilon})} \leq p \leq x} \frac{1 + r(p)}{p} = o(1).
\]
Since the product in (4.6) lies between 0 and 1 we conclude that
\[
\sum_{n \leq x} \frac{f^*(n)}{n} \geq \left( 1 - 2 \log(1 + \sqrt{e}) + 4 \int_1^{\sqrt{e}} \frac{\log t}{t+1} dt \right) \log 2 + o(1),
\]
and for equality to be possible here we must have \( (4.8) \), and in addition that the product in (4.6) is \( 1 + o(1) \). These conditions may be written as
\[
\sum_{3 \leq p \leq x^{1/(1+\sqrt{\varepsilon})}} \sum_{k=1}^\infty \frac{1 - f^*(p^k)}{p^k} + \sum_{x^{1/(1+\sqrt{\varepsilon})} \leq p \leq x} \frac{1 - f^*(p)}{p} = o(1).
\]

If the above condition holds then, by (3.5), \( \sum_{n \leq x} g(n) \gg x \log x \) and so for equality to hold in (4.5) we must have \( H_0 = o(1 / \log x) \). Thus equality in (4.9) is only possible if
\[
\left( \sum_{k=1}^\infty \frac{1 + f^*(2^k)}{2^k} \right) \log x + \sum_{3 \leq p \leq x^{1/(1+\sqrt{\varepsilon})}} \sum_{k=1}^\infty \frac{1 - f^*(p^k)}{p^k} + \sum_{x^{1/(1+\sqrt{\varepsilon})} \leq p \leq x} \frac{1 - f^*(p)}{p} = o(1).
\]

Conversely, if the above is true then equality holds in (4.5), (4.6), and (4.7) giving equality in (4.9). This proves Theorem 2.
References


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